

Weak separation properties for closed subgroups of locally compact groups

by

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Abstract. Three separation properties for a closed subgroup H of a locally compact group G are studied: (1) the existence of a bounded approximate indicator for H , (2) the existence of a completely bounded invariant projection $\text{VN}(G) \rightarrow \text{VN}_H(G)$, and (3) the approximability of the characteristic function χ_H by functions in $M_{cb}A(G)$ with respect to the weak* topology of $M_{cb}A(G_d)$. We show that the H -separation property of Kaniuth and Lau is characterized by the existence of certain bounded approximate indicators for H and that a discretized analogue of the H -separation property is equivalent to (3). Moreover, we give a related characterization of amenability of H in terms of any group G containing H as a closed subgroup. The weak amenability of G or the fact that G_d satisfies the approximation property, in combination with the existence of a natural projection (in the sense of Lau and Ülger), are shown to suffice to deduce (3). Several consequences of (2) involving the cb-multiplier completion of $A(G)$ are given. Finally, a convolution technique for averaging over the closed subgroup H is developed and used to weaken a condition for the existence of a bounded approximate indicator for H .

1. Introduction. Our objective is to study connections between various forms of amenability for a locally compact group G and certain separation properties for closed subgroups, and moreover to establish relationships between these separation properties. Following the influential work of Ruan [Rua], much work on the homology of the Fourier algebra $A(G)$ as a completely contractive Banach algebra has affirmed this as the appropriate category in which to consider $A(G)$ and the related algebras of abstract harmonic analysis (e.g. [ARS, CT, FKLS, FRS]). Motivated by the success of this perspective, we focus on completely bounded projections, operator amenability, and the completely bounded multiplier algebra of $A(G)$. We consider the following separation properties for a closed subgroup H of G :

2010 *Mathematics Subject Classification*: Primary 43A15; Secondary 43A22, 43A30, 46L07, 22D25.

Key words and phrases: locally compact group, group von Neumann algebra, Fourier algebra, completely bounded multiplier, invariant projection.

Received 9 October 2016; revised 2 March 2017.

Published online 28 April 2017.

- (1) There exists a bounded approximate indicator for H .
- (2) There exists a completely bounded $A(G)$ -bimodule projection of $\text{VN}(G)$ onto $\text{VN}_H(G)$.
- (3) The characteristic function χ_H may be approximated by functions in $B(G)$ or $M_{cb}A(G)$ in the weak* topology on the corresponding algebra of the discretized group.

Condition (3) is also considered for subsets of G that are not necessarily closed subgroups.

Bounded approximate indicators for closed subgroups were introduced in [ARS] as a means of obtaining invariant projections. They have subsequently been shown to have an intimate connection with homological properties of $A(G)$ and its completion in the cb-multipliers, $A_{cb}(G)$ [CT]. A result of Granirer and Leinert [GL, Theorem B₂] yields bounded approximate indicators in $B(G)$ from weaker conditions than given in Definition 2.2. This useful tool is unavailable for nets in $M_{cb}A(G)$, and Section 6 develops a convolution technique relative to the closed subgroup H that recovers the weakened condition in the cb-multiplier setting.

The existence of invariant projections has been studied by several authors in connection with other separation properties and the existence of approximate identities for ideals in $A(G)$ [ARS, CT, D, For 1, FKLS, KaL]. In particular, in [KaL] it is shown that if G has the H -separation property, then an invariant projection $\text{VN}(G) \rightarrow \text{VN}_H(G)$ exists. We show in Section 4 that, in fact, the H -separation property is equivalent to the existence of a bounded approximate indicator for H consisting of positive definite functions that are identically one on H . An analogue of the H -separation property is moreover shown to characterize the $B(G)$ -approximability of χ_H . Condition (3) was first studied in [ARS], where it was claimed that a bounded approximate indicator for H exists whenever χ_H is $B(G)$ -approximable. This argument was later found to contain a gap [ARST]. We give examples in Section 3 showing that the cb-multiplier analogue is false.

Conditions (1) to (3) are related to amenability properties of G and to homological properties of $A(G)$, and it is this connection that is the main focus of the present article. We show in Section 4 that H is already amenable when χ_H is $A(G)$ -approximable for any locally compact group G containing H as a closed subgroup. In the case that G is amenable, the algebra $A(G)$ has a bounded approximate identity and [For 1, Proposition 6.4] then asserts that an invariant projection $\text{VN}(G) \rightarrow \text{VN}_H(G)$ exists exactly when the ideal $I_{A(G)}(H)$ in $A(G)$ has a bounded approximate identity. By [FKLS], the latter occurs for every closed subgroup of G and it follows that an approximate indicator exists for every closed subgroup, since $(1_G - e_\alpha)_\alpha$ is an approximate indicator for H when $(e_\alpha)_\alpha$ is a bounded approximate identity

for $I_{A(G)}(H)$. Thus all closed subgroups of an amenable locally compact group are separated in the strongest sense that we consider. For generic locally compact groups the situation is more complicated, although some strong connections are known to hold in general. For example, using the identity $A(G) \widehat{\otimes} A(G) = A(G \times G)$, it is routine to show that an approximate indicator for the diagonal G_Δ in $A(G \times G)$ (and bounded there) is exactly a bounded approximate diagonal for $A(G)$, the existence of which characterizes amenability of G [Rua]. Moreover, the existence of an invariant projection $\text{VN}(G \times G) \rightarrow \text{VN}_{G_\Delta}(G \times G)$ characterizes the operator biflatness of $A(G)$, a weaker homological condition than operator amenability. Contractive operator biflatness, which asks that this invariant projection be a complete contraction, was recently shown to be equivalent to the existence of a contractive approximate indicator for G_Δ in $B(G \times G)$ [CT].

A bounded approximate indicator for H always yields the approximability of χ_H in the corresponding algebra, but it is unclear when the latter follows from the existence of an invariant projection $\text{VN}(G) \rightarrow \text{VN}_H(G)$ alone. In Section 3 we show that if H satisfies certain weak forms of amenability, then the existence of a bounded map $\text{VN}(G) \rightarrow \text{VN}_H(G)$ satisfying $\lambda(s) \mapsto \chi_H(s)\lambda(s)$ —a weaker condition than the existence of an invariant projection onto $\text{VN}_H(G)$ —implies the $M_{cb}A(G)$ -approximability of χ_H . We give an example in which the former condition fails while the latter holds. Establishing relations amongst the conditions (1) to (3) in the presence of amenability type conditions on H or G is the second goal of this article.

2. Preliminaries. For a locally compact group G , the following algebras were defined by Eymard [E], who established the basic properties we outline below. The space of coefficient functions of strongly continuous unitary representations of G ,

$$(2.1) \quad B(G) = \{\langle \pi(\cdot)\xi \mid \eta \rangle : \pi : G \rightarrow B(\mathcal{H}) \text{ a representation, } \xi, \eta \in \mathcal{H}\},$$

forms a commutative completely contractive Banach algebra, the *Fourier–Stieltjes algebra* of G , under pointwise multiplication and norm

$$\|u\|_{B(G)} = \inf\{\|\xi\| \|\eta\| : u = \langle \pi(\cdot)\xi \mid \eta \rangle \text{ with } \pi, \xi, \eta \text{ as in (2.1)}\}.$$

The operator space structure on $B(G)$ arises from its identification with the dual space of the universal enveloping C^* -algebra of $L^1(G)$ —the *group C^* -algebra* $C^*(G)$ of G —via the duality

$$\langle u, \pi(f) \rangle_{B(G), C^*(G)} = \int_G fu \quad (u \in B(G), f \in L^1(G)).$$

We refer to [ER] for the theory of operator spaces and completely contractive Banach algebras.

The *positive definite functions* in $B(G)$ are those that correspond to positive functionals on $C^*(G)$; they are denoted $P(G)$. For $u \in P(G)$ we have $\|u\|_{B(G)} = \|u\|_{L^\infty(G)} = u(e)$. The adjoint on $B(G)$ is given by

$$u^*(s) = \overline{u(s^{-1})} \quad (u \in B(G), s \in G)$$

and the self-adjoint functions in $B(G)$ correspond to the self-adjoint bounded functionals on $C^*(G)$. Consequently, given $u \in B(G)$ self-adjoint, there are $u^\pm \in P(G)$ such that $u = u^+ - u^-$ and $\|u\|_{B(G)} = \|u^+\|_{B(G)} + \|u^-\|_{B(G)}$.

The *Fourier algebra* of G is the closed ideal $A(G)$ of $B(G)$ given by the coefficients of the left regular representation $\lambda : G \rightarrow B(L^2(G))$, which is defined by

$$\lambda(s)\xi(t) = \xi(s^{-1}t) \quad (s, t \in G, \xi \in L^2(G)).$$

When necessary, we denote this representation of G by λ_G . The Fourier algebra coincides with the closure of the compactly supported functions in $B(G)$, is closed under the adjoint, and is regular in the sense that for any $K \subset G$ compact and $U \supset K$ open, there is a function $v \in A(G)$ with $v(K) = 1$ and $v(G \setminus U) = 0$. The *group von Neumann algebra* of G is the weak operator topology closure $\text{VN}(G)$ of $\text{span } \lambda(G)$ in $B(L^2(G))$ and is identified with the dual of the Fourier algebra via

$$\langle \lambda(s), u \rangle_{\text{VN}(G), A(G)} = u(s) \quad (s \in G, u \in A(G)).$$

Given an algebra \mathcal{A} of functions on G and a subset E of G , we denote by $I_{\mathcal{A}}(E)$ the ideal of functions in \mathcal{A} vanishing on E . For a closed subgroup H of G , the annihilator $I_{A(G)}(H)^\perp$ coincides with the von Neumann algebra $\text{VN}_H(G)$ generated by $\lambda_G(H)$ [TT, Theorem 6], which is identified with $\text{VN}(H)$ via the normal $*$ -isomorphism defined by $\lambda_H(h) \mapsto \lambda_G(h)$ for $h \in H$. The preadjoint of this normal $*$ -isomorphism is the restriction map $r_H : A(G) \rightarrow A(H)$, which is thus a complete quotient. The closed subgroups of G are sets of *spectral synthesis* for $A(G)$ [H], meaning that the ideal $I_{A(G)}(H)$ coincides with the closure of the functions in $A(G)$ that have compact support disjoint from H .

The *completely bounded (cb-) multiplier algebra* $M_{cb}A(G)$ of $A(G)$ is the algebra of functions m on G for which $mA(G) \subset A(G)$ and the map $M_m : \text{VN}(G) \rightarrow \text{VN}(G)$ is completely bounded, where M_m is the adjoint of the multiplication map $u \mapsto mu$ on $A(G)$. Such functions are continuous and bounded, and form a completely contractive Banach algebra under pointwise multiplication and norm $\|m\|_{M_{cb}A(G)} = \|M_m\|_{cb}$. The cb-multipliers of $A(G)$ admit the following representation theorem [J].

THEOREM 2.1 (Gilbert's representation theorem). *Let G be a locally compact group. A function m on G is in $M_{cb}A(G)$ if and only if there exists*

a Hilbert space \mathcal{H} and bounded continuous maps $P, Q : G \rightarrow \mathcal{H}$ such that

$$m(s^{-1}t) = \langle P(t) \mid Q(s) \rangle \quad (s, t \in G).$$

The norm $\|m\|_{M_{cb}A(G)}$ is the infimum of the quantities $\|P\|_\infty \|Q\|_\infty$ taken over all such maps P and Q and Hilbert spaces \mathcal{H} .

It follows from Gilbert's representation theorem that $B(G) \subset M_{cb}A(G)$ and that $\|\cdot\|_{M_{cb}A(G)} \leq \|\cdot\|_{B(G)}$ on $B(G)$. The restriction $r_H : M_{cb}A(G) \rightarrow M_{cb}A(H)$ is a well defined complete contraction [DCH, Proposition 1.12]. The norm closure of $A(G)$ in the potentially smaller cb-multiplier norm is denoted $A_{cb}(G)$ and forms a completely contractive Banach subalgebra of the cb-multipliers with spectrum G [FRS, Proposition 2.2].

Since cb-multipliers of $A(G)$ lie in $L^\infty(G)$, we may consider $L^1(G)$ as a subspace of the dual of $M_{cb}A(G)$. Taking the completion of $L^1(G)$ with respect to the norm given, for $f \in L^1(G)$, by

$$\|f\|_{Q(G)} = \sup \left\{ \left| \int_G fm \right| : m \in M_{cb}A(G) \text{ with } \|m\|_{M_{cb}A(G)} \leq 1 \right\}$$

yields a predual $Q(G)$ for $M_{cb}A(G)$ [DCH, Proposition 1.10]. With this predual, the cb-multipliers $M_{cb}A(G)$ form a completely contractive dual Banach algebra in the sense of Runde [Run 1]. It follows from Theorem 2.1 that $\|\cdot\|_{L^\infty(G)} \leq \|\cdot\|_{M_{cb}A(G)}$ and consequently $\|\cdot\|_{Q(G)} \leq \|\cdot\|_{L^1(G)}$ on $L^1(G)$. We let $\mathcal{C}_c(G)$ and $\mathcal{C}_b(G)$ denote respectively the continuous compactly supported functions and the continuous bounded functions on G . We have

$$A(G) \subset A_{cb}(G), \quad A(G) \subset B(G) \subset M_{cb}A(G) \subset \mathcal{C}_b(G).$$

The second containment is strict unless G is compact. An unpublished result of Ruan asserts that $A(G)$ is closed in $M_{cb}A(G)$ exactly when G is amenable, so that that the first and third containments are strict unless G is amenable. For $u \in A(G)$ and $v \in B(G)$,

$$\begin{aligned} \|u\|_\infty &\leq \|u\|_{A_{cb}(G)} = \|u\|_{M_{cb}A(G)} \leq \|u\|_{A(G)} = \|u\|_{B(G)}, \\ \|v\|_\infty &\leq \|v\|_{M_{cb}A(G)} \leq \|v\|_{B(G)}, \end{aligned}$$

so the first, third, and fourth inclusions are in general contractive while the second is isometric.

A locally compact group G equipped with the discrete topology is denoted G_d . The inclusions $B(G) \subset B(G_d)$ and $M_{cb}A(G) \subset M_{cb}A(G_d)$ are complete isometries ([E] and [Sp, Corollary 6.3], respectively). For each $s \in G$ the point mass $\delta_s \in \ell^1(G_d)$ is contained in $C^*(G_d)$ and in $Q(G_d)$ as the evaluation functional at s , from which it follows that convergence in the weak* topology of $B(G_d)$ or $M_{cb}A(G_d)$ implies pointwise convergence. It will be important for us that, on bounded sets, the converse holds (see [E])

regarding $B(G_d)$, and [FRS, Lemma 2.6] or the useful Appendix A of [Kn] regarding $M_{cb}A(G_d)$).

The separation properties for closed subgroups that we discuss are defined as follows.

DEFINITION 2.2. Let G be a locally compact group and H a closed subgroup.

- (1) A *bounded approximate indicator* for H is a bounded net $(m_\alpha)_\alpha$ in $M_{cb}A(G)$ satisfying
 - (a) $\|ur_H(m_\alpha) - u\|_{A(H)} \rightarrow 0$ for all $u \in A(H)$, and
 - (b) $\|um_\alpha\|_{A(G)} \rightarrow 0$ for all $u \in I_{A(G)}(H)$.
 If $(m_\alpha)_\alpha$ is in $B(G)$ and is also bounded there, then we refer to a bounded approximate indicator in $B(G)$.
- (2) A completely bounded projection $VN(G) \rightarrow VN_H(G)$ is *invariant* if it is an $A(G)$ -bimodule map.
- (3) Given a norm closed subalgebra \mathcal{A} of $B(G)$ or $M_{cb}A(G)$ and $E \subset G$, the characteristic function χ_E is called *\mathcal{A} -approximable* if χ_E is in the weak* closure of \mathcal{A} in $B(G_d)$ or $M_{cb}A(G_d)$, respectively.

The *operator amenability* of the Fourier algebra means the existence of a *bounded approximate diagonal* in the operator space projective tensor product $A(G) \widehat{\otimes} A(G)$. This is a bounded net $(d_\alpha)_\alpha$ in the tensor product for which we have the norm convergence

$$u \cdot d_\alpha - d_\alpha \cdot u \rightarrow 0 \quad \text{and} \quad \Delta(d_\alpha)u \rightarrow u \quad (u \in A(G)),$$

where $\Delta : A(G) \widehat{\otimes} A(G) \rightarrow A(G)$ is the completely bounded linearization of multiplication and the $A(G)$ action on the tensor product is given by $u \cdot (v \otimes w) = uv \otimes w$ and $(v \otimes w) \cdot u = v \otimes wu$ for $u, v, w \in A(G)$. The product map Δ is a complete quotient and the *operator biflatness* of $A(G)$ means the existence of a completely bounded $A(G)$ -bimodule left inverse to its adjoint $\Delta^* : VN(G) \rightarrow VN(G) \overline{\otimes} VN(G)$, where the $A(G)$ action on $VN(G)$ is the dual action. The identification $A(G) \widehat{\otimes} A(G) = A(G \times G)$ (see [ER, Theorem 7.2.4]) yields $\ker \Delta = I_{A(G \times G)}(G_\Delta)$, from which it follows that such a left inverse is exactly an invariant projection $VN(G \times G) \rightarrow VN_{G_\Delta}(G \times G)$. A thorough account of the homological conditions we consider is given in [Run 2].

Leptin's [Le] classical result states that amenability of a locally compact group G is characterized by the existence of a bounded approximate identity in $A(G)$. We call a locally compact group G *weakly amenable* when $A_{cb}(G)$ has a bounded approximate identity. This weaker notion was introduced by De Cannière and Haagerup [DCH], who showed that the free group on two generators is weakly amenable. The weak amenability of G is equivalent to the assertion that every cb-multiplier is the weak* limit of a bounded net in

$A(G)$. When $A(G)$ is merely weak* dense in $M_{cb}A(G)$, the group G is said to have the *approximation property* (see [HK]).

3. Approximability of characteristic functions. In this section, we investigate when characteristic functions of subsets of a locally compact group G are approximable. For a closed subgroup H of G , the assertion that χ_H is approximable may be viewed as a very weak form of subgroup separation. In [ARS], the *discretized Fourier–Stieltjes algebra* $B^d(G)$ is defined to be the weak* closure of $B(G)$ in $B(G_d)$. We make the analogous definition for the cb-multipliers of G .

DEFINITION 3.1. The *discretized cb-multiplier algebra* $M_{cb}^dA(G)$ of a locally compact group G is the weak* closure of $M_{cb}A(G)$ in $M_{cb}A(G_d)$. The predual $Q(G_d)/M_{cb}^d(G)_\perp$ of $M_{cb}^dA(G)$ is denoted $Q^d(G)$. Let $A^d(G)$ and $A_{cb}^d(G)$ denote the weak* closures of $A(G)$ in $B(G_d)$ and in $M_{cb}A(G_d)$, respectively.

Given $E \subset G$, for χ_E to be approximable, it must already be the case that $\chi_E \in M_{cb}A(G_d)$, and the subsets of G for which this occurs are not well understood when G_d is not amenable. In the amenable case, the algebras $M_{cb}A(G_d)$ and $B(G_d)$ coincide [Lo] and the Cohen–Host idempotent theorem provides a complete description of the subsets of G with characteristic function in $B(G_d)$. For discrete groups G , determining the approximable characteristic functions is exactly the problem of determining the sets with characteristic function in the cb-multipliers. When G is moreover weakly amenable, [CT, Corollary 5.4] together with [FRS, Corollary 3.5] implies that such sets E are exactly those for which the ideal $I_{A(G)}(E)$ has a cb-multiplier bounded approximate identity.

EXAMPLE 3.2. Let G be a locally compact group and H a closed subgroup for which a bounded approximate indicator $(m_\alpha)_\alpha$ exists. We show that χ_H is approximable. If $s \in H$, then we may find $u \in A(H)$ with $u(s) = 1$, in which case

$$|m_\alpha(s) - 1| \leq \|ur_H(m_\alpha) - u\|_{L^\infty(H)} \leq \|ur_H(m_\alpha) - u\|_{A(H)} \rightarrow 0.$$

If $s \in G \setminus H$, then [E, Lemme 3.2] asserts that we may find $w \in I_{A(G)}(H)$ with $w(s) = 1$, and

$$|m_\alpha(s)| \leq \|wm_\alpha\|_{L^\infty(G)} \leq \|wm_\alpha\|_{A(G)} \rightarrow 0.$$

Since $M_{cb}A(G)$ is contained in $M_{cb}A(G_d)$ and weak* and pointwise convergence coincide on bounded subsets of $M_{cb}A(G_d)$, it follows that $(m_\alpha)_\alpha$ has weak* limit χ_H in $M_{cb}A(G_d)$.

The algebra $M_{cb}^dA(G)$ is a weak* closed subalgebra of $M_{cb}A(G_d)$, and thus has separately weak* continuous multiplication. If we impose a rather

weak condition on the discrete group G_d , then every function in $M_{cb}A(G_d)$ may be approximated in the weak* topology by functions in $A(G)$:

PROPOSITION 3.3. *Let G be a locally compact group. The inclusion $A_{cb}(G_d) \subset A_{cb}^d(G)$ always holds. Consequently, if G_d has the approximation property, then $M_{cb}A(G_d) = A_{cb}^d(G)$.*

Proof. By [BKLS, Proposition 1] we have $A(G_d) \subset A^d(G)$ and, because weak* convergence in $B(G_d)$ implies weak* convergence in $M_{cb}A(G_d)$, it follows that $A(G_d) \subset A_{cb}^d(G)$. Taking norm closures in $M_{cb}A(G_d)$ yields $A_{cb}(G_d) \subset A_{cb}^d(G)$. If $(e_\alpha)_\alpha$ is a net in $A(G_d)$ converging weak* to 1_G in $M_{cb}A(G_d)$ and if $m \in M_{cb}A(G_d)$, then $me_\alpha \in A(G_d)$ for each α and $me_\alpha \xrightarrow{w^*} m$ in $M_{cb}A(G_d)$, implying that $m \in A_{cb}^d(G)$. ■

In [ARS, Section 3], that the Fourier–Stieltjes algebra is the dual of a C^* -algebra is noted to imply that the unique weak* continuous extension of the inclusion $B(G) \subset B^d(G)$ is a quotient map $B(G)^{**} \rightarrow B^d(G)$. We provide a more concrete construction of the analogous canonical map $M_{cb}A(G)^{**} \rightarrow M_{cb}^dA(G)$ that exploits the relation between $M_{cb}A(G)$ and $M_{cb}^dA(G)$. Let $\iota_d : M_{cb}A(G) \rightarrow M_{cb}A(G_d)$ and $\kappa_Q : Q(G_d) \rightarrow Q(G_d)^{**}$ denote the inclusion maps. By the bipolar theorem $M_{cb}^dA(G)_\perp = M_{cb}A(G)_\perp$ and we have

$$\kappa_Q(M_{cb}A(G)_\perp) \subset M_{cb}A(G)_\perp^\perp = (\text{im } \iota_d)^\perp = \ker \iota_d^*$$

which together imply that the composition

$$Q(G_d) \xrightarrow{\kappa_Q} Q(G_d)^{**} \xrightarrow{\iota_d^*} Q(G)^{**}$$

induces a map $Q^d(G) \rightarrow Q(G)^{**}$. Denote the adjoint of this induced map by $\tau : M_{cb}A(G)^{**} \rightarrow M_{cb}^dA(G)$. It is straightforward to verify that τ extends the inclusion $M_{cb}A(G) \subset M_{cb}^dA(G)$, so that $\tau(m)(s) = m(s)$ for $m \in M_{cb}A(G)$ and $s \in G$. It follows that if a net $(m_\alpha)_\alpha$ in $M_{cb}A(G)$ converges weak* to $\omega \in M_{cb}A(G)^{**}$, then

$$\tau(\omega)(s) = \lim_\alpha m_\alpha(s) \quad (s \in G),$$

so that the map τ extracts pointwise limits from nets in $M_{cb}A(G)$ that are weak* convergent in the bidual. Moreover, the range of τ consists of exactly those functions in $M_{cb}^dA(G)$ that are limits of bounded nets in $M_{cb}A(G)$.

PROPOSITION 3.4. *Let G be a locally compact group and $E \subset G$. If there is a bounded map $\Psi : \text{VN}(G) \rightarrow \text{VN}(G)$ satisfying $\Psi(\lambda(s)) = \chi_E(s)\lambda(s)$ for all $s \in G$, then $\chi_E A(G) \subset A_{cb}^d(G)$. If moreover $\chi_E \in M_{cb}A(G_d)$ and 1_G is $A_{cb}(G)$ -approximable, then χ_E is $A_{cb}(G)$ -approximable.*

Proof. Let $\kappa_A : A(G) \rightarrow A(G)^{**}$ and $\iota_A : A(G) \rightarrow M_{cb}A(G)$ be the inclusions and let σ denote the composition

$$A(G) \xrightarrow{\kappa_A} A(G)^{**} \xrightarrow{\Psi^*} A(G)^{**} \xrightarrow{\iota_A^{**}} M_{cb}A(G)^{**} \xrightarrow{\tau} M_{cb}^dA(G).$$

For $u \in A(G)$ and $s \in G$, with δ_s denoting the point mass at s in $\ell^1(G_d) \subset Q(G_d) \subset Q^d(G)$, we have

$$\begin{aligned} \sigma(u)(s) &= \langle \sigma(u), \delta_s \rangle_{M_{cb}^d A(G), Q^d(G)} = \langle \iota_A^{**} \Psi^* \kappa_A(u), \iota_d^* \kappa_Q(\delta_s) \rangle_{M_{cb} A(G)^{**}, M_{cb} A(G)^*} \\ &= \langle \Psi^* \kappa_A(u), \lambda(s) \rangle_{A(G)^{**}, \text{VN}(G)} = \langle \Psi(\lambda(s)), u \rangle_{\text{VN}(G), A(G)} \\ &= \chi_E(s)u(s). \end{aligned}$$

Thus $\chi_E u = \sigma(u) \in M_{cb}^d A(G)$ for all $u \in A(G)$. Since τ extends the inclusion of $M_{cb} A(G)$ into $M_{cb}^d A(G)$, it maps $A(G)$ into $A_{cb}^d(G)$, which together with the weak* continuity of $\tau \iota_A^{**}$ implies that σ in fact has range in $A_{cb}^d(G)$. If $(e_\alpha)_\alpha$ is a net in $A(G)$ converging weak* to 1_G in $M_{cb} A(G_d)$, then $\chi_E \in M_{cb} A(G_d)$ implies $\chi_E e_\alpha \xrightarrow{w^*} \chi_E$, by weak* continuity of multiplication in $M_{cb} A(G_d)$. Since $\chi_E e_\alpha \in A_{cb}^d(G)$, we conclude that χ_E is $A_{cb}(G)$ -approximable. ■

For a subgroup H of a locally compact group G , it is straightforward to verify that χ_H is a positive definite function on G_d , so that $\chi_H \in B(G_d) \subset M_{cb} A(G_d)$ and the second part of Proposition 3.4 is applicable to characteristic functions of subgroups. It is shown in Lemma 3.8 below that when $\chi_H A(G) \subset A_{cb}^d(G)$, we need only require 1_H to be $A_{cb}(H)$ -approximable to deduce that χ_H is $A_{cb}(G)$ -approximable.

Lau and Ülger [LÜ] define a projection Ψ on $\text{VN}(G)$ to be *natural* if $\Psi(\lambda(s)) = \chi_E(s)\lambda(s)$ for some subset E of G . We may interpret Proposition 3.4 as imposing restrictions on which subsets of G can arise from a natural projection.

EXAMPLE 3.5. Let G be a locally compact group and H a closed subgroup. We show that an invariant projection $\Psi : \text{VN}(G) \rightarrow \text{VN}_H(G)$ is natural. It is clear that $\Psi(\lambda(s)) = \lambda(s)$ for $s \in H$. Let $s \in G \setminus H$ and let $T_\alpha \in \text{span } \lambda(H)$ converge weak* to $\Psi(\lambda(s))$ in $\text{VN}(G)$. If $u \in A(G)$ with $u(s) = 1$ and $u|_H = 0$, then

$$\langle u \cdot \lambda(s), v \rangle = \langle \lambda(s), vu \rangle = v(s)u(s) = v(s) = \langle \lambda(s), v \rangle \quad (v \in A(G)),$$

so $u \cdot \lambda(s) = \lambda(s)$. If $S = \sum_j \alpha_j \lambda(s_j) \in \text{span } \lambda(H)$, then

$$\langle u \cdot S, v \rangle = \sum_j \alpha_j v(s_j)u(s_j) = 0 \quad (v \in A(G)),$$

so that $u \cdot S = 0$. Thus $0 = u \cdot T_\alpha \xrightarrow{w^*} u \cdot \Psi(\lambda(s)) = \Psi(u \cdot \lambda(s)) = \Psi(\lambda(s))$ and $\Psi(\lambda(s)) = 0$.

Let G be a locally compact group. A bounded net $(e_\alpha)_\alpha$ in $A(G)$ or $A_{cb}(G)$ is called a Δ -weak bounded approximate identity if it converges pointwise to 1_G . This notion was introduced in [JL] and shown in [LÜ] to be closely related to the existence of natural projections. Reasoning as in Example 3.2 shows that a bounded approximate identity for $A_{cb}(G)$ is a Δ -weak

one, so that $A_{cb}(G)$ has Δ -weak bounded approximate identity whenever G is weakly amenable. The function 1_G is $A_{cb}(G)$ -approximable when $A_{cb}(G)$ has a Δ -weak bounded approximate identity.

The construction of the map τ above and the proof of Proposition 3.4 may be carried out with $M_{cb}A(G)$ replaced by $B(G)$, but to conclude that χ_H is $B(G)$ -approximable using this result, we require 1_G to be in the weak* closure of $A(G)$ in $B(G_d)$. The proof of Theorem 4.6 shows that the canonical map $A(G)^{**} \rightarrow A^d(G)$ extending the inclusion $A(G) \subset A^d(G)$ is surjective, so that 1_G is then the weak* limit of a bounded net, which is then a Δ -weak bounded approximate identity for $A(G)$, implying that G is already amenable [KÜ, Theorem 5.1]. It is the availability of Δ -weak bounded approximate identities in $A_{cb}(G)$ for a larger class of groups—containing at least the weakly amenable groups—that is responsible for the utility of Proposition 3.4. Whether the existence of a Δ -weak bounded approximate identity for $A_{cb}(G)$ implies weak amenability of G appears to be an open question.

For a locally compact group G and a closed subgroup H , let

$$r_H : M_{cb}A(G_d) \rightarrow M_{cb}A(H_d), \quad e_H : M_{cb}A(H_d) \rightarrow M_{cb}A(G_d)$$

denote respectively the restriction map and the extension by zero. For a function f on H let \mathring{f} denote its extension by zero to G . The restriction r_H is a complete quotient and the extension e_H is a complete isometry ([Sp, Corollary 6.3] or [St, Proposition 4.1]), and it is clear that $r_H e_H = \text{id}_{M_{cb}A(H_d)}$ and $e_H r_H = M_{\chi_H}$, the multiplication by χ_H .

LEMMA 3.6. *Let G be a locally compact group and H a closed subgroup. The maps r_H and e_H are weak* continuous.*

Proof. If $f = \sum_{j=1}^n \alpha_j \delta_{x_j} \in \mathcal{C}_c(H_d)$, then $\mathring{f} \in \mathcal{C}_c(G_d)$ and

$$\langle r_H^*(f), m \rangle = \sum_{j=1}^n \alpha_j m(x_j) = \langle \mathring{f}, m \rangle \quad (m \in M_{cb}A(G_d)),$$

showing that $r_H^*(\mathcal{C}_c(H_d)) \subset Q(G_d)$. Since $\mathcal{C}_c(H_d)$ is dense in $Q(H_d)$, it follows that r_H is weak* continuous.

Now if $f = \sum_{j=1}^n \alpha_j \delta_{x_j} \in \mathcal{C}_c(G_d)$, then, for $m \in M_{cb}A(H_d)$,

$$\langle e_H^*(f), m \rangle = \langle \mathring{m}, f \rangle = \sum_{j=1}^n \alpha_j \chi_H(x_j) m(x_j) = \left\langle \sum_{j=1}^n \alpha_j \chi_H(x_j) \delta_{x_j}, m \right\rangle,$$

and so $\sum_{j=1}^n \alpha_j \chi_H(x_j) \delta_{x_j} \in \mathcal{C}_c(H_d)$. Therefore $e_H^*(\mathcal{C}_c(G_d)) \subset Q(H_d)$ and the claim follows by density, as above. ■

LEMMA 3.7. *Let G be a locally compact group and H a closed subgroup. The restriction r_H maps $M_{cb}^d A(G)$ into $M_{cb}^d A(H)$. In addition, the following*

are equivalent:

- (1) $\chi_H A(G) \subset A_{cb}^d(G)$.
- (2) $e_H(A_{cb}^d(H)) \subset A_{cb}^d(G)$.
- (3) $A_{cb}^d(G) = I_{A_{cb}^d(G)}(G \setminus H) \oplus I_{A_{cb}^d(G)}(H)$ (algebraic direct sum).

Proof. Since the restriction of a cb-multiplier of G to the closed subgroup H yields a cb-multiplier of H [DCH, Proposition 1.12], the first claim follows from weak* continuity of r_H .

(1) \Rightarrow (2). If $\chi_H A(G) \subset A_{cb}^d(G)$, then, because $A(H) = r_H(A(G))$,

$$e_H(A(H)) = e_H(r_H(A(G))) = \chi_H A(G) \subset A_{cb}^d(G),$$

and (2) follows by weak* continuity of e_H .

(2) \Rightarrow (3). If $e_H(A_{cb}^d(H)) \subset A_{cb}^d(G)$, then given $m \in A_{cb}^d(G)$, the weak* continuity of r_H implies $r_H(m) \in A_{cb}^d(H)$ and it follows that $\chi_H m = e_H r_H(m) \in A_{cb}^d(G)$, whence $\chi_{G \setminus H} m = m - \chi_H m \in A_{cb}^d(G)$, and therefore $m = \chi_H m + \chi_{G \setminus H} m \in I_{A_{cb}^d(G)}(G \setminus H) + I_{A_{cb}^d(G)}(H)$. These ideals clearly have trivial intersection.

(3) \Rightarrow (1). Write $m \in A(G)$ as $m = m_1 + m_2$ for $m_1 \in I_{A_{cb}^d(G)}(G \setminus H)$ and $m_2 \in I_{A_{cb}^d(G)}(H)$, in which case $\chi_H m = m_1 \in A_{cb}^d(G)$. ■

When the equivalent conditions of Lemma 3.7 hold, condition (2) implies that $e_H(A_{cb}^d(H)) = I_{A_{cb}^d(G)}(G \setminus H)$, and because e_H is isometric, condition (3) asserts that $A_{cb}^d(G) = A_{cb}^d(H) \oplus I_{A_{cb}^d(G)}(H)$.

LEMMA 3.8. *Let G be a locally compact group and H a closed subgroup. If $\chi_H A(G) \subset A_{cb}^d(G)$ and 1_H is $A_{cb}(H)$ -approximable, then χ_H is $A_{cb}(G)$ -approximable.*

Proof. It follows from Lemma 3.7(2) that $\chi_H = e_H(1_H) \in A_{cb}^d(G)$. ■

Combining the results of this section, we obtain the following.

THEOREM 3.9. *The characteristic function of a closed subgroup H of a locally compact group G is $A_{cb}(G)$ -approximable when either of the following conditions is satisfied:*

- (1) G_d has the approximation property.
- (2) There is a bounded map $\Psi : \text{VN}(G) \rightarrow \text{VN}(G)$ such that $\Psi(\lambda(s)) = \chi_H(s)\lambda(s)$ for $s \in G$, which is satisfied if Ψ is a natural or invariant projection onto $\text{VN}_H(G)$, and 1_H is $A_{cb}(H)$ -approximable, which occurs when H is weakly amenable or H_d has the approximation property.

Proof. (1) When G_d has the approximation property, Proposition 3.3 asserts that $A_{cb}^d(G) = M_{cb}A(G_d)$. Since $M_{cb}A(G_d)$ contains the characteristic functions of all subgroups of G , we have $\chi_H \in A_{cb}^d(G)$.

(2) If $\Psi : \text{VN}(G) \rightarrow \text{VN}_H(G)$ is a bounded map such that $\Psi(\lambda(s)) = \chi_H(s)\lambda(s)$ for $s \in G$ and 1_H is $A_{cb}(H)$ -approximable, then $\chi_H A(G) \subset A_{cb}^d(G)$ by Proposition 3.4. Lemma 3.8 then asserts that χ_H is $A_{cb}(G)$ -approximable. Example 3.5 shows that invariant projections $\text{VN}(G) \rightarrow \text{VN}_H(G)$ are natural, and the latter are by definition bounded maps satisfying (2). It was noted above that weak amenability of H implies $1_H \in A_{cb}^d(H)$, while Proposition 3.3 implies $1_H \in A_{cb}^d(H)$ when H_d has the approximation property. ■

Theorem 3.7 of [ARS] claims that a bounded approximate indicator in $B(G)$ for a closed subgroup H of the locally compact group G exists whenever χ_H is $B(G)$ -approximable. The argument establishing this result was found to contain an error [ARST] and it is not known whether the claim holds. The following examples show that χ_H may be $M_{cb}A(G)$ -approximable even when no invariant projection onto $\text{VN}_H(G)$ exists, in which case no bounded approximate indicator for H exists either, by Proposition 5.1.

EXAMPLE 3.10. The locally compact group $G = \text{SL}(2, \mathbb{R})$ contains $H = \mathbb{F}_2$ as a closed subgroup. It has recently been shown that G_d is weakly amenable [KnL], so that χ_H is $A_{cb}(G)$ -approximable by Proposition 3.3. Since G is connected, its group von Neumann algebra is injective [Co, Corollary 6.9(c)], so there exists a completely bounded projection $B(L^2(G)) \rightarrow \text{VN}(G)$. If a completely bounded projection $\text{VN}(G) \rightarrow \text{VN}_H(G)$ existed, then composition would yield a completely bounded projection $B(L^2(G)) \rightarrow \text{VN}_H(G)$, implying that $\text{VN}_H(G) = \text{VN}(H)$ is an injective von Neumann algebra [CS]. It would follow that the discrete group H is amenable [Pat, (2.35)], which is false.

EXAMPLE 3.11. Let $G = \text{SL}(2, \mathbb{R})$ and consider the diagonal subgroup G_Δ of $G \times G$. The Fourier algebra $A(G)$ is not operator biflat by [CT, Corollary 3.7], meaning that no invariant projection $\text{VN}(G \times G) \rightarrow \text{VN}_{G_\Delta}(G \times G)$ exists. But the weak amenability of G_d , noted in the preceding example, implies the weak amenability of $(G \times G)_d$, so that χ_{G_Δ} is $A_{cb}(G \times G)$ -approximable, again by Proposition 3.3.

4. The discretized H -separation property. In this section, we characterize the approximability of the characteristic function of a closed subgroup H of a locally compact group G in the spirit of the H -separation property of Kaniuth and Lau. For a closed subgroup H of G , let $P_H(G)$ denote the norm closed convex set $\{u \in P(G) : u(H) = 1\}$.

DEFINITION 4.1 ([KaL]). A locally compact group G is said to have the H -separation property for a closed subgroup H if, for each $s \in G \setminus H$, there exists $u \in P_H(G)$ such that $u(s) \neq 1$.

It is routine to verify that G has the H -separation property for any open, compact, or normal subgroup H , and it was shown by Forrest [For 2] that if G is a SIN group, then G has the H -separation property for every closed subgroup H . In [KaL], a fixed point argument is used to show that an invariant projection $\text{VN}(G) \rightarrow \text{VN}_H(G)$ exists when the locally compact group G has the H -separation property (it is noted in Proposition 5.1 below that the projections arising this way are completely positive, in particular completely bounded). In fact, the following stronger result holds.

PROPOSITION 4.2. *Let G be a locally compact group and H a closed subgroup. Then G has the H -separation property if and only if there exists a bounded approximate indicator for H in $P_H(G)$.*

Proof. Suppose that G has the H -separation property. The proof of [KaL, Proposition 3.1] constructs an invariant projection $P : \text{VN}(G) \rightarrow \text{VN}_H(G)$ that is the weak* operator topology limit of a net $(M_{u_\alpha})_\alpha$, where $u_\alpha \in P_H(G)$ and $M_{u_\alpha} : \text{VN}(G) \rightarrow \text{VN}(G)$ is the adjoint of the multiplication map $u \mapsto u_\alpha u$ on $A(G)$. Let $r_H : A(G) \rightarrow A(H)$ be the restriction map, which is a surjection satisfying $r_H^*(\text{VN}(H)) \subset \text{VN}_H(G)$. Given $u \in A(H)$, let $\tilde{u} \in A(G)$ with $r_H(\tilde{u}) = u$, so that for $T \in \text{VN}(H)$,

$$\begin{aligned} \langle T, ur_H(u_\alpha) \rangle &= \langle T, r_H(\tilde{u}u_\alpha) \rangle = \langle M_{u_\alpha}(r_H^*(T)), \tilde{u} \rangle \\ &\rightarrow \langle P(r_H^*(T)), \tilde{u} \rangle = \langle r_H^*(T), \tilde{u} \rangle = \langle T, u \rangle. \end{aligned}$$

If $w \in I_{A(G)}(H)$, then

$$\langle T, wu_\alpha \rangle = \langle M_{u_\alpha}(T), w \rangle \rightarrow \langle P(T), w \rangle = 0 \quad (T \in \text{VN}(G))$$

since $P(T) \in \text{VN}_H(G) = I_{A(G)}(H)^\perp$. Therefore $ur_H(u_\alpha) \rightarrow u$ weakly in $A(H)$ for all $u \in A(H)$ and $wu_\alpha \rightarrow 0$ weakly in $A(G)$ for all $w \in I_{A(G)}(H)$. Passing to convex combinations yields a bounded approximate indicator for H which remains in the convex set $P_H(G)$.

Conversely, if $(u_\alpha)_\alpha$ is a bounded approximate indicator for H in $P_H(G)$, then, given $s \in G \setminus H$, choose $w \in I_{A(G)}(H)$ with $w(s) = 1$, in which case

$$|u_\alpha(s)| = |u_\alpha(s)w(s)| \leq \|u_\alpha w\|_{L^\infty(G)} \leq \|u_\alpha w\|_{A(G)} \rightarrow 0$$

implies $u_\alpha(s) \neq 1$ eventually. ■

For a closed subgroup H of a locally compact group G , we now show that a weaker form of the H -separation property, replacing the algebra $B(G)$ with $B^d(G)$, characterizes when χ_H is $B(G)$ -approximable.

DEFINITION 4.3. Let G be a locally compact group and H a closed subgroup. Then G is said to have the *discretized H -separation property* if, for any $s \in G \setminus H$, there exists $u \in B^d(G) \cap P_H(G_d)$ such that $u(s) \neq 1$.

PROPOSITION 4.4. *Let G be a locally compact group and H a closed subgroup. Then G has the discretized H -separation property if and only if χ_H is $B(G)$ -approximable.*

Proof. Suppose that G has the discretized H -separation property and for each $s \in G \setminus H$ let $u_s \in B^d(G) \cap P_H(G_d)$ with $u_s(s) \neq 1$. Replacing u_s by $\frac{1}{2}(1_G + u_s)$, which remains in $B^d(G) \cap P_H(G_d)$, we may assume that $|u_s(s)| < 1$. Then the sequence $(u_s^n)_{n \geq 1}$ is in $B^d(G) \cap P_H(G_d)$ with $\|u_s^n\|_{B(G_d)} = u_s^n(e) = 1$, and thus has a weak* cluster point u_s^0 in the unit ball of $B^d(G)$. Then $u_s^0|_H = 1$ and $|u_s^0(s)| \leq \limsup_n |u_s(s)|^n = 0$, so $u_s^0(s) = 0$. Let \mathcal{F} be the collection of finite subsets of G and for each $F \in \mathcal{F}$ let $u_F = \prod_{s \in F} u_s^0$. Ordering \mathcal{F} by inclusion, we have $u_F|_H = 1$ and $u_F(s) = 0$ eventually for each $s \in G \setminus H$, so that $u_F \xrightarrow{ptw} \chi_H$ and by boundedness $u_F \xrightarrow{w^*} \chi_H$ in $B(G_d)$, whence $\chi_H \in B^d(G)$. The converse is clear, given that the characteristic function of a subgroup is always in $P_H(G_d)$. ■

When the locally compact group G is second countable, the H -separation property may also be characterized in terms of a single function on G .

THEOREM 4.5. *Let G be a second countable locally compact group. For a closed subgroup H , the following are equivalent:*

- (1) G has the H -separation property.
- (2) There is $u \in B(G)$ of norm one with $\{s \in G : u(s) = 1\} = H$.
- (3) There is $u \in P(G)$ with $\{s \in G : u(s) = 1\} = H$.

Proof. (1) \Rightarrow (2). For $s \in G \setminus H$, let $u_s \in P_H(G)$ with $u_s(s) \neq 1$ and choose an open neighborhood U_s of s with $1 \notin u_s(U_s)$. Then $(U_s)_{s \in G \setminus H}$ is an open cover of $G \setminus H$, so has a countable subcover $(U_{s_n})_{n \geq 1}$ by σ -compactness of the open set $G \setminus H$ in G . The function $u = \sum_{n \geq 1} 2^{-n} u_{s_n}$ is in the norm closed convex set $P_H(G)$, so that $\|u\|_{B(G)} = u(e) = 1$. Given $s \in G \setminus H$, choose n such that $s \in U_{s_n}$, so $u_{s_n}(s) \neq 1$. Since $\|u_{s_n}\|_{L^\infty(G)} = 1$, we have $\operatorname{Re} u_{s_n}(s) < 1$, implying that $\operatorname{Re} u(s) = \sum_{n \geq 1} 2^{-n} \operatorname{Re} u_{s_n}(s) < 1$ and hence that $u(s) \neq 1$.

(2) \Rightarrow (3). Replacing u by $\frac{1}{2}(u + u^*)$, where $u^*(s) = \overline{u(s^{-1})}$, we obtain a function of $B(G)$ -norm one (because the $B(G)$ -norm dominates the $L^\infty(G)$ -norm) for which we may write $u = u^+ - u^-$ with $u^\pm \in P(G)$ satisfying $\|u\|_{B(G)} = \|u^+\|_{B(G)} + \|u^-\|_{B(G)}$. Then

$$\begin{aligned} u^+(e) - u^-(e) &= u(e) = 1 = \|u\|_{B(G)} \\ &= \|u^+\|_{B(G)} + \|u^-\|_{B(G)} = u^+(e) + u^-(e), \end{aligned}$$

and consequently $\|u^-\|_{B(G)} = u^-(e) = 0$, so that $u = u^+ \in P_H(G)$.

(3) \Rightarrow (1). This is clear. ■

The amenability of H is known to imply the existence of an invariant projection $\text{VN}(G) \rightarrow \text{VN}_H(G)$ for any locally compact group G containing H as a closed subgroup [Cr, Corollary 3.7] (see also [D]). We now show that the former condition may be characterized in terms of a separation property relative to any such G .

THEOREM 4.6. *A locally compact group H is amenable if and only if χ_H is $A(G)$ -approximable for some (equivalently, any) locally compact group G containing H as a closed subgroup.*

Proof. Fix a locally compact group G that contains H as a closed subgroup.

Suppose that H is amenable. Let $(e_\alpha)_\alpha$ be a bounded approximate identity for $A(H)$ and let $\Psi : \text{VN}(G) \rightarrow \text{VN}_H(G)$ be an invariant projection. Example 3.5 shows that $\Psi(\lambda_G(s)) = \chi_H(s)\lambda_G(s)$ for all $s \in G$ and the argument of Example 3.2 shows that $e_\alpha \xrightarrow{ptw} 1_H$. Recall that the adjoint of the restriction $r_H : A(G) \rightarrow A(H)$ is a $*$ -isomorphism $\tau = r_H^* : \text{VN}(H) \rightarrow \text{VN}_H(G)$ taking $\lambda_H(s)$ to $\lambda_G(s)$ for all $s \in H$. The composition

$$\text{VN}(H)^* \xrightarrow{(\tau^*)^{-1}} \text{VN}_H(G)^* \xrightarrow{\Psi^*} \text{VN}(G)^*$$

satisfies, for $s \in G$,

$$\begin{aligned} \langle \Psi^*(\tau^*)^{-1}(e_\alpha), \lambda_G(s) \rangle &= \langle (\tau^*)^{-1}(e_\alpha), \chi_H(s)\lambda_G(s) \rangle \\ &= \langle e_\alpha, \tau^{-1}(\chi_H(s)\lambda_G(s)) \rangle \\ &= \begin{cases} \langle e_\alpha, \lambda_H(s) \rangle, & s \in H, \\ 0, & s \notin H \end{cases} \\ &\rightarrow \chi_H(s). \end{aligned}$$

Let E be a weak* cluster point of the bounded net $(\Psi^*(\tau^*)^{-1}(e_\alpha))_\alpha$ in $\text{VN}(G)^*$, so that $\langle E, \lambda_G(s) \rangle = \chi_H(s)$ for all $s \in G$ by the above computation. Letting $(u_\alpha)_\alpha$ be a bounded net in $A(G)$ converging weak* to E , we have $u_\alpha \xrightarrow{ptw} \chi_H$, and therefore $u_\alpha \xrightarrow{w^*} \chi_H$ in $B(G_d)$ by boundedness.

Conversely, suppose that $(u_\alpha)_\alpha$ is a net in $A(G)$ such that $u_\alpha \xrightarrow{w^*} \chi_H$ in $B(G_d)$. Analogously to the proof of Lemma 3.6, the restriction $r_H : B(G_d) \rightarrow B(H_d)$ is weak* continuous, so that $r_H(u_\alpha) \xrightarrow{w^*} 1_H$ and 1_H is in the weak* closure of $A(H)$ in $B(H_d)$. Viewing λ_H as a representation of H_d , the universal property of $C^*(H_d)$ yields a quotient $*$ -homomorphism $C^*(H_d) \rightarrow C_\delta^*(H) : \omega_{H_d}(s) \mapsto \lambda_H(s)$, where ω_{H_d} denotes the universal representation of H_d and $C_\delta^*(H)$ the C^* -algebra generated by $\lambda_H(H)$ in $B(L^2(H))$, a subalgebra of $\text{VN}(H)$. Composing with the inclusion, we obtain a $*$ -homomorphism $\Psi : C^*(H_d) \rightarrow \text{VN}(H)$. The adjoint $\Psi^* : A(H)^{**} \rightarrow B(H_d)$ is the weak* continuous extension of the inclusion $A(H) \subset B(H_d)$, so that $\ker \Psi = A(H)_\perp$ (preannihilator taken with respect to the $B(H_d)$ - $C^*(H_d)$

duality). Then $A(H)_\perp = A^d(H)_\perp$ is an ideal in $C^*(H_d)$ and Ψ drops to an injective $*$ -homomorphism $C^*(H_d)/A^d(H)_\perp \rightarrow \text{VN}(H)$. This injective $*$ -homomorphism is isometric, so that its adjoint $A(H)^{**} \rightarrow A^d(H)$ is a quotient map, and given $\epsilon > 0$, the function 1_H is the weak* limit in $B(H_d)$ of a net $(v_\alpha)_\alpha$ in $A(H)$ of bound $1 + \epsilon$.

The adjoint on $B(H_d)$ is a weak* continuous isometry preserving $A(H)$, so that we may replace v_α by $\frac{1}{2}(v_\alpha + v_\alpha^*)$ and assume that $v_\alpha = v_\alpha^+ - v_\alpha^-$ for $v_\alpha^\pm \in P(H) \cap A(H)$ with $\|v_\alpha\|_{B(H)} = \|v_\alpha^+\|_{B(H)} + \|v_\alpha^-\|_{B(H)}$. Passing to subnets, let v^\pm be weak* limits of $(v_\alpha^\pm)_\alpha$ in $B(H_d)$. Then

$$1 = \lim_\alpha v_\alpha(e) = \lim_\alpha v_\alpha^+(e) - \lim_\alpha v_\alpha^-(e) = v^+(e) - v^-(e)$$

and so

$$v^-(e) = v^+(e) - 1 = \lim_\alpha v_\alpha^+(e) - 1 = \lim_\alpha \|v_\alpha^+\|_{B(H)} - 1 \leq \epsilon.$$

Let ω_ϵ be a weak* cluster point of $(v_\alpha^+)_\alpha$ in $A(H)^{**}$, so that ω_ϵ is a state on $\text{VN}(H)$ satisfying, for $s \in H$,

$$\begin{aligned} |\langle \omega_\epsilon, \lambda_H(s) \rangle - 1| &= \left| \lim_\alpha v_\alpha^+(s) - \lim_\alpha v_\alpha(s) \right| = \lim_\alpha |v_\alpha^-(s)| \\ &\leq \lim_\alpha v_\alpha^-(e) = v^-(e) \leq \epsilon. \end{aligned}$$

Letting ω be a weak* cluster point of the states $(\omega_\epsilon)_{\epsilon > 0}$ on $\text{VN}(H)$, we have $\langle \omega, \lambda_H(s) \rangle = 1$ for all $s \in H$, and any extension of ω to a state on $B(L^2(H))$ still takes the value 1 on the unitaries $\lambda_H(s)$ for $s \in H$. The amenability of H follows: by a Cauchy–Schwarz argument, any extension of ω to a state on $B(L^2(H))$ is invariant under the conjugation action of the unitaries $\lambda_H(s)$ for $s \in H$, whence λ_H is an amenable representation of H (see [B]). ■

COROLLARY 4.7. *Let G be a locally compact group and H a closed subgroup. Then H is amenable if and only if G has the discretized H -separation property witnessed by functions in $A^d(G) \cap P_H(G_d)$.*

Proof. In the argument establishing Proposition 4.4, substituting $\frac{1}{2}(u_s^2 + u_s)$ for the function $\frac{1}{2}(1_G + u_s)$ yields the proof that G has the desired property if and only if χ_H is $A(G)$ -approximable. ■

5. Invariant projections and bounded approximate indicators.

In this section we establish some consequences of the existence of a bounded approximate indicator for a closed subgroup of a locally compact group. We first provide the well known argument that this stronger separation property indeed yields invariant projections. For a commutative completely contractive Banach algebra \mathcal{A} , let $\text{CB}_\mathcal{A}(\mathcal{A}^*)$ denote the completely bounded \mathcal{A} -bimodule maps on \mathcal{A}^* . This space has compact unit ball when given the

weak* operator topology, which is determined by the seminorms

$$\Psi \mapsto |\langle \Psi(\varphi), a \rangle| \quad (\varphi \in \mathcal{A}^*, a \in \mathcal{A}).$$

PROPOSITION 5.1. *Let G be a locally compact group and H a closed subgroup. If there is a bounded approximate indicator for H , then there is a completely bounded invariant projection $\text{VN}(G) \rightarrow \text{VN}_H(G)$. If there is a bounded approximate indicator for H consisting of positive-definite functions, then there is a completely positive invariant projection $\text{VN}(G) \rightarrow \text{VN}_H(G)$.*

Proof. Let $(m_\alpha)_\alpha$ be a bounded approximate indicator for H , so that the net $(M_{m_\alpha})_\alpha$ of multiplication maps in $\text{CB}_{A(G)}(\text{VN}(G))$ is then bounded and thus has a weak* operator topology cluster point $\Psi \in \text{CB}(\text{VN}(G))$. Passing to a subnet if necessary, we may assume that Ψ is the limit of this net. For $u, v \in A(G)$ and $T \in \text{VN}(G)$,

$$\langle \Psi(v \cdot T), u \rangle = \lim_\alpha \langle T, m_\alpha uv \rangle = \langle \Psi(T), uv \rangle = \langle v \cdot \Psi(T), u \rangle,$$

showing that Ψ is invariant. Given $T \in \text{VN}_H(G)$, so that $T = r_H^*(S)$ for some $S \in \text{VN}(H)$, we calculate for $u \in A(G)$ that

$$\langle \Psi(T), u \rangle = \lim_\alpha \langle S, r_H(uu_\alpha) \rangle = \lim_\alpha \langle S, r_H(u)r_H(u_\alpha) \rangle = \langle S, r_H(u) \rangle = \langle T, u \rangle,$$

whence Ψ is the identity on $\text{VN}_H(G)$. For $T \in \text{VN}(G)$ and $u \in I_{A(G)}(H)$ we have $\langle \Psi(T), u \rangle = \lim_\alpha \langle S, uu_\alpha \rangle = 0$, so that Ψ maps into $I_{A(G)}(H)^\perp = \text{VN}_H(G)$, and is thus a projection onto $\text{VN}_H(G)$.

If the functions m_α are in $P(G)$, then the maps M_{m_α} are completely positive [DCH, Proposition 4.2], and by [Pau, Theorem 7.4] their weak* operator topology cluster point Ψ is also completely positive. ■

From the preceding we obtain an analogous result for $A_{cb}(G)$, at least when G is a weakly amenable locally compact group.

PROPOSITION 5.2. *Let G be a weakly amenable locally compact group and H a closed subgroup. If there is a bounded approximate indicator for H , then there is a completely bounded invariant projection $A_{cb}(G)^* \rightarrow I_{A_{cb}(G)}(H)^\perp$.*

Proof. Let $(m_\alpha)_\alpha$ be an approximate indicator for H . Since $A(G)$ is an ideal in $M_{cb}A(G)$, so too is its closure $A_{cb}(G)$, so that multiplication by m_α is a completely bounded $A_{cb}(G)$ -module map on $A_{cb}(G)$. Denote its adjoint by M_{m_α} . Passing to a subnet, we may assume that $(M_{m_\alpha})_\alpha$ has a weak* operator topology limit $\Psi \in \text{CB}(A_{cb}(G)^*)$, and passing to a further subnet we may assume that the net $(M_{m_\alpha})_\alpha$ of maps in $\text{CB}_{A(G)}(\text{VN}(G))$ also has a weak* operator topology limit Ψ_A , which is an invariant projection $\text{VN}(G) \rightarrow \text{VN}_H(G)$ by the argument of Proposition 5.1. For $u, v \in A_{cb}(G)$

and $T \in A_{cb}(G)^*$, we have

$$\langle \Psi(v \cdot T), u \rangle = \lim_{\alpha} \langle T, m_{\alpha}uv \rangle = \langle \Psi(T), uv \rangle = \langle v \cdot \Psi(T), u \rangle,$$

so Ψ is invariant.

Let $\iota : A(G) \rightarrow A_{cb}(G)$ be the inclusion. If $T \in A_{cb}(G)^*$ and $u \in A(G)$, then

$$\begin{aligned} \langle \Psi_{A\iota^*}(T), u \rangle &= \lim_{\alpha} \langle \iota^*(T), um_{\alpha} \rangle = \lim_{\alpha} \langle T, \iota(u)m_{\alpha} \rangle \\ &= \langle \Psi(T), \iota(u) \rangle = \langle \iota^*\Psi(T), u \rangle, \end{aligned}$$

and $\Psi_{A\iota^*} = \iota^*\Psi$ by density of $A(G)$ in $A_{cb}(G)$. It follows that

$$\iota^*\Psi^2 = \Psi_{A\iota^*}\Psi = \Psi_{A\iota^*}^2\iota^* = \Psi_{A\iota^*}\iota^* = \iota^*\Psi,$$

which, together with injectivity of ι^* , implies $\Psi^2 = \Psi$. If $T \in I_{A_{cb}(G)}(H)^{\perp}$, then $\iota^*(T) \in I_{A(G)}(H)^{\perp}$ and

$$\langle \Psi(T), \iota(u) \rangle = \langle \Psi_{A\iota^*}(T), u \rangle = \langle \iota^*(T), u \rangle = \langle T, \iota(u) \rangle \quad (u \in A(G)),$$

whence $\Psi(T) = T$, again by density of $A(G)$. Therefore $I_{A_{cb}(G)}(H)^{\perp}$ is contained in the range of Ψ . Finally, for any $T \in A_{cb}(G)^*$, if $u \in I_{A(G)}(H)$, then

$$\langle \Psi(T), \iota(u) \rangle = \langle \Psi_{A\iota^*}(T), u \rangle = 0, \text{ so } \Psi(T) \in I_{A(G)}(H)^{\perp} = \overline{I_{A(G)}(H)}^{A_{cb}(G)\text{-cl}}.$$

That $A_{cb}(G)$ has a bounded approximate identity implies every set of synthesis for $A(G)$ is one for $A_{cb}(G)$ [FRS, Proposition 3.1], and because compactly supported functions in $A_{cb}(G)$ are in $A(G)$, that H is of spectral synthesis for $A_{cb}(G)$ is exactly the assertion that $\overline{I_{A(G)}(H)}^{A_{cb}(G)\text{-cl}} = I_{A_{cb}(G)}(H)$. Therefore Ψ has range in $I_{A_{cb}(G)}(H)^{\perp}$. ■

Note that the arguments of the preceding two propositions yield projections of completely bounded norm which does not exceed the bound on an approximate indicator for the subgroup. Let Λ_G denote the *Cowling–Haagerup constant*, that is, the infimum of bounds on approximate identities for $A_{cb}(G)$.

COROLLARY 5.3. *Let G be a weakly amenable locally compact group and H a closed subgroup for which an approximate indicator of bound C exists. Then:*

- (1) $I_{A_{cb}(G)}(H)$ has an approximate identity of bound $(1 + C)\Lambda_G$.
- (2) There exists an approximate indicator for H of bound $1 + (1 + C)\Lambda_G$ that takes the value one on H .
- (3) There exists an approximate indicator for H of bound $C\Lambda_G$ in $A_{cb}(G)$.

Proof. (1) The argument of Proposition 5.2 yields an invariant projection $A_{cb}(G)^* \rightarrow I_{A_{cb}(G)}(H)^{\perp}$ of norm at most C , and because the Banach algebra $A_{cb}(G)$ has an approximate identity of bound Λ_G , it follows from [For 1,

Proposition 6.4] and its proof that the ideal $I_{A_{cb}(G)}(H)$ has an approximate identity of bound $(1 + C)A_G$.

(2) If $(e_\alpha)_\alpha$ is an approximate identity for $I_{A_{cb}(G)}(H)$ of bound $(1 + C)A_G$, then $(1_G - e_\alpha)_\alpha$ is an approximate indicator for H with the claimed norm bound.

(3) Let $(e_\alpha)_{\alpha \in A}$ be a bounded approximate identity for $A_{cb}(G)$, let $(m_\beta)_{\beta \in B}$ be a bounded approximate indicator for H , and for $\gamma = (\beta, (\alpha_{\beta'})_{\beta' \in B}) \in B \times A^B$ set $u_\gamma = m_\beta e_{\alpha_\beta}$, which is in the ideal $A_{cb}(G)$ of $M_{cb}A(G)$. Giving $B \times A^B$ the product order, for $u \in A(H)$ and $w \in I_{A(G)}(H)$ we have the norm limits

$$\begin{aligned} \lim_{\gamma \in B \times A^B} r_H(u_\gamma)u &= \lim_{\beta} \lim_{\alpha} r_H(m_\beta e_\alpha)u = \lim_{\beta} r_H(m_\beta)u = u, \\ \lim_{\gamma \in B \times A^B} u_\gamma w &= \lim_{\beta} \lim_{\alpha} m_\beta e_\alpha w = \lim_{\beta} m_\beta w = 0, \end{aligned}$$

by [Ke, p. 69], hence $(u_\gamma)_{\gamma \in B \times A^B}$ is a bounded approximate indicator for H of norm bound $\sup_\alpha \|e_\alpha\|_{M_{cb}A(G)} \sup_\beta \|m_\beta\|_{M_{cb}A(G)}$. ■

6. Convergence of cb-multipliers and averaging over closed subgroups. Fix a locally compact group G and a closed subgroup H . It is folklore that the convergence properties of nets of cb-multipliers can be improved by convolving them with probability measures in $\mathcal{C}_c(G)$. For example, Knudby recently recorded the following, the second part of which originates in an argument of Cowling and Haagerup [CH, Proposition 1.1]. If a net $(m_\alpha)_\alpha$ of functions on a topological space converges uniformly on compact sets to a function m , we write $m_\alpha \xrightarrow{ucs} m$.

THEOREM 6.1 ([Kn, Lemma B.2]). *Let $(m_\alpha)_\alpha$ be a bounded net in $M_{cb}A(G)$, $m \in M_{cb}A(G)$, and let $f \in \mathcal{C}_c(G)$ be such that $f \geq 0$ and $\int_G f = 1$. Convolution on the left with f is a contraction on $M_{cb}A(G)$, and:*

- (1) *If $m_\alpha \xrightarrow{w^*} m$ in $M_{cb}A(G)$, then $f * m_\alpha \xrightarrow{ucs} f * m$.*
- (2) *If $m_\alpha \xrightarrow{ucs} m$, then $\|(f * m_\alpha)u - (f * m)u\|_{A(G)} \rightarrow 0$ for all $u \in A(G)$.*

In this section, we develop an analogue of the convolution technique relative to a closed subgroup. Fix a function $f \in \mathcal{C}_c(H)$ such that $f \geq 0$ and $\int_H f = 1$. For any function f on G and $s, t \in G$, let ${}_s f(t) = f(st)$.

DEFINITION 6.2. For $u \in \mathcal{C}_b(G)$, define a function $\Omega_f(u)$ on G by

$$\Omega_f(u)(s) = \int_H f(h)u(h^{-1}s) dh \quad (s \in G).$$

We will show that Ω_f defines a bounded map on $M_{cb}A(G)$. For a Hilbert space \mathcal{H} , let $\mathcal{C}_c(G, \mathcal{H})$ and $\mathcal{C}_b(G, \mathcal{H})$ denote the continuous functions $G \rightarrow \mathcal{H}$ that are compactly supported and bounded, respectively.

LEMMA 6.3. *Let \mathcal{H} be a Hilbert space. If $u \in \mathcal{C}_c(G, \mathcal{H})$ then for any $\epsilon > 0$ there exists an open neighborhood U of the identity e such that $\sup_{t \in G} \|u(st) - u(t)\| < \epsilon$ for all $s \in U$.*

Proof. The standard proof in the case that $\mathcal{H} = \mathbb{C}$ (see for example [Fol, Proposition 2.6]) works for any Hilbert space. ■

LEMMA 6.4. *Let \mathcal{H} be a Hilbert space. If $u \in \mathcal{C}_b(G, \mathcal{H})$, $s_0 \in G$, and $\epsilon > 0$, then there exists an open neighborhood U of s_0 in G such that $\sup_{h \in H} \|f(h)u(sh) - f(h)u(s_0h)\| < \epsilon$ for all $s \in U$.*

Proof. If $u = 0$, then the claim trivially holds, so assume $u \neq 0$. Since H is closed in G , the function f extends to a continuous compactly supported function f' on G . Assume that $s_0 = e$. Since $f'u$ is compactly supported, Lemma 6.3 yields an open neighborhood U of e such that

$$\sup_{t \in G} \|f'u(st) - f'u(t)\| < \epsilon/2, \quad \sup_{t \in G} |f'(st) - f'(t)| < \frac{\epsilon}{2\|u\|_\infty}$$

for all $s \in U$. Then

$$\begin{aligned} \sup_{h \in H} \|f(h)u(sh) - f(h)u(h)\| &\leq \sup_{t \in G} \|f'(t)u(st) - f'(t)u(t)\| \\ &\leq \sup_{t \in G} (\|f'(t)u(st) - f'u(st)\| + \|f'u(st) - f'(t)u(t)\|) \\ &< \|u\|_\infty \sup_{t \in G} |f'(st) - f'(t)| + \epsilon/2 < \epsilon \end{aligned}$$

for all $s \in U$. For $s_0 \neq e$, the above argument with u replaced by s_0u yields a neighborhood U of e , and s_0U is then the desired neighborhood of s_0 . ■

PROPOSITION 6.5. *If $u \in M_{cb}A(G)$, then we have $\Omega_f(u) \in M_{cb}A(G)$ with $\|\Omega_f(u)\|_{M_{cb}A(G)} \leq \|u\|_{M_{cb}A(G)}$ and $r_H \Omega_f(u) = f * r_H(u)$.*

Proof. Let $u \in M_{cb}A(G)$ and apply Gilbert's theorem to obtain a Hilbert space \mathcal{H} and functions $P, Q \in \mathcal{C}_b(G, \mathcal{H})$ such that $u(s^{-1}t) = \langle P(t) \mid Q(s) \rangle$ for all $s, t \in G$. Then, for $s, t \in G$,

$$\Omega_f(u)(s^{-1}t) = \int_H f(h)u(h^{-1}s^{-1}t) dh = \left\langle P(t) \mid \int_H f(h)Q(sh) dh \right\rangle.$$

We show that $q(s) = \int_H f(h)Q(sh) dh$ defines a bounded continuous function on G , from which it will follow that $\Omega_f(u)$ is in $M_{cb}A(G)$, again by Gilbert's theorem. Define $Q' : G \rightarrow L^1(H, \mathcal{H})$ by $Q'(s) = f(s)Q$, which maps into $L^1(H, \mathcal{H})$ since f has compact support. Set $K = \text{supp } f$ and let $|K|$ denote the Haar measure of K , which is nonzero and finite by continuity of the nonzero, compactly supported function f . Given $s_0 \in G$ and $\epsilon > 0$, Lemma 6.4 yields an open neighborhood U of s_0 in G such that

$$\|Q'(s) - Q'(s_0)\|_{L^\infty(H, \mathcal{H})} = \sup_{h \in H} \|f(h)Q(sh) - f(h)Q(s_0h)\| < \epsilon/|K|$$

for all $s \in U$. Since $Q'(s)$ is supported in K for every $s \in G$, it follows that

$$\begin{aligned} \|Q'(s) - Q'(s_0)\|_{L^1(H, \mathcal{H})} &= \|\chi_K(Q'(s) - Q'(s_0))\|_{L^1(H, \mathcal{H})} \\ &\leq \|\chi_K\|_{L^1(H, \mathcal{H})} \|Q'(s) - Q'(s_0)\|_{L^\infty(H, \mathcal{H})} < \epsilon \end{aligned}$$

for all $s \in U$. Thus Q' is continuous, and so too is q , being the composition of Q' with the bounded map $L^1(H, \mathcal{H}) \rightarrow \mathcal{H} : g \mapsto \int_H g$.

Because f is nonnegative with mass one, if $s \in G$, then $\|q(s)\| \leq \int_H f(h)\|Q(sh)\| dh \leq \|Q\|_\infty$, so that q is bounded with $\|q\|_\infty \leq \|Q\|_\infty$. By the norm characterization of Gilbert's theorem, $\|\Omega_f(u)\|_{M_{cb}A(G)} \leq \|P\|_\infty \|q\|_\infty \leq \|P\|_\infty \|Q\|_\infty$, and since P , Q , and \mathcal{H} are an arbitrary representation of u , we conclude that $\|\Omega_f(u)\|_{M_{cb}A(G)} \leq \|u\|_{M_{cb}A(G)}$. Finally, for $s \in H$ we have

$$r_H \Omega_f(u)(s) = \int_H f(h)u(h^{-1}s) dh = \int_H f(h)r_H(u)(h^{-1}s) dh = f * r_h(u)(s),$$

as required. ■

THEOREM 6.6. *Let $(m_\alpha)_\alpha$ be a bounded net in $M_{cb}A(G)$ and let $m \in M_{cb}A(H)$. Then:*

- (1) *If $r_H(m_\alpha) \xrightarrow{w^*} m$ in $M_{cb}A(H)$, then $r_H \Omega_f(m_\alpha) \xrightarrow{ucs} f * m$.*
- (2) *If $r_H(m_\alpha) \xrightarrow{ucs} m$, then $\|r_H \Omega_f(m_\alpha)u - (f * m)u\|_{A(H)} \rightarrow 0$ for all $u \in A(H)$.*

Proof. These follow immediately from Theorem 6.1 and Proposition 6.5. ■

In our applications, the preceding theorem will be applied with $m = 1_H$, which is fixed under convolution with f on the left. We list some additional properties that the map Ω_f enjoys:

- (1) An argument very similar to that establishing Proposition 6.5 shows that $\Omega_f(u)$ is bounded and continuous for any bounded continuous function u on G .
- (2) If $u = \langle \pi(\cdot)\xi \mid \eta \rangle$ is a coefficient of the unitary representation π of G , then

$$\Omega_f(u)(s) = \int_H f(h)\langle \pi(s)\xi \mid \pi(h)\eta \rangle dh = \left\langle \pi(s)\xi \mid \left(\int_H f(h)\pi(h) dh \right) \eta \right\rangle,$$

so $\Omega_f(u) \in B(G)$, and from $\|\int_H f(h)\pi(h) dh\| \leq \int_H f(h)\|\pi(h)\| dh = 1$ it follows that $\|\Omega_f(u)\|_{B(G)} \leq \|u\|_{B(G)}$. Thus Ω_f restricts to a contraction on $B(G)$, and moreover it restricts to a contraction on $A(G)$, since $\Omega_f(u)$ is a coefficient of the same representation as u .

- (3) An argument similar to that establishing the weak* continuity of the map Φ_f in the proof of [HK, Lemma 1.16] shows that Ω_f is weak* continuous on $M_{cb}A(G)$ with preadjoint mapping $g \in L^1(G)$ to the $L^1(G)$ function $s \mapsto \int_H f(h)g(hs) dh$.

Say that a net $(m_\alpha)_\alpha$ of functions on a topological space X converges locally eventually to zero on $A \subset X$, and write $m_\alpha \xrightarrow{le} 0$, if for any compact subset K of A there is an index α_0 such that $m_\alpha|_K = 0$ for all $\alpha \geq \alpha_0$.

PROPOSITION 6.7. *If $(m_\alpha)_\alpha$ is a bounded net in $M_{cb}A(G)$ and $m'_\alpha = \Omega_f(m_\alpha)$, then the net $(m'_\alpha)_\alpha$ has the same norm bound as $(m_\alpha)_\alpha$ and*

- (1) *if $r_H(m_\alpha) \xrightarrow{ucs} 1_H$, then $\|u \cdot r_H(m'_\alpha) - u\|_{A(H)} \rightarrow 0$ for all $u \in A(H)$;*
- (2) *if $m_\alpha \xrightarrow{le} 0$ on $G \setminus H$, then $m'_\alpha \xrightarrow{le} 0$ on $G \setminus H$.*

If the bounded net $(m_\alpha)_\alpha$ satisfies the hypotheses of both (1) and (2), then $(m'_\alpha)_\alpha$ is a bounded approximate indicator for H .

Proof. The claim regarding norm bounds holds since the map Ω_f of Section 6 is a contraction on $M_{cb}A(G)$.

(1) If $r_H(m_\alpha) \xrightarrow{ucs} 1_H$, then since restriction is a contraction from $M_{cb}A(G)$ into $M_{cb}A(H)$, the net $(r_H(m_\alpha))_\alpha$ is bounded, and (1) follows from Theorem 6.6.

(2) Suppose that $m_\alpha \xrightarrow{le} 0$ on $G \setminus H$. Let $K \subset G \setminus H$ be compact and choose α_0 such that $\alpha \geq \alpha_0$ implies $m_\alpha = 0$ on the compact set $(\text{supp } f)^{-1}K$. For $\alpha \geq \alpha_0$, if $s \in K$ and $h \in H$, then $f(h)m_\alpha(h^{-1}s) = 0$ since either $h \notin \text{supp } f$ or $h^{-1}s \in (\text{supp } f)^{-1}K$, implying that

$$m'_\alpha(s) = \int_H f(h)m_\alpha(h^{-1}s) dh = 0.$$

Therefore $m'_\alpha = 0$ on K for all $\alpha \geq \alpha_0$.

If $(m_\alpha)_\alpha$ satisfies the hypotheses of both (1) and (2), then $(m'_\alpha)_\alpha$ satisfies the first condition of Definition 2.2. If $u \in I_{A(G)}(H)$ has compact support, then $m'_\alpha u = 0$ eventually by (2), so certainly $\|um'_\alpha\|_{A(G)} \rightarrow 0$. Since H is of spectral synthesis in $A(G)$, if $u \in I_{A(G)}(H)$ is arbitrary then given $\epsilon > 0$ we can choose $u_0 \in I_{A(G)}(H)$ of compact support with $\|u - u_0\|_{A(G)} < \epsilon$. For sufficiently large α ,

$$\|um'_\alpha\|_{A(G)} \leq \|u_0m'_\alpha\|_{A(G)} + \|u_0m'_\alpha - um'_\alpha\|_{A(G)} < \epsilon\|m'_\alpha\|_{M_{cb}A(G)},$$

and thus $\|um'_\alpha\|_{A(G)} \rightarrow 0$ by boundedness of $(m'_\alpha)_\alpha$. ■

Proposition 6.7 allows one to obtain approximate indicators consisting of cb-multipliers by verifying the same conditions that yielded approximate indicators in [ARS].

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