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Historical remarks on equivalence of functions

by

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Summary. A theorem of E. Szpilrajn (Marczewski) (1936) on equivalent functions is recalled and its relation to subsequent results published by M. Morayne and C. Ryll-Nardzewski (1999) and M. Kysiak (2005) is briefly discussed.

More than 80 years ago Edward Szpilrajn, who later changed his name to Marczewski, published a fundamental paper [8]. In that paper he defined the very useful notion of the characteristic function of a sequence of sets, nowadays often called the Marczewski function (see, e.g., [1, p. 9]). He used it to study the equivalence of sequences of sets, following a suggestion of Stanisław Ulam.

It is less known that in the same paper he also introduced and studied the notion of equivalence for functions (see [8, 2.1]). To recall the corresponding definition, let X, Y and Z be sets and let $g\colon X \to Z$ and $h\colon Y \to Z$. The functions g and h are called *equivalent* if there exists a bijection $\varphi\colon X \to Y$ such that $g(x) = h(\varphi(x))$ for every $x \in X$. It is shown in [8, 2.4(i)] that g and h are equivalent if and only if $g^{-1}(z)$ and $h^{-1}(z)$ are equipotent for every $z \in Z$.

Let now X be an uncountable Polish space and let $\boldsymbol{F}$ be a class of functions on X with values in the unit interval I. Consider the following two conditions on $\boldsymbol{F}$:

(a) For every $f \in \boldsymbol{F}$ there exists a perfect subset P of X such that $f|P$ is continuous.
(b) There is a subset K of X such that both K and $X \setminus K$ have cardinality $\mathfrak{c}$ and for every $f\colon X \to I$ we have $f \in \boldsymbol{F}$ whenever $f|K$ is Borel measurable.

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As noted in [8, 3.5], these conditions are satisfied by the class of Lebesgue measurable functions (on $X = I^n$) and the class of functions having the Baire property (on $X = I^n$). The same is true for the class of functions satisfying condition (s) (see [7] for definition), which are nowadays also called Marczewski measurable. The continuum hypothesis assumed in [8] in this last case turned out later to be superfluous (see [5, Chapter 5, Section I, Theorem 36]).

Let U be a set of cardinality $\mathfrak{c}$. Then $f\colon U \to I$ is said to satisfy *condition* (A) if either $f(U)$ contains a perfect set or $f^{-1}(y_0)$ has cardinality $\mathfrak{c}$ for some $y_0 \in I$.

With this notation, the following theorem holds (see [8, 3.4(iii)]):

Let $\boldsymbol{F}$ *be a class of functions satisfying conditions* (a) *and* (b) *and let* $f\colon U \to I$. *Then* f *is equivalent to a function from* $\boldsymbol{F}$ *if and only if it satisfies condition* (A).

More than 60 years later a special case of Marczewski's theorem, which applies to the class of Lebesgue measurable functions and that of functions having the Baire property, was rediscovered by Morayne and Ryll-Nardzewski [4, Theorem]. That special case was then generalized by Kysiak [3, Theorem 5] to include the class of functions having property (s). Kysiak's generalization is also a consequence of Marczewski's theorem. (Condition (a) above appears in [3] as the *Weak Continuous Restriction Property.*) Another related paper [2] does refer to [8], but mainly in connection with the definition of equivalent functions. In particular, the priority of [8] over [3] and [4] has been overlooked by the authors of [2].

In [3] an essential role is played by the class $\mathcal{H}(\mathcal{A})$ of sets that belong hereditarily to a given class $\mathcal{A}$ of sets. It is worth noting that, in another context, the same class already appeared in [6].

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