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Eligible integers represented by positive ternary quadratic forms

by

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1. Introduction. Let $f(x_1, x_2, x_3)$ be a positive definite integral ternary quadratic form. Let

$$\theta(z; f) = \sum_{n=0}^{\infty} a(n; f)q^n = \sum_{n=0}^{\infty} a(n; f)e^{2\pi inz}$$

be its theta function, where $z \in \mathbb{C}$, $\mathrm{Im} z > 0$ and $a(n; f)$ is the number of the solutions of $f(x_1, x_2, x_3) = n$ with $(x_1, x_2, x_3) \in \mathbb{Z}^3$. Let $A_f := \big(\frac{\partial^2 f}{\partial x_i \partial x_j}\big)_{3\times 3}$ be the Gram matrix of $f(x_1, x_2, x_3)$. Let N_f (the level of $f(x_1, x_2, x_3)$) be the least positive integer such that all the entries of the matrix $N_f A_f^{-1}$ are integers and the entries on the diagonal are even. Let $\chi_f = \big(\frac{2 \det A_f}{\cdot}\big)$. It is well known that $\theta(z; f)$ is a modular form of weight $3/2$ with character χ_f for $\Gamma_0(N_f)$ (see [WP2, Theorem 10.1]). For more about modular forms of half-integral weights, see Shimura [Sh].

One of the classical questions of number theory is to find all numbers n which are represented by f, i.e. which have $a(n; f) > 0$. The local-global principle (Hasse–Minkowski) says that n is represented by f locally everywhere if and only if it is represented by some quadratic form in the genus of f. Following Kaplansky, we call such a non-negative integer n *eligible* for the ternary quadratic form f. Duke and Schulze-Pillot [DS] established that if a large integer is represented by the spinor genus of a form then it is represented by the form itself. We use $\mathrm{gen}(f)$ and $\mathrm{spn}(f)$ to denote the genus and the spinor genus of f respectively. So if $\mathrm{spn}(f) = \mathrm{gen}(f)$, then all but finitely many eligible integers are represented by f.

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Ono and Soundararajan [OS] considered Ramanujan's form $\phi_1 = x_1^2 + x_2^2 + 10x_3^2$. This form is in a genus consisting of two classes, and $\phi_2 = 2x_1^2+2x_2^2+3x_3^2-2x_1x_3$ is a representative of the other class. Since ϕ_1 and ϕ_2 are in the same spinor genus, there are finitely many eligible integers which are not represented by ϕ_1. Assuming the Generalized Riemann Hypothesis (GRH), they proved the following conjecture:

Conjecture 1.1. *The eligible integers which are not of the form $x_1^2 + x_2^2 + 10x_3^2$ are:*

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719.$$

In order to prove this conjecture, first, Ono and Soundararajan reduced the question to the set of integers M which are coprime to 10. Then they obtained the following result, which reduced the problem to dealing with square-free eligible integers:

Proposition 1.2. *If M with $(M, 10) = 1$ is an eligible integer which is not square-free, then it is of the form $x_1^2 + x_2^2 + 10x_3^2$.*

Let E be the elliptic curve $y^2 = x^3 + x^2 + 4x + 4$. For every integer D, let $E(D)$ denote the D-quadratic twist of E, $y^2 = x^3 + Dx^2 + 4D^2x + 4D^3$. For square-free M, they have the following result:

Proposition 1.3. *Let M be a square-free eligible integer. If M is not of the form $x_1^2 + x_2^2 + 10x_3^2$, then*

$$h^2(-40M) = \frac{4\sqrt{M}}{\Omega(E(-10))} L(E(-10M), 1), \tag{1.1}$$

where $\Omega(E(-10)) \sim 0.71915$ is the real period of $E(-10)$.

Assuming GRH, Ono and Soundararajan obtained a new upper bound for $L(E(-10M), 1)$. Combining this upper bound, Littlewood's lower bound for $h(-40M)$ and equation (1.1), they proved that all eligible integers larger than $2 \cdot 10^{10}$ are represented by Ramanujan's form. W. Galway verified that there are no further exceptions below $2 \cdot 10^{10}$, so this provides a conditional answer to Conjecture 1.1.

In this paper, we will prove that Proposition 1.2 holds in general. The following theorem is our main result:

Theorem 1.4. *Assume that f is a positive definite integral ternary quadratic form. Let N_f denote the level of f. Assume that there are exactly two classes in $\mathrm{gen}(f)$ and let g be a representative of the other class. Assume further that f and g are in the same spinor genus. If M with $(M, N_f) = 1$ is an eligible integer which is not square-free, then it can be represented by f, i.e. $a(M; f) > 0$.*

Ramanujan's form $\phi_1 = x_1^2 + x_2^2 + 10x_3^2$ satisfies the assumptions in our theorem, so Proposition 1.2 is a special case of our result. Also, if we let $f = x_1^2+7x_2^2+7x_3^2$, then Theorem 3 of Wang and Pei [WP1] is a consequence of Theorem 1.4. Another example is $f = x_1^2+x_2^2+7x_3^2$ (see Kelley [Ke, Theorem 1.1]).

REMARK 1.5. If there are more than two classes in the spinor genus, then this does not hold in general (even if the genus and the spinor genus coincide). For some other results with class number $h \geq 2$, one can see Kane [Ka]. We thank the referee for pointing out these.

2. Proof of Theorem 1.4. Let

$$\theta_1 = \sum_{n=0}^{\infty} a_1(n)q^n \quad \text{and} \quad \theta_2 = \sum_{n=0}^{\infty} a_2(n)q^n$$

be the theta functions associated to f and g respectively. Set $o(f) = \#\{S \in M_3(\mathbb{Z}) \mid SA_fS^T = A_f\}$ and $o(g) = \#\{S \in M_3(\mathbb{Z}) \mid SA_gS^T = A_g\}$ (the isometry numbers of f and g respectively). Define the theta function of $\mathrm{gen}(f)$ as

$$\theta_g := \sum_{n\geq 0} a_g(n)q^n = l_1\theta_1(z) + l_2\theta_2(z) = \left(\frac{1}{o(f)} + \frac{1}{o(g)}\right)^{-1}\left(\frac{\theta_1}{o(f)} + \frac{\theta_2}{o(g)}\right).$$

Define $\vartheta := \sum_{n=0}^{\infty} b(n)q^n = \theta_1 - \theta_2$. By a theorem of Schiemann [S], $\vartheta \neq 0$. Let $\mathscr{M}(N_f, 3/2, \chi_f)$, $\mathscr{E}(N_f, 3/2, \chi_f)$ and $\mathscr{S}(N_f, 3/2, \chi_f)$ denote the vector spaces of modular forms, of Eisenstein series and of cusp forms, of weight 3/2, with character χ_f for $\Gamma_0(N_f)$, respectively. It is well known that θ_g is an Eisenstein series (see [WP2, Theorem 10.3]) and ϑ is a cusp form (see [WP2, p. 365]).

Let $\mathscr{M}(f)$ denote the $\mathbb{C}$-linear space spanned by θ_1 and θ_2. Then $\mathscr{M}(f) = \mathbb{C}\theta_1 + \mathbb{C}\theta_2 = \mathbb{C}\theta_g + \mathbb{C}\vartheta$. Define

$$\mathscr{E}(f) = \mathscr{M}(f) \cap \mathscr{E}(N_f, 3/2, \chi_f) \quad \text{and} \quad \mathscr{S}(f) = \mathscr{M}(f) \cap \mathscr{S}(N_f, 3/2, \chi_f).$$

Then $\mathscr{E}(f) = \mathbb{C}\theta_g$ and $\mathscr{S}(f) = \mathbb{C}\vartheta$. Let p with $(p, N_f) = 1$ be a prime. By Eichler's Commutation Relation (see [Sc1, Hilfssatz 1]), $\mathscr{M}(f)$ is stable under the Hecke operator T_{p^2}. Thus $\mathscr{E}(f)$ and $\mathscr{S}(f)$ are stable under T_{p^2}. Since $\mathscr{E}(f)$ and $\mathscr{S}(f)$ are vector spaces of dimension 1 generated by θ_g and ϑ respectively, θ_g and ϑ are eigenfunctions of T_{p^2}. It is well known that $T_{p^2}\theta_g = (p+1)\theta_g$ (see [Sc1, Satz 1]). Assume that $T_{p^2}\vartheta = \omega_p\vartheta$.

Let t with $(t, N_f) = 1$ be a square-free positive eligible integer, and let p be a prime with $(p, N_f) = 1$. Our aim is to prove that $a_1(tp^2) > 0$. Since $\theta_1 = \theta_g + l_2\vartheta$, we have $a_1(n) = a_g(n) + l_2b(n)$ for all $n \geq 0$. So

$$a_1(tp^2) = a_g(tp^2) + l_2b(tp^2) \quad \text{and} \quad a_1(t) = a_g(t) + l_2b(t).$$

Since θ_g and ϑ are eigenfunctions of T_{p^2}, by [Sh, Corollary 1.8] we have

$$a_g(tp^2) = a_g(t)\left(p+1-\chi_f(p)\left(\frac{-t}{p}\right)\right), \quad b(tp^2) = b(t)\left(\omega_p - \chi_f(p)\left(\frac{-t}{p}\right)\right).$$

If $a_1(t) > 0$, then $a_1(tp^2) \ge a_1(t) > 0$. If $a_1(t) = 0$, then $a_g(t) = -l_2 b(t)$ and

$$a_1(tp^2) = a_g(t)(p+1-\omega_p). \tag{2.1}$$

Since t is eligible, $a_g(t) > 0$. Now we have

$$a_1(tp^2) > 0 \quad \text{if and only if} \quad \omega_p \ne p+1.$$

So we are going to prove that $\omega_p \ne p+1$.

Let s be a square-free positive integer. Shimura [Sh] constructed a linear map Φ_s from $\mathscr{S}(N_f, 3/2, \chi_f)$ to $\mathscr{M}(N_f/2, 2, \chi_f^2)$, now called the *Shimura s-lifting*. By the theory of the Shimura lifting, $\Phi_s(\vartheta) \in \mathscr{M}(N_f/2, 2, \chi_f^2)$. Since f and g are in the same spinor genus, by [Sc1, Satz 4], $\Phi_s(\vartheta)$ is in fact a cusp form, i.e.,

$$\Phi_s(\vartheta) \in \mathscr{S}(N_f/2, 2, \chi_f^2).$$

By [C, Proposition 5.1], the Shimura lifting commutes with the action of Hecke operators, so $\Phi_s(T_{p^2}.\vartheta) = T_p.\Phi_s(\vartheta)$. Now we have

$$T_p.\Phi_s(\vartheta) = \omega_p \Phi_s(\vartheta).$$

Since $\vartheta \ne 0$, we can find a square-free positive integer s such that $\Phi_s(\vartheta) \ne 0$. Now $\Phi_s(\vartheta) \ne 0$ is an eigenfunction of the Hecke operator T_p with eigenvalue ω_p. By Rankin's estimate on the coefficients of cusp forms, we have the following lemma which asserts that $\omega_p \ne p+1$.

LEMMA 2.1. *Let $h \ne 0$ be a cusp form of weight 2 with character χ for $\Gamma_0(N)$. Let p with $(p, N) = 1$ be a prime. If h is an eigenfunction of the Hecke operator T_p with eigenvalue ω_p, then $\omega_p \ne p+1$.*

Proof. Let $h = \sum_{n=0}^{\infty} c(n)q^n$. Since $h \ne 0$, there exists a square-free positive integer s with $c(s) \ne 0$. By assumption, we have

$$\omega_p c(s) = c(sp), \quad \omega_p c(sp^{k+1}) = c(sp^{k+2}) + \chi(p)pc(sp^k) \quad \text{for all } k \ge 0.$$

Let α and β be the two distinct roots of $x^2 - \omega_p x + \chi(p)p = 0$ and $|\alpha| \ge |\beta|$. Then

$$c(sp^k) = \left(\frac{\alpha^k - \beta^k}{\alpha - \beta}\omega_p - \frac{\alpha^k\beta - \beta^k\alpha}{\alpha - \beta}\right)c(s) \quad \text{for all } k \ge 0.$$

If $|\alpha| > |\beta|$, then

$$c(sp^k) = O^*(\alpha^k), \quad k \to \infty.$$

If $\omega_p = p+1$, then $|\alpha| \ge p$ and

$$\lim_{k\to\infty} \frac{c(sp^k)}{sp^k} \gg 0.$$

This contradicts Rankin's estimate $c(n) = O(n^{4/5})$. So $\omega_p \neq p+1$ and the lemma is proved.

We continue the proof of our main theorem. By Lemma 2.1, if t with $(t, N_f) = 1$ is a square-free positive eligible integer, and p with $(p, N_f) = 1$ is a prime, then $a_1(tp^2) > 0$. If M with $(M, N_f) = 1$ is an eligible integer which is not square-free, then there exist a square-free positive eligible integer t, a prime p and a positive integer n_0 such that $M = tp^2n_0^2$. So $a_1(M) \geq a_1(tp^2) > 0$. This completes the proof of Theorem 1.4.

REMARK 2.2. (1). When f and g are in different spinor genera, for example $f = x_1^2 + 3x_2^2 + 36x_3^2$ and $g = 3x_1^2 + 4x_2^2 + 9x_3^2$, the Shimura lifting $\Phi_s(\vartheta)$ is an Eisenstein series of weight 2 and there may exist primes p with $\omega_p = p+1$. For such primes p, $a_1(tp^2) = 0$, and our theorem does not hold. The set of numbers that are represented by $\mathrm{gen}(f)$ but not represented by f or g (the spinor exceptions of the genus) has been determined by Schulze-Pillot [Sc2]. In particular, in this case, if an eligible integer is not spinor exceptional, then it can be represented by f and also by g. One can consult Schulze-Pillot [Sc3, Theorem 4.3] or [Sc2] for the results about the spinor exceptional square classes, and Earnest–Hsia–Hung [EHH] for primitive spinor exceptions.

(2) In his proof of the Weil conjectures on algebraic varieties over finite fields, Deligne gave a better bound than Rankin's estimate: see Deligne [D] or Kilford [Ki, Theorem 5.21].

3. Applications. Sun [Su] discussed questions of representation of positive integers as certain sums involving polygonal numbers and generalized polygonal numbers. In this section, we use Theorem 1.4 to prove two propositions conjectured by Sun.

PROPOSITION 3.1 (see [Su, Remark 3.4]). *Let $n > 1$ be a positive integer with $(n, 6) = 1$. Then there are $x, y, z \in \mathbb{Z}$ with x even such that $n^2 = x^2 + 3y^2 + 6z^2$.*

Proof. Let $f = 3x_1^2 + 4x_2^2 + 6x_3^2$. Then f satisfies the conditions in Theorem 1.4 and $g = x_1^2 + 6x_2^2 + 12x_3^2$. Take $M = n^2$ in Theorem 1.4. It is clear that n^2 is represented by g, so n^2 is eligible for f. Since $N_f = 48$ and $(n, 6) = 1$, we have $(n^2, N_f) = 1$. Since $n > 1$, n^2 is not square-free. By Theorem 1.4, there are $a, b, c \in \mathbb{Z}$ such that $n^2 = 3a^2 + 4b^2 + 6c^2$. Let $x = 2b, y = a, z = c$. Then $n^2 = x^2 + 3y^2 + 6z^2$ and x is even. ∎

Similarly, we can prove:

PROPOSITION 3.2 (see [Su, Remark 1.8(i)]). *Let $n > 1$ be a positive integer with $(n, 10) = 1$. Then there are $x, y, z \in \mathbb{Z}$ with x even such that $n^2 = x^2 + 5y^2 + 5z^2$.*

In this proposition, we can take

$$f = 4x_1^2 + 5x_2^2 + 5x_3^2 \quad \text{and} \quad g = x_1^2 + 5x_2^2 + 20x_3^2.$$

This can be explicitly computed using standard computer algebra systems (for example, there is now a built-in package in Sage).

Let $m \geq 3$ be a positive integer. The *m-gonal number* is defined by

$$p_m(n) := (m-2)\binom{n}{2} + n = \frac{(m-2)n^2 - (m-4)n}{2} \quad \text{for } n \in \mathbb{Z}_{>0}.$$

An integer is called a *generalized m-gonal number* if it is of the form $p_m(x)$ for some $x \in \mathbb{Z}$. For $i, j, k \geq 3$, we say that the sum $p_i + p_j + p_k$ is *universal* over $\mathbb{Z}$ if for every positive integer n there exist $x, y, z \in \mathbb{Z}$ such that $n = p_i(x) + p_j(y) + p_k(z)$. Using Proposition 3.2, we will prove the following result:

THEOREM 3.3. *The sum $p_3 + p_3 + p_{12}$ is universal over $\mathbb{Z}$.*

Proof. Let $n \in \mathbb{Z}_{>0}$. By [Su, proof of Theorem 1.7(iv)], if there are $x, y, z \in \mathbb{Z}$ such that $20n + 21 = 5x^2 + 5y^2 + 4z^2$, then $n = p_3(x) + p_3(y) + p_{12}(z)$. If $20n + 21$ is not a square, the result follows from [Su]; if $20n + 21$ is a square, it follows from Proposition 3.2. ■

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