

*EXISTENCE OF CYCLE-FINITE ALGEBRAS OF INFINITE
REPRESENTATION TYPE WITHOUT DIRECTING PROJECTIVE
OR INJECTIVE MODULES*

BY

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Dedicated to Daniel Simson on the occasion of his 75th birthday

Abstract. We solve an open problem concerning the existence of cycle-finite algebras of infinite representation type for which all indecomposable projective modules and indecomposable injective modules are nondirecting (lie on oriented cycles of indecomposable modules). We prove that there exist such algebras having large numbers of almost acyclic Auslander–Reiten components with finite cyclic multisections.

1. Introduction. Throughout the paper, by an *algebra* we mean a basic indecomposable finite-dimensional algebra over a fixed algebraically closed field K . For an algebra A , we denote by $\text{mod } A$ the category of finite-dimensional right A -modules and by $\text{ind } A$ the full subcategory of $\text{mod } A$ formed by the indecomposable modules. The radical rad_A of $\text{mod } A$ is the ideal generated by all nonisomorphisms between modules in $\text{ind } A$, and the infinite radical rad_A^∞ is the intersection of all powers rad_A^i , $i \geq 1$, of rad_A . By a result of Auslander [4], $\text{rad}_A^\infty = 0$ if and only if A is of finite representation type, that is, $\text{ind } A$ admits only a finite number of pairwise nonisomorphic modules. On the other hand, if A is of infinite representation type, then $(\text{rad}_A^\infty)^2 \neq 0$, by a result proved in [7]. Moreover, we denote by Γ_A the Auslander–Reiten quiver of A and by τ_A the Auslander–Reiten translation $D \text{Tr}$. We do not distinguish between a module in $\text{ind } A$ and the vertex of Γ_A corresponding to it.

A prominent role in the representation theory of algebras is played by cycles of indecomposable modules. Recall that a *cycle* in $\text{ind } A$ is a sequence

$$M_0 \xrightarrow{f_1} M_1 \rightarrow \cdots \rightarrow M_{r-1} \xrightarrow{f_r} M_r = M_0,$$

2010 *Mathematics Subject Classification*: Primary 16G10, 16G70; Secondary 16G60.

Key words and phrases: tame algebra, tubular algebra, replicated algebra, cycle of modules, Auslander–Reiten quiver, component quiver.

Received 10 January 2017.

Published online 24 April 2017.

$r \geq 1$, of nonzero nonisomorphisms in $\text{ind } A$ [21], and such a cycle is said to be *finite* if the homomorphisms f_1, \dots, f_r do not belong to rad_A^∞ (see [2, 3]). We note that every cycle of irreducible homomorphisms between modules in $\text{ind } A$ is finite.

Following Ringel [21], a module M in $\text{ind } A$ that does not lie on a cycle in $\text{ind } A$ is said to be *directing*. It has been proved independently in [18, 26] that the Auslander–Reiten quiver Γ_A of an algebra A admits at most finitely many τ_A -orbits containing directing modules. Moreover, if all modules in $\text{ind } A$ are directing, then A is of finite representation type [21].

In general, in order to obtain information on a module M in $\text{ind } A$, it is natural to study properties of cycles in $\text{ind } A$ containing M . Following [2, 3], by a *cycle-finite algebra* we mean an algebra A for which all cycles in $\text{ind } A$ are finite. The cycle-finite algebras form a wide class of algebras of tame representation type containing the following distinguished classes of algebras: the algebras of finite representation type, the tame tilted algebras [10, 21], the tame double tilted algebras [19], the tame generalized double tilted algebras [20], the tubular algebras [21], the tame quasi-tilted algebras [30], the tame generalized multicoil algebras [15], and the strongly simply connected algebras of polynomial growth [29]. The study of cycle-finite algebras has attracted much attention. We refer to [5, 6, 10, 11, 12, 13, 16, 28, 29, 32, 34, 35, 36] for some results on cycle-finite algebras and their module categories.

Let A be an algebra. Following [14], we denote by ${}_c\Gamma_A$ the *cyclic quiver* of A obtained from Γ_A by removing all modules not lying on oriented cycles and the arrows attached to them. Then the connected components of the translation quiver ${}_c\Gamma_A$ are called *cyclic components* of Γ_A . It has been proved in [14] that two modules X and Y in $\text{ind } A$ belong to the same cyclic component of Γ_A if and only if there is an oriented cycle in Γ_A passing through X and Y . It is known that every finite cyclic component of Γ_A contains a projective module and an injective module (see [13]). We also note that if A is a cycle-finite algebra then every cycle in $\text{ind } A$ has a refinement to a cycle of irreducible homomorphisms in $\text{ind } A$, and consequently consists of modules of a cyclic component of Γ_A . We refer to [13] for the structure of the support algebras of cyclic components over cycle-finite algebras. A subquiver Γ of Γ_A is said to be *almost acyclic* if its cyclic part ${}_c\Gamma$ is finite.

It has been proved recently in [5] that every cycle-finite algebra of infinite representation type contains infinitely many directing modules. Then the following question occurred naturally: does every cycle-finite algebra of infinite representation type contain at least one directing projective module or a directing injective module? The main aim of the paper is to exhibit a family of cycle-finite algebras of infinite representation type with all

indecomposable projective modules and indecomposable injective modules nondirecting. Moreover, we show that there are such algebras with an arbitrarily large number of almost acyclic Auslander–Reiten components having finite cyclic multisections.

The following theorem is the main result of the paper.

THEOREM. *Let s, m_1, \dots, m_s be positive integers and A_0, A_1, \dots, A_s triangular algebras of finite representation type. Then there exist a tubular algebra B and a cycle-finite algebra $A = A(m_1, \dots, m_s, B, A_0, A_1, \dots, A_s)$ such that:*

- (i) *Every indecomposable projective module in $\text{mod } A$ is nondirecting.*
- (ii) *Every indecomposable injective module in $\text{mod } A$ is nondirecting.*
- (iii) *For each $k \in \{1, \dots, s\}$, the m_k -replicated algebra $B^{(m_k)}$ of B is a quotient algebra of A , and $\Gamma_{B^{(m_k)}}$ is a full translation subquiver of Γ_A .*
- (iv) *For each $t \in \{0, 1, \dots, s\}$, A_t is a quotient algebra of A , and Γ_{A_t} is a full translation subquiver of Γ_A .*
- (v) *$\Gamma_A = \mathcal{C}_0 \cup \mathcal{X}_1 \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_{s-1} \cup \mathcal{X}_s \cup \mathcal{C}_s$, where:*
 - (1) *For each $k \in \{1, \dots, s\}$, \mathcal{X}_k is the cyclic part ${}_c\Gamma_{B^{(m_k)}}$ of $\Gamma_{B^{(m_k)}}$ containing one projective noninjective module, one injective nonprojective module, and $10(m_k - 1)$ projective-injective modules.*
 - (2) *For each $t \in \{0, 1, \dots, s\}$, \mathcal{C}_t is a generalized standard almost acyclic component having a finite cyclic multisection $\Delta^{(t)}$ containing Γ_{A_t} .*
 - (3) *For each $t \in \{0, \dots, s - 1\}$, the unique postprojective component $\mathcal{P}^{B^{(m_{t+1})}}$ of $\Gamma_{B^{(m_{t+1})}}$ is a full translation subquiver of \mathcal{C}_t , which is closed under successors in \mathcal{C}_t .*
 - (4) *For each $t \in \{1, \dots, s\}$, the unique preinjective component $\mathcal{Q}^{B^{(m_t)}}$ of $\Gamma_{B^{(m_t)}}$ is a full translation subquiver of \mathcal{C}_t , which is closed under predecessors in \mathcal{C}_t .*
 - (5) *$\mathcal{C}_0 = \Delta^{(0)} \cup \mathcal{P}^{B^{(m_1)}}$, $\mathcal{C}_s = \mathcal{Q}^{B^{(m_s)}} \cup \Delta^{(s)}$, and $\mathcal{C}_t = \mathcal{Q}^{B^{(m_t)}} \cup \Delta^{(t)} \cup \mathcal{P}^{B^{(m_{t+1})}}$ for $t \in \{1, \dots, s - 1\}$.*
- (vi) *The component quiver Σ_A of A is acyclic, \mathcal{C}_0 is a unique source of Σ_A , and \mathcal{C}_s is a unique sink of Σ_A .*
- (vii) *For each $t \in \{1, \dots, s - 1\}$, \mathcal{C}_t weakly separates $\bigcup_{j=1}^t (\mathcal{C}_{j-1} \cup \mathcal{X}_j)$ from $\bigcup_{k=t+1}^s (\mathcal{X}_k \cup \mathcal{C}_k)$.*

The construction of the algebras $A = A(m_1, \dots, m_s, B, A_0, A_1, \dots, A_s)$ occurring in the above theorem depends heavily on the discovery of an exceptional tubular algebra B of tubular type $(2, 3, 6)$, for which we may apply finite cyclic enlargements and gluing techniques developed in our paper [13]. In particular, all these algebras have tubular quotient algebras, and consequently are algebras of infinite Krull–Gabriel dimension [8]. We also mention

that, by the main result of [32], the class of cycle-finite algebras of finite Krull–Gabriel dimension coincides with the class of cycle-finite domestic algebras, or equivalently, with the class of cycle-finite algebras with nilpotent infinite radical (see [28]).

We ask the following.

QUESTION 1. *Let A be a cycle-finite algebra of finite Krull–Gabriel dimension. Is it true that $\text{ind } A$ admits a directing projective module or a directing injective module?*

It would also be interesting to know the following (see Section 9).

QUESTION 2. *Let A be a cycle-finite algebra of global dimension at most four. Is it true that $\text{ind } A$ admits a directing projective module or a directing injective module?*

This paper is organized as follows. In Sections 2 and 3 we recall some concepts and facts from representation theory, which are necessary for further considerations. In Section 4 we introduce mono sink-source triangular algebras of finite representation type, forming one of the constituents of our construction of algebras occurring in the main theorem. Sections 5 and 6 are devoted to proving some general results on the structure of the Auslander–Reiten quivers of finite cyclic enlargements and cyclic gluings of admissible algebras, using the mono sink-source triangular algebras of finite representation type. In Section 7 we introduce the exceptional tubular algebra B of type $(2, 3, 6)$ and describe the structure of the Auslander–Reiten quivers of the replicated algebras of B . In Section 8 we present the proof of the main result of the paper. The aim of the final Section 9 is to present examples illustrating the main theorem and the questions posed.

For basic background on representation theory relevant to the paper we refer to the books [1, 21, 22, 23].

2. Preliminaries. Let A be an algebra. Since A is assumed to be basic and indecomposable, it is isomorphic to a bound quiver algebra KQ/I , where $Q = Q_A$ is a finite connected quiver, called the *Gabriel quiver* of A , and I is an admissible ideal in the path algebra KQ of Q over K (see [1, Chapter II]). For each vertex i of Q , we denote by $P_A(i)$, $I_A(i)$, $S_A(i)$ respectively the indecomposable projective module, the indecomposable injective module, and the simple module in $\text{mod } A$ associated to i (see [1, Chapter III]). We will also write P_i , I_i , S_i instead of $P_A(i)$, $I_A(i)$, $S_A(i)$, in case no confusion is created. Moreover, A is said to be *triangular* if the quiver Q_A is acyclic.

Following [21], a component \mathcal{T} of an Auslander–Reiten quiver Γ_A without multiple arrows is said to be a *tube* if its geometric realization $|\mathcal{T}|$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}_0^+$, where \mathbb{S}^1 is the unit circle and \mathbb{R}_0^+ is the set of nonnegative real numbers. A tube \mathcal{T} is said to be *stable* if it contains neither

a projective module nor an injective module. It is known that every stable tube \mathcal{T} of Γ_A is of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ for some positive integer r , called the *rank* of \mathcal{T} . A tube \mathcal{T} of Γ_A without injective modules is said to be a *ray tube*. Dually, a tube \mathcal{T} of Γ_A without projective modules is said to be a *coray tube*. We note that the cyclic part ${}_c\mathcal{T}$ of a tube in Γ_A is infinite and cofinite in \mathcal{T} .

Following [25], a component \mathcal{C} of an Auslander–Reiten quiver Γ_A is said to be *generalized standard* if $\text{rad}_A^\infty(X, Y) = 0$ for all modules X and Y in \mathcal{C} . We note that if \mathcal{C} is a generalized standard component of Γ_A and $f : X \rightarrow Y$ is a nonisomorphism between two modules X and Y in \mathcal{C} then f is a finite sum of compositions of irreducible homomorphisms between indecomposable modules in \mathcal{C} . Moreover, by [25, Theorem 2.3], every generalized standard component \mathcal{C} of Γ_A is almost periodic, that is, all but finitely many τ_A -orbits in \mathcal{C} are periodic.

Let A be an algebra and $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$ be a family of components in Γ_A . Then \mathcal{C} is said to be a *weakly separating family* in $\text{mod } A$ if the components of Γ_A split into three disjoint families \mathcal{P}^A , $\mathcal{C}^A = \mathcal{C}$ and \mathcal{Q}^A such that the following conditions are satisfied:

- (S1) \mathcal{C}^A is a family of pairwise orthogonal generalized standard components;
- (S2) $\text{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\text{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0$, $\text{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0$;
- (S3) any homomorphism from \mathcal{P}^A to \mathcal{Q}^A in $\text{mod } A$ factors through the additive category $\text{add}(\mathcal{C}^A)$ of \mathcal{C}^A .

We note that a weakly separating family $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$ of components in Γ_A is a separating family in $\text{mod } A$ in the sense of [15] if the family \mathcal{C} is *sincere*, that is, every simple module in $\text{mod } A$ is a composition factor of a module in \mathcal{C} .

We end this section with the concept of the *component quiver* Σ_A of an algebra A , introduced in [27]. The vertices of Σ_A are the components of Γ_A , and two components \mathcal{C} and \mathcal{D} of Γ_A are connected in Σ_A by an arrow $\mathcal{C} \rightarrow \mathcal{D}$ if $\text{rad}_A^\infty(X, Y) \neq 0$ for some modules X in \mathcal{C} and Y in \mathcal{D} . Observe that a component \mathcal{C} of Γ_A is generalized standard if and only if Σ_A has no loop at \mathcal{C} . Moreover, every algebra A with acyclic component quiver Σ_A is a cycle-finite algebra. However, there are cycle-finite algebras A for which the component quiver Σ_A is not acyclic (see [11, Example 7.4]).

3. Almost acyclic Auslander–Reiten components. Let A be an algebra and \mathcal{C} be a component of Γ_A . Following [20], a full connected subquiver Δ of \mathcal{C} is said to be a *multisection* in \mathcal{C} if the following conditions are satisfied:

- (1) Δ is almost acyclic.
- (2) Δ is convex in \mathcal{C} .

- (3) For each τ_A -orbit \mathcal{O} in \mathcal{C} , we have $1 \leq |\Delta \cap \mathcal{O}| < \infty$.
- (4) $|\Delta \cap \mathcal{O}| = 1$ for all but finitely many τ_A -orbits \mathcal{O} in \mathcal{C} .
- (5) No proper full convex subquiver of Δ satisfies (1)–(4).

The following characterization of almost acyclic Auslander–Reiten components was established in [20, Theorem 2.5].

THEOREM 3.1. *Let A be an algebra and \mathcal{C} be a component of Γ_A . Then \mathcal{C} is almost acyclic if and only if \mathcal{C} admits a multisection Δ .*

Let A be an algebra, \mathcal{C} a component of Γ_A , and Δ be a multisection in \mathcal{C} . The following full subquivers of \mathcal{C} have been introduced in [20, Section 2]:

$$\begin{aligned} \Delta'_l &= \{X \in \Delta; \text{there is a nonsectional path in } \mathcal{C} \\ &\qquad\qquad\qquad \text{from } X \text{ to a projective module } P\}, \\ \Delta'_r &= \{X \in \Delta; \text{there is a nonsectional path in } \mathcal{C} \\ &\qquad\qquad\qquad \text{from an injective module } I \text{ to } X\}, \\ \Delta''_l &= \{X \in \Delta'_l; \tau_A^{-1}X \notin \Delta'_l\}, \quad \Delta''_r = \{X \in \Delta'_r; \tau_AX \notin \Delta'_r\}, \\ \Delta_l &= (\Delta \setminus \Delta'_r) \cup \tau_A\Delta''_r, \quad \Delta_r = (\Delta \setminus \Delta'_l) \cup \tau_A^{-1}\Delta''_l, \quad \Delta_c = \Delta'_l \cap \Delta'_r. \end{aligned}$$

Then Δ_l is called the *left part* of Δ , Δ_r the *right part* of Δ , and Δ_c the *core* of Δ . We note that Δ_l and Δ_r are not necessarily subquivers of Δ .

The following proposition from [20, Proposition 2.4] describes basic properties of multisections.

PROPOSITION 3.2. *Let A be an algebra and \mathcal{C} be a component in Γ_A with a multisection Δ . Then:*

- (i) *Every cycle in \mathcal{C} lies in Δ_c .*
- (ii) *Δ_c is finite.*
- (iii) *Every indecomposable module X in \mathcal{C} lies in Δ_c , or is a predecessor of Δ_l in \mathcal{C} , or is a successor of Δ_r in \mathcal{C} .*

We also mention the following fact proved in [20, Proposition 2.11].

PROPOSITION 3.3. *Let A be an algebra, \mathcal{C} a component of Γ_A , and Δ, Σ multisections in \mathcal{C} . Then $\Delta_c = \Sigma_c$.*

It would be interesting to describe the structure of almost acyclic components \mathcal{C} of an Auslander–Reiten quiver Γ_A which admit a cyclic multisection Δ . Observe that then \mathcal{C} is a component with finitely many τ_A -orbits, $\Delta = \Delta_c$, and hence Δ is a unique multisection of \mathcal{C} . In particular, it is important to know whether there exist algebras A whose Auslander–Reiten quiver Γ_A admits a large number of components with cyclic multisections, being also all finite components of ${}_c\Gamma_A$. The main result of this paper provides a solution of this problem.

4. Mono sink-source triangular algebras. An algebra Λ is said to be a *mono sink-source triangular algebra* if the quiver Q_Λ is acyclic and has one sink and one source.

LEMMA 4.1. *Let Λ be a mono sink-source triangular algebra, with a the sink and b the source of Q_Λ . Then for any module M in $\text{ind } \Lambda$ there exists a path in $\text{ind } \Lambda$ from S_a to S_b passing through M .*

Proof. Let c be the vertex of Q_Λ from the support of M . Since Q_Λ is acyclic, there are in Q_Λ paths

$$\begin{aligned} c &= a_m \rightarrow a_{m-1} \rightarrow \cdots \rightarrow a_1 \rightarrow a_0 = a, \\ b &= b_0 \rightarrow b_1 \rightarrow \cdots \rightarrow b_{n-1} \rightarrow b_n = c. \end{aligned}$$

Then there exist in $\text{ind } \Lambda$ paths

$$\begin{aligned} S_a &= P_{a_0} \rightarrow P_{a_1} \rightarrow \cdots \rightarrow P_{a_{m-1}} \rightarrow P_{a_m} = P_c, \\ I_c &= I_{b_n} \rightarrow I_{b_{n-1}} \rightarrow \cdots \rightarrow I_{b_1} \rightarrow I_{b_0} = S_b. \end{aligned}$$

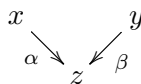
Clearly, we have $\text{Hom}_\Lambda(P_c, M) \neq 0$ and $\text{Hom}_\Lambda(M, I_c) \neq 0$. Therefore, there exists a path in $\text{ind } \Lambda$ from S_a to S_b passing through M . ■

PROPOSITION 4.2. *Let A be a triangular algebra of finite representation type. Then there exists a mono sink-source triangular algebra Λ of finite representation type satisfying the following conditions:*

- (i) A is a quotient algebra of Λ .
- (ii) Q_A is a full convex subquiver of Q_Λ .
- (iii) Γ_A is a full convex translation subquiver of Γ_Λ .

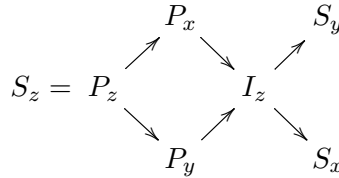
Proof. Let $A = KQ_A/I_A$ be a bound quiver presentation of A . Since A is a triangular algebra, the quiver Q_A contains at least one sink and at least one source. We denote by $u(A)$ the number of sinks of Q_A , by $v(A)$ the number of sources of Q_A , and set $w(A) = u(A) + v(A)$. Note that if $w(A) = 2$ then A is a mono sink-source algebra. Therefore, assume that $w(A) \geq 3$. We will prove that there exists a triangular algebra B of finite representation type such that $w(B) = w(A) - 1$, Q_A is a full convex subquiver of Q_B , and Γ_A is a full convex translation subquiver of Γ_B . We have two cases to consider.

(1) Assume that $u(A) \geq 2$. Take two arbitrary sinks of Q_A , say x and y . We define the algebra B as follows. The quiver Q_B of B is obtained from the quiver Q_A by adding a vertex z and two arrows α and β as follows:



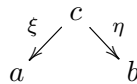
Then the algebra B is the bound quiver algebra KQ_B/I_B , where the ideal I_B is generated by all relations in KQ_A generating the ideal I_A and by the following paths of length 2: $\gamma\alpha$ for all arrows γ in Q_A with sink in x , and $\sigma\beta$

for all arrows σ in Q_A with sink in y . Then B is a triangular algebra of finite representation type whose Auslander–Reiten quiver Γ_B is obtained from the Auslander–Reiten quiver Γ_A of A by adding the following translation quiver:

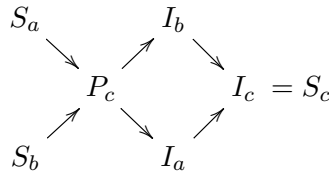


with $P_x = \tau_B S_y$, $P_y = \tau_B S_x$, $P_z = \tau_B I_z$. We also note that the sinks x and y of Q_A are replaced in Q_B by the sink z , and hence $u(B) = u(A) - 1$.

(2) Assume that $v(A) \geq 2$. Take two arbitrary sources of Q_A , say a and b . We define the algebra B as follows. The quiver Q_B of B is obtained from the quiver Q_A by adding a vertex c and two arrows ξ and η as follows:



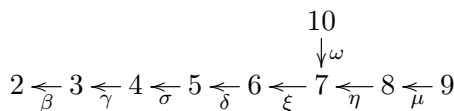
Then the algebra B is the bound quiver algebra KQ_B/I_B , where the ideal I_B is generated by all relations in KQ_A generating the ideal I_A and by the following paths of length 2: $\xi\delta$ for all arrows δ in Q_A with source in a , and $\eta\omega$ for all arrows ω in Q_A with source in b . Then B is a triangular algebra of finite representation type whose Auslander–Reiten quiver Γ_B is obtained from the Auslander–Reiten quiver Γ_A of A by adding the following translation quiver:



with $I_b = \tau_B^{-1} S_a$, $I_a = \tau_B^{-1} S_b$, $I_c = \tau_B^{-1} P_c$. We also note that the sources a and b of Q_A are replaced in Q_B by the source c , and hence $v(B) = v(A) - 1$.

In both cases, we have $w(B) = w(A) - 1$. Then, repeating the above constructions, we conclude that there exists a mono sink-source triangular algebra Λ of finite representation type satisfying the required conditions (i) and (ii). ■

5. Finite cyclic enlargements of components. Let H be the path algebra $K\Omega$ of the quiver Ω of the form



and H' be the path algebra $K\Omega'$ of the quiver Ω' of the form

$$\begin{array}{ccccccccccc}
 & & & & 1' & & & & & & \\
 & & & & \uparrow \alpha' & & & & & & \\
 2' & \xleftarrow{\beta'} & 3' & \xleftarrow{\gamma'} & 4' & \xleftarrow{\sigma'} & 5' & \xleftarrow{\delta'} & 6' & \xleftarrow{\xi'} & 7' & \xleftarrow{\eta'} & 8' & \xleftarrow{\mu'} & 9'
 \end{array}$$

We note that H and H' are hereditary algebras of Euclidean type $\widetilde{\mathbb{E}}_8$, and H' is isomorphic to the opposite algebra H^{op} of H .

An algebra C is said to be *left admissible* if the hereditary algebra H is a quotient algebra of C and the preinjective component \mathcal{Q}^H of Γ_H is a component of Γ_C . Dually, an algebra D is said to be *right admissible* if the hereditary algebra H' is a quotient algebra of D and the postprojective component $\mathcal{P}^{H'}$ of $\Gamma_{H'}$ is a component of Γ_D .

THEOREM 5.1. *Let C be a left admissible algebra and Λ be a mono sink-source triangular algebra of finite representation type. Then there exists an algebra $A = C * \Lambda$ such that the Auslander–Reiten quiver Γ_A of A has a decomposition*

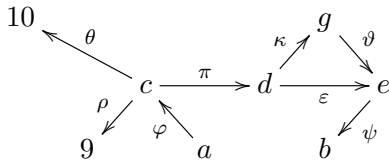
$$\Gamma_A = \Gamma_C^{(l)} \cup \mathcal{C},$$

and the following statement hold:

- (i) C and Λ are quotient algebras of A .
- (ii) $\Gamma_C^{(l)}$ consists of all components of Γ_C different from the preinjective component \mathcal{Q}^H of the Auslander–Reiten quiver Γ_H of the hereditary quotient algebra H of C .
- (iii) \mathcal{C} is an almost acyclic component with a cyclic multisection Δ such that the following conditions are satisfied:
 - (1) The left part Δ_l of Δ consists of the indecomposable injective H -modules, and the preinjective component \mathcal{Q}^H of Γ_H is a full translation subquiver of \mathcal{C} formed by all predecessors of Δ_l in \mathcal{C} .
 - (2) The right part Δ_r of Δ is empty.
 - (3) The Auslander–Reiten quiver Γ_Λ of Λ is a full translation subquiver of the cyclic part $\Delta = \Delta_c$ of \mathcal{C} .
- (iv) \mathcal{C} is a sink of the component quiver Σ_A of A .

Proof. Let $C = KQ_C/I_C$ be a bound quiver presentation of C . Since C is a left admissible algebra, the quiver Q_H of the hereditary algebra H is a full convex subquiver of Q_C , and clearly $Q_H \cap I_C = 0$. Moreover, let $\Lambda = KQ_\Lambda/I_\Lambda$ be a bound quiver presentation of Λ , and let a be the unique sink and b be the unique source of Q_Λ . Observe that the Auslander–Reiten quiver Γ_Λ of Λ is a finite translation quiver having a unique source S_a and a unique sink S_b . The quiver Q_A of $A = C * \Lambda$ is obtained from the disjoint

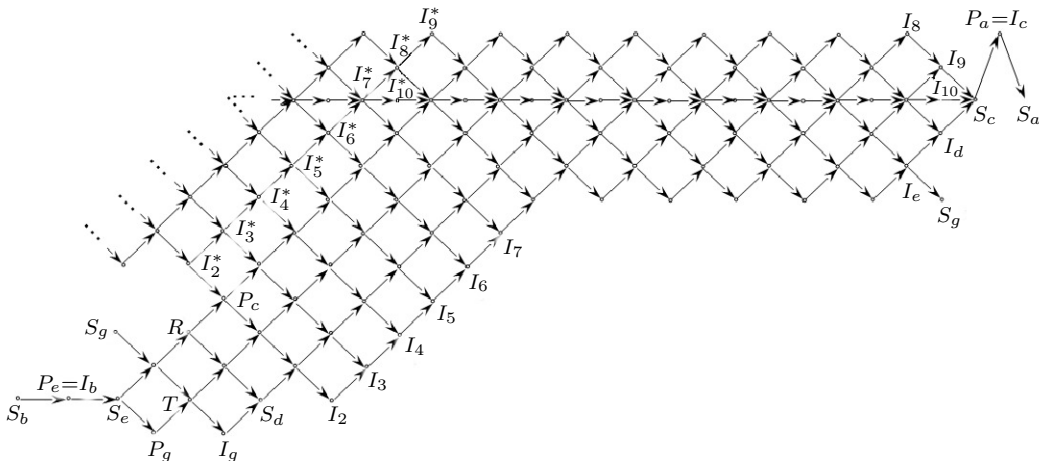
union of the quivers Q_C and Q_A by adding the following connecting quiver:



Then the algebra A is defined as the bound quiver algebra $A = KQ_A/I_A$, where I_A is the ideal in KQ_A generated by the generating elements of I_C and I_A , and the following elements: $\theta\omega - \rho\mu\eta$, $\varphi\rho$, $\varphi\theta$, $\varphi\pi$, $\pi\kappa$, $\kappa\vartheta$, $\varepsilon\psi$, $\vartheta\psi$, $\iota\varphi$ for all arrows ι in Q_A ending at a , and $\psi\varepsilon$ for all arrows ε in Q_A starting from b . We also observe that $\Gamma_C = \Gamma_C^{(l)} \cup \mathcal{Q}^H$, and \mathcal{Q}^H is a sink of the component quiver Σ_C of C . Then, applying [13, Section 6], we conclude that Γ_A has a decomposition

$$\Gamma_A = \Gamma_C^{(l)} \cup \mathcal{C},$$

where \mathcal{C} is the almost acyclic component obtained by gluing the Auslander–Reiten quiver Γ_A of A with the following translation quiver at the common simple modules S_a and S_b :



where $I_t^* = I_C(t) = I_H(t)$ and $I_t = I_A(t)$ for $t \in \{2, \dots, 10\}$, $P_r = P_A(r)$ and $S_r = S_A(r)$ for $r \in \{a, b, c, d, e, g\}$.

We note that the preinjective component \mathcal{Q}^H of Γ_H is the full translation subquiver of \mathcal{C} given by all predecessors of the indecomposable injective H -modules I_2^*, \dots, I_{10}^* in \mathcal{C} . Moreover, the Auslander–Reiten quiver Γ_A is a full translation subquiver of \mathcal{C} . Further, it follows from Lemma 4.1 that for any module M in Γ_A there is a path in Γ_A , and hence in \mathcal{C} , from S_a to S_b passing through M . Then we conclude that all modules of \mathcal{C} which do not belong to \mathcal{Q}^H form a finite cyclic component Δ of ${}_c\Gamma_A$. Summing up, we deduce that Δ is a cyclic multisection of \mathcal{C} with $\Delta = \Delta_c$, Δ_r empty, and Δ_l given by

the indecomposable injective H -modules I_t^* , $t \in \{2, \dots, 10\}$. Clearly, Γ_A is a full translation subquiver of Δ , and \mathcal{C} is a sink of the component quiver Σ_A of A . ■

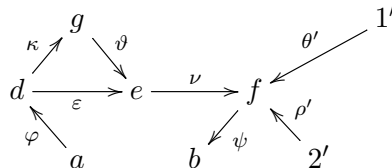
THEOREM 5.2. *Let D be a right admissible algebra and Λ be a mono sink-source triangular algebra of finite representation type. Then there exists an algebra $A = \Lambda * D$ such that the Auslander–Reiten quiver Γ_A of A has a decomposition*

$$\Gamma_A = \mathcal{C} \cup \Gamma_D^{(r)},$$

and the following statements hold:

- (i) Λ and D are quotient algebras of A .
- (ii) $\Gamma_D^{(r)}$ consists of all components of Γ_D different from the postprojective component $\mathcal{P}^{H'}$ of the Auslander–Reiten quiver $\Gamma_{H'}$ of the hereditary quotient algebra H' of D .
- (iii) \mathcal{C} is an almost acyclic component with a cyclic multisection Δ such that the following conditions are satisfied:
 - (1) The right part Δ_r of Δ consists of the indecomposable projective H' -modules, and the postprojective component $\mathcal{P}^{H'}$ of $\Gamma_{H'}$ is a full translation subquiver of \mathcal{C} formed by all successors of Δ_r in \mathcal{C} .
 - (2) The left part Δ_l of Δ is empty.
 - (3) The Auslander–Reiten quiver Γ_Λ of Λ is a full translation subquiver of the cyclic part $\Delta = \Delta_c$ of \mathcal{C} .
- (iv) \mathcal{C} is a source of the component quiver Σ_A of A .

Proof. Let $D = KQ_D/I_D$ be a bound quiver presentation of D . Since D is a right admissible algebra, the quiver $Q_{H'}$ of the hereditary algebra H' is a full convex subquiver of Q_D , and clearly $Q_{H'} \cap I_D = 0$. Moreover, let $\Lambda = KQ_\Lambda/I_\Lambda$ be a bound quiver presentation of Λ , and let a be the unique sink and b be the unique source of Q_Λ . Then the Auslander–Reiten quiver Γ_Λ of Λ is a finite translation quiver having a unique source S_a and a unique sink S_b . The quiver Q_A of $A = \Lambda * D$ is obtained from the disjoint union of the quivers Q_Λ and Q_D by adding the following connecting quiver:

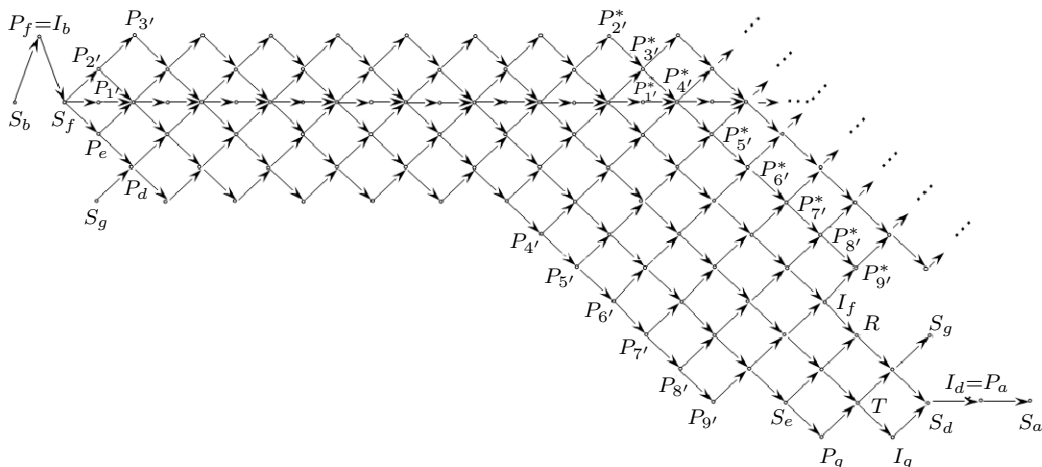


Then the algebra A is defined as the bound quiver algebra $A = KQ_A/I_A$, where I_A is the ideal in KQ_A generated by the generating elements of I_Λ and I_D , and by the elements $\varphi\kappa$, $\varphi\varepsilon$, $\kappa\vartheta$, $\vartheta\nu$, $\nu\psi$, $\theta'\psi$, $\rho'\psi$, $\alpha'\theta' - \gamma'\beta'\rho'$, $\iota\varphi$

for all arrows ι in Q_A ending at a , and $\psi\epsilon$ for all arrows ϵ in Q_A starting from b . We also observe that $\Gamma_D = \mathcal{P}^{H'} \cup \Gamma_D^{(r)}$, and $\mathcal{P}^{H'}$ is a source in the component quiver Σ_D of D . Then, applying [13, Section 6], we conclude that Γ_A has a decomposition

$$\Gamma_A = \mathcal{C} \cup \Gamma_D^{(r)},$$

where \mathcal{C} is the almost acyclic component obtained by gluing the Auslander–Reiten quiver Γ_A of A and the following translation quiver via the common simple modules S_a and S_b :



where $P_t^* = P_D(t) = P_{H'}(t)$ and $P_t = P_A(t)$ for $t \in \{1', \dots, 9'\}$, $I_r = I_A(r)$ and $S_r = S_A(r)$ for $r \in \{a, b, d, e, f, g\}$.

We note that the postprojective component $\mathcal{P}^{H'}$ of $\Gamma_{H'}$ is the full translation subquiver of \mathcal{C} given by all successors of the indecomposable projective H' -modules $P_{1'}^*, \dots, P_{9'}^*$ in \mathcal{C} . Moreover, the Auslander–Reiten quiver Γ_A is a full translation subquiver of \mathcal{C} . Further, it follows from Lemma 4.1 that for any module M in Γ_A there is a path in Γ_A , and hence in \mathcal{C} , from S_a to S_b passing through M . Then we conclude that all modules in \mathcal{C} which do not belong to $\mathcal{P}^{H'}$ form a finite cyclic component Δ of ${}_c\Gamma_A$. Summing up, we deduce that Δ is a cyclic multisection of \mathcal{C} with $\Delta = \Delta_c, \Delta_l$ empty, and Δ_r given by the indecomposable projective H' -modules $P_t^*, t \in \{1', \dots, 9'\}$. Clearly, Γ_A is a full translation subquiver of Δ , and \mathcal{C} is a source of the component quiver Σ_A of A . ■

6. Finite cyclic gluings of components. The main aim of this section is to prove the following theorem.

THEOREM 6.1. *Let C be a left admissible algebra, D be a right admissible algebra, and Λ be a mono sink-source triangular algebra of finite representation type. Then there exists an algebra $A = C * \Lambda * D$ such that the*

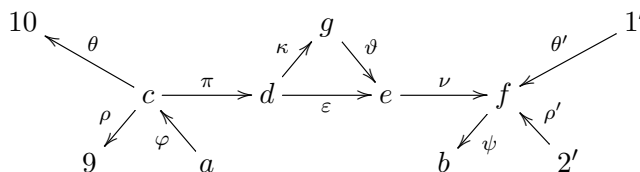
Auslander–Reiten quiver Γ_A of A has a decomposition

$$\Gamma_A = \Gamma_C^{(l)} \cup \mathcal{C} \cup \Gamma_D^{(r)},$$

and the following statement hold:

- (i) C , Λ and D are quotient algebras of A .
- (ii) $\Gamma_C^{(l)}$ consists of all components of Γ_C different from the preinjective component \mathcal{Q}^H of the Auslander–Reiten quiver Γ_H of the hereditary quotient algebra H of C .
- (iii) $\Gamma_D^{(r)}$ consists of all components of Γ_D different from the postprojective component $\mathcal{P}^{H'}$ of the Auslander–Reiten quiver $\Gamma_{H'}$ of the hereditary quotient algebra H' of D .
- (iv) \mathcal{C} is an almost cyclic component with a cyclic multisection Δ such that the following conditions are satisfied:
 - (1) The left part Δ_l of Δ consists of the indecomposable injective H -modules, and the preinjective component \mathcal{Q}^H of Γ_H is a full translation subquiver of \mathcal{C} formed by all predecessors of Δ_l in \mathcal{C} .
 - (2) The right part Δ_r of Δ consists of the indecomposable projective H' -modules, and the postprojective component $\mathcal{P}^{H'}$ of $\Gamma_{H'}$ is a full translation subquiver of \mathcal{C} formed by all successors of Δ_r in \mathcal{C} .
 - (3) The Auslander–Reiten quiver Γ_Λ of Λ is a full translation subquiver of the cyclic part $\Delta = \Delta_c$ of \mathcal{C} .
- (v) \mathcal{C} weakly separates $\Gamma_C^{(l)}$ from $\Gamma_D^{(r)}$.

Proof. Let $C = KQ_C/I_C$ and $D = KQ_D/I_D$ be bound quiver presentations of the algebras C and D . Since C is left admissible, the quiver Q_H of the hereditary quotient algebra H of C is a full convex subquiver of Q_C , and $Q_H \cap I_C = 0$. Similarly, since D is right admissible, the quiver $Q_{H'}$ of the hereditary quotient algebra H' of D is a full convex subquiver of Q_D , and $Q_{H'} \cap I_D = 0$. Further, let $\Lambda = KQ_\Lambda/I_\Lambda$ be a bound quiver presentation of Λ , and a be the unique sink and b be the unique source of Q_Λ . Then the Auslander–Reiten quiver Γ_Λ of Λ is a finite translation quiver having a unique source S_a and a unique sink S_b . The quiver Q_A of $A = C * \Lambda * D$ is obtained from the disjoint union of the quivers Q_C , Q_D and Q_Λ by adding the following connecting quiver:



sink S_b of Γ_Λ . Moreover, we have $I_s^* = I_C(s) = I_H(s)$ and $I_s = I_A(s)$ for $s \in \{2, \dots, 10\}$, $P_t^* = P_D(t) = P_{H'}(t)$ and $P_t = P_A(t)$ for $t \in \{1', \dots, 9'\}$, $P_r = P_A(r)$, $I_r = I_A(r)$ and $S_r = S_A(r)$ for $r \in \{a, b, c, d, e, f, g\}$.

We observe that the preinjective component \mathcal{Q}^H of Γ_H is the full translation subquiver of \mathcal{C} formed by all predecessors of the indecomposable injective H -modules I_2^*, \dots, I_{10}^* in \mathcal{C} , and the postprojective component $\mathcal{P}^{H'}$ of $\Gamma_{H'}$ is the full translation subquiver of \mathcal{C} formed by all successors of the indecomposable projective H' -modules $P_{1'}^*, \dots, P_{9'}^*$ in \mathcal{C} . Moreover, the Auslander–Reiten quiver Γ_Λ of Λ is a full translation subquiver of \mathcal{C} . Applying Lemma 4.1 again, we conclude that every indecomposable module M in Γ_Λ lies on a path from S_a to S_b in Γ_Λ , and hence in \mathcal{C} . Then we find that all modules in \mathcal{C} which do not belong to $\mathcal{Q}^H \cup \mathcal{P}^{H'}$ form a finite cyclic component Δ of ${}_c\Gamma_A$. Therefore, Δ is a cyclic multisection of \mathcal{C} with $\Delta = \Delta_c, \Delta_l$ given by the indecomposable injective H -modules $I_s^*, s \in \{2, \dots, 10\}$, and Δ_r given by the indecomposable projective H' -modules $P_t^*, t \in \{1', \dots, 9'\}$. Clearly, Γ_Λ is a full translation subquiver of Δ . We also observe that the quivers Q_C and Q_D are disjoint full convex subquivers of Q_A , and hence $\text{Hom}_A(X, Y) = 0$ and $\text{Hom}_A(Y, X) = 0$ for any modules X in $\text{mod } C$ and Y in $\text{mod } D$. We conclude that \mathcal{C} weakly separates $\Gamma_C^{(l)}$ from $\Gamma_D^{(r)}$. ■

7. Replicated algebras. Let B be a triangular algebra and $1_B = e_1 + \dots + e_n$ a decomposition of the identity of B into a sum of pairwise orthogonal primitive idempotents. Following [9], the *repetitive category* of B is the category \widehat{B} with the objects $i_m = (m, i)$, $m \in \mathbb{Z}$, $i \in \{1, \dots, n\}$, and the morphism spaces

$$\widehat{B}(i_m, j_r) = \begin{cases} e_j B e_i, & r = m, \\ D(e_i B e_j), & r = m + 1, \\ 0, & \text{otherwise.} \end{cases}$$

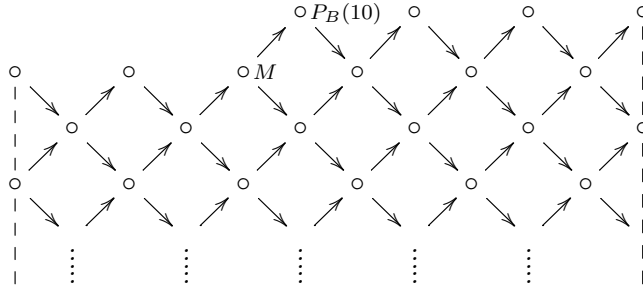
Observe that $e_j B e_i = \text{Hom}_B(e_i B, e_j B)$ and $D(e_i B e_j) = e_j D(B) e_i$ for $i, j \in \{1, \dots, n\}$. The repetitive category \widehat{B} is a locally bounded selfinjective K -category. We denote by $\nu_{\widehat{B}}$ the Nakayama automorphism of \widehat{B} defined by $\nu_{\widehat{B}}(i_m) = i_{m+1}$ for all $(m, i) \in \mathbb{Z} \times \{1, \dots, n\}$. We identify B with the full subcategory B_0 of \widehat{B} given by the objects $i_0, i \in \{1, \dots, n\}$. For each $m \in \mathbb{Z}$, we denote by B_m the full subcategory of \widehat{B} given by the objects $i_m, i \in \{1, \dots, n\}$. For a sink i of $Q_B = \{1, \dots, n\}$, the reflection $S_i^+ B$ of B at i is the full subcategory of \widehat{B} given by the objects $j_0, j \in \{1, \dots, n\} \setminus \{i\}$, and $i_1 = \nu_{\widehat{B}}(i_0)$. Moreover, we denote by $T_i^+ B$ the full subcategory of \widehat{B} given by the objects $i_0, i \in \{1, \dots, n\}$, and $i_1 = \nu_{\widehat{B}}(i_0)$. For each integer $p \geq 1$, we denote by $B_{0,p}$ the full subcategory of \widehat{B} given by the objects of

THEOREM 7.1. *The Auslander–Reiten quiver Γ_B of B has a decomposition*

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{T}_0^B \cup \left(\bigcup_{q \in \mathbb{Q}_1^0} \mathcal{T}_q^B \right) \cup \mathcal{T}_1^B \cup \mathcal{Q}^B,$$

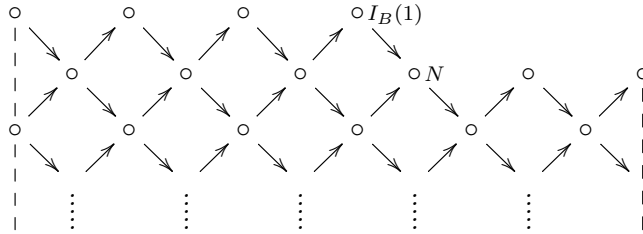
where:

- (1) \mathcal{P}^B is the postprojective component \mathcal{P}^{H_0} of Γ_{H_0} ;
- (2) \mathcal{T}_0^B is the $\mathbb{P}_1(K)$ -family $(\mathcal{T}_{0,\lambda}^B)_{\lambda \in \mathbb{P}_1(K)}$ of ray tubes obtained from the $\mathbb{P}_1(K)$ -family \mathcal{T}^{H_0} of stable tubes of tubular type $(2, 3, 5)$ by inserting in the unique stable tube of rank 5, say $\mathcal{T}_{0,0}^{H_0}$, one ray as follows:



where the vertices along the dashed vertical lines have to be identified.

- (3) For each $q \in \mathbb{Q}_1^0$, \mathcal{T}_q^B is a $\mathbb{P}_1(K)$ -family $(\mathcal{T}_{q,\lambda}^B)_{\lambda \in \mathbb{P}_1(K)}$ of stable tubes of tubular type $(2, 3, 6)$.
- (4) \mathcal{T}_1^B is the $\mathbb{P}_1(K)$ -family $(\mathcal{T}_{1,\lambda}^B)_{\lambda \in \mathbb{P}_1(K)}$ of coray tubes obtained from the $\mathbb{P}_1(K)$ -family \mathcal{T}^{H_1} of stable tubes of tubular type $(2, 3, 5)$ by inserting in the unique stable tube of rank 5, say $\mathcal{T}_{1,0}^{H_1}$, one coray as follows:



where the vertices along the dashed vertical lines have to be identified.

- (5) \mathcal{Q}^B is the preinjective component \mathcal{Q}^{H_1} of Γ_{H_1} .

The tubular algebra B is said to be the *exceptional tubular algebra of type $(2, 3, 6)$* .

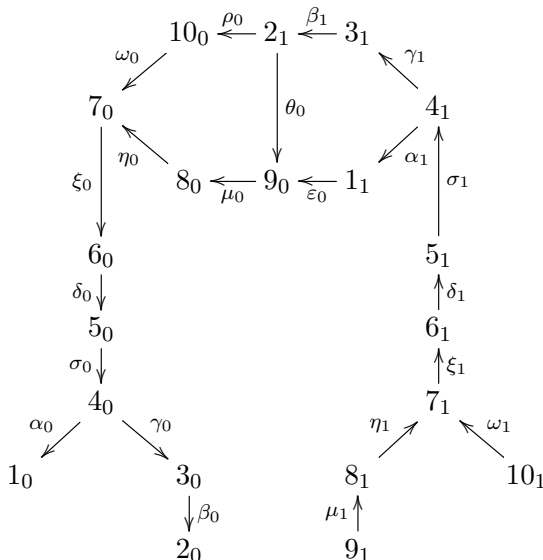
We will now describe the structure of the Auslander–Reiten quivers of the m -replicated algebras $B^{(m)}$ of B for all $m \geq 2$. This will follow from the description of the Auslander–Reiten quiver of the duplicated algebra

$$B^{(2)} = \left\{ \begin{bmatrix} b_1 & 0 \\ f_1 & b_2 \end{bmatrix}; b_1, b_2 \in B, f_1 \in D(B) \right\}$$

of B . We first note that the quiver Q of B has the reflection sequence of sinks $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, and consequently

$$B^{(2)} = T_{10}^+ T_9^+ T_8^+ T_7^+ T_6^+ T_5^+ T_4^+ T_3^+ T_2^+ T_1^+ B.$$

Therefore $B^{(2)} = KQ^{(2)}/I^{(2)}$, where $Q^{(2)}$ is the quiver



and $I^{(2)}$ is the ideal in $KQ^{(2)}$ generated by the following elements:

$$\omega_0 \xi_0 \delta_0 \sigma_0 \alpha_0, \varepsilon_0 \mu_0 \eta_0 \xi_0 \delta_0 \sigma_0 \gamma_0, \rho_0 \omega_0 - \theta_0 \mu_0 \eta_0, \beta_1 \rho_0 \omega_0 \xi_0 \delta_0 \sigma_0 \gamma_0 \beta_0, \gamma_1 \beta_1 \theta_0 - \alpha_1 \varepsilon_0, \sigma_1 \alpha_1 \varepsilon_0 \mu_0 \eta_0 \xi_0 \delta_0 \sigma_0, \delta_1 \sigma_1 \alpha_1 \varepsilon_0 \mu_0 \eta_0 \xi_0 \delta_0, \xi_1 \delta_1 \sigma_1 \alpha_1 \varepsilon_0 \mu_0 \eta_0 \xi_0, \eta_1 \xi_1 \delta_1 \sigma_1 \gamma_1 \beta_1 \rho_0, \mu_1 \eta_1 \xi_1 \delta_1 \sigma_1 \alpha_1 \varepsilon_0 \mu_0, \omega_1 \xi_1 \delta_1 \sigma_1 \alpha_1.$$

Moreover, the following iterated reflection algebras of B form a complete family of tubular convex subcategories of $B^{(2)}$ inside the repetitive category \widehat{B} :

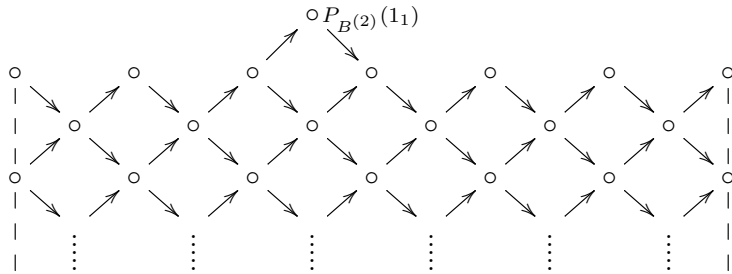
- $B = B_0$, given by the objects $1_0, 2_0, 3_0, 4_0, 5_0, 6_0, 7_0, 8_0, 9_0, 10_0$;
- $S_1^+ B$, given by the objects $2_0, 3_0, 4_0, 5_0, 6_0, 7_0, 8_0, 9_0, 10_0, 1_1$;
- $S_3^+ S_2^+ S_1^+ B$, given by the objects $4_0, 5_0, 6_0, 7_0, 8_0, 9_0, 10_0, 1_1, 2_1, 3_1$;
- $S_7^+ S_6^+ S_5^+ S_4^+ S_3^+ S_2^+ S_1^+ B$, given by the objects $8_0, 9_0, 10_0, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1$;
- $S_9^+ S_8^+ S_7^+ S_6^+ S_5^+ S_4^+ S_3^+ S_2^+ S_1^+ B$, given by the objects $10_0, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1, 8_1, 9_1$;
- $S_{10}^+ S_9^+ S_8^+ S_7^+ S_6^+ S_5^+ S_4^+ S_3^+ S_2^+ S_1^+ B = B_1 = \nu_{\widehat{B}}(B_0)$, given by the objects $1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1, 8_1, 9_1, 10_1$.

Then, applying [17, Section 3], we conclude that the Auslander–Reiten quiver $\Gamma_{B^{(2)}}$ of $B^{(2)}$ has a decomposition

$$\Gamma_{B^{(2)}} = \mathcal{P}^{B^{(2)}} \cup \left(\bigcup_{q \in \overline{\mathbb{Q}}_6^0} \mathcal{T}_q^{B^{(2)}} \right) \cup \mathcal{Q}^{B^{(2)}},$$

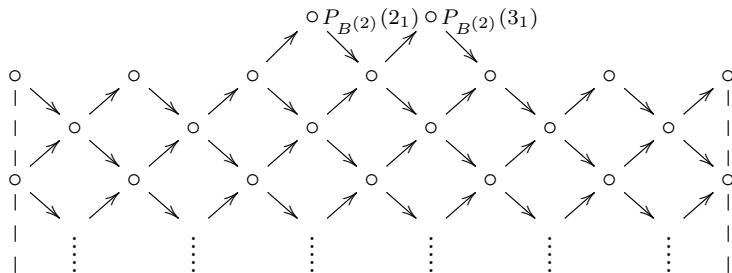
where:

- (1) $\mathcal{P}^{B^{(2)}}$ is the postprojective component \mathcal{P}^{B_0} of Γ_{B_0} .
- (2) $\mathcal{Q}^{B^{(2)}}$ is the preinjective component \mathcal{Q}^{B_1} of Γ_{B_1} .
- (3) For $q \in \overline{\mathbb{Q}}_6^0 \setminus \{0, 1, 2, 3, 4, 5, 6\}$, $\mathcal{T}_q^{B^{(2)}}$ is a $\mathbb{P}_1(K)$ -family $(\mathcal{T}_{q,\lambda}^{B^{(2)}})_{\lambda \in \mathbb{P}_1(K)}$ of stable tubes of tubular type $(2, 3, 6)$.
- (4) $\mathcal{T}_0^{B^{(2)}}$ is the $\mathbb{P}_1(K)$ -family $\mathcal{T}_0^{B_0}$ of ray tubes of Γ_{B_0} containing the indecomposable projective (noninjective) module $P_B(10) = P_{B^{(2)}}(10_0)$, described above.
- (5) $\mathcal{T}_6^{B^{(2)}}$ is the $\mathbb{P}_1(K)$ -family $\mathcal{T}_1^{B_1}$ of coray tubes of Γ_{B_1} containing the indecomposable injective (nonprojective) module $I_{B_1}(1_1)$, described above.
- (6) $\mathcal{T}_1^{B^{(2)}}$ is a $\mathbb{P}_1(K)$ -family of tubes $(\mathcal{T}_{1,\lambda}^{B^{(2)}})_{\lambda \in \mathbb{P}_1(K)}$ containing a unique nonstable tube $\mathcal{T}_{1,0}^{B^{(2)}}$ of the form



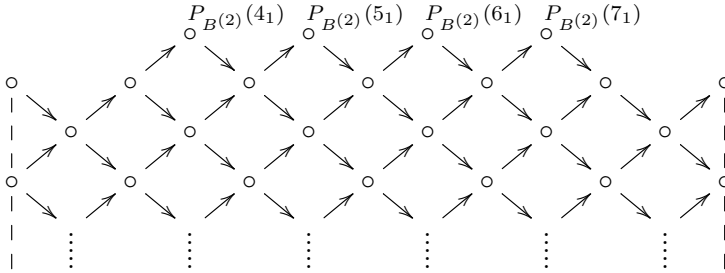
with $P_{B^{(2)}}(1_1) = I_{B^{(2)}}(1_0)$ projective-injective, one stable tube of rank 2, one stable tube of rank 3, and all remaining tubes being stable tubes of rank 1.

- (7) $\mathcal{T}_2^{B^{(2)}}$ is a $\mathbb{P}_1(K)$ -family of tubes $(\mathcal{T}_{2,\lambda}^{B^{(2)}})_{\lambda \in \mathbb{P}_1(K)}$ containing a unique nonstable tube $\mathcal{T}_{2,0}^{B^{(2)}}$ of the form



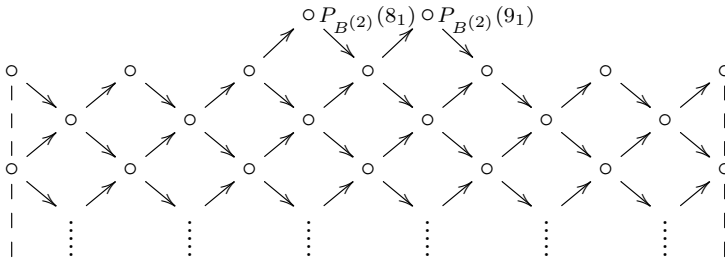
with $P_{B(2)}(2_1) = I_{B(2)}(2_0)$ and $P_{B(2)}(3_1) = I_{B(2)}(3_0)$ projective-injective, one stable tube of rank 2, one stable tube of rank 3, and all remaining tubes being stable tubes of rank 1;

- (8) $\mathcal{T}_3^{B(2)}$ is a $\mathbb{P}_1(K)$ -family of tubes $(\mathcal{T}_{3,\lambda}^{B(2)})_{\lambda \in \mathbb{P}_1(K)}$ containing a unique nonstable tube $\mathcal{T}_{3,0}^{B(2)}$ of the form



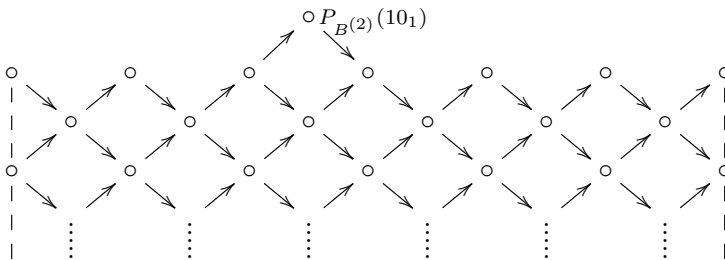
with $P_{B(2)}(4_1) = I_{B(2)}(4_0)$, $P_{B(2)}(5_1) = I_{B(2)}(5_0)$, $P_{B(2)}(6_1) = I_{B(2)}(6_0)$ and $P_{B(2)}(7_1) = I_{B(2)}(7_0)$ projective-injective, one stable tube of rank 2, one stable tube of rank 3, and all remaining tubes being stable tubes of rank 1.

- (9) $\mathcal{T}_4^{B(2)}$ is a $\mathbb{P}_1(K)$ -family of tubes $(\mathcal{T}_{4,\lambda}^{B(2)})_{\lambda \in \mathbb{P}_1(K)}$ containing a unique nonstable tube $\mathcal{T}_{4,0}^{B(2)}$ of the form



with $P_{B(2)}(8_1) = I_{B(2)}(8_0)$ and $P_{B(2)}(9_1) = I_{B(2)}(9_0)$ projective-injective, one stable tube of rank 2, one stable tube of rank 3, and all remaining tubes being stable tubes of rank 1.

- (10) $\mathcal{T}_5^{B(2)}$ is a $\mathbb{P}_1(K)$ -family of tubes $(\mathcal{T}_{5,\lambda}^{B(2)})_{\lambda \in \mathbb{P}_1(K)}$ containing a unique nonstable tube $\mathcal{T}_{5,0}^{B(2)}$ of the form



with $P_{B^{(2)}}(10_1) = I_{B^{(2)}}(10_0)$ projective-injective, one stable tube of rank 2, one stable tube of rank 3, and all remaining tubes being stable tubes of rank 1.

THEOREM 7.2. *Let $m \geq 2$ be a natural number and $B^{(m)}$ be the m -replicated algebra of the exceptional tubular algebra B of type $(2, 3, 6)$. Then the Auslander–Reiten quiver $\Gamma_{B^{(m)}}$ of $B^{(m)}$ has a decomposition*

$$\Gamma_{B^{(m)}} = \mathcal{P}^{B^{(m)}} \cup \left(\bigcup_{q \in \overline{\mathbb{Q}}_{5m-4}^0} \mathcal{T}_q^{B^{(m)}} \right) \cup \mathcal{Q}^{B^{(m)}},$$

where:

- (1) $\mathcal{P}^{B^{(m)}}$ is the postprojective component \mathcal{P}^{B_0} of Γ_{B_0} .
- (2) $\mathcal{Q}^{B^{(m)}}$ is the preinjective component \mathcal{Q}^{B_m} of Γ_{B_m} .
- (3) $\mathcal{T}_0^{B^{(m)}}$ is the $\mathbb{P}_1(K)$ -family $\mathcal{T}_0^{B_0}$ of ray tubes of Γ_{B_0} .
- (4) $\mathcal{T}_{5m-4}^{B^{(m)}}$ is the $\mathbb{P}_1(K)$ -family $\mathcal{T}_1^{B_m}$ of coray tubes of $\Gamma_{B^{(m)}}$.
- (5) $\mathcal{T}_{p+5k}^{B^{(m)}} = \nu_{\widehat{B}}^k(\mathcal{T}_p^{B^{(2)}})$ for $p \in \mathbb{Q} \cap (0, 5]$ and $k \in \{0, 1, \dots, m-2\}$.
- (6) $\mathcal{T}_{p+5(m-1)}^{B^{(m)}} = \nu_{\widehat{B}}^{m-1}(\mathcal{T}_p^{B_0})$ for $p \in \mathbb{Q}_1^0$.

Moreover:

- (i) For $q \in \overline{\mathbb{Q}}_{5m-4}^0$, the family $\mathcal{T}_q^{B^{(m)}}$ weakly separates $\mathcal{P}^{B^{(m)}} \cup (\bigcup_{p < q} \mathcal{T}_p^{B^{(m)}})$ from $(\bigcup_{p > q} \mathcal{T}_p^{B^{(m)}}) \cup \mathcal{Q}^{B^{(m)}}$.
- (ii) The component quiver $\Sigma_{B^{(m)}}$ of $B^{(m)}$ is acyclic, $\mathcal{P}^{B^{(m)}}$ is a unique source of $\Sigma_{B^{(m)}}$, and $\mathcal{Q}^{B^{(m)}}$ is a unique sink of $\Sigma_{B^{(m)}}$.
- (iii) The directing projective modules belong to $\mathcal{P}^{B^{(m)}}$.
- (iv) The directing injective modules belong to $\mathcal{Q}^{B^{(m)}}$.

We obtain the following consequence of Theorems 7.1 and 7.2, crucial for the construction of the algebra A occurring in the main theorem of the paper.

COROLLARY 7.3. *Let m be a positive natural number and B be the exceptional tubular algebra of type $(2, 3, 6)$. Then the m -replicated algebra $B^{(m)}$ of B is a left admissible and a right admissible algebra.*

8. Proof of the main theorem. Let s, m_1, \dots, m_s be positive integers and $\Lambda_0, \Lambda_1, \dots, \Lambda_s$ triangular algebras of finite representation type. It follows from Proposition 4.2 that there exist mono sink-source triangular algebras $\overline{\Lambda}_0, \overline{\Lambda}_1, \dots, \overline{\Lambda}_s$ of finite representation type such that

- Λ_k is a quotient algebra of $\overline{\Lambda}_k$,
- Q_{Λ_k} is a full convex subquiver of $Q_{\overline{\Lambda}_k}$,
- Γ_{Λ_k} is a full convex translation subquiver of $\Gamma_{\overline{\Lambda}_k}$,

for any $k \in \{0, 1, \dots, s\}$.

\mathcal{P}^{H_0} of the hereditary algebra $H_0 = KQ^{(0)}$ of Euclidean type $\widetilde{\mathbb{E}}_8$ given by the subquiver $Q^{(0)}$ of Σ given by the vertices 1, 2, 3, 4, 5, 6, 7, 8, 9.

- \mathcal{C}_1 is an almost acyclic component of the form $\mathcal{Q}^B \cup \Delta^{(1)}$ with a cyclic multisection $\Delta^{(1)}$ such that the right part $\Delta_r^{(1)}$ of $\Delta^{(1)}$ is empty, and the left part $\Delta_l^{(1)}$ of $\Delta^{(1)}$ is given by the indecomposable injective modules of the preinjective component \mathcal{Q}^B of Γ_B , being the preinjective component \mathcal{Q}^{H_1} of the hereditary algebra $H_1 = KQ^{(1)}$ of Euclidean type $\widetilde{\mathbb{E}}_8$ given by the subquiver $Q^{(1)}$ of Σ given by the vertices 2, 3, 4, 5, 6, 7, 8, 9, 10.

A direct checking shows that $\text{gl.dim } A = 5$, $\text{pd}_A S_a = 5 = \text{id}_A S_a$ and $\text{pd}_A S_b = 5 = \text{id}_A S_b$.

Consider the algebra $A_0 = A_0 * H_0$. Then A_0 is isomorphic to the bound quiver algebra $K\Omega^{(0)}/J^{(0)}$, where $\Omega^{(0)}$ is the full convex subquiver of Σ given by all vertices except 10, b, c, h, i, j , and $J^{(0)} = J \cap K\Omega^{(0)}$. Then it follows from Theorem 5.2 that the Auslander–Reiten quiver Γ_{A_0} of A_0 has a decomposition

$$\Gamma_{A_0} = \mathcal{C}_0 \cup \mathcal{T}^{H_0} \cup \mathcal{Q}^{H_0},$$

where \mathcal{C}_0 is the almost acyclic component of Γ_A described above, \mathcal{T}^{H_0} is the $\mathbb{P}_1(K)$ -family of stable tubes of type $(2, 3, 5)$ of Γ_{H_0} , and \mathcal{Q}^{H_0} is the preinjective component of Γ_{H_0} . In particular, the Auslander–Reiten quiver Γ_{H_0} of H_0 is a cofinite full translation subquiver of Γ_{A_0} , which is closed under successors in Γ_{A_0} . Therefore, A_0 is a domestic cycle-finite algebra for which all indecomposable projective modules are nondirecting but all the indecomposable injective modules lying in the preinjective component \mathcal{Q}^{H_0} are directing. We also note that there are some nondirecting indecomposable injective A_0 -modules lying in the cyclic part $\Delta^{(0)} = \Delta_c^{(0)}$ of \mathcal{C}_0 . Moreover, A_0 is an algebra with $\text{gl.dim } A_0 = 5$ and $\text{pd}_{A_0} S_a = 5 = \text{id}_{A_0} S_a$.

Consider the algebra $A_1 = H_1 * A_1$. Then A_1 is isomorphic to the bound quiver algebra $K\Omega^{(1)}/J^{(1)}$, where $\Omega^{(1)}$ is the full convex subquiver of Σ given by all vertices except 1, a, d, e, f, g , and $J^{(1)} = J \cap K\Omega^{(1)}$. Then it follows from Theorem 5.1 that the Auslander–Reiten quiver Γ_{A_1} of A_1 has a decomposition

$$\Gamma_{A_1} = \mathcal{P}^{H_1} \cup \mathcal{T}^{H_1} \cup \mathcal{C}_1,$$

where \mathcal{P}^{H_1} is the postprojective component of Γ_{H_1} , \mathcal{T}^{H_1} is the $\mathbb{P}_1(K)$ -family of stable tubes of type $(2, 3, 5)$ of Γ_{H_1} , and \mathcal{C}_1 is the almost acyclic component of Γ_A described above. In particular, the Auslander–Reiten quiver Γ_{H_1} of H_1 is a cofinite full translation subquiver of Γ_{A_1} , which is closed under predecessors in Γ_{A_1} . Therefore, A_1 is a domestic cycle-finite algebra for which all indecomposable injective modules are nondirecting but all the indecomposable projective modules lying in the postprojective component \mathcal{P}^{H_1} are directing. We also note that there are some nondirecting indecom-

posable projective A_1 -modules lying in the cyclic part $\Delta^{(1)} = \Delta_c^{(1)}$ of \mathcal{C}_1 . Moreover, $\text{gl.dim } A_1 = 5$ and $\text{pd}_{A_1} S_b = 5 = \text{id}_{A_1} S_b$.

We refer to [33] for the structure of arbitrary algebras with all indecomposable projective (respectively, injective) modules directing.

Acknowledgements. The research has been initiated during the visit of P. Malicki and A. Skowroński at the Centro de Investigación en Matemáticas (CIMAT) in Guanajuato (June 2015). These authors would like to thank J. A. de la Peña and CIMAT for the warm hospitality and wonderful conditions for the successful realization of this joint research project.

This work was completed with the support of the research grant DEC-2011/02/A/ST1/00216 from the National Science Centre, Poland, and of the CIMAT, Guanajuato, México.

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