STUDIA MATHEMATICA 238 (2) (2017)

Sharp bounds for oscillatory singular integrals on Hardy spaces

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Abstract. We establish the optimal bound for oscillatory singular integrals with polynomial phase functions of arbitrary degrees on the Hardy space $H^1(\mathbb{R}^n)$. We also prove that such a bound remains valid in the setting of weighted Hardy spaces.

1. Introduction. Let $n \in \mathbb{N}$, $d \in \mathbb{N} \cup \{0\}$, and let $\mathcal{P}(n, d)$ denote the set of all polynomials on $\mathbb{R}^n$ with real coefficients and degrees not exceeding d.

Let $K(x)$ be a Calderón–Zygmund kernel on $\mathbb{R}^n$. For any $P \in \mathcal{P}(n, d)$, we consider the following oscillatory singular integral operator:

$$T_P : f \mapsto \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x-y)} K(x-y) f(y)\, dy. \tag{1}$$

Thanks to the work of Ricci–Stein [10] and Chanillo–Christ [4], it has been known for quite some time that T_P is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, with bounds on $\|T_P\|_{L^p \to L^p}$ and $\|T_P\|_{L^1 \to L^{1,\infty}}$ dependent on K, n and d, but not on the coefficients of P. However, an example given in [9] revealed that analogous uniform boundedness of T_P does not hold on the space $H^1(\mathbb{R}^n)$.

Various estimates for $\|T_P\|_{H^1 \to H^1}$, initially for phase polynomials which have no first order terms and subsequently for more general phases, have been obtained in [9], [8] and [1]. The following theorem from [1] provides an estimate on $\|T_P\|_{H^1 \to H^1}$ for each P in $\mathcal{P}(n, d)$ in terms of its coefficients.

THEOREM 1.1 ([1]). *Let $n \in \mathbb{N}$, $d \in \mathbb{N} \cup \{0\}$ and $P(x) = \sum_{0 \le |\alpha| \le d} a_\alpha x^\alpha \in \mathcal{P}(n, d)$. Let K be a Calderón–Zygmund kernel and T_P be given as in* (1).

2010 *Mathematics Subject Classification*: Primary 42B20; Secondary 42B30.
Key words and phrases: Hardy spaces, oscillatory integrals, singular integrals.
Received 9 January 2016; revised 14 February 2017.
Published online 21 April 2017.

DOI: 10.4064/sm8496-2-2017

Then there exists a positive constant C such that

$$\|T_P f\|_{H^1(\mathbb{R}^n)} \le C\Big(1 + \frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{2\le|\alpha|\le d}|a_\alpha|^{1/|\alpha|}}\Big)\|f\|_{H^1(\mathbb{R}^n)} \tag{2}$$

for all $f \in H^1(\mathbb{R}^n)$. The constant C may depend on n, d and K, but is independent of the coefficients $\{a_\alpha\}$ of P.

In exploring the best possible bound on $\|T_P\|_{H^1\to H^1}$, an example was given in [1] to show that, as $A := \sum_{|\alpha|=1}|a_\alpha| / \sum_{2\le|\alpha|\le d}|a_\alpha|^{1/|\alpha|} \to \infty$, any bound on $\|T_P\|_{H^1\to H^1}$ must increase at least at the rate of $\log(A)$. This naturally led to the speculation on whether

$$\|T_P f\|_{H^1(\mathbb{R}^n)} \le C_{n,d}\Big(1 + \log^+\Big(\frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{2\le|\alpha|\le d}|a_\alpha|^{1/|\alpha|}}\Big)\Big)\|f\|_{H^1(\mathbb{R}^n)} \tag{3}$$

in fact holds for all $f \in H^1(\mathbb{R}^n)$.

Recently, the above bound was confirmed to be valid when the phase polynomial P is quadratic (i.e. $d = 2$):

THEOREM 1.2 ([2]). *Let $n \in \mathbb{N}$ and $P(x) = \sum_{0\le|\alpha|\le 2} a_\alpha x^\alpha$ be a quadratic polynomial in $\mathbb{R}^n$ with real coefficients. Let K be a Calderón–Zygmund kernel and T_P be given as in (1). Then there exists a positive constant C such that*

$$\|T_P f\|_{H^1(\mathbb{R}^n)} \le C\Big(1 + \log^+\Big(\frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{|\alpha|=2}|a_\alpha|^{1/2}}\Big)\Big)\|f\|_{H^1(\mathbb{R}^n)} \tag{4}$$

for all $f \in H^1(\mathbb{R}^n)$. The constant C may depend on n and K, but is independent of the coefficients $\{a_\alpha\}$ of P.

The main purpose of this paper is to establish (3) for phase polynomials of arbitrary degrees. The arguments used here are more flexible than those in [2]: not only do they allow the lifting of the restriction $d = 2$, but also the extension to weighted Hardy spaces with Muckenhoupt's A_1 weights. Namely, we have

THEOREM 1.3. *Let $n \in \mathbb{N}$, $d \in \mathbb{N}\cup\{0\}$, $w \in A_1(\mathbb{R}^n)$ and suppose $P(x) = \sum_{0\le|\alpha|\le d} a_\alpha x^\alpha \in \mathcal{P}(n,d)$. Let K be a Calderón–Zygmund kernel and T_P be given as in (1). Then there exists a positive constant C such that*

$$\|T_P f\|_{H^1_w(\mathbb{R}^n)} \le C\Big(1 + \log^+\Big(\frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{2\le|\alpha|\le d}|a_\alpha|^{1/|\alpha|}}\Big)\Big)\|f\|_{H^1_w(\mathbb{R}^n)} \tag{5}$$

for all $f \in H^1_w(\mathbb{R}^n)$. The constant C may depend on n, d, K and the A_1 constant of w, but is independent of the coefficients $\{a_\alpha\}$ of P. The bound given in (5) is the best possible in the sense that the logarithmic function cannot be replaced by any function with a slower rate of growth.

2. Some lemmas. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x with radius r, and $|B(x,r)|$ the Euclidean volume of $B(x,r)$.

The statement below is [3, Corollary 7.3], with the cube $[0,1]^n$ replaced by $[-1,1]^n$.

LEMMA 2.1. *For $n,d \in \mathbb{N}$, there exists an absolute constant $C_{n,d} > 0$ such that*

$$\Big| \int_{[-1,1]^n} e^{iP(x)}\,dx \Big| \le C_{n,d}\Big(\sum_{1\le|\alpha|\le d} |a_\alpha| \Big)^{-1/d} \tag{6}$$

for all $P(x) = \sum_{0\le|\alpha|\le d} a_\alpha x^\alpha \in \mathcal{P}(n,d)$.

The next lemma is from [10].

LEMMA 2.2. *For $n,d \in \mathbb{N}$ and $0 < \sigma < 1/d$, there exists an absolute constant $C_{n,d,\sigma} > 0$ such that*

$$\int_{B(0,1)} |P(x)|^{-\sigma} \le C_{n,d,\sigma}\Big(\sum_{0\le|\alpha|\le d} |a_\alpha| \Big)^{-\sigma} \tag{7}$$

for all $P(x) = \sum_{0\le|\alpha|\le d} a_\alpha x^\alpha \in \mathcal{P}(n,d)$.

LEMMA 2.3. *For $n \in \mathbb{N}$, $d \ge 2$ and $P(x) = \sum_{0\le|\alpha|\le d} a_\alpha x^\alpha \in \mathcal{P}(n,d)$, let the operator G_P be given by*

$$G_P f(x) = \int_{B(0,1)} e^{iP(x-y)} f(y)\,dy.$$

Then, for each $p \in [2,\infty)$, there exists a $C_{n,d,p} > 0$ such that

$$\|G_P f\|_{L^p(B(0,t))} \le C_{n,d,p} t^{(2nd-1)/(2pd)} \Big(\sum_{2\le|\alpha|\le d} |a_\alpha|^{1/|\alpha|} \Big)^{-1/(pd)} \|f\|_{L^p(\mathbb{R}^n)} \tag{8}$$

for all $f \in L^p(\mathbb{R}^n)$ and $t \ge 1$. The constant $C_{n,d,p}$ is independent of t and the coefficients of P.

Proof. Since

$$\|G_P f\|_{L^\infty(B(0,t))} \le |B(0,1)|\,\|f\|_{L^\infty(\mathbb{R}^n)},$$

it suffices to show that

$$\|G_P f\|_{L^2(B(0,t))} \le C_{n,d} t^{(2nd-1)/(4d)} \Big(\sum_{2\le|\alpha|\le d} |a_\alpha|^{1/|\alpha|} \Big)^{-1/(2d)} \|f\|_{L^2(\mathbb{R}^n)} \tag{9}$$

for all $f \in L^2(\mathbb{R}^n)$.

For $x, y \in \mathbb{R}^n$ and $t \geq 1$, let

$$(10)\quad \Omega_t(x,y) = \int_{[-t,t]^n} e^{i(P(z-x)-P(z-y))}\,dz = t^n \int_{[-1,1]^n} e^{i(P(tz-x)-P(tz-y))}\,dz.$$

Using standard multi-index notation, we have

$$\begin{aligned} &P(tz-x) - P(tz-y) \\ &\qquad = \sum_{0\le|\alpha|\le d} a_\alpha \Big(\sum_{\beta\le\alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} (tz)^\beta (x^{\alpha-\beta} - y^{\alpha-\beta}) \Big) \\ &\qquad = \sum_{0\le|\beta|\le d} t^{|\beta|} \Big(\sum_{0\le|\gamma|\le d-|\beta|} \binom{\beta+\gamma}{\beta} (-1)^{|\gamma|} a_{\beta+\gamma} (x^\gamma - y^\gamma) \Big) z^\beta. \end{aligned}$$

Let $\alpha_0 \in (\mathbb{N}\cup\{0\})^n$ be such that $2 \le |\alpha_0| \le d$ and

$$(11)\qquad |a_{\alpha_0}|^{1/|\alpha_0|} = \max\{|a_\alpha|^{1/|\alpha|} : \alpha \in (\mathbb{N}\cup\{0\})^n \text{ and } 2 \le |\alpha| \le d\}.$$

Then there exist $\beta_0, \gamma_0 \in (\mathbb{N}\cup\{0\})^n$ such that $1 \le |\beta_0| \le d-1$, $|\gamma_0| = 1$ and

$$(12)\qquad \beta_0 + \gamma_0 = \alpha_0.$$

It follows from Lemma 2.1 that

$$|\Omega_t(x,y)| \le Ct^n \Big| t^{|\beta_0|} \Big(\sum_{0\le|\gamma|\le d-|\beta_0|} \binom{\beta_0+\gamma}{\beta_0} (-1)^{|\gamma|} a_{\beta_0+\gamma} (x^\gamma - y^\gamma) \Big) \Big|^{-1/d}.$$

By (10), (12), $1/d < 1/(d-|\beta_0|)$ and Lemma 2.2, we have

$$(13)\qquad \sup_{y\in\mathbb{R}^n} \int_{B(0,1)} |\Omega_t(x,y)|\,dx \le \min\{Ct^n (t^{|\beta_0|}|a_{\alpha_0}|)^{-1/d}, 2^n t^n\}.$$

By $0 < 1/|\alpha_0| < 1$, $|\beta_0|/|\alpha_0| \ge 1/2$, $t \ge 1$, (11) and (13), we get

$$\sup_{y\in\mathbb{R}^n} \int_{B(0,1)} |\Omega_t(x,y)|\,dx \le Ct^n t^{-1/(2d)} \Big(\sum_{2\le|\alpha|\le d} |a_\alpha|^{1/|\alpha|} \Big)^{-1/d}.$$

Similarly,

$$\sup_{x\in\mathbb{R}^n} \int_{B(0,1)} |\Omega_t(x,y)|\,dy \le Ct^n t^{-1/(2d)} \Big(\sum_{2\le|\alpha|\le d} |a_\alpha|^{1/|\alpha|} \Big)^{-1/d}.$$

Thus,

$$\begin{aligned}
&\|G_P f\|^2_{L^2(B(0,t))} \\
&\quad \le \int_{[-t,t]^n} \Big(\int_{B(0,1)} e^{iP(z-x)} f(x)\,dx \Big) \Big(\int_{B(0,1)} e^{-iP(z-y)} \overline{f(y)}\,dy \Big)\,dz \\
&\quad = \int_{B(0,1)} \int_{B(0,1)} \Omega_t(x,y) f(x) \overline{f(y)}\,dx\,dy \\
&\quad \le \Big(\sup_{y\in\mathbb{R}^n} \int_{B(0,1)} |\Omega_t(x,y)|\,dx \Big)^{1/2} \Big(\sup_{x\in\mathbb{R}^n} \int_{B(0,1)} |\Omega_t(x,y)|\,dy \Big)^{1/2} \|f\|^2_{L^2(\mathbb{R}^n)} \\
&\quad \le C t^n t^{-1/(2d)} \Big(\sum_{2\le|\alpha|\le d} |a_\alpha|^{1/|\alpha|} \Big)^{-1/d} \|f\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}$$

which implies (9). ■

Next, let us recall the definition of A_p weights for $1 \le p < \infty$. Let $w(\cdot)$ be a nonnegative, locally integrable function on $\mathbb{R}^n$ and

$$w(E) = \int_E w(y)\,dy$$

for every measurable set $E \subseteq \mathbb{R}^n$.

DEFINITION 2.1. For $1 < p < \infty$, w is said to be in the *Muckenhoupt weight class* $A_p(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that

$$\Big(\frac{1}{|B|} \int_B w(y)\,dy \Big) \Big(\frac{1}{|B|} \int_B w(y)^{-1/(p-1)}\,dy \Big)^{p-1} \le C \tag{14}$$

for all balls B in $\mathbb{R}^n$. Further, w is said to be in $A_1(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that

$$\frac{w(B)}{|B|} \le C w(x) \tag{15}$$

for all balls B and a.e. $x \in B$.

The smallest such constant C in (14) or (15) is the corresponding A_p constant of w.

We recall that $A_{p_1}(\mathbb{R}^n) \subset A_{p_2}(\mathbb{R}^n)$ when $p_1 < p_2$ and

LEMMA 2.4 ([6]). *For every $w \in A_1(\mathbb{R}^n)$, there exists a $\delta \in (0,1)$ such that $w^{1+\delta} \in A_1(\mathbb{R}^n)$. Both δ and the A_1 constant of $w^{1+\delta}$ depend on n and the A_1 constant of w only.*

Let ϕ be a function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. For each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we let

$$M_\phi f(x) = \sup_{s>0} |(f * \phi_s)(x)|$$

where $\phi_s(x) = s^{-n}\phi(x/s)$.

Definition 2.2. For a nonnegative, locally integrable function w on $\mathbb{R}^n$, we define the *weighted Hardy space* $H^1_w(\mathbb{R}^n)$ by

$$H^1_w(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|M_\phi f\|_{L^1_w(\mathbb{R}^n)} < \infty\}$$

and we set $\|f\|_{H^1_w(\mathbb{R}^n)} = \|M_\phi f\|_{L^1_w(\mathbb{R}^n)} = \int_{\mathbb{R}^n} M_\phi f(x) w(x)\,dx$.

Definition 2.3. Let $w \in A_1(\mathbb{R}^n)$. A measurable function ψ on $\mathbb{R}^n$ is called an H^1_w *atom* if there exist $\zeta \in \mathbb{R}^n$ and $r > 0$ such that

$$\mathrm{supp}(\psi) \subseteq B(\zeta, r), \tag{16}$$

$$\|\psi\|_\infty \le \frac{1}{w(B(\zeta,r))}, \tag{17}$$

$$\int_{\mathbb{R}^n} \psi(x)\,dx = 0. \tag{18}$$

Lemma 2.5 ([5], [6], [12]). *Let $w \in A_1(\mathbb{R}^n)$ and $f \in H^1_w(\mathbb{R}^n)$. Then there exist H^1_w atoms $\{\psi_\nu\}$ and coefficients $\{b_\nu\}$ such that*

$$f = \sum_\nu b_\nu \psi_\nu$$

and

$$C_1 \|f\|_{H^1_w(\mathbb{R}^n)} \le \inf\Big\{\sum_\nu |b_\nu| : f = \sum_\nu b_\nu \psi_\nu\Big\} \le C_2 \|f\|_{H^1_w(\mathbb{R}^n)},$$

where C_1, C_2 are positive constants which may depend on the A_1 constant of w, but are independent of f.

Definition 2.4. A C^1 function $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called a *Calderón–Zygmund kernel* if

$$\int_{a<|x|<b} K(x)\,dx = 0 \tag{19}$$

for all $0 < a < b$ and there exists a $C_K > 0$ such that

$$|K(x)| + |x|\,|\nabla K(x)| \le C_K |x|^{-n} \tag{20}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.

We shall refer to the smallest C_K in (20) as the *CZ constant* of K.

3. Proof of the main theorem. We shall let C denote an absolute constant whose value may change from line to line.

We begin with a proposition concerning the action of T_P on a particular class of atoms.

PROPOSITION 3.1. *Let n, d, $K(x)$, $w(x)$, $P(x)$ and T_P be as in Theorem 1.3. Then there exists a positive constant C such that*

$$\|T_P\psi\|_{L^1_w(\mathbb{R}^n)} \le C\left(1+\log^+\left(\frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{2\le|\alpha|\le d}|a_\alpha|^{1/|\alpha|}}\right)\right) \tag{21}$$

for every H^1_w atom ψ which satisfies (16)–(18) with $\zeta = 0$ and $r = 1$. The constant C depends on the following parameters only: n, d, the CZ constant of K and the A_1 constant of w.

Proof. Since (21) holds trivially when $\deg(P) \le 1$, we may assume that $\deg(P) \ge 2$. Let

$$\eta(P) = \sum_{|\alpha|=1}|a_\alpha|, \qquad \theta(P) = \sum_{2\le|\alpha|\le d}|a_\alpha|^{1/|\alpha|}.$$

We also let

$$\mu = \max\{2, (\eta(P))^{-1}\}, \qquad \lambda = \max\{\mu, (\theta(P))^{-2}\}.$$

Suppose that ψ satisfies (16)–(18) with $\zeta = 0$ and $r = 1$. By (15), for $s > 1$ and $t > 0$,

$$\begin{aligned} w(B(0,st)) &\le C|B(0,st)| \operatorname*{ess\,inf}_{x\in B(0,st)} w(x) \\ &\le Cs^n|B(0,t)| \operatorname*{ess\,inf}_{x\in B(0,t)} w(x) \le Cs^n w(B(0,t)). \end{aligned} \tag{22}$$

To obtain the desired estimate for $\|T_P\psi\|_{L^1_w(\mathbb{R}^n)}$, we shall split the domain of integration into four regions: $B(0,2)$, $B(0,\mu)\setminus B(0,2)$, $B(0,\lambda)\setminus B(0,\mu)$ and $B(0,\lambda)^c$, and treat each region separately.

Since $w \in A_1(\mathbb{R}^n) \subset A_2(\mathbb{R}^n)$, it follows from (16)–(17) and the uniform boundedness of T_P on $L^2_w(\mathbb{R}^n)$ that

$$\begin{aligned} \|T_P\psi\|_{L^1_w(B(0,2))} &\le (w(B(0,2)))^{1/2}\|T_P\psi\|_{L^2_w(\mathbb{R}^n)} \\ &\le C(w(B(0,1)))^{1/2}\|\psi\|_{L^2_w(B(0,1))} \le C. \end{aligned} \tag{23}$$

By Lemma 2.4, there exists a $\delta \in (0,1)$ such that $w^{1+\delta} \in A_1(\mathbb{R}^n)$. Let

$$Q(x) = P(0) + \sum_{2\le|\alpha|\le d} a_\alpha x^\alpha.$$

It follows from [9, Theorem 1] that

$$\|T_Q\psi\|_{L^1_w(\mathbb{R}^n)} \le C. \tag{24}$$

Let $N_0 = [\log_2 \mu]$. By (16)–(17) and (20) we have

$$\begin{aligned}(25)\quad &\int_{2\le|x|\le\mu} |T_P\psi(x)-T_Q\psi(x)|w(x)\,dx \\ &\le \int_{2\le|x|\le\mu}\int_{|y|\le1} |e^{iP(x-y)}-e^{iQ(x-y)}|\,|K(x-y)|\,|\psi(y)|w(x)\,dy\,dx \\ &\le \frac{C\eta(P)}{w(B(0,1))}\int_{2\le|x|\le\mu}\frac{w(x)dx}{|x|^{n-1}} \\ &\le \frac{C\eta(P)}{w(B(0,1))}\Big(\int_{2\le|x|\le\mu}\frac{dx}{|x|^{n-1}}\Big)^{\delta/(1+\delta)}\Big(\int_{2\le|x|\le\mu}\frac{(w(x))^{1+\delta}\,dx}{|x|^{n-1}}\Big)^{1/(1+\delta)} \\ &\le C\frac{\eta(P)(\mu-2)^{\delta/(1+\delta)}}{w(B(0,1))}\Big(\sum_{j=1}^{N_0}2^{-j(n-1)}\int_{B(0,2^{j+1})}(w(x))^{1+\delta}\,dx\Big)^{1/(1+\delta)} \\ &\le C\frac{\eta(P)(\mu-2)^{\delta/(1+\delta)}}{w(B(0,1))}\Big(\sum_{j=1}^{N_0}2^{j}\operatorname*{ess\,inf}_{x\in B(0,2^{j+1})}(w(x))^{1+\delta}\Big)^{1/(1+\delta)} \\ &\le C\frac{\eta(P)(\mu-2)^{\delta/(1+\delta)}}{w(B(0,1))}\cdot 2^{N_0/(1+\delta)}\operatorname*{ess\,inf}_{x\in B(0,1)}w(x) \\ &\le C\mu^{1/(1+\delta)}(\mu-2)^{\delta/(1+\delta)}\eta(P)\le C.\end{aligned}$$

It follows from (24) and (25) that

$$\begin{aligned}(26)\quad &\|T_P\psi\|_{L^1_w(B(0,\mu)\setminus B(0,2))} \\ &\le \int_{B(0,\mu)\setminus B(0,2)} |T_P\psi(x)-T_Q\psi(x)|w(x)\,dx+\|T_Q\psi\|_{L^1_w(\mathbb{R}^n)}\le C.\end{aligned}$$

For $x\in\mathbb{R}^n$ and $y\in B(0,1)$, we have

$$(27)\qquad |e^{iP(x-y)}-e^{i(\sum_{|\alpha|=1}a_\alpha x^\alpha+Q(x-y))}|\le\min\{2,\eta(P)\}\le 4\mu^{-1}.$$

By (24) and (27), we have

$$\begin{aligned}&\|T_P\psi\|_{L^1_w(B(0,\lambda)\setminus B(0,\mu))} \\ &\quad\le \int_{\mu\le|x|\le\lambda}|e^{i(\sum_{|\alpha|=1}a_\alpha x^\alpha)}T_Q\psi(x)|w(x)\,dx \\ &\qquad+\int_{\mu\le|x|\le\lambda}|T_P\psi(x)-e^{i(\sum_{|\alpha|=1}a_\alpha x^\alpha)}T_Q\psi(x)|w(x)\,dx \\ &\quad\le C+\frac{C\mu^{-1}}{w(B(0,1))}\int_{\mu\le|x|\le\lambda}\frac{w(x)\,dx}{|x|^n} \\ &\quad= C+\frac{C\mu^{-1}}{w(B(0,1))}\Big[\frac{1}{\lambda^n}\int_{\mu\le|x|\le\lambda}w(x)\,dx+n\int_\mu^\lambda\Big(\int_{\mu\le|x|\le t}w(x)\,dx\Big)\frac{dt}{t^{n+1}}\Big]\end{aligned}$$

$$\le C + \frac{C\mu^{-1}}{w(B(0,1))}\Big(\lambda^{-n} w(B(0,\lambda)) + n\int_{\mu}^{\lambda} \frac{w(B(0,t))\,dt}{t^{n+1}}\Big)$$

$$\le C + \frac{C\mu^{-1}}{w(B(0,1))}\Big(Cw(B(0,1)) + Cw(B(0,1))\int_{\mu}^{\lambda} \frac{dt}{t}\Big)$$

$$\le C(1 + \mu^{-1}\log(\lambda/\mu)).$$

If $\mu \ge (\theta(P))^{-2}$, then $\lambda = \mu$ and $\mu^{-1}\log(\lambda/\mu) = 0$.

If $\mu < (\theta(P))^{-2}$, then $\lambda = (\theta(P))^{-2}$ and

$$\frac{\lambda}{\mu^2} \le \frac{(\theta(P))^{-2}}{(\eta(P))^{-2}} = \left(\frac{\eta(P)}{\theta(P)}\right)^2.$$

In this case,

$$\mu^{-1}\log\left(\frac{\lambda}{\mu}\right) \le \sup_{t\ge 2}\left(\frac{\log t}{t}\right) + \mu^{-1}\log\left(\frac{\lambda}{\mu^2}\right) \le C\left(1 + \log^+\left(\frac{\eta(P)}{\theta(P)}\right)\right).$$

It follows that

$$\|T_P\psi\|_{L^1_w(B(0,\lambda)\setminus B(0,\mu))} \le C\left(1 + \log^+\left(\frac{\eta(P)}{\theta(P)}\right)\right). \tag{28}$$

By (20), we have

$$\int_{|x|\ge\lambda}\Big|T_P\psi(x) - K(x)\int_{B(0,1)} e^{iP(x-y)}\psi(y)\,dy\Big|w(x)\,dx \tag{29}$$

$$\le \int_{|x|\ge 2}\int_{|y|\le 1} |K(x-y) - K(x)|\,|\psi(y)|w(x)\,dy\,dx$$

$$\le \frac{C}{w(B(0,1))}\int_{|x|\ge 2}\frac{w(x)\,dx}{|x|^{n+1}}$$

$$\le \frac{C}{w(B(0,1))}\sum_{j=1}^{\infty} 2^{-(n+1)j} w(B(0,2^{j+1})\setminus B(0,2^j))$$

$$\le C\sum_{j=1}^{\infty} 2^{-(n+1)j}2^{n(j+1)} \le C.$$

Since $(1+\delta)/\delta > 2$, it follows from (16)–(17), (20), (22) and Lemma 2.3 that

$$\int_{|x|\ge\lambda}\Big|K(x)\int_{B(0,1)} e^{iP(x-y)}\psi(y)\,dy\Big|w(x)\,dx \tag{30}$$

$$\le C\int_{|x|\ge\lambda} |G_P\psi(x)|w(x)\,\frac{dx}{|x|^n} = n\int_{\lambda}^{\infty}\Big(\int_{B(0,t)\setminus B(0,\lambda)} |G_P\psi(x)|w(x)\,dx\Big)\frac{dt}{t^{n+1}}$$

$$\le C\int_{\lambda}^{\infty}\|G_P\psi\|_{L^{(1+\delta)/\delta}(B(0,t))}\Big(\int_{B(0,t)}(w(x))^{1+\delta}\,dx\Big)^{1/(1+\delta)}\frac{dt}{t^{n+1}}$$
$$\le C(\theta(P))^{-\delta/(d(1+\delta))}\|\psi\|_{L^{(1+\delta)/\delta}(\mathbb{R}^n)}$$
$$\times\int_{\lambda}^{\infty}t^{\delta(2nd-1)/(2d(1+\delta))}t^{n/(1+\delta)}w(B(0,1))t^{-n-1}\,dt$$
$$\le C(\theta(P))^{-\delta/(d(1+\delta))}\int_{\lambda}^{\infty}t^{-1-\delta/(2d(1+\delta))}\,dt\le C(\lambda(\theta(P))^2)^{-\delta/(2d(1+\delta))}\le C.$$

By (29) and (30), we have

$$\|T_P\psi\|_{L^1_w(\mathbb{R}^n\setminus B(0,\lambda))}\le C. \tag{31}$$

Combining (23), (26), (28) and (31), we obtain (21). ∎

We recall the following result concerning Riesz transforms and Hardy spaces:

Proposition 3.2 ([7], [11], [13]). *For $1\le j\le n$, let R_j denote the jth Riesz transform, i.e.*

$$\widehat{R_jf}(\xi)=\frac{i\xi_j}{|\xi|}\hat{f}(\xi).$$

Then, for every $w\in A_1(\mathbb{R}^n)$, there exist $C_w, C_{1,w}, C_{2,w}>0$ such that

$$\|R_jf\|_{H^1_w(\mathbb{R}^n)}\le C_w\|f\|_{H^1_w(\mathbb{R}^n)} \tag{32}$$

for $1\le j\le n$ and

$$C_{1,w}\|f\|_{H^1_w(\mathbb{R}^n)}\le\|f\|_{L^1_w(\mathbb{R}^n)}+\sum_{j=1}^{n}\|R_jf\|_{L^1_w(\mathbb{R}^n)}\le C_{2,w}\|f\|_{H^1_w(\mathbb{R}^n)} \tag{33}$$

for all $f\in H^1_w(\mathbb{R}^n)$. The constants $C_w, C_{1,w}$ and $C_{2,w}$ depend on n and the A_1 constant of w only.

Proof of Theorem 1.3. Let $f\in H^1_w(\mathbb{R}^n)$, $\{b_\nu\}$ be a sequence of complex numbers and $\{\psi_\nu\}$ be a sequence of H^1_w atoms such that

$$f=\sum_{\nu}b_\nu\psi_\nu.$$

For each ν, let $\zeta_\nu\in\mathbb{R}^n$ and $r_\nu>0$ be such that $\mathrm{supp}(\psi_\nu)\subseteq B(\zeta_\nu,r_\nu)$ and $\|\psi_\nu\|_\infty\le 1/w(B(\zeta_\nu,r_\nu))$.

Let $P_\nu(x)=P(r_\nu x)$, $K_\nu(x)=r_\nu^nK(r_\nu x)$, $\tilde{\psi}_\nu(x)=r_\nu^n\psi_\nu(r_\nu x+\zeta_\nu)$ and $w_\nu(x)=w(r_\nu x+\zeta_\nu)$. Observe that, for each ν, K_ν satisfies (19)–(20) with the same CZ constant as K, and $\tilde{\psi}_\nu$ satisfies (16)–(18) with $\zeta=0$, $r=1$, and w_ν in place of w. We also point out that w_ν is an A_1 weight with the same A_1 constant as w.

By Proposition 3.1,

$$\|T_P\psi_\nu\|_{L^1_w(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} e^{iP_\nu(x-y)}K_\nu(x-y)\tilde{\psi}_\nu(y)\,dy\Big|w_\nu(x)\,dx \le C\bigg(1+\log^+\bigg(\frac{\eta(P_\nu)}{\theta(P_\nu)}\bigg)\bigg) = C\bigg(1+\log^+\bigg(\frac{\eta(P)}{\theta(P)}\bigg)\bigg),$$

which implies

$$\|T_Pf\|_{L^1_w(\mathbb{R}^n)} \le C\bigg(1+\log^+\bigg(\frac{\eta(P)}{\theta(P)}\bigg)\bigg)\sum_\nu |b_\nu|.$$

It follows from Lemma 2.5 that

$$\|T_Pf\|_{L^1_w(\mathbb{R}^n)} \le C\bigg(1+\log^+\bigg(\frac{\eta(P)}{\theta(P)}\bigg)\bigg)\|f\|_{H^1_w(\mathbb{R}^n)}. \tag{34}$$

By the translation invariance of T_P, and by (32) and (34), we have

$$\begin{aligned}\sum_{j=1}^n \|R_jT_Pf\|_{L^1_w(\mathbb{R}^n)} &= \sum_{j=1}^n \|T_PR_jf\|_{L^1_w(\mathbb{R}^n)} \\ &\le C\bigg(1+\log^+\bigg(\frac{\eta(P)}{\theta(P)}\bigg)\bigg)\sum_{j=1}^n \|R_jf\|_{H^1_w(\mathbb{R}^n)} \\ &\le C\bigg(1+\log^+\bigg(\frac{\eta(P)}{\theta(P)}\bigg)\bigg)\|f\|_{H^1_w(\mathbb{R}^n)}.\end{aligned} \tag{35}$$

Applying (33)–(35), we obtain (5). ∎

References

[1] H. Al-Qassem, L. Cheng and Y. Pan, *Estimates for oscillatory singular integrals on Hardy spaces*, Studia Math. 224 (2014), 277–289.

[2] H. Al-Qassem, L. Cheng and Y. Pan, *Logarithmic bounds for oscillatory singular integrals on Hardy spaces*, J. Funct. Spaces 2016, art. ID 1570109.

[3] A. Carbery, M. Christ and J. Wright, *Multidimensional van der Corput and sublevel set estimates*, J. Amer. Math. Soc. 12 (1999), 981–1015.

[4] S. Chanillo and M. Christ, *Weak* $(1,1)$ *bounds for oscillatory singular integrals*, Duke Math. J. 55 (1987), 141–155.

[5] R. R. Coifman, *A real variable characterization of* H^p, Studia Math. 51 (1974), 269–274.

[6] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250.

[7] C. Fefferman and E. M. Stein, H^p *spaces of several variables*, Acta Math. 129 (1972), 137–193.

[8] M. Folch-Gabayet and J. Wright, *Weak-type* $(1,1)$ *bounds for oscillatory singular integrals with rational phases*, Studia Math. 210 (2012), 57–76.

[9] Y. Hu and Y. Pan, *Boundedness of oscillatory singular integrals on Hardy spaces*, Ark. Mat. 30 (1992), 311–320.

[10] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals. I. Oscillatory integrals*, J. Funct. Anal. 73 (1987), 179–194.
[11] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.
[12] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. 1381, Springer, 1989.
[13] R. L. Wheeden, *A boundary value characterization of weighted H^1*, Enseign. Math. 22 (1976), 121–134.

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