COLLOQUIUM MATHEMATICUM

VOL. 149 2017 NO. 1

ON DIPPER–MATHAS'S MORITA EQUIVALENCES

BY

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Abstract. Dipper and Mathas have proved that every Ariki–Koike algebra (i.e., nondegenerate cyclotomic Hecke algebra of type $G(\ell,1,n)$) is Morita equivalent to a direct sum of tensor products of some smaller Ariki–Koike algebras which have q-connected parameter sets. They proved this result by explicitly constructing a progenerator which induces this equivalence. In this paper we use the nondegenerate affine Hecke algebra $\mathcal{H}_n^{\mathrm{aff}}$ to derive Dipper–Mathas's Morita equivalence as a consequence of an equivalence between the block $\mathcal{H}_n^{\mathrm{aff}}\text{-mod}[\boldsymbol{\gamma}]$ of the category of finite-dimensional modules over $\mathcal{H}_n^{\mathrm{aff}}$ and the block $\mathcal{H}_{n_1}^{\mathrm{aff}}\otimes\cdots\otimes\mathcal{H}_{n_r}^{\mathrm{aff}}\text{-mod}[(\boldsymbol{\gamma}^{(1)},\dots,\boldsymbol{\gamma}^{(r)})]$ of the category of finite-dimensional modules over the parabolic subalgebra $\mathcal{H}_{n_1}^{\mathrm{aff}}\otimes\cdots\otimes\mathcal{H}_{n_r}^{\mathrm{aff}}$ under certain conditions on $\boldsymbol{\gamma},\boldsymbol{\gamma}^{(1)},\dots,\boldsymbol{\gamma}^{(r)}$. Similar results for the degenerate versions of these algebras are also obtained.

1. Introduction. Throughout this paper, let F be an algebraically closed field and $F^{\times} := F\setminus\{0\}$. Let $q\in F^{\times}$, $\ell,n\in\mathbb{N}$ and $Q_1,\dots,Q_\ell\in F$. We define the multi-set $\mathbf{Q} := \{Q_1,\dots,Q_\ell\}$.

Definition 1.1. The *Ariki–Koike algebra*

$$\mathscr{H}_{\ell,n}(\mathbf{Q}) := \mathscr{H}_{\ell,n,F}(Q_1,\dots,Q_\ell)$$

with *Hecke parameter* q and *cyclotomic parameters* $Q_1,\dots,Q_\ell$ is defined to be the unital associative F-algebra with generators $T_0,T_1,\dots,T_{n-1}$ and relations

$$\begin{aligned}&(T_0-Q_1)\cdots(T_0-Q_\ell)=0, \qquad (T_a+1)(T_a-q)=0,\\ &T_sT_{s+1}T_s=T_{s+1}T_sT_{s+1}, \qquad T_0T_1T_0T_1=T_1T_0T_1T_0,\\ &T_rT_s=T_sT_r \quad \text{if } |r-s|>1,\end{aligned}$$

where $1\le a<n$, $0\le r<n$ and $1\le s<n-1$.

For each $1\le i\le n$, let $L_i := q^{1-i}T_{i-1}\cdots T_1T_0T_1\cdots T_{i-1}$. The elements $\{L_1,\dots,L_n\}$ are called the *Jucys–Murphy operators* of $\mathscr{H}_{\ell,n}(\mathbf{Q})$.

2010 *Mathematics Subject Classification*: Primary 05E10; Secondary 20C08.
Key words and phrases: affine Hecke algebras, Morita equivalence, Ariki–Koike algebras.
Received 15 July 2015; revised 11 July 2016.

DOI: 10.4064/cm6711-7-2016

Theorem 1.2 ([DM]). *Suppose $q \in F^\times$ and $\mathbf{Q} = \mathbf{Q}_1 \sqcup \cdots \sqcup \mathbf{Q}_r$ is a disjoint union of r multi-sets such that*

$$Q_i = Q_j q^k \text{ for some } k\in\mathbb{Z}, \text{ and } Q_i \in \mathbf{Q}_\alpha, Q_j \in \mathbf{Q}_\beta \text{ only if } \alpha = \beta. \tag{1.1}$$

Set $\ell_i := |\mathbf{Q}_1| + \cdots + |\mathbf{Q}_i|$ for any $1 \le i \le r$. Then $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$ is Morita equivalent to the following algebra:

$$\bigoplus_{\substack{n_1+\cdots+n_r=n\\ n_i\in\mathbb{N},\forall i}} \mathscr{H}_{\ell_1,n_1,F}(\mathbf{Q}_1) \otimes \mathscr{H}_{\ell_2-\ell_1,n_2,F}(\mathbf{Q}_2) \otimes \cdots \otimes \mathscr{H}_{\ell_r-\ell_{r-1},n_r,F}(\mathbf{Q}_r).$$

Note that we allow $q = 1$ as in [DM, Theorem 1.1]. Dipper and Mathas proved the above theorem by explicitly constructing a progenerator which induces the required Morita equivalence; see also [DR] for a special case of this Morita equivalence. In this paper we give an alternative approach to Dipper–Mathas's Morita equivalence via the theory of affine Hecke algebras. To state our result, we need some notation and definitions.

Definition 1.3. Suppose $q \in F^\times$. The *nondegenerate affine Hecke algebra of type A* is the unital associative F-algebra $\mathcal{H}_n^{\mathrm{aff}}$ with generators $T_1, \ldots, T_{n-1}$, $X_1, \ldots, X_n$ and relations

$$\begin{aligned}
&(T_i - q)(T_i + 1) = 0, \quad 1 \le i < n,\\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad 1 \le i \le n-2,\\
&T_iT_k = T_kT_i, \quad |i-k| > 1, \quad X_iX_k = X_kX_i, \quad 1 \le i,k \le n,\\
&T_iX_iT_i = qX_{i+1}, \quad 1 \le i < n, \quad T_iX_k = X_kT_i, \quad k \ne i, i+1.
\end{aligned}$$

Note that there is a natural surjective homomorphism from the nondegenerate affine Hecke algebra $\mathcal{H}_n^{\mathrm{aff}}$ onto the Ariki–Koike algebra $\mathscr{H}_{\ell,n}(\mathbf{Q})$ which sends X_i and T_j to L_i and T_j respectively for any $1 \le i \le n, 1 \le j < n$.

Let $\{t_k \mid 1 \le k \le n\}$ be a set of n algebraically independent indeterminates over F. Clearly, there is a natural left action of $\mathfrak{S}_n$ on $F[t_1, \ldots, t_n]$.

Following [HL, Section 2], for any $f \in F[t_1, \ldots, t_n]$, $1 \le r < n$ and $1 \le k \le n$, we define

$$\begin{cases} X_k \cdot f := t_k f,\\ T_r \cdot f := (t_{r+1} - qt_r)\dfrac{s_r(f) - f}{t_{r+1} - t_r} + qf. \end{cases}$$

Then it is easy to check that the above rules extend uniquely to a representation ρ of $\mathcal{H}_n^{\mathrm{aff}}$ on $F[t_1, \ldots, t_n]$, and the image under ρ of the elements of

$$\{T_w X_1^{a_1} \cdots X_n^{a_n} \mid w \in \mathfrak{S}_n,\ a_1, \ldots, a_n \in \mathbb{N}\} \tag{1.2}$$

are F-linearly independent (cf. [M]). It follows that (1.2) forms a basis of $\mathcal{H}_n^{\mathrm{aff}}$ and ρ is a faithful representation of $\mathcal{H}_n^{\mathrm{aff}}$.

In [G, Proposition 4.1], Grojnowski works with a bigger affine Hecke algebra $\widetilde{\mathcal{H}_n^{\mathrm{aff}}}$ which contains $X_1^{-1}, \ldots, X_n^{-1}$. Let $F[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ (resp.,

$F[X_1^{\pm 1},\dots,X_n^{\pm 1}]^{\mathfrak{S}_n}$) be the F-subalgebra generated by all the symmetric polynomials (resp., Laurent polynomials) in $X_1,\dots,X_n$. By a result of Bernstein (see [G, Proposition 4.1]), the center of $\widetilde{\mathcal{H}_n^{\text{aff}}}$ is $F[X_1^{\pm 1},\dots,X_n^{\pm 1}]^{\mathfrak{S}_n}$. Now the basis (1.2) implies that same argument used in the proof of [G, Proposition 4.1] applies equally well for our affine Hecke algebra $\mathcal{H}_n^{\text{aff}}$, so that one can show that $F[X_1,\dots,X_n]^{\mathfrak{S}_n}$ is the center of $\mathcal{H}_n^{\text{aff}}$.

For any F-algebra A, we denote by A-mod the category of finite-dimensional left A-modules, and by $K(A\text{-mod})$ the *Grothendieck group* of A-mod. For any $M \in A$-mod, let $[M]$ be the element in $K(A\text{-mod})$ represented by M.

Let $\mathcal{P}_n$ be the commutative F-subalgebra of $\mathcal{H}_n^{\text{aff}}$ generated by $X_1,\dots,X_n$. For each $a \in F$, let $L(a)$ be the one-dimensional $\mathcal{P}_1$-module defined by $X_1 v = av$ for any $v \in L(a)$. As $\mathcal{P}_n \cong \mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_1$, we get a one-dimensional $\mathcal{P}_n$-module by taking the outer tensor product $L(a_1) \boxtimes \cdots \boxtimes L(a_n)$ for any $a_1,\dots,a_n \in F$.

Let $\underline{a} := (a_1,\dots,a_n) \in F^n$. For any $M \in \mathcal{P}_n$-mod, we use $M_{\underline{a}}$ to denote the simultaneous generalized eigenspace for $X_1,\dots,X_n$ corresponding to the eigenvalues $a_1,\dots,a_n$ respectively. That is,

$$M_{\underline{a}} = \{v \in M \mid (X_i - a_i)^k v = 0 \text{ for all } 1 \le i \le n \text{ and } k \gg 0\}.$$

In $K(\mathcal{P}_n\text{-mod})$ we have

$$[M] = \sum_{\underline{a} \in F^n} r_{\underline{a}} [L(a_1) \boxtimes \cdots \boxtimes L(a_n)],$$

where $r_{\underline{a}} = \dim_F M_{\underline{a}}$ for each $\underline{a} = (a_1,\dots,a_n) \in F^n$.

Let $n \in \mathbb{N}$. A *composition* $\mu = (\mu_1,\mu_2,\dots)$ of n is a sequence of nonnegative integers with sum n. If the sequence is nonincreasing, then μ is a *partition* of n. For a composition μ, we use $|\mu|$ to denote the sum of its components.

Definition 1.4. For any composition $\mu = (\mu_1,\dots,\mu_r)$ of n, we define $\bar{\mu}_0 := 0$ and $\bar{\mu}_k := \mu_1 + \cdots + \mu_k$, $1 \le k \le r$.

Let $\mathfrak{S}_n$ be the symmetric group of $\{1,\dots,n\}$, where $w \in \mathfrak{S}_n$ acts on the numbers on the left. For any composition $\mu = (\mu_1,\dots,\mu_r)$ of n, let $\mathfrak{S}_\mu := \mathfrak{S}_{\{1,\dots,\bar{\mu}_1\}} \times \cdots \times \mathfrak{S}_{\{\bar{\mu}_{r-1}+1,\dots,n\}}$ be the corresponding standard Young subgroup of $\mathfrak{S}_n$. Let $\mathcal{H}_\mu^{\text{aff}}$ be the subalgebra of $\mathcal{H}_n^{\text{aff}}$ generated by

$$\{T_k \mid \bar{\mu}_{i-1} + 1 \le k < \bar{\mu}_i,\ 1 \le i \le r\}$$

and the subalgebra $\mathcal{P}_n$. In particular, $\mathcal{H}_{(1^n)}^{\text{aff}} = \mathcal{P}_n \subseteq \mathcal{H}_\mu^{\text{aff}}$, where $(1^n) := (1,\dots,1)$ (n copies). We have the natural restriction functor $\operatorname{res}_{(1^n)}^{\mu}$ from the category $\mathcal{H}_\mu^{\text{aff}}$-mod to the category $\mathcal{P}_n$-mod. For any $M \in \mathcal{H}_\mu^{\text{aff}}$-mod, we define the *formal character*

$$\operatorname{ch} M := [\operatorname{res}^{\mu}_{(1^n)} M] \in K(\mathcal{P}_n\text{-mod}).$$

Let $\mu = (\mu_1, \dots, \mu_r)$ be a composition of n. Following [G, Section 5.3], we define the *q-discriminant* $\Delta^{\mu}_q : F^n \to F$ by

$$\Delta^{\mu}_q(a_1, \dots, a_n) := \prod (a_i - qa_j), \tag{1.3}$$

where the product runs over all pairs (i, j) such that $1 \leq i, j \leq n$ and there is no $k \geq 1$ such that both i and j lie in the same interval $[\bar{\mu}_{k-1} + 1, \bar{\mu}_{k-1} + 2, \dots, \bar{\mu}_k]$. Since we allow q to take the value 1, the element $\Delta^{\mu}_1(a_1, \dots, a_n)$ is defined too.

For any F-algebra A, let $Z(A)$ be the center of A. A *central character* of A is a unital F-algebra homomorphism $\chi : Z(A) \to F$. Suppose that $M \in A$-mod. For any central character χ of A, let $M[\chi]$ be the corresponding generalized eigenspace over $Z(A)$, that is,

$$M[\chi] = \{v \in M \mid (a - \chi(a))^k v = 0 \text{ for all } a \in Z(A) \text{ and } k \gg 0\}.$$

Since $Z(A)$ is central in A, it is clear that $M[\chi]$ is an A-submodule of M.

Definition 1.5. For any central character χ of A, we define A-mod$[\chi]$ to be the full subcategory of A-mod consisting of all modules $M \in A$-mod satisfying $M = M[\chi]$.

The subcategory A-mod$[\chi]$ is called a *block* of A-mod. In the following several paragraphs, we shall describe the blocks of $\mathcal{H}^{\text{aff}}_n$-mod and $\mathcal{H}^{\text{aff}}_{\mu}$-mod, where μ is a composition of n.

For any $\underline{a} \in F^n$, let $\chi_{\underline{a}}$ be the associated central character defined by

$$\chi_{\underline{a}} : Z(\mathcal{H}^{\text{aff}}_n) \to F, \qquad f(X_1, \dots, X_n) \mapsto f(a_1, \dots, a_n).$$

Since $X_1, \dots, X_n$ are n algebraically independent elements in $\mathcal{P}_n$, the F-algebra homomorphism $\chi_{\underline{a}}$ is well-defined.

Let $\mathfrak{S}_n$ act on F^n by place permutations, that is, for $\underline{a} = (a_1, \dots, a_n) \in F^n$ and $w \in \mathfrak{S}_n$, $w \cdot \underline{a} = (a_{w^{-1}1}, \dots, a_{w^{-1}n})$. We write $\underline{a} \sim \underline{b}$ if there exists $w \in \mathfrak{S}_n$ such that $\underline{b} = w \cdot \underline{a}$. It is easy to see that "$\sim$" is an equivalence relation. Let $F^n/{\sim}$ be the set of $\mathfrak{S}_n$-orbits on F^n. It is well-known that for any $\underline{a}, \underline{b} \in F^n$, $\chi_{\underline{a}} = \chi_{\underline{b}}$ if and only if $\underline{a} \sim \underline{b}$.

If $\boldsymbol{\gamma} \in F^n/{\sim}$, we set $\chi_{\boldsymbol{\gamma}} := \chi_{\underline{a}}$ for any $\underline{a} \in \boldsymbol{\gamma}$. For any $M \in \mathcal{H}^{\text{aff}}_n$-mod and $\boldsymbol{\gamma} \in F^n/{\sim}$, we set

$$M[\boldsymbol{\gamma}] := M[\chi_{\boldsymbol{\gamma}}], \qquad \mathcal{H}^{\text{aff}}_n\text{-mod}[\boldsymbol{\gamma}] := \mathcal{H}^{\text{aff}}_n\text{-mod}[\chi_{\boldsymbol{\gamma}}],$$

where the second equality used the notation in Definition 1.5. It is clear that $M[\boldsymbol{\gamma}] = \bigoplus_{\underline{a} \in \boldsymbol{\gamma}} M_{\underline{a}}$.

Corollary 1.6 ([G, Section 4]). *For any* $M \in \mathcal{H}_n^{\mathrm{aff}}$-mod,

$$M = \bigoplus_{\gamma \in F^n/\sim} M[\gamma]$$

as an $\mathcal{H}_n^{\mathrm{aff}}$*-module.*

Suppose $\mu = (\mu_1, \dots, \mu_r)$ is a composition of n. Then $Z(\mathcal{H}_\mu^{\mathrm{aff}})$ is the tensor product $F[X_1, \dots, X_{\bar{\mu}_1}]^{\mathfrak{S}_{\{1,2,\dots,\bar{\mu}_1\}}} \otimes \cdots \otimes F[X_{\bar{\mu}_{r-1}+1}, \dots, X_n]^{\mathfrak{S}_{\{\bar{\mu}_{r-1}+1,\dots,n\}}}$. Because $\mathfrak{S}_\mu$ is a subgroup of $\mathfrak{S}_n$, $\mathfrak{S}_\mu$ also acts on F^n. We use "$\sim_\mu$" to denote the equivalence relation defined by the action of $\mathfrak{S}_\mu$ on F^n, and $F^n/{\sim_\mu}$ to denote the set of $\mathfrak{S}_\mu$-orbits. Suppose $\underline{a} = (a_1, \dots, a_n) \in F^n$, $\underline{a}^{(i)} := (a_{\bar{\mu}_{i-1}+1}, \dots, a_{\bar{\mu}_i}) \in F^{\mu_i}$ for each $1 \le i \le r$. The associated central character of $\mathcal{H}_\mu^{\mathrm{aff}}$ is

$$\widehat{\chi}_{\underline{a}} : Z(\mathcal{H}_\mu^{\mathrm{aff}}) \to F, \quad f_1 \otimes \cdots \otimes f_r \mapsto \prod_{i=1}^r (\chi_{\underline{a}^{(i)}}(f_i)) = \prod_{i=1}^r f_i(a_{\bar{\mu}_{i-1}+1}, \dots, a_{\bar{\mu}_i}).$$

where $f_i \in F[X_{\bar{\mu}_{i-1}+1}, \dots, X_{\bar{\mu}_i}]^{\mathfrak{S}_{\{\bar{\mu}_{i-1}+1,\dots,\bar{\mu}_i\}}}$, $1 \le i \le r$. For each $1 \le i \le r$, we denote the $\mathfrak{S}_{\{\bar{\mu}_{i-1}+1,\dots,\bar{\mu}_i\}}$-orbit $\mathfrak{S}_{\{\bar{\mu}_{i-1}+1,\dots,\bar{\mu}_i\}} \cdot \underline{a}^{(i)} \in F^{\mu_i}/\sim$ by $\boldsymbol{\gamma}^{(i)}$. Then the $\mathfrak{S}_\mu$-orbit $\mathfrak{S}_\mu \cdot \underline{a} \in F^n/{\sim_\mu}$ is just the r-tuple $(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})$. It is easy to check that for any $\underline{a}, \underline{b} \in F^n$, $\widehat{\chi}_{\underline{a}} = \widehat{\chi}_{\underline{b}}$ if and only if $\underline{a} \sim_\mu \underline{b}$. For any $\underline{a} \in (\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})$, we set $\widehat{\chi}_{(\boldsymbol{\gamma}^{(1)},\dots,\boldsymbol{\gamma}^{(r)})} := \widehat{\chi}_{\underline{a}}$. For any $M \in \mathcal{H}_\mu^{\mathrm{aff}}$-mod, let

$$\begin{aligned} M[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})] &:= M[\widehat{\chi}_{(\boldsymbol{\gamma}^{(1)},\dots,\boldsymbol{\gamma}^{(r)})}], \\ \mathcal{H}_\mu^{\mathrm{aff}}\text{-mod}[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})] &:= \mathcal{H}_\mu^{\mathrm{aff}}\text{-mod}[\widehat{\chi}_{(\boldsymbol{\gamma}^{(1)},\dots,\boldsymbol{\gamma}^{(r)})}], \end{aligned}$$

where the second equality uses the notation in Definition 1.5. It is clear that

$$M[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})] = \bigoplus_{\underline{a} \in (\boldsymbol{\gamma}^{(1)},\dots,\boldsymbol{\gamma}^{(r)})} M_{\underline{a}}. \tag{1.4}$$

Corollary 1.7. *Let* $\mu = (\mu_1, \dots, \mu_r)$ *be a composition of* n*. Then as an* $\mathcal{H}_\mu^{\mathrm{aff}}$*-module,* M *equals* $\bigoplus_{(\boldsymbol{\gamma}^{(1)},\dots,\boldsymbol{\gamma}^{(r)})} M[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})]$ *, where the subscript in the sum runs over* $F^n/{\sim_\mu}$*. In particular,*

$$\mathcal{H}_\mu^{\mathrm{aff}}\text{-mod} = \bigoplus_{(\boldsymbol{\gamma}^{(1)},\dots,\boldsymbol{\gamma}^{(r)}) \in F^n/\sim_\mu} \mathcal{H}_\mu^{\mathrm{aff}}\text{-mod}[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})].$$

Theorem 1.8. *Let* $q \in F^\times$ *and* $\mu = (\mu_1, \dots, \mu_r)$ *be a composition of* n*. Suppose* $\underline{a} = (a_1, \dots, a_n) \in F^n$ *is such that both* $\Delta_1^\mu(\underline{a})$ *and* $\Delta_q^\mu(\underline{a})$ *are nonzero. For each* $1 \le i \le r$*, let* $\underline{a}^{(i)} := (a_{\bar{\mu}_{i-1}+1}, \dots, a_{\bar{\mu}_i})$ *be a* μ_i*-tuple. Set*

$$\begin{aligned} \boldsymbol{\gamma}^{(i)} &:= \mathfrak{S}_{\{\bar{\mu}_{i-1}+1,\bar{\mu}_{i-1}+2,\dots,\bar{\mu}_i\}} \cdot \underline{a}^{(i)}, \quad 1 \le i \le r, \\ \boldsymbol{\gamma} &:= \mathfrak{S}_n \cdot \underline{a}, \\ (\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)}) &:= \mathfrak{S}_\mu \cdot \underline{a}. \end{aligned}$$

Then there exists an equivalence of categories

$$\mathcal{H}_n^{\mathrm{aff}}\text{-mod}[\gamma] \simeq \mathcal{H}_\mu^{\mathrm{aff}}\text{-mod}[(\gamma^{(1)}, \dots, \gamma^{(r)})].$$

The proof of the above theorem, which relies on certain intertwining elements in the affine Hecke algebras, will be given in Section 2. In Section 3 we show that Theorem 1.2 can be deduced from Theorem 1.8, thus giving an alternative approach to Dipper–Mathas's Morita equivalence. The Ariki–Koike algebra $\mathscr{H}_{\ell,n}(\mathbf{Q})$ is also called the nondegenerate cyclotomic Hecke algebra of type $G(\ell, 1, n)$ in the literature. In Section 4, we apply the same approach to give the corresponding results for the degenerate affine Hecke algebras and the degenerate cyclotomic Hecke algebra of type $G(\ell, 1, n)$. Furthermore, we expect similar arguments can be applied to the affine Hecke algebras of types other than A, affine BMW algebras and their finite-dimensional cyclotomic quotients so that we can get some new Morita equivalences.

2. Proof of Theorem 1.8. In this section, we shall give the proof of Theorem 1.8.

DEFINITION 2.1. Let $\mu = (\mu_1, \dots, \mu_r)$ be a composition of n. Suppose that $\underline{a}^{(j)} = (a_1^{(j)}, \dots, a_{\mu_j}^{(j)}) \in F^{\mu_j}$, $1 \le j \le r$. We say that $\underline{c} = (c_1, \dots, c_n) \in F^n$ is obtained by *shuffling* $\underline{a}^{(1)}, \dots, \underline{a}^{(r)}$ if for each $1 \le j \le r$ there exist $1 \le u_1^{(j)} < \dots < u_{\mu_j}^{(j)} \le n$ such that $(c_{u_1^{(j)}}, \dots, c_{u_{\mu_j}^{(j)}}) = (a_1^{(j)}, \dots, a_{\mu_j}^{(j)})$, and

$$\Delta_1^\mu(u_1^{(1)}, \dots, u_{\mu_1}^{(1)}, \dots, u_1^{(r)}, \dots, u_{\mu_r}^{(r)}) \neq 0.$$

Note that in the above definition the sequence $u_1^{(j)}, \dots, u_{\mu_j}^{(j)}$ is not necessarily uniquely determined even if we know that $\underline{c} = (c_1, \dots, c_n) \in F^n$ is obtained by shuffling $\underline{a}^{(1)}, \dots, \underline{a}^{(r)}$.

Let $\mu = (\mu_1, \dots, \mu_r)$ be a composition of n. The *Young diagram* of μ is the set $[\mu] = \{(i, j) \mid 1 \le j \le \mu_i, i \ge 1\}$. A *$\mu$-tableau* is a bijection $\mathfrak{t} : [\mu] \to \{1, \dots, n\}$. We call the μ-tableau $\mathfrak{t}$ a *row-standard tableau* if for any i, $\mathfrak{t}(i, j) \le \mathfrak{t}(i, l)$ whenever $j \le l$. Let $\mathfrak{t}^\mu$ be the unique row-standard μ-tableau that has the numbers $1, \dots, n$ entered in order, from left to right, and then top to bottom, along the rows of μ. If $w \in \mathfrak{S}_n$ and $\mathfrak{t}$ is a μ-tableau, then the map composition $w\mathfrak{t}$ is again a μ-tableau. If we fix a μ-tableau $\mathfrak{t}$, it is easy to check that $\{w\mathfrak{t} \mid w \in \mathfrak{S}_n\}$ is the set of all μ-tableaux.

PROPOSITION 2.2. *Let* $\mu = (\mu_1, \dots, \mu_r)$ *be a composition of* n. *Suppose* $\underline{a}^{(j)} = (a_1^{(j)}, \dots, a_{\mu_j}^{(j)}) \in F^{\mu_j}, 1 \le j \le r$. *Let*

$$\underline{a} = (a_1^{(1)}, \dots, a_{\mu_1}^{(1)}, \dots, a_1^{(r)}, \dots, a_{\mu_r}^{(r)}).$$

If $\varDelta_1^\mu(\underline{a}) \neq 0$, *then the number of different* n-*tuples obtained by shuffling* $\underline{a}^{(1)}, \dots, \underline{a}^{(r)}$ *is equal to the number of row-standard* μ-*tableaux.*

Proof. We need to give a one-to-one correspondence between the set of n-tuples obtained by shuffling $\underline{a}^{(1)}, \dots, \underline{a}^{(r)}$ and the set of row-standard μ-tableaux. In fact, if $\underline{c}$ is an n-tuple obtained by shuffling $\underline{a}^{(1)}, \dots, \underline{a}^{(r)}$, then because $\varDelta_1^\mu(\underline{a}) \neq 0$, for each $1 \le j \le r$ there exists a unique sequence $1 \le u_1^{(j)} < \dots < u_{\mu_j}^{(j)} \le n$ such that $(c_{u_1^{(j)}}, \dots, c_{u_{\mu_j}^{(j)}}) = (a_1^{(j)}, \dots, a_{\mu_j}^{(j)})$. Furthermore,

$$\varDelta_1^\mu(u_1^{(1)}, \dots, u_{\mu_1}^{(1)}, \dots, u_1^{(r)}, \dots, u_{\mu_r}^{(r)}) \neq 0.$$

Therefore, we get a unique row-standard μ-tableau $\mathfrak{t}_{\underline{c}}$ by entering the numbers $u_1^{(j)}, \dots, u_{\mu_j}^{(j)}$ into the jth row of $\mathfrak{t}_{\underline{c}}$. It is easy to check that the map $\underline{c} \mapsto \mathfrak{t}_{\underline{c}}$ is a one-to-one correspondence. ■

Remark 2.3. In the proof of the previous proposition, suppose $\underline{c}$ is an n-tuple obtained by shuffling $\underline{a}^{(1)}, \dots, \underline{a}^{(r)}$, and $\underline{c}$ corresponds to the row-standard μ-tableau $\mathfrak{t}_{\underline{c}}$ under the one-to-one correspondence. Suppose $\mathfrak{t}_{\underline{c}} = w\mathfrak{t}^\mu$. One can easily check that $\underline{c} = w \cdot \underline{a}$.

For any composition μ of n, let D_μ be the set of distinguished minimal length left coset representatives of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$. Let μ, ν be two compositions of n. For each $w \in D_{\mu,\nu} := D_\mu^{-1} \cap D_\nu$, let $\mu \cap w\nu$ and $w^{-1}\mu \cap \nu$ be the compositions of n such that $\mathfrak{S}_\mu \cap w\mathfrak{S}_\nu w^{-1} = \mathfrak{S}_{\mu\cap w\nu}$ and $w^{-1}\mathfrak{S}_\mu w \cap \mathfrak{S}_\nu = \mathfrak{S}_{w^{-1}\mu\cap\nu}$. There exists an algebra isomorphism

$$\varphi_{w^{-1}} : \mathcal{H}^{\mathrm{aff}}_{\mu\cap w\nu} \to \mathcal{H}^{\mathrm{aff}}_{w^{-1}\mu\cap\nu}$$

with $\varphi_{w^{-1}}(T_y) = T_{w^{-1}yw}$, $\varphi_{w^{-1}}(X_i) = X_{w^{-1}i}$ for $y \in \mathfrak{S}_{\mu\cap w\nu}$ and $1 \le i \le n$. If M is a left $\mathcal{H}^{\mathrm{aff}}_{w^{-1}\mu\cap\nu}$-module, then by twisting the action with the isomorphism $\varphi_{w^{-1}}$ we get a left $\mathcal{H}^{\mathrm{aff}}_{\mu\cap w\nu}$-module, which will be denoted by $w^{-1}M$. For more details, we refer the reader to [G, §5.2] and [K, §3.5].

Let μ, ν be two compositions of n such that $\mathfrak{S}_\nu$ is a subgroup of $\mathfrak{S}_\mu$. We define the restriction functor

$$\mathrm{res}^\mu_\nu : \mathcal{H}^{\mathrm{aff}}_\mu\text{-mod} \to \mathcal{H}^{\mathrm{aff}}_\nu\text{-mod}$$

by regarding any $M \in \mathcal{H}^{\mathrm{aff}}_\mu$-mod naturally as an $\mathcal{H}^{\mathrm{aff}}_\nu$-module, and the induction functor

$$\mathrm{ind}^\mu_\nu : \mathcal{H}^{\mathrm{aff}}_\nu\text{-mod} \to \mathcal{H}^{\mathrm{aff}}_\mu\text{-mod}, \quad N \mapsto \mathcal{H}^{\mathrm{aff}}_\mu \otimes_{\mathcal{H}^{\mathrm{aff}}_\nu} N.$$

In particular, if $\mu = (n)$ then we write

$$\mathrm{res}^n_\nu := \mathrm{res}^{(n)}_\nu, \qquad \mathrm{ind}^n_\nu := \mathrm{ind}^{(n)}_\nu .$$

Theorem 2.4 ([G, K, Mackey Theorem]). *Suppose* μ, ν *are compositions of* n. *Let* $M \in \mathcal{H}^{\mathrm{aff}}_\nu$-mod. *Then* $\mathrm{res}^n_\mu \mathrm{ind}^n_\nu M$ *admits a filtration with* *subquotients isomorphic to*

$$\operatorname{ind}^{\mu}_{\mu\cap w\nu} w^{-1}(\operatorname{res}^{\nu}_{w^{-1}\mu\cap\nu} M),$$

one for each $w \in D_{\mu,\nu}$. *Moreover,* $\operatorname{ind}^{\mu}_{\mu\cap\nu}(\operatorname{res}^{\nu}_{\mu\cap\nu} M)$ *is the bottom section of this filtration.*

We need modified forms of the intertwining elements introduced in [Lu], [R] and [BK2].

Definition 2.5. For each $1 \leq k < n$, we define

$$\Theta_k := T_k(X_{k+1} - X_k) + (1-q)X_{k+1} \in \mathcal{H}^{\mathrm{aff}}_n.$$

One can check directly that

$$\Theta_k^2 = (X_{k+1} - qX_k)(X_k - qX_{k+1}), \tag{2.1}$$

$$\Theta_k X_k = X_{k+1}\Theta_k,\ \Theta_k X_{k+1} = X_k\Theta_k,\ \Theta_k X_l = X_l\Theta_k,\ l \neq k, k+1, \tag{2.2}$$

$$\Theta_k\Theta_l = \Theta_l\Theta_k, \quad \Theta_k\Theta_{k+1}\Theta_k = \Theta_{k+1}\Theta_k\Theta_{k+1}, \quad |k-l| > 1. \tag{2.3}$$

According to (2.2), one can easily check that

$$\Theta_{i_1}\cdots\Theta_{i_m}X_i = X_{wi}\Theta_{i_1}\cdots\Theta_{i_m},$$

where $w = s_{i_1}\cdots s_{i_m}$.

Proof of Theorem 1.8. There is the functor of induction

$$\operatorname{ind}^n_\mu : \mathcal{H}^{\mathrm{aff}}_\mu\text{-mod}[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})] \to \mathcal{H}^{\mathrm{aff}}_n\text{-mod}[\boldsymbol{\gamma}].$$

In fact, let $M \in \mathcal{H}^{\mathrm{aff}}_\mu\text{-mod}[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})]$. By the Mackey Theorem, $\operatorname{res}^n_{(1^n)}\operatorname{ind}^n_\mu M$ admits a filtration with subquotients isomorphic to $w^{-1}(\operatorname{res}^n_{(1^n)} M)$, one for each $w \in D_\mu$. Therefore, the simultaneous generalized eigenvalues of X_i $(1 \leq i \leq n)$ on $\operatorname{res}^n_{(1^n)}\operatorname{ind}^n_\mu M$ belong to $\boldsymbol{\gamma}$, hence our functor of induction is well-defined.

There is also the functor of restriction

$$\operatorname{res}^n_\mu : \mathcal{H}^{\mathrm{aff}}_n\text{-mod}[\boldsymbol{\gamma}] \to \mathcal{H}^{\mathrm{aff}}_\mu\text{-mod}.$$

According to Corollary 1.7, it is reasonable to define the functor $p'_{\boldsymbol{\gamma}}$ of natural projection from $\mathcal{H}^{\mathrm{aff}}_\mu$-mod onto the block $\mathcal{H}^{\mathrm{aff}}_\mu\text{-mod}[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})]$ such that $p'_{\boldsymbol{\gamma}}(M) = M[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})]$ for any $M \in \mathcal{H}^{\mathrm{aff}}_\mu$-mod. Let $\underline{b} = (b_1, \dots, b_n) \in F^n$. By abuse of notation, we still use $p'_{\boldsymbol{\gamma}}$ to define a $\mathbb{Z}$-linear map from $K(\mathcal{P}_n)$ to $K(\mathcal{P}_n)$ such that

$$p'_{\boldsymbol{\gamma}}([L(b_1)\boxtimes\cdots\boxtimes L(b_n)]) := \begin{cases} [L(b_1)\boxtimes\cdots\boxtimes L(b_n)] & \text{if } \underline{b} \in (\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)}), \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $N \in \mathcal{H}^{\mathrm{aff}}_\mu$-mod we have $p'_{\boldsymbol{\gamma}}(\operatorname{ch} N) = \operatorname{ch}(p'_{\boldsymbol{\gamma}} N)$.

Let $M \in \mathcal{H}^{\mathrm{aff}}_\mu\text{-mod}[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})]$. We first prove $M \cong p'_{\boldsymbol{\gamma}}\operatorname{res}^n_\mu\operatorname{ind}^n_\mu M$. By the Mackey Theorem, M is isomorphic to a submodule of $\operatorname{res}^n_\mu\operatorname{ind}^n_\mu M$ and

$p'_\gamma(M) = M$. Note that $D^{-1}_{(1^n)} \cap D_\mu = D_\mu$, and for $w \in D_\mu$, $(1^n) \cap w\mu = (1^n)$ and $w^{-1}(1^n) \cap \mu = (1^n)$. By the Mackey Theorem, $\operatorname{res}^n_{(1^n)} \operatorname{ind}^n_\mu M$ admits a filtration with subquotients isomorphic to $w^{-1}(\operatorname{res}^n_{(1^n)} M)$, one for each $w \in D_\mu$. By the definition of ch,

$$\begin{aligned}\operatorname{ch} \operatorname{res}^n_\mu \operatorname{ind}^n_\mu M &= \operatorname{ch} \operatorname{res}^n_{(1^n)} \operatorname{ind}^n_\mu M \\ &= \operatorname{ch}(\operatorname{res}^n_{(1^n)} \mathrm{M}) + \sum_{w\neq 1,\, w\in D_\mu} \operatorname{ch} w^{-1}(\operatorname{res}^n_{(1^n)} M).\end{aligned}$$

Suppose that

$$\operatorname{ch} M = \sum_{\underline{c}\in F^n} r_{\underline{c}}[L(c_1) \boxtimes \cdots \boxtimes L(c_{\bar\mu_1}) \boxtimes \cdots \boxtimes L(c_{\bar\mu_{r-1}+1}) \boxtimes \cdots \boxtimes L(c_n)],$$

where $(c_{\bar\mu_{j-1}+1}, \dots, c_{\bar\mu_j}) \in \gamma^{(j)}$ for each $1 \le j \le r$. Then

$$\begin{aligned}\operatorname{ch} w^{-1}(\operatorname{res}^n_{(1^n)} M) = \sum_{\underline{c}\in F^n} r_{\underline{c}}[L(c_{w^{-1}1}) \boxtimes \cdots \boxtimes L(c_{w^{-1}\bar\mu_1}) \boxtimes \cdots \\ \boxtimes L(c_{w^{-1}(\bar\mu_{r-1}+1)}) \boxtimes \cdots \boxtimes L(c_{w^{-1}n})]\end{aligned}$$

for every $w \in D_\mu$. Note that $w \notin \mathfrak{S}_\mu$ whenever $1 \neq w \in D_\mu$. Therefore, there exists some $1 \le j \le r$ such that $(c_{w^{-1}(\bar\mu_{j-1}+1)}, \dots, c_{w^{-1}\bar\mu_j}) \notin \gamma^{(j)}$ by the assumption that $\Delta^\mu_1(\underline{a}) \neq 0$. It follows that

$$\operatorname{ch}(p'_\gamma(\operatorname{res}^n_\mu \operatorname{ind}^n_\mu M/M)) = p'_\gamma(\operatorname{ch}(\operatorname{res}^n_\mu \operatorname{ind}^n_\mu M/M)) = 0,$$

which forces $p'_\gamma \operatorname{res}^n_\mu \operatorname{ind}^n_\mu M \cong M$, as required.

Next, let $M \in \mathcal{H}^{\mathrm{aff}}_n$-mod$[\gamma]$. We want to show that $M \cong \operatorname{ind}^n_\mu p'_\gamma \operatorname{res}^n_\mu M$. Let $M_{\underline{c}}$ be a simultaneous generalized eigenspace of $X_1, \dots, X_n$ with basis $v_1, \dots, v_s$, where $\underline{c} = (c_1, \dots, c_n) \in \gamma$. Suppose that $c_k \neq q^{\pm 1}c_{k+1}$ for some $1 \le k < n$. Recall $s_k \cdot \underline{c} = (c_{s_k 1}, \dots, c_{s_k k}, c_{s_k(k+1)}, \dots, c_{s_k n})$. We claim $M_{s_k\cdot\underline{c}}$ is a nonzero simultaneous generalized eigenspace of $X_1, \dots, X_n$ with basis $\Theta_k v_1, \dots, \Theta_k v_s$, where Θ_k is defined in (2.5).

In fact, by (2.2), it is easy to check that $\Theta_k v_1, \dots, \Theta_k v_s$ belong to $M_{s_k\cdot\underline{c}}$. On the other hand, we can assume that the matrix of X_i $(1 \le i \le n)$ under $v_1, \dots, v_s$ is triangular with c_i on the diagonal. Therefore, by (2.1), the matrix of Θ_k^2 under $v_1, \dots, v_s$ is triangular with $(c_{k+1} - qc_k)(c_k - qc_{k+1}) \neq 0$ on the diagonal because $c_k \neq q^{\pm 1}c_{k+1}$. It follows that $\Theta_k^2 v_1, \dots, \Theta_k^2 v_s$ are linearly independent. Hence so are $\Theta_k v_1, \dots, \Theta_k v_s$. Conversely, if $w_1, \dots, w_s$ is a basis of $M_{s_k\cdot\underline{c}}$, then $\Theta_k w_1, \dots, \Theta_k w_s$ is a set of linearly independent elements in $M_{\underline{c}}$. Therefore, $M_{s_k\cdot\underline{c}} \cong M_{\underline{c}}$ as a vector space with basis $\Theta_k v_1, \dots, \Theta_k v_s$. In particular, $\Theta_k \cdot M_{\underline{c}} = M_{s_k\cdot\underline{c}}$. This proves the claim.

According to the decomposition (1.4), we can assume

$$\begin{aligned}p'_\gamma \operatorname{res}^n_\mu M &= (\operatorname{res}^n_\mu M)_{(\underline{a}^{(1,1)},\dots,\underline{a}^{(1,r)})} \oplus \cdots \oplus (\operatorname{res}^n_\mu M)_{(\underline{a}^{(t,1)},\dots,\underline{a}^{(t,r)})} \\ &= M_{(\underline{a}^{(1,1)},\dots,\underline{a}^{(1,r)})} \oplus \cdots \oplus M_{(\underline{a}^{(t,1)},\dots,\underline{a}^{(t,r)})},\end{aligned}$$

where $\underline{a}^{(i,j)} \in \boldsymbol{\gamma}^{(j)}$ for any $1 \leq i \leq t, 1 \leq j \leq r$ and $\{(\underline{a}^{(i,1)}, \dots, \underline{a}^{(i,r)}) \mid 1 \leq i \leq t\}$ coincides with the $\mathfrak{S}_\mu$-orbit $(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})$. Let $\underline{c} \in F^n$ with $M_{\underline{c}} \neq 0$. Because $\Delta_1^\mu(\underline{a}) \neq 0$, there exists a unique $\underline{b} = (\underline{b}^{(1)}, \dots, \underline{b}^{(r)}) \in (\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})$ such that $\underline{c}$ is a shuffling of $(\underline{b}^{(1)}, \dots, \underline{b}^{(r)})$ where $\underline{b}^{(j)} \in \boldsymbol{\gamma}^{(j)}$, $1 \leq j \leq r$. By the discussion in the last paragraph and the assumption that $\Delta_q^\mu(\underline{a}) \neq 0$, we can choose a sequence of simple reflections $s_{i_1}, \dots, s_{i_k}$ such that $\underline{c} = (s_{i_1} \cdots s_{i_k}) \cdot \underline{b}$, and for each $1 \leq j \leq k$, $\Theta_{i_j}\Theta_{i_{j+1}} \cdots \Theta_{i_k} M_{\underline{b}} = M_{(s_{i_j}s_{i_{j+1}} \cdots s_{i_k}) \cdot \underline{b}}$ and $\dim_F M_{(s_{i_j}s_{i_{j+1}} \cdots s_{i_k}) \cdot \underline{b}} = \dim_F M_{(s_{i_{j+1}} \cdots s_{i_k}) \cdot \underline{b}}$, where by convention we set $s_{i_{j+1}} \cdots s_{i_k} := 1$ when $j = k$. Therefore $\Theta_{i_1} \cdots \Theta_{i_k} M_{\underline{b}} = M_{\underline{c}}$ and $\dim_F M_{\underline{c}} = \dim_F M_{\underline{b}}$. Hence $M_{\underline{b}} \neq 0$ and there exists a unique $1 \leq i \leq t$ such that $\underline{b} = (\underline{a}^{(i,1)}, \dots, \underline{a}^{(i,r)})$. For each $1 \leq i \leq t$, by Proposition 2.2 and the assumption $\Delta_1^\mu(\underline{a}) \neq 0$, the number of different n-tuples which are obtained by shuffling $\underline{a}^{(i,1)}, \dots, \underline{a}^{(i,r)}$ is equal to the number of row-standard μ-tableaux, which is equal to $|D_\mu|$ by [DJ, Lemma 1.1]. Also, if $1 \leq i \neq j \leq t$, then the n-tuples obtained by shuffling $\underline{a}^{(i,1)}, \dots, \underline{a}^{(i,r)}$ are different from the n-tuples obtained by shuffling $\underline{a}^{(j,1)}, \dots, \underline{a}^{(j,r)}$. Therefore, we deduce that $\dim_F M = |D_\mu| \dim_F p'_\gamma \operatorname{res}^n_\mu M$ and M is generated by $p'_\gamma \operatorname{res}^n_\mu M$ as a left $\mathcal{H}_n^{\mathrm{aff}}$-module.

By Frobenius reciprocity, there exists an isomorphism

$$\operatorname{Hom}_{\mathcal{H}_n^{\mathrm{aff}}}(\operatorname{ind}^n_\mu p'_\gamma \operatorname{res}^n_\mu M, M) \cong \operatorname{Hom}_{\mathcal{H}_\mu^{\mathrm{aff}}}(p'_\gamma \operatorname{res}^n_\mu M, \operatorname{res}^n_\mu M).$$

Hence there exists a homomorphism $f : \operatorname{ind}^n_\mu p'_\gamma \operatorname{res}^n_\mu M \to M$ such that $p'_\gamma \operatorname{res}^n_\mu M \subset \operatorname{Im} f$. By the last sentence in the previous paragraph, f is surjective. On the other hand,

$$\dim_F M = |D_\mu| \dim_F p'_\gamma \operatorname{res}^n_\mu M = \dim_F \operatorname{ind}^n_\mu p'_\gamma \operatorname{res}^n_\mu M.$$

Hence f must be an isomorphism. This completes the proof of the theorem.

3. Dipper–Mathas's Morita equivalence. In this section we shall derive Theorem 1.2 from Theorem 1.8. In fact, we shall give a refined version of Theorem 1.2 (which is actually a block version of Theorem 1.2 if $q \neq 1$) under the assumption that $q \in F^\times$ and $\mathbf{Q} = \mathbf{Q}_1 \sqcup \cdots \sqcup \mathbf{Q}_r$ is a disjoint union of r multi-sets such that the condition (1.1) holds.

A *multicomposition* of n with ℓ components is an ordered ℓ-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ of compositions $\lambda^{(k)}$ such that $|\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}|$ equals to n; $\boldsymbol{\lambda}$ is a *multipartition* whenever its components are partitions and we write $\boldsymbol{\lambda} \vdash n$. The *Young diagram* associated to $\boldsymbol{\lambda}$ is the set $[\boldsymbol{\lambda}] = \{(i, j, k) \mid 1 \leq k \leq \ell, 1 \leq j \leq \lambda_i^{(k)}, i \geq 1\}$. Elements in $[\boldsymbol{\lambda}]$ are called the *nodes* of $\boldsymbol{\lambda}$. A $\boldsymbol{\lambda}$-*tableau* is a bijection $\mathfrak{t} : [\boldsymbol{\lambda}] \to \{1, \dots, n\}$. If $\boldsymbol{\lambda}$ is a multipartition, a *standard* $\boldsymbol{\lambda}$-*tableau* is a $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$ satisfying $\mathfrak{t}(i, j, k) \leq \mathfrak{t}(i', j', k)$ whenever $i \leq i'$ and $j \leq j'$ for all $1 \leq k \leq \ell$.

Let $\Lambda_{\ell,n}^{+}$ be the set of multipartitions $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(\ell)})$ of n with ℓ-components. For each $\boldsymbol{\lambda}\in\Lambda_{\ell,n}^{+}$, let $\operatorname{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux.

Let A be a node which lies in the rth row and cth column of the lth component. The *residue* of the node A is then defined to be $\operatorname{res}(A)=q^{c-r}Q_l$. If $\mathfrak{t}$ is a $\boldsymbol{\lambda}$-tableau and $1\le k\le n$ then the *residue* of k in $\mathfrak{t}$ is $\operatorname{res}_{\mathfrak{t}}(k)=\operatorname{res}(A)$ whenever k occupies the place which A occupies in $[\boldsymbol{\lambda}]$. Let

$$\operatorname{Res}(\Lambda_{\ell,n}^{+}):=\{\operatorname{res}(x)\mid x\in[\boldsymbol{\lambda}]\text{ for some }\boldsymbol{\lambda}\in\Lambda_{\ell,n}^{+}\}$$

be the set of all possible residues. Two multipartitions $\boldsymbol{\lambda},\boldsymbol{\mu}$ are said to be *residue equivalent* if

$$\#\bigl\{x\in[\boldsymbol{\lambda}]\mid\operatorname{res}(x)=f\bigr\}=\#\bigl\{x\in[\boldsymbol{\mu}]\mid\operatorname{res}(x)=f\bigr\}\text{ for any }f\in\operatorname{Res}(\Lambda_{\ell,n}^{+}).$$

Recall the Jucys–Murphy operators $\{L_1,\dots,L_n\}$ of $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$ introduced in the paragraph below Definition 1.1. Applying [DJM, (2.1)], we see that any symmetric polynomial in $L_1,\dots,L_n$ is in the center of $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$. Let $\boldsymbol{\lambda}\in\Lambda_{\ell,n}^{+}$. By [GL, Proposition 2.6], any $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$-module endomorphism of the Specht module $S^{\boldsymbol{\lambda}}$ is a scalar. Therefore, applying [JM, Proposition 3.7], we can deduce that any symmetric polynomial $f(L_1,\dots,L_n)$ in $L_1,\dots,L_n$ acts as the scalar $f(\operatorname{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(1),\dots,\operatorname{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(n))$ on the Specht module $S^{\boldsymbol{\lambda}}$ (and hence on any simple composition factors of $S^{\boldsymbol{\lambda}}$). As a result, if two Specht modules $S^{\boldsymbol{\lambda}},S^{\boldsymbol{\mu}}$ belong to the same block as $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$-modules then $\boldsymbol{\lambda},\boldsymbol{\mu}$ are residue equivalent.

REMARK 3.1. By the main result of [LM], if $q\neq1$ then the converse of the above statement is also true. In other words, in the case when $q\neq1$, the Specht modules $S^{\boldsymbol{\lambda}},S^{\boldsymbol{\mu}}$ belong to the same block as $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$-modules if and only if $\boldsymbol{\lambda},\boldsymbol{\mu}$ are residue equivalent. However, we do not need this in our new proof of Dipper–Mathas's Morita equivalence.

We first prove the following useful lemma.

LEMMA 3.2. *Let* $1\le t\le n$. *Then in* $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$,

$$\Bigl(\prod_{k=1}^{\ell}(L_t-Q_k)\Bigr)\Bigl(\prod_{j=1}^{t-1}(L_t-qL_j)(L_t-q^{-1}L_j)\Bigr)=0.$$

Proof. Let $v,\hat{Q}_1,\dots,\hat{Q}_\ell$ be $\ell+1$ indeterminates over $\mathbb{Z}$. We set

$$\mathcal{A}:=\mathbb{Z}[v^{\pm1},\hat{Q}_1,\dots,\hat{Q}_\ell],\quad \mathcal{K}:=\mathbb{Q}(v,\hat{Q}_1,\dots,\hat{Q}_\ell),\quad \hat{\mathbf{Q}}:=\{\hat{Q}_1,\dots,\hat{Q}_\ell\}.$$

Let $\mathscr{H}_{\ell,n,\mathcal{A}}(\hat{\mathbf{Q}})$ and $\mathscr{H}_{\ell,n,\mathcal{K}}(\hat{\mathbf{Q}})$ be the Ariki–Koike algebra with Hecke parameter v and cyclotomic parameters $\hat{Q}_1,\dots,\hat{Q}_\ell$ and defined over $\mathcal{A}$ and $\mathcal{K}$ respectively. We put a hat on the corresponding symbols (such as Hecke generators, Jucys–Murphy operators and residue etc.) for $\mathscr{H}_{\ell,n,\mathcal{A}}(\hat{\mathbf{Q}})$ and $\mathscr{H}_{\ell,n,\mathcal{K}}(\hat{\mathbf{Q}})$. The map $v\mapsto q$, $\hat{Q}_i\mapsto Q_i$ (for $1\le i\le\ell$), $\hat{T}_k\mapsto T_k$ (for

$1 \le k < n$) and $\hat{L}_a \mapsto L_a$ (for $1 \le a \le n$) can be extended naturally to a $\mathbb{Z}$-algebra homomorphism from $\mathscr{H}_{\ell,n,\mathcal{A}}(\hat{\mathbf{Q}})$ to $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$. Therefore, to prove the lemma, it suffices to show that

$$\Big(\prod_{k=1}^{\ell}(\hat{L}_t - \hat{Q}_k)\Big)\Big(\prod_{j=1}^{t-1}(\hat{L}_t - v\hat{L}_j)(\hat{L}_t - v^{-1}\hat{L}_j)\Big) = 0. \tag{3.1}$$

On the other hand, since $\mathscr{H}_{\ell,n,\mathcal{A}}(\hat{\mathbf{Q}}) \hookrightarrow \mathscr{H}_{\ell,n,\mathcal{K}}(\hat{\mathbf{Q}})$, it suffices to check (3.1) inside the semisimple $\mathcal{K}$-algebra $\mathscr{H}_{\ell,n,\mathcal{K}}(\hat{\mathbf{Q}})$.

Following [Ma, 2.4], let $\{f_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \Lambda^+_{\ell,n}\}$ be the seminormal basis of $\mathscr{H}_{\ell,n,\mathcal{K}}(\hat{\mathbf{Q}})$. By [Ma, 2.6], we have $\hat{L}_k f_{\mathfrak{s}\mathfrak{t}} = \hat{\mathrm{res}}_{\mathfrak{s}}(k) f_{\mathfrak{s}\mathfrak{t}}$ and $f_{\mathfrak{s}\mathfrak{t}}\hat{L}_k = \hat{\mathrm{res}}_{\mathfrak{t}}(k) f_{\mathfrak{s}\mathfrak{t}}$ for any $1 \le k \le n$. Now since $\mathfrak{s}$ is a standard tableau, the integer t sits either in the first row and the first column of the kth component for some $1 \le k \le \ell$, or there is an integer $1 \le j < t$ such that j sits adjacent to t. It follows that

$$\Big(\prod_{k=1}^{\ell}(\hat{\mathrm{res}}_{\mathfrak{s}}(t) - \hat{Q}_k)\Big)\Big(\prod_{j=1}^{t-1}(\hat{\mathrm{res}}_{\mathfrak{s}}(t) - v\hat{\mathrm{res}}_{\mathfrak{s}}(j))(\hat{\mathrm{res}}_{\mathfrak{s}}(t) - v^{-1}\hat{\mathrm{res}}_{\mathfrak{s}}(j))\Big) = 0.$$

Hence

$$\Big(\prod_{k=1}^{\ell}(\hat{L}_t - \hat{Q}_k)\Big)\Big(\prod_{j=1}^{t-1}(\hat{L}_t - v\hat{L}_j)(\hat{L}_t - v^{-1}\hat{L}_j)\Big) f_{\mathfrak{s}\mathfrak{t}} = 0$$

for any $\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and any $\boldsymbol{\lambda} \in \Lambda^+_{\ell,n}$. As a result, we can deduce that (3.1) holds. This completes the proof of the lemma. ∎

Definition 3.3. Let $\mathbf{Q} = \mathbf{Q}_1 \sqcup \cdots \sqcup \mathbf{Q}_r$ be a disjoint union of r nonempty multi-sets with elements in F, where

$$\mathbf{Q}_j = \{Q_{\ell_{j-1}+1}, Q_{\ell_{j-1}+2}, \dots, Q_{\ell_j}\}, \qquad 1 \le j \le r,$$

$\ell_0 := 0$, $\ell_t := |\mathbf{Q}_1| + \cdots + |\mathbf{Q}_t|$ for $1 \le t \le r$ with $\ell_r = \ell$, such that

$$Q_i = Q_j q^k \text{ for some } k \in \mathbb{Z}, \text{ and } Q_i \in \mathbf{Q}_\alpha, Q_j \in \mathbf{Q}_\beta \text{ only if } \alpha = \beta. \tag{3.2}$$

Let $F[L_1, \dots, L_n]^{\mathfrak{S}_n}$ be the F-subalgebra generated by all the symmetric polynomials in $L_1, \dots, L_n$. Set

$$\Sigma := \{(\mathrm{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(1), \dots, \mathrm{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(n)) \mid \boldsymbol{\lambda} \in \Lambda^+_{\ell,n}\}.$$

Recall the equivalence relation "$\sim$" on F^n introduced in Section 1. This induces an equivalence relation on the subset Σ of F^n. We use $\Sigma/\!\sim$ to denote the corresponding set of equivalence classes. The discussion in the paragraph preceding Remark 3.1 implies that for any $\boldsymbol{\lambda} \in \Lambda^+_{\ell,n}$, the map

$$f(L_1, \dots, L_n) \mapsto f(\mathrm{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(1), \dots, \mathrm{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(n)),\ \forall f(L_1, \dots, L_n) \in F[L_1, \dots, L_n]^{\mathfrak{S}_n},$$

is a well-defined F-algebra homomorphism from $F[L_1, \dots, L_n]^{\mathfrak{S}_n}$ to F. Furthermore, this homomorphism depends only on the equivalence class

$\boldsymbol{\gamma}$ of $(\mathrm{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(1),\dots,\mathrm{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(n))$ in Σ. Thus we can denote it by $\chi_{\boldsymbol{\gamma}}$. Since $F[L_1,\dots,L_n]^{\mathfrak{S}_n}$ may not be the full center of $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$, it might be possible that $\chi_{\boldsymbol{\gamma}}$ is not a center character of $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$. Nevertheless, as $F[L_1,\dots,L_n]^{\mathfrak{S}_n}$ is a subalgebra of the center of $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$ and $\chi_{\boldsymbol{\gamma}} : F[L_1,\dots,L_n]^{\mathfrak{S}_n} \to F$ is an F-algebra homomorphism, we can define a submodule category $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$-mod$[\chi_\gamma]$ in a similar way to Definition 1.5.

Let $\boldsymbol{\gamma} \in \Sigma/{\sim}$. By some abuse of notation, for any $M \in \mathscr{H}_{\ell,n,F}(\mathbf{Q})$-mod, let $M[\boldsymbol{\gamma}]$ be the generalized eigenspace over $F[L_1,\dots,L_n]^{\mathfrak{S}_n}$ associated to $\chi_{\boldsymbol{\gamma}}$, that is,

$$M[\boldsymbol{\gamma}] = \{v \in M \mid (f-\chi_{\boldsymbol{\gamma}}(f))^k v = 0 \text{ for all } f \in F[L_1,\dots,L_n]^{\mathfrak{S}_n} \text{ and } k \gg 0\}.$$

Then $M = \bigoplus_{\boldsymbol{\gamma}\in\Sigma/\sim} M[\boldsymbol{\gamma}]$. We define $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$-mod$[\boldsymbol{\gamma}]$ to be the full subcategory with modules $M \in \mathscr{H}_{\ell,n,F}(\mathbf{Q})$-mod satisfying $M = M[\boldsymbol{\gamma}]$. In particular, for the regular $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$-module $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$, we set

$$\mathscr{H}_{\ell,n,F}(\mathbf{Q},\boldsymbol{\gamma}) := \mathscr{H}_{\ell,n,F}(\mathbf{Q})[\boldsymbol{\gamma}],$$

which is an F-subalgebra of $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$. Then

$$\mathscr{H}_{\ell,n,F}(\mathbf{Q}) = \bigoplus_{\boldsymbol{\gamma}\in\Sigma/\sim} \mathscr{H}_{\ell,n,F}(\mathbf{Q},\boldsymbol{\gamma}).$$

We can write $1 = \sum_{\boldsymbol{\gamma}\in\Sigma/\sim} e(\boldsymbol{\gamma})$, where $e(\boldsymbol{\gamma}) \in \mathscr{H}_{\ell,n,F}(\mathbf{Q},\boldsymbol{\gamma})$ for each $\boldsymbol{\gamma}$. It is clear that $\{e(\boldsymbol{\gamma}) | \boldsymbol{\gamma} \in \Sigma/{\sim}\}$ is a set of pairwise orthogonal central idempotents (but not necessarily primitive). Moreover, $\mathscr{H}_{\ell,n,F}(\mathbf{Q},\boldsymbol{\gamma}) = e(\boldsymbol{\gamma})\mathscr{H}_{\ell,n,F}(\mathbf{Q})$ and $e(\boldsymbol{\gamma})$ is the unit element of the subalgebra $\mathscr{H}_{\ell,n,F}(\mathbf{Q},\boldsymbol{\gamma})$. It follows that $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$-mod$[\boldsymbol{\gamma}] = \mathscr{H}_{\ell,n,F}(\mathbf{Q},\boldsymbol{\gamma})$-mod. Note that there is a unique equivalence class in $F^n/{\sim}$ which contains $\boldsymbol{\gamma}$, and we will still denote it by $\boldsymbol{\gamma}$ (by some abuse of notation). Then $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$-mod$[\boldsymbol{\gamma}]$ can be identified with the full subcategory of $\mathcal{H}_n^{\mathrm{aff}}$-mod$[\boldsymbol{\gamma}]$ which consists of modules annihilated by $(X_1 - Q_1)\cdots(X_1 - Q_\ell)$.

Let $\mathbf{Q}$ be the multi-set defined in Definition 3.3. Let $\boldsymbol{\gamma} \in \Sigma/{\sim}$. Then property (3.2) determines a composition $\mu = (\mu_1,\dots,\mu_r)$ of n such that we can choose an n-tuple $\underline{a} = (a_1,\dots,a_n) \in \boldsymbol{\gamma}$ with the property that, for each $1 \le j \le r$,

$$\{\bar{\mu}_{j-1}+1,\dots,\bar{\mu}_j\} = \bigl\{1 \le s \le n \bigm| a_s \in \{Q_i q^k \mid Q_i \in \mathbf{Q}_j,\ k \in \mathbb{Z}\}\bigr\}.$$

Henceforth we fix such an n-tuple $\underline{a}$. For each $1 \le j \le r$, we define $\underline{a}^{(j)} := (a_{\bar{\mu}_{j-1}+1},\dots,a_{\bar{\mu}_j})$ and

$$\begin{aligned}\boldsymbol{\gamma}^{(j)} := \bigl\{&(\mathrm{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(1),\dots,\mathrm{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(\mu_j)) \mid \boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(\ell_j-\ell_{j-1})}) \vdash \mu_j\bigr\}\\ &\cap \mathfrak{S}_{\{\bar{\mu}_{j-1}+1,\dots,\bar{\mu}_j\}} \cdot \underline{a}^{(j)},\end{aligned}$$

where, for each $1 \le k \le \mu_j$, $\mathrm{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(k) = q^{b-a}Q_{\ell_{j-1}+c}$ whenever k lies in the ath row and bth column of the cth component in $\mathfrak{t}^{\boldsymbol{\lambda}}$. As before, there is a unique equivalence class in $F^{\mu_j}/\sim$ which contains $\boldsymbol{\gamma}^{(j)}$, and we still denote it by $\boldsymbol{\gamma}^{(j)}$ (by some abuse of notation). For each $1 \leq j \leq r$, we use $\mathcal{H}_{\mu_j}^{\mathrm{aff}}$ (by some abuse of notation) to denote the subalgebra of $\mathcal{H}_n^{\mathrm{aff}}$ generated by $\{T_i \mid \bar{\mu}_{j-1}+1 \leq i < \bar{\mu}_j\}$ and $X_{\bar{\mu}_{j-1}+1}, \dots, X_{\bar{\mu}_j}$. Let $\mathcal{I}_j$ be the two-sided ideal of $\mathcal{H}_{\mu_j}^{\mathrm{aff}}$ generated by $\prod_{Q_k \in \mathbf{Q}_j}(X_{\bar{\mu}_{j-1}+1} - Q_k)$. We set

$$\mathcal{I}_\mu := \sum_{1 \leq j \leq r} \mathcal{H}_{\mu_1}^{\mathrm{aff}} \otimes \cdots \otimes \mathcal{H}_{\mu_{j-1}}^{\mathrm{aff}} \otimes \mathcal{I}_j \otimes \mathcal{H}_{\mu_{j+1}}^{\mathrm{aff}} \otimes \cdots \otimes \mathcal{H}_{\mu_r}^{\mathrm{aff}}.$$

Then $(\mathscr{H}_{\ell_1,\mu_1,F}(\mathbf{Q}_1, \boldsymbol{\gamma}^{(1)}) \otimes \cdots \otimes \mathscr{H}_{\ell_r-\ell_{r-1},\mu_r,F}(\mathbf{Q}_r, \boldsymbol{\gamma}^{(r)}))$-mod can be identified with the full subcategory of $\mathcal{H}_\mu^{\mathrm{aff}}$-mod$[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})]$ which consists of all modules annihilated by $\mathcal{I}_\mu$. The following theorem is a refined version of Theorem 1.2.

THEOREM 3.4. *With the notation as in Definition* 3.3*, we have the following Morita equivalence:*

$$\mathscr{H}_{\ell,n,F}(\mathbf{Q}, \boldsymbol{\gamma}) \overset{\text{Morita}}{\sim} \mathscr{H}_{\ell_1,\mu_1,F}(\mathbf{Q}_1, \boldsymbol{\gamma}^{(1)}) \otimes \cdots \otimes \mathscr{H}_{\ell_r-\ell_{r-1},\mu_r,F}(\mathbf{Q}_r, \boldsymbol{\gamma}^{(r)}).$$

Proof. The assumption in Definition 3.3 implies that we are in a position to apply Theorem 1.8 to get an equivalence of categories

$$\theta : \mathcal{H}_n^{\mathrm{aff}}\text{-mod}[\boldsymbol{\gamma}] \simeq \mathcal{H}_\mu^{\mathrm{aff}}\text{-mod}[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})].$$

We have the following diagram of functors:

$$\begin{tikzcd}
\mathscr{H}_{\ell,n,F}(\mathbf{Q}, \boldsymbol{\gamma})\text{-mod} \arrow[d, "\iota"'] & \big(\mathscr{H}_{\ell_1,\mu_1,F}(\mathbf{Q}_1, \boldsymbol{\gamma}^{(1)}) \otimes \cdots \otimes \mathscr{H}_{\ell_r-\ell_{r-1},\mu_r,F}(\mathbf{Q}_r, \boldsymbol{\gamma}^{(r)})\big)\text{-mod} \arrow[d, "\iota_1"] \\
\mathcal{H}_n^{\mathrm{aff}}\text{-mod}[\boldsymbol{\gamma}] \arrow[r, "\sim", "\theta"'] & \mathcal{H}_\mu^{\mathrm{aff}}\text{-mod}[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})]
\end{tikzcd}$$

where the two vertical functors are natural embeddings induced by the two surjective homomorphisms

$$\mathcal{H}_n^{\mathrm{aff}} \twoheadrightarrow \mathscr{H}_{\ell,n,F}(\mathbf{Q}), \qquad \mathcal{H}_\mu^{\mathrm{aff}} \twoheadrightarrow \mathscr{H}_{\ell_1,\mu_1,F}(\mathbf{Q}_1) \otimes \cdots \otimes \mathscr{H}_{\ell_r-\ell_{r-1},\mu_r,F}(\mathbf{Q}_r),$$

and the functor $\theta = p'_{\boldsymbol{\gamma}} \operatorname{res}_\mu^n$ defined in the proof of Theorem 1.8.

We first prove that the image of $\theta^{-1} \circ \iota_1$ is contained in the image of ι, where θ^{-1} is the induction functor ind_μ^n.

Let $N \in (\mathscr{H}_{\ell_1,\mu_1,F}(\mathbf{Q}_1, \boldsymbol{\gamma}^{(1)}) \otimes \cdots \otimes \mathscr{H}_{\ell_r-\ell_{r-1},\mu_r,F}(\mathbf{Q}_r, \boldsymbol{\gamma}^{(r)}))$-mod. We claim that

$$\Big(\prod_{k=1}^{\ell}(X_1 - Q_k)\Big) \operatorname{ind}_\mu^n \iota_1(N) = 0. \tag{3.3}$$

Let $M=\operatorname{ind}_\mu^n\iota_1(N)$. Because $p'_{\boldsymbol{\gamma}}\operatorname{res}_\mu^n M\cong\iota_1(N)$, we have

$$\Big(\prod_{k=\ell_{j-1}+1}^{\ell_j}(X_{\bar\mu_{j-1}+1}-Q_k)\Big)p'_{\boldsymbol{\gamma}}\operatorname{res}_\mu^n M=0 \tag{3.4}$$

for every $1\le j\le r$. As in the proof of Theorem 1.8, we can write $p'_{\boldsymbol{\gamma}}\operatorname{res}_\mu^n M=M_{(\underline{a}^{(1,1)},\dots,\underline{a}^{(1,r)})}\oplus\cdots\oplus M_{(\underline{a}^{(t,1)},\dots,\underline{a}^{(t,r)})}$, where $\underline{a}^{(i,j)}\in\boldsymbol{\gamma}^{(j)}$ for any $1\le i\le t$ and $1\le j\le r$. For every simultaneous generalized eigenspace $M_{\underline{c}}$, $\underline{c}\in F^n$, there exists a unique i $(1\le i\le t)$ such that $\underline{c}$ is a shuffling of $(\underline{a}^{(i,1)},\dots,\underline{a}^{(i,r)})$, and we can choose a sequence of simple reflections $s_{i_1},\dots,s_{i_k}$ such that

$$M_{\underline{c}}=M_{w\cdot(\underline{a}^{(i,1)},\dots,\underline{a}^{(i,r)})}=\Theta_{i_1}\cdots\Theta_{i_k}\cdot M_{(\underline{a}^{(i,1)},\dots,\underline{a}^{(i,r)})}$$

where $w=s_{i_1}\cdots s_{i_k}$. By Remark 2.3, $w\mathfrak{t}^\mu$ is a row-standard μ-tableau, hence $w^{-1}(1)=\bar\mu_{j-1}+1$ for some $1\le j\le r$. According to (2.2), we have

$$\begin{aligned}\Big(\prod_{k=1}^{\ell}(X_1-Q_k)\Big)M_{\underline{c}}&=\Big(\prod_{k=1}^{\ell}(X_1-Q_k)\Big)\Theta_{i_1}\cdots\Theta_{i_k}\cdot M_{(\underline{a}^{(i,1)},\dots,\underline{a}^{(i,r)})}\\&=\Theta_{i_1}\cdots\Theta_{i_k}\Big(\prod_{k=1}^{\ell}(X_{w^{-1}(1)}-Q_k)\Big)\cdot M_{(\underline{a}^{(i,1)},\dots,\underline{a}^{(i,r)})}\\&=\Theta_{i_1}\cdots\Theta_{i_k}\Big(\prod_{k=1}^{\ell}(X_{\bar\mu_{j-1}+1}-Q_k)\Big)\cdot M_{(\underline{a}^{(i,1)},\dots,\underline{a}^{(i,r)})}=0,\end{aligned} \tag{3.5}$$

where the last equality follows from (3.4). This completes the proof of our claim (3.3).

To finish the proof of the theorem, it remains to show that the image of $\theta\circ\iota$ is contained in the image of ι_1.

Let $M\in\mathscr{H}_{\ell,n,F}(\mathbf{Q},\boldsymbol{\gamma})$-mod. We claim that for each $1\le j\le r$,

$$\Big(\prod_{k=\ell_{j-1}+1}^{\ell_j}(X_{\bar\mu_{j-1}+1}-Q_k)\Big)(p'_{\boldsymbol{\gamma}}\operatorname{res}_\mu^n\iota(M))=0. \tag{3.6}$$

By the definition of $p'_{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}^{(j)}$, for each integer i with $\bar\mu_{j-1}+1\le i\le\bar\mu_j$, every eigenvalue of X_i on $p'_{\boldsymbol{\gamma}}\operatorname{res}_\mu^n\iota(M)$ is of the form q^aQ_k for some $a\in\mathbb{Z}$ and $\ell_{j-1}<k\le\ell_j$. We are going to use Lemma 3.2 to prove (3.6). By Lemma 3.2, we have

$$\Big(\prod_{k=1}^{\ell}(L_{\bar\mu_{j-1}+1}-Q_k)\Big)\Big(\prod_{i=1}^{\bar\mu_{j-1}}(L_{\bar\mu_{j-1}+1}-qL_i)(L_{\bar\mu_{j-1}+1}-q^{-1}L_i)\Big)=0.$$

As a result,

$$\Big(\prod_{k=1}^{\ell}(X_{\bar\mu_{j-1}+1}-Q_k)\Big)\Big(\prod_{i=1}^{\bar\mu_{j-1}}(X_{\bar\mu_{j-1}+1}-qX_i)(X_{\bar\mu_{j-1}+1}-q^{-1}X_i)\Big)$$

acts as zero on $p'_\gamma \operatorname{res}^n_\mu \iota(M)$. By the sentence at the beginning of this paragraph, for any $1\le i\le \bar\mu_{j-1}$, every eigenvalue of X_i on $p'_\gamma \operatorname{res}^n_\mu \iota(M)$ is of the form q^aQ_k for some $a\in\mathbb{Z}$, $1\le k\le \ell_{j-1}$, and every eigenvalue of $X_{\bar\mu_{j-1}+1}$ is of the form q^bQ_s for some $b\in\mathbb{Z}$, $\ell_{j-1}+1\le s\le \ell_j$. It follows that every eigenvalue of $X_{\bar\mu_{j-1}+1}-q^{\pm1}X_i$ on $p'_\gamma \operatorname{res}^n_\mu \iota(M)$ is of the form $q^bQ_s-q^{a\pm1}Q_k\neq 0$. In particular, $X_{\bar\mu_{j-1}+1}-q^{\pm1}X_i$ restricts to an invertible linear transformation on $p'_\gamma \operatorname{res}^n_\mu \iota(M)$. It follows that $\prod_{k=1}^{\ell}(X_{\bar\mu_{j-1}+1}-Q_k)$ acts as zero on $p'_\gamma \operatorname{res}^n_\mu \iota(M)$.

Let a be any integer such that $1\le a\le \ell_{j-1}$ or $\ell_j<a\le\ell$. By the beginning of the last paragraph, every eigenvalue of $X_{\bar\mu_{j-1}+1}-Q_a$ on $p'_\gamma \operatorname{res}^n_\mu \iota(M)$ is of the form $q^cQ_s-Q_a\neq 0$ for some $c\in\mathbb{Z}$ and $s\in\{\ell_{j-1}+1,\dots,\ell_j\}$. In particular, $X_{\bar\mu_{j-1}+1}-Q_a$ restricts to an invertible linear transformation on $p'_\gamma \operatorname{res}^n_\mu \iota(M)$. It follows that $\prod_{k=\ell_{j-1}+1}^{\ell_j}(X_{\bar\mu_{j-1}+1}-Q_k)$ acts as zero on $p'_\gamma \operatorname{res}^n_\mu \iota(M)$. This proves (3.6), completing the proof of the theorem. ∎

As a consequence of Theorem 3.4, we reprove Dipper–Mathas's Morita equivalence Theorem 1.2. Finally, we remark that Lyle and Mathas's result in [LM] implies that if $q\neq 1$ then each subalgebra $\mathscr{H}_{\ell,n,F}(\mathbf{Q},\gamma)$ is actually a block of $\mathscr{H}_{\ell,n,F}(\mathbf{Q})$.

4. The degenerate case.

In this section, we prove the analogus of Theorems 1.8 and 3.4 in the degenerate case.

DEFINITION 4.1. The *degenerate affine Hecke algebra* of type A is the unital associative F-algebra H_n^{aff} with generators $s_1,\dots,s_{n-1},x_1,\dots,x_n$ and relations

$$\begin{aligned}
&s_i^2=1,\quad 1\le i<n,\\
&s_is_{i+1}s_i=s_{i+1}s_is_{i+1},\quad 1\le i\le n-2,\\
&s_is_k=s_ks_i,\quad |i-k|>1,\quad x_ix_k=x_kx_i,\quad 1\le i,k\le n,\\
&s_ix_is_i=x_{i+1}-s_i,\quad 1\le i<n,\quad s_ix_k=x_ks_i,\quad k\neq i,i+1.
\end{aligned}$$

By [K, Theorem 3.3.1], the center of H_n^{aff} consists of all symmetric polynomials in $x_1,\dots,x_n$. Let $Q_1,\dots,Q_\ell\in F$ and define the multi-set $\mathbf{Q}:=\{Q_1,\dots,Q_\ell\}$.

DEFINITION 4.2. The *degenerate cyclotomic Hecke algebra* $H_{\ell,n}(\mathbf{Q}):=H_{\ell,n,F}(\mathbf{Q})$ of type $G(\ell,1,n)$ is defined to be the quotient of H_n^{aff} by the two-sided ideal generated by $(x_1-Q_1)\cdots(x_1-Q_\ell)$.

For each $1 \leq i \leq n$, we define L_i to be the image of x_i in $H_{\ell,n}(\mathbf{Q})$. The elements $L_1, \ldots, L_n$ are called the *Jucys–Murphy operators* of $H_{\ell,n}(\mathbf{Q})$.

Let $\mu = (\mu_1, \ldots, \mu_r)$ be a composition of n. For any $j \in \mathbb{Z}$, we define the jth *degenerate discriminant* $\underline{\varDelta}_j^{\mu} : F^n \to F$ by

$$\underline{\varDelta}_j^{\mu}(a_1, \ldots, a_n) = \prod (a_s - a_t + j \cdot 1_F),$$

where the product runs over all pairs (s,t) such that $1 \leq s,t \leq n$ and there is no $k \geq 1$ such that both s and t lie in the same interval $[\bar{\mu}_{k-1}+1, \ldots, \bar{\mu}_k]$.

For any composition $\mu = (\mu_1, \ldots, \mu_r)$ of n, let H_{μ}^{aff} be the subalgebra of H_n^{aff} generated by all $s_i \in \mathfrak{S}_{\mu}$ and all x_i, $1 \leq i \leq n$. Then $H_{\mu}^{\mathrm{aff}} \cong H_{\mu_1}^{\mathrm{aff}} \otimes \cdots \otimes H_{\mu_r}^{\mathrm{aff}}$. If ν is a composition of n such that $\mathfrak{S}_{\nu} \subseteq \mathfrak{S}_{\mu}$, then we can define the restriction functor res_{ν}^{μ} and the induction functor ind_{ν}^{μ} in a similar way as in the nondegenerate case.

Following [K, Section 3.8], we need the intertwining elements as follows:

$$\varPhi_k := s_k(x_k - x_{k+1}) + 1_F \in H_n^{\mathrm{aff}}, \quad 1 \leq k < n. \tag{4.1}$$

By [K, Section 3.8], we have the following relations:

$$\varPhi_k^2 = (x_k - x_{k+1} - 1_F)(x_{k+1} - x_k - 1_F), \qquad 1 \leq k < n, \tag{4.2}$$

$$\varPhi_k x_k = x_{k+1}\varPhi_k, \quad \varPhi_k x_{k+1} = x_k \varPhi_k, \quad \varPhi_k x_l = x_l \varPhi_k, \qquad l \neq k, k+1, \tag{4.3}$$

$$\varPhi_k \varPhi_l = \varPhi_l \varPhi_k,\ \varPhi_k \varPhi_{k+1} \varPhi_k = \varPhi_{k+1} \varPhi_k \varPhi_{k+1}, \qquad |k-l| > 1. \tag{4.4}$$

Let $\mu = (\mu_1, \ldots, \mu_r)$ be a composition of n and let $\underline{a} = (a_1, \ldots, a_n) \in F^n$. For each $1 \leq i \leq r$, let $\underline{a}^{(i)} = (a_{\bar{\mu}_{i-1}+1}, \ldots, a_{\bar{\mu}_i})$ be a μ_i-tuple. Set

$$\boldsymbol{\gamma}^{(i)} := \mathfrak{S}_{\{\bar{\mu}_{i-1}+1, \ldots, \bar{\mu}_i\}} \cdot \underline{a}^{(i)}, \quad 1 \leq i \leq r,$$
$$\boldsymbol{\gamma} := \mathfrak{S}_n \cdot \underline{a}, \quad (\boldsymbol{\gamma}^{(1)}, \ldots, \boldsymbol{\gamma}^{(r)}) := \mathfrak{S}_{\mu} \cdot \underline{a}.$$

All the definitions and results in the nondegenerate case can be rephrased in the degenerate setting if we replace $\mathcal{H}_n^{\mathrm{aff}}$, X_k, T_i and $\varTheta_k$ by H_n^{aff}, x_k, s_i and $\varPhi_k$ respectively. In particular, we have the subcategory H_n^{aff}-mod$[\boldsymbol{\gamma}]$ which is the block of H_n^{aff}-mod corresponding to $\boldsymbol{\gamma}$; and the subcategory H_{μ}^{aff}-mod$[(\boldsymbol{\gamma}^{(1)}, \ldots, \boldsymbol{\gamma}^{(r)})]$, which is the block of H_{μ}^{aff}-mod corresponding to $(\boldsymbol{\gamma}^{(1)}, \ldots, \boldsymbol{\gamma}^{(r)})$. The next theorem is the degenerate version of Theorem 1.8.

THEOREM 4.3. *With the notation and definitions as above, assume that both* $\underline{\varDelta}_0^{\mu}(\underline{a})$ *and* $\underline{\varDelta}_1^{\mu}(\underline{a})$ *are nonzero in* F. *Then there exists an equivalence of categories*

$$H_n^{\mathrm{aff}}\text{-mod}[\boldsymbol{\gamma}] \simeq H_{\mu}^{\mathrm{aff}}\text{-mod}[(\boldsymbol{\gamma}^{(1)}, \ldots, \boldsymbol{\gamma}^{(r)})].$$

Proof. As in the nondegenerate case, we have the induction functor

$$\mathrm{ind}_{\mu}^{n} : H_{\mu}^{\mathrm{aff}}\text{-mod}[(\boldsymbol{\gamma}^{(1)}, \ldots, \boldsymbol{\gamma}^{(r)})] \to H_n^{\mathrm{aff}}\text{-mod}[\boldsymbol{\gamma}]$$

and the restriction functor

$$\operatorname{res}_{\mu}^{n} : H_n^{\mathrm{aff}}\text{-mod}[\boldsymbol{\gamma}] \to H_{\mu}^{\mathrm{aff}}\text{-mod}.$$

Let P_n be the subalgebra of H_n^{aff} generated by $x_1, \dots, x_n$. Let $p'_{\boldsymbol{\gamma}}$ be the functor of natural projection from H_{μ}^{aff}-mod onto H_{μ}^{aff}-mod$[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})]$. Let $\underline{b} = (b_1, \dots, b_n) \in F^n$. By abuse of notation, we still use $p'_{\boldsymbol{\gamma}}$ to define a $\mathbb{Z}$-linear map from $K(P_n)$ to $K(P_n)$ such that

$$p'_{\boldsymbol{\gamma}}([L(b_1) \boxtimes \cdots \boxtimes L(b_n)]) := \begin{cases} [L(b_1) \boxtimes \cdots \boxtimes L(b_n)] & \text{if } \underline{b} \in (\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)}), \\ 0 & \text{otherwise.} \end{cases}$$

For any $N \in H_{\mu}^{\mathrm{aff}}$-mod we have $p'_{\boldsymbol{\gamma}}(\operatorname{ch} N) = \operatorname{ch}(p'_{\boldsymbol{\gamma}} N)$. Now using exactly the same argument as in the proof of Theorem 1.8 and the Mackey Theorem [K, §3.5], we can prove that $M \cong p'_{\boldsymbol{\gamma}} \operatorname{res}_{\mu}^{n} \operatorname{ind}_{\mu}^{n} M$ for any M in H_{μ}^{aff}-mod$[(\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(r)})]$.

It remains to show that $M \cong \operatorname{ind}_{\mu}^{n} p'_{\boldsymbol{\gamma}} \operatorname{res}_{\mu}^{n} M$ for any $M \in H_n^{\mathrm{aff}}$-mod$[\boldsymbol{\gamma}]$. This follows from exactly the same argument as in the proof of Theorem 1.8 after replacing X_k, Θ_k and $(c_{k+1} - qc_k)(c_k - qc_{k+1})$ with x_k, Φ_k and $(c_k - c_{k+1} - 1_F)(c_{k+1} - c_k - 1_F)$ respectively. ∎

Let x be an indeterminate over F, and $\mathcal{O} := F[x]$ be the ring of polynomials in x with coefficients in F. Let $F(x)$ be the quotient field of $\mathcal{O}$. Define the multi-set $\hat{\mathbf{Q}} := \{x + Q_1, x^2 + Q_2, \dots, x^{\ell} + Q_{\ell}\}$. Let $H_{\ell,n,F(x)}$ be the degenerate cyclotomic Hecke algebra of type $G(\ell, 1, n)$ with cyclotomic parameters multi-set $\hat{\mathbf{Q}}$ and defined over $F(x)$. Let $H_{\ell,n,\mathcal{O}}$ be the degenerate cyclotomic Hecke algebra of type $G(\ell, 1, n)$ with cyclotomic parameters multi-set $\hat{\mathbf{Q}}$ and defined over $\mathcal{O}$. Let $\pi : \mathcal{O} \to F$ be the canonical homomorphism defined by specializing x to 0. So F can be regarded as an $\mathcal{O}$-algebra via π. In particular,

$$F \otimes_{\mathcal{O}} H_{\ell,n,\mathcal{O}} \cong H_{\ell,n,F}, \qquad H_{\ell,n,\mathcal{O}} \hookrightarrow H_{\ell,n,F(x)}.$$

We need the notions of residue and content in our degenerate setting. Let $\boldsymbol{\lambda} \in \Lambda_{\ell,n}^{+}$. Let A be a node which lies in the rth row and cth column of the lth component. We define the *residue* of the node A to be $\underline{\operatorname{res}}(A) := (c-r)\cdot 1_F + Q_l \in F$, while the *content* of the node A is $\underline{\operatorname{cont}}(A) := (c-r)\cdot 1_F + x^l + Q_l \in F[x]$. If $\mathfrak{t}$ is a $\boldsymbol{\lambda}$-tableau and $1 \le k \le n$ then the *residue* of k in $\mathfrak{t}$ is $\underline{\operatorname{res}}_{\mathfrak{t}}(k) = \underline{\operatorname{res}}(A)$, and similarly the content of k in $\mathfrak{t}$ is $\underline{\operatorname{cont}}_{\mathfrak{t}}(k) = \underline{\operatorname{cont}}(A)$ whenever k occupies the place which A occupies in $[\boldsymbol{\lambda}]$. In particular, $\pi(\underline{\operatorname{cont}}_{\mathfrak{t}}(k)) = \underline{\operatorname{res}}_{\mathfrak{t}}(k)$. We can define the set $\{(\underline{\operatorname{res}}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(1), \dots, \underline{\operatorname{res}}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(n)) \mid \boldsymbol{\lambda} \in \Lambda_{\ell,n}^{+}\}/\!\sim$ as in the nondegenerate case (see the two paragraphs below (3.2)).

Applying the semisimplicity criterion of [AMR, Theorem 6.11], we find that $H_{\ell,n,F(x)}$ is split semisimple. Let $\{g_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \vdash n\}$ be the seminormal basis of $H_{\ell,n,F(x)}$ introduced in [AMR, Definition 6.7] (denoted by

$f_{\mathfrak{st}}$ there). Then for any $1 \le k \le n$, $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \vdash n$,

$$g_{\mathfrak{st}} L_k = \underline{\mathrm{cont}}_{\mathfrak{t}}(k) g_{\mathfrak{st}}, \qquad L_k g_{\mathfrak{st}} = \underline{\mathrm{cont}}_{\mathfrak{s}}(k) g_{\mathfrak{st}}.$$

Lemma 4.4. *Let $1 \le t \le n$. Then in $H_{\ell,n,F}(\mathbf{Q})$,*

$$\Big(\prod_{k=1}^{\ell}(L_t - Q_k)\Big)\Big(\prod_{j=1}^{t-1}(L_t - L_j - 1_F)(L_t - L_j + 1_F)\Big) = 0.$$

Proof. Follow the proof of Lemma 3.2 with $f_{\mathfrak{st}}, \mathcal{A}, \mathcal{K}, \widehat{\mathrm{res}}_{\mathfrak{t}}(k)$ and $v^{\pm 1}\widehat{\mathrm{res}}_{\mathfrak{t}}(k)$ replaced by $g_{\mathfrak{st}}$, $\mathcal{O}$, $F(x)$, $\underline{\mathrm{cont}}_{\mathfrak{t}}(k)$ and $\underline{\mathrm{cont}}_{\mathfrak{t}}(k) \pm 1_F$ respectively. ∎

Definition 4.5. Let $\mathbf{Q} = \mathbf{Q}_1 \sqcup \cdots \sqcup \mathbf{Q}_r$ be a disjoint union of r nonempty multi-sets with elements in F, where

$$\mathbf{Q}_j = \{Q_{\ell_{j-1}+1}, Q_{\ell_{j-1}+2}, \dots, Q_{\ell_j}\}, \qquad 1 \le j \le r,$$

$\ell_0 := 0$, $\ell_t := |\mathbf{Q}_1| + \cdots + |\mathbf{Q}_t|$ for $1 \le t \le r$ with $\ell_r = \ell$, such that

$$\begin{aligned} &Q_i = Q_j + k \cdot 1_F \text{ for some } k \in \mathbb{Z}, \\ &Q_i \in \mathbf{Q}_\alpha,\ Q_j \in \mathbf{Q}_\beta \text{ only if } \alpha = \beta. \end{aligned} \tag{4.5}$$

Let $\mathbf{Q}$ be as defined in Definition 4.5. Let $\boldsymbol{\gamma} \in \big\{(\underline{\mathrm{res}}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(1), \dots, \underline{\mathrm{res}}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(n)) \mid \boldsymbol{\lambda} \in \Lambda^+_{\ell,n}\big\}/{\sim}$. Then property (4.5) determines a composition $\mu = (\mu_1, \dots, \mu_r)$ of n such that we can choose an n-tuple $\underline{a} = (a_1, \dots, a_n) \in \boldsymbol{\gamma}$ with the property that, for each $1 \le j \le r$,

$$\{\bar{\mu}_{j-1} + 1, \dots, \bar{\mu}_j\} = \big\{1 \le s \le n \;\big|\; a_s \in \{Q_i + k \cdot 1_F \mid Q_i \in \mathbf{Q}_j,\ k \in \mathbb{Z}\}\big\}.$$

We fix such an $\underline{a}$. For each $1 \le j \le r$, we define $\underline{a}^{(j)} := (a_{\bar{\mu}_{j-1}+1}, \dots, a_{\bar{\mu}_j})$ and

$$\begin{aligned} \boldsymbol{\gamma}^{(j)} := \{&(\underline{\mathrm{res}}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(1), \dots, \underline{\mathrm{res}}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(\mu_j)) \mid \boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell_j - \ell_{j-1})}) \vdash \mu_j\} \\ &\cap \mathfrak{S}_{\{\bar{\mu}_{j-1}+1, \dots, \bar{\mu}_j\}} \cdot \underline{a}^{(j)}, \end{aligned}$$

where, for each $1 \le k \le \mu_j$, $\underline{\mathrm{res}}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(k) = (b - a) \cdot 1_F + Q_{\ell_{j-1}+c}$ whenever k lies in the ath row and bth column of the cth component in $\mathfrak{t}^{\boldsymbol{\lambda}}$. Let $H_{\ell,n,F}(\mathbf{Q}, \boldsymbol{\gamma})$ be the (nonunital) subalgebra of $H_{\ell,n,F}(\mathbf{Q})$ corresponding to $\boldsymbol{\gamma}$ defined much as in the nondegenerate case (see the two paragraphs below (3.2)).

The following theorem is a degenerate and refined version of Dipper–Mathas's Morita equivalence. In the case when $F = \mathbb{C}$ (the complex numbers field), this was proved by Brundan–Kleshchev [BK1, §5.5] via Schur–Weyl duality of higher levels between degenerate cyclotomic Hecke algebras and a certain parabolic BGG category $\mathcal{O}$ of general linear Lie algebras.

Theorem 4.6. *With the notation as in Definition 4.5 and above, we have the following Morita equivalence:*

$$H_{\ell,n,F}(\mathbf{Q}, \boldsymbol{\gamma}) \overset{\text{Morita}}{\sim} H_{\ell_1,\mu_1,F}(\mathbf{Q}_1, \boldsymbol{\gamma}^{(1)}) \otimes \cdots \otimes H_{\ell_r - \ell_{r-1},\mu_r,F}(\mathbf{Q}_r, \boldsymbol{\gamma}^{(r)}).$$

Proof. Follow the proof of Theorem 3.4 with Theorem 1.8, Lemma 3.2, Assumption 3.2, X_k, $\varTheta_k$ and q^aQ_k replaced by Theorem 4.3, Lemma 4.4, Assumption 4.5, x_k, $\varPhi_k$ and $a\cdot 1_F+Q_k$ respectively. ∎

COROLLARY 4.7. *Suppose* $\mathbf{Q}=\mathbf{Q}_1\sqcup\cdots\sqcup\mathbf{Q}_r$ *be as given in Definition* 4.5. *Then* $H_{\ell,n,F}(\mathbf{Q})$ *is Morita equivalent to the following algebra:*

$$\bigoplus_{\substack{n_1+\cdots+n_r=n\\ n_i\in\mathbb{N},\,\forall i}} H_{\ell_1,n_1,F}(\mathbf{Q}_1)\otimes H_{\ell_2-\ell_1,n_2,F}(\mathbf{Q}_2)\otimes\cdots\otimes H_{\ell_r-\ell_{r-1},n_r,F}(\mathbf{Q}_r).$$

Acknowledgements. The research was supported by the National Natural Science Foundation of China (No. 11525102, 11471315) and the research fund from Zhejiang University. Both authors are grateful to the anonymous referees for their useful comments and suggestions.

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