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On the X-coordinates of Pell equations which are Tribonacci numbers

by

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1. Introduction. Let $d > 1$ be a positive integer which is not a perfect square. Consider the Pell equation

$$X^2 - dY^2 = \pm 1. \tag{1}$$

All its positive integer solutions (X, Y) are given by

$$X_n + Y_n\sqrt{d} = (X_1 + Y_1\sqrt{d})^n$$

for some positive integer n, where (X_1, Y_1) is the smallest positive solution. In several recent papers, the following problem was investigated. Let $\mathbf{U} = \{U_n\}_{n\geq 0}$ be some interesting sequence of positive integers. What can one say about the square-free integers d such that $X_n \in \mathbf{U}$ has at least two solutions n? For most sequences, one expects the answer to be that $X_n \in \mathbf{U}$ has at most one positive integer solution n for any given d except maybe for a few (finitely many) values of d. In [3], this was shown to be so when $\mathbf{U}$ is the sequence of all base 10-repdigits, that is, numbers of the form $c(10^m - 1)/9$ for some positive integers $m \geq 1$ and $c \in \{1, \dots, 9\}$. The only exceptional d's in this case were $d = 2, 3$. For each of these two values of d, the equation $X_n \in \mathbf{U}$ has two solutions n. In [5], it was shown, more generally, that if $b \geq 2$ is any fixed positive integer, $\mathbf{U}$ is the sequence of base b-repdigits, and d is such that $X_n \in \mathbf{U}$ has two solutions n, then

$$d < \exp((10b)^{10^5}).$$

In [7], it was shown that if $\mathbf{U}$ is the sequence of Fibonacci numbers, then

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$X_n \in \mathbf{U}$ has at most one positive integer solution n, except when $d = 2$ for which there are exactly two solutions.

In this paper, we consider the same problem for the sequence $\mathbf{U} := \mathbf{T}$ of Tribonacci numbers given by $T_0 = 0,\ T_1 = T_2 = 1$ and $T_{m+3} = T_{m+2} + T_{m+1} + T_m$ for all $m \geq 0$. Our result is the following.

THEOREM 1.1. *Let $d \geq 2$ be square-free. The Diophantine equation*

$$X_n = T_m \tag{2}$$

has at most one solution (n, m) in positive integers with the following exceptions:

- $(n_1, m_1) = (1, 3)$ *and* $(n_2, m_2) = (2, 5)$ *in the* $+1$ *case,*
- $(n_1, m_1) = (1, 1)$, $(n_2, m_2) = (1, 2)$ *and* $(n_3, m_3) = (3, 5)$ *in the* -1 *case.*

A few words about our method. For the arguments in [3], [5] and [7], the arithmetical properties of the members of $\mathbf{U}$ played an important role. For example, it was important to know all the solutions of the equation $U_m = 2X^2 - 1$ in positive integers (m, X), which are easy to find when $\mathbf{U}$ is the sequence of Fibonacci numbers or base 10-repdigits. It was also important that $\gcd(U_m, U_n)$ was closely related to $U_{\gcd(m,n)}$, which is the case both when $\mathbf{U}$ is the sequence of Fibonacci numbers and the sequence of repdigits. In contrast, the sequence of Tribonacci numbers does not display similar properties. For example, the equation $T_m = 2X^2 - 1$ in positive integers (m, X) is unsolved and there is no general method that would allow one to solve it (albeit some tricky elementary arguments might solve such equations), and $\gcd(T_m, T_n)$ is not related in any obvious way to $T_{\gcd(m,n)}$. Our method consists in applying Baker's theory of linear forms in logarithms three times to three different linear forms in order to get an absolute bound on all the variables, after which we reduce our bounds to some reasonable values and carry on the computations in the remaining range. Our method works not only for the Tribonacci sequence but also for other linearly recurrent sequences satisfying certain technical conditions. For example, it works for sequences $(u_m)_{m\geq 1}$ which are linearly recurrent, nondegenerate, have a simple dominant root $\alpha > 1$ and all other roots of absolute value smaller than 1, and furthermore if a is the coefficient of α^m in the Binet formula for u_m, then $\log(2a)$ and $\log\alpha$ are linearly dependent over $\mathbb{Q}$ (which ensures that the analog of the left-hand side of (16) is nonzero).

2. The Tribonacci sequence. Here, we recall a few important properties of the Tribonacci sequence $\{T_n\}_{n\geq 0}$. The characteristic equation

$$x^3 - x^2 - x - 1 = 0$$

has roots $\alpha,\ \beta,\ \gamma = \overline{\beta}$, where

$$\alpha = \frac{1+\omega_1+\omega_2}{3}, \qquad \beta = \frac{2-\omega_1-\omega_2+\sqrt{3}i(\omega_1-\omega_2)}{6},$$

and

$$\omega_1 = \sqrt[3]{19+3\sqrt{33}}, \qquad \omega_2 = \sqrt[3]{19-3\sqrt{33}}.$$

Further, Binet's formula is

$$T_m = a\alpha^m + b\beta^m + c\gamma^m \quad \text{for all } m \geq 0, \tag{3}$$

where

$$a = \frac{1}{(\alpha-\beta)(\alpha-\gamma)}, \quad b = \frac{1}{(\beta-\alpha)(\beta-\gamma)}, \quad c = \frac{1}{(\gamma-\alpha)(\gamma-\beta)} = \overline{b} \tag{4}$$

(see [9]). Numerically,

$$\begin{aligned} &1.83 < \alpha < 1.84,\\ &0.73 < |\beta| = |\gamma| = \alpha^{-1/2} < 0.74,\\ &0.18 < a < 0.19,\\ &0.35 < |b| = |c| < 0.36. \end{aligned} \tag{5}$$

Further,

$$\alpha^{m-2} \leq T_m \leq \alpha^{m-1} \quad \text{for all } m \geq 2 \tag{6}$$

(see [1]).

3. Linear forms in logarithms. We need some results from the theory of lower bounds in nonzero linear forms in logarithms of algebraic numbers. We start by recalling [2, Theorem 9.4], which is a modified version of a result of Matveev [8]. Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $d_1, \ldots, d_l$ be nonzero integers. We set

$$D = \max\{|d_1|, \ldots, |d_l|, 3\}, \qquad \Gamma = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\} \quad \text{for } j = 1, \ldots, l,$$

where for an algebraic number η with minimal polynomial

$$f(X) = a_0(X-\eta^{(1)})\cdots(X-\eta^{(k)}) \in \mathbb{Z}[X]$$

over the integers with positive a_0, we write $h(\eta)$ for its Weil height given by

$$h(\eta) = \frac{1}{k}\Big(\log a_0 + \sum_{j=1}^{k} \max\{0, \log|\eta^{(j)}|\}\Big).$$

The following consequence of Matveev's theorem is [2, Theorem 9.4].

Theorem 3.1. *If $\Gamma \neq 0$ and $\mathbb{L} \subseteq \mathbb{R}$, then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 \cdots A_l.$$

When $k = 2$ and η_1, η_2 are positive and multiplicatively independent, we can do better. Namely, let in this case B_1, B_2 be real numbers larger than 1 such that

$$\log B_i \geq \max\{h(\eta_i), |\log \eta_i|/d_{\mathbb{L}}, 1/d_{\mathbb{L}}\} \quad i = 1, 2,$$

and let

$$b' = \frac{|d_1|}{d_{\mathbb{L}} \log B_2} + \frac{|d_2|}{d_{\mathbb{L}} \log B_1}.$$

Set

$$\Lambda = d_1 \log \eta_1 + d_2 \log \eta_2.$$

Note that $\Lambda \neq 0$ when η_1 and η_2 are multiplicatively independent. The following inequality is [6, Corollary 2].

Theorem 3.2. *With the above notation, assuming that $k = 2$, $\mathbb{L}$ is real, and η_1, η_2 are positive and multiplicatively independent, we have*

$$\log|\Lambda| > -24.34 d_{\mathbb{L}}^4 (\max\{\log b' + 0.14, 21/d_{\mathbb{L}}, 1/2\})^2 \log B_1 \log B_2. \tag{7}$$

4. The Baker–Davenport lemma. We recall the Baker–Davenport reduction method (see [4, Lemma 5a]), which will be useful to reduce the bounds arising from applying Theorems 3.1 and 3.2.

Lemma 4.1. *Let $\kappa \neq 0$ and μ be real numbers. Assume that M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$, and set*

$$\xi = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\| \cdot \|$ denotes the distance from the nearest integer. If $\xi > 0$, then there is no solution of the inequality

$$0 < |m\kappa - n + \mu| < AB^{-k}$$

in positive integers m, n and k with

$$\frac{\log(AQ/\xi)}{\log B} \leq k \quad \text{and} \quad m \leq M.$$

5. Bounding the variables. We assume that (X_1, Y_1) is the minimal solution of the Pell equation (1). Setting

$$X_1^2 - dY_1^2 =: \varepsilon, \qquad \varepsilon \in \{\pm 1\},$$

we define

$$\delta = X_1 + \sqrt{d}\, Y_1 \quad \text{and} \quad \eta = X_1 - \sqrt{d}\, Y_1 = \varepsilon \delta^{-1}.$$

Then

$$X_n = \tfrac{1}{2}(\delta^n + \eta^n). \tag{8}$$

Since $\delta \geq 1 + \sqrt{2}$, it follows that

$$\delta^n/\alpha \leq X_n < \delta^n \quad \text{for all } n \geq 1. \tag{9}$$

We now assume that (n_1, m_1) and (n_2, m_2) are pairs of positive integers such that

$$X_{n_1} = T_{m_1} \quad \text{and} \quad X_{n_2} = T_{m_2}.$$

To fix ideas, we assume that $n_1 < n_2$, so $m_1 < m_2$. Setting $(n, m) := (n_i, m_i)$, for $i \in \{1, 2\}$ and using inequalities (6) and (9), we get

$$\alpha^{m-2} \leq T_m = X_n < \delta^n \quad \text{and} \quad \delta^n/\alpha \leq X_n = T_m \leq \alpha^{m-1}. \tag{10}$$

Hence,

$$nc_1 \log \delta \leq m \leq nc_1 \log \delta + 2, \qquad c_1 := 1/\log \alpha. \tag{11}$$

Next, using (3) and (8), we get

$$\tfrac{1}{2}(\delta^n + \eta^n) = a\alpha^m + b\beta^m + c\gamma^m,$$

so

$$\delta^n (2a)^{-1} \alpha^{-m} - 1 = -(2a)^{-1} \alpha^{-m} \eta^n + (b/a)(\beta\alpha^{-1})^m + (c/a)(\gamma\alpha^{-1})^m.$$

Hence, using (5), and assuming that $m > 100$, we obtain

$$\begin{aligned} |\delta^n (2a)^{-1} \alpha^{-m} - 1| &\leq \frac{1}{2a\alpha^m \delta^n} + \frac{|b|\,|\beta|^m}{a\alpha^m} + \frac{|c|\,|\gamma|^m}{a\alpha^m} < \frac{1}{2a\alpha^m\delta^n} + \frac{2|b|}{a\alpha^{3m/2}} \\ &< \frac{\alpha^3}{2a\alpha^{2m}} + \frac{2|b|}{a\alpha^{3m/2}} < \frac{4.5}{\alpha^{3m/2}}. \end{aligned}$$

In the above, we have used the fact that $|b|/a < 2$ (see (5)) and $\alpha^{m/2} > \alpha^3/(2a)$ for $m > 100$. Since $\alpha^{3m/2} > 6$, it follows that the last number above is $< 1/2$. Thus,

$$|\delta^n (2a)^{-1} \alpha^{-m} - 1| < \frac{4.5}{\alpha^{3m/2}}. \tag{12}$$

Set

$$\Lambda = n \log \delta - \log 2a - m \log \alpha.$$

Since $|e^\Lambda - 1| < 1/2$, it follows that

$$|\Lambda| < 2|e^\Lambda - 1| < \frac{9}{\alpha^{3m/2}}.$$

Recalling that $(m, n) = (m_i, n_i)$, we get

$$|n_i \log \delta - \log 2a - m_i \log \alpha| < \frac{9}{\alpha^{3m_i/2}} \quad \text{for } i = 1, 2, \tag{13}$$

where $m_2 > m_1 > 100$. We apply Matveev's theorem on the left-hand side of (12). First we need to check that

$$\Gamma := e^{\Lambda} - 1 = \delta^n (2a)^{-1} \alpha^{-m} - 1$$

is nonzero. Indeed, if it were zero, then $\delta^n = (2a)\alpha^m$. The right-hand side belongs to $\mathbb{Q}[\alpha]$ which is a field of degree 3, while the left-hand side belongs to $\mathbb{Q}[\sqrt{d}]$ which is a quadratic field. The intersection of these two fields is $\mathbb{Q}$. Hence, $\delta^n \in \mathbb{Q}$. Since δ is an algebraic integer and $n \geq 1$, it follows that $\delta^n \in \mathbb{Z}$. Since δ is a unit, we get $\delta^n = 1$, so $n = 0$, a contradiction.

Thus, $\Gamma \neq 0$, and we can apply Matveev's theorem. We take

$$l = 3,\ \eta_1 = \delta,\ \eta_2 = 2a,\ \eta_3 = \alpha,\ d_1 = n,\ d_2 = -1,\ d_3 = -m$$

and $\mathbb{L} = \mathbb{Q}[\sqrt{d}, \alpha]$, which has degree $d_{\mathbb{L}} = 6$. Since $\delta \geq 1 + \sqrt{2} > \alpha$, the second inequality (10) tells us right away that $n < m$, so we take $D = m$. We have $h(\eta_1) = (1/2)\log\delta$ and $h(\eta_3) = (1/3)\log\alpha$. Further,

$$a = \frac{\alpha}{\alpha^2 + 2\alpha + 3},$$

and the minimal polynomial of $2a$ is $11X^3 + 4X - 2$ and has roots $2a,\ 2b,\ 2c$. Moreover, $\max\{|2a|, |2b|, |2c|\} < 1$ by (5). Hence, $h(\eta_2) = (1/3)\log 11$. Hence, we can take

$$A_1 = 3\log\delta, \qquad A_2 = 2\log 11, \qquad A_3 = 2\log 1.84.$$

Now Theorem 3.1 tells us that

$$\begin{aligned}\log|\Gamma| &> -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2 (1 + \log 6)(1 + \log m)(3\log\delta)(2\log 11)(2\log 1.84)\\ &> -2.6 \cdot 10^{14} (\log\delta)(1 + \log m).\end{aligned}$$

Comparing the above inequality with (12), we get

$$1.5m\log\alpha - \log 4.5 < 2.6 \cdot 10^{14} (\log\delta)(1 + \log m).$$

Thus,

$$m\log\alpha < 1.8 \cdot 10^{14} (\log\delta)(1 + \log m).$$

Since $\alpha^m > \delta^n$ (see the right part of (10)), we get

$$n < 1.8 \cdot 10^{14} (1 + \log m). \tag{14}$$

Further, since $\alpha > 1.83$, we get

$$m < 3 \cdot 10^{14} (\log\delta)(1 + \log m). \tag{15}$$

Let us record what we have proved so far.

Lemma 5.1. *If $X_n = T_m$ and $m > 100$, then*

$$n < 1.8 \cdot 10^{14} (1 + \log m) \quad \textit{and} \quad m < 3 \cdot 10^{14} (\log\delta)(1 + \log m).$$

Next, we return to the two inequalities given by (13). Multiply the one for $i = 1$ by n_2 and the one for $i = 2$ by n_1, subtract the results and apply the triangle inequality to get

$$\begin{aligned}(16)\qquad & |(n_2-n_1)\log 2a+(n_2m_1-n_1m_2)\log\alpha|\\ & = |n_2(n_1\log\delta-\log 2a-m_1\log\alpha)-n_1(n_2\log\delta-\log 2a-m_2\log\alpha)|\\ & \le n_2|n_1\log\delta-\log 2a-m_1\log\alpha|+n_1|n_2\log\delta-\log 2a-m_2\log\alpha|\\ & \le \frac{9n_2}{\alpha^{3m_1/2}}+\frac{9n_1}{\alpha^{3m_2/2}}<\frac{18n_2}{\alpha^{3m_1/2}}.\end{aligned}$$

We are now ready to apply Theorem 3.2 with

$$l=2,\quad \eta_1=2a,\quad \eta_2=\alpha,\quad d_1=n_2-n_1,\quad d_2=n_2m_1-m_2n_1.$$

The fact that η_1 and η_2 are multiplicatively independent follows because the norm of η_1 is $2/11$ while η_2 is a unit. Observe that $n_2-n_1<n_2$, while by the absolute value inequality in (16), we have

$$|n_2m_1-n_1m_2|\le(n_2-n_1)\frac{|\log 2a|}{\log\alpha}+\frac{12n_2}{\alpha^{3m_1/2}\log\alpha}<2n_2,$$

because $m_1>100$. We have $\mathbb{L}=\mathbb{Q}[\alpha]$ with $d_{\mathbb{L}}=3$. So, we can take

$$\begin{aligned}\log B_1&=\max\{h(\eta_1),|\log\eta_1|/3,1/3\}=(\log 11)/3,\\ \log B_2&=\max\{h(\eta_2),(\log\eta_2)/3,1/3\}=1/3.\end{aligned}$$

Thus,

$$b'=\frac{(n_2-n_1)}{3\cdot(1/3)}+\frac{|n_2m_1-n_1m_2|}{3\cdot\log(11)/3}<2n_2.$$

Now Theorem 3.2 tells us that with

$$\Lambda:=(n_2-n_1)\log 2a+(n_2m_1-n_1m_2)\log\alpha,$$

we have

$$\log|\Lambda|>-24.34\cdot 3^4\max\{\log 2n_2+0.14,7\}^2\cdot(1/3)\cdot\log(11)/3.$$

Thus,

$$\log|\Lambda|>-526(\max\{\log 2n_2+0.14,7\})^2.$$

Combining this with (16), we get

$$1.5m_1\log\alpha-\log 18n_2<526(\max\{\log 2n_2+0.14,7\})^2.$$

If $\log 2n_2+0.14\le 7$, then $n_2\le 476$. The above inequality then gives

$$1.5m_1\log\alpha<526\cdot 7^2+\log 12\cdot 476,$$

which implies $m_1\le 28444$. Hence, $n_1<n_2\le 476$ and $m_1\le 28444$ in this case. Assume next that $n_2>476$. Then

$$1.5m_1\log\alpha<526(\log 2n_2+0.14)^2+\log 18n_2<528(1+\log n_2)^2,$$

which gives

$$(17)\qquad m_1<583(1+\log n_2)^2.$$

Since $\alpha^{m_1} > \delta^{n_1} \geq \delta$ (see the right part of (10)), we get

$$\log \delta < m_1 \log \alpha < 356(1 + \log n_2)^2.$$

Combining this with the second inequality of Lemma 5.1 with $(n, m) = (n_2, m_2)$, together with the fact that $n_2 < m_2$, we get

$$m_2 < 3 \cdot 10^{14} \cdot 356(1 + \log m_2)^3,$$

giving $m_2 < 1.6 \cdot 10^{22}$. Inserting this into the first inequality of Lemma 5.1, we get $n_2 < 10^{16}$, which together with (17) gives $m_1 < 835000$.

Let us summarize what we have proved.

Lemma 5.2. *If $X_{n_i} = T_{m_i}$ for $i = 1, 2$ with $m_1 < m_2$ (so $n_1 < n_2$), then*

$$m_1 < 835000, \qquad n_2 < 10^{16}, \qquad m_2 < 1.6 \cdot 10^{22}.$$

To lower these bounds we use continued fractions on (16), and Baker–Davenport reduction on (13).

6. The final computations. Set $\chi = -\log 2a/\log \alpha$. Inequality (16) implies

$$|(n_2 - n_1)\chi - (n_2 m_1 - n_1 m_2)| < \frac{18 n_2}{\alpha^{3m_1/2} \log \alpha}. \tag{18}$$

Since

$$\frac{18 n_2}{\alpha^{3m_1/2} \log \alpha} < \frac{1}{2(n_2 - n_1)}, \tag{19}$$

it follows that $(n_2 m_1 - n_1 m_2)/(n_2 - n_1)$ is a convergent of $-\log 2a/\log \alpha$. Indeed, $\log \alpha < 0.61$ and $m_1 > 100$, together with Lemma 5.2, imply

$$\alpha^{3m_1/2} > 6 \cdot 10^{33} > 60 n_2^2 > 60(n_2 - n_1) n_2 > \frac{36}{\log \alpha}(n_2 - n_1) n_2, \tag{20}$$

which immediately leads to (19).

Obviously, $n_2 - n_1 < n_2 < 10^{16}$. Let

$$[a_0, a_1, a_2, \ldots] = [1, 1, 1, 1, 6, 1, 1, 22, 1, \ldots]$$

be the continued fraction expansion of χ, and let p_k/q_k be its kth convergent. A computer calculation yields

$$4999601640630812 = q_{33} < 10^{16} < 24351826693265967 = q_{34};$$

further, the maximum of a_i ($i = 0, 1, \ldots, 34$) is $22 = a_7$. Hence,

$$\frac{1}{24 n_2} < \frac{1}{24(n_2 - n_1)} < |(n_2 - n_1)\chi - (n_2 m_1 - n_1 m_2)| < \frac{18 n_2}{\alpha^{3m_1/2} \log \alpha},$$

and comparing the leftmost and rightmost expressions gives $m_1 \leq 87.8$ by Lemma 5.2. Since we have assumed that $m_1 > 100$, we conclude that $m_1 \leq 100$. Now (11) gives $n_1 < 69.2$.

These upper bounds (on n_1 and m_1) make it possible to compute all existing n_1 and m_1. Define

$$P_n^+(X) = \frac{(X+\sqrt{X^2-1})^n + (X-\sqrt{X^2-1})^n}{2},$$
$$P_n^-(X) = \frac{(X+\sqrt{X^2+1})^n + (X-\sqrt{X^2+1})^n}{2}.$$

A computer search on the equations

$$P_{n_1}^+(X_1) = T_{m_1} \quad \text{and} \quad P_{n_1}^-(X_1) = T_{m_1}$$

with $1 \le m_1 \le 100$ and $1 \le n_1 \le 69$, where $n_1 < m_1$, results in only the following possibilities:

Besides the trivial case $n_1 = 1$ (for both equations), which implies $X_1 = T_{m_1}$, only

$$(n_1, m_1, X_1) = (2,5,2) \quad \text{and} \quad (n_1, m_1, X_1) = (3,5,1)$$

are the only nontrivial solutions in the first and in the second case, respectively.

In order to check the trivial cases $n_1 = 1$, $X_1 = T_{m_1}$, we have used a brute force algorithm which essentially coincides with the treatment of the nontrivial cases that will be discussed later. For any $1 \le m_1 \le 100$ we determined the decomposition $T_{m_1}^2 - \varepsilon = dY_1^2$, where d is squarefree. In this way we find $\delta_{m_1} = X_1 + \sqrt{d}\,Y_1$. Then we considered the first convergents of the continued fraction expansions of

$$\frac{\log \delta_{m_1}}{\log \alpha} \tag{21}$$

such that the denominator is larger than $M = 6 \cdot 10^{16}$, and the ξ value in Lemma 4.1 is positive. The upper bounds on m_2 are always less than 100, which contradicts the assertion $m_2 > 100$. Thus, only the cases $m_2 \le 100$ remain to verify. And we get $(n_2, m_2) = (2,5)$ if $(n_1, m_1) = (1,3)$, and further $(n_2, m_2) = (3,5)$ if $(n_1, m_1) = (1,1)$ or $(1,2)$.

To illustrate the treatment, take $\varepsilon = 1$, $m_1 = 17$. Now $T_{17} = 10609$, $T_{17}^2 - 1 = 112550880 = 7034430 \cdot 4^2$, therefore $\delta_{m_{17}} = 10609 + 4\sqrt{7034430}$. The first denominator of the continued fractions corresponding to (21), which is larger than M, is q_{29}, but the first denominator with positive ξ is q_{31} ($\xi > 0.276$). Lemma 4.1 implies $m_2 \le 50$. However, since we have assumed that $m_2 > 100$, we get $m_2 \le 100$. But the equations $P_n^{\pm}(X) = T_m$ have already been solved for $m \le 100$, so we get no further solutions.

The nontrivial solutions lead to $(d, Y_1) = (3,1)$ and $(d, Y_1) = (2,1)$, respectively. Now, applying (13) and Lemma 4.1 we verify the possible new solutions to (2). First observe that

$$\left|n_2\frac{\log\delta}{\log\alpha}-m_2+\chi\right|<\frac{9}{\alpha^{(3/2)m_2}\log\alpha}<14.8\cdot 2.4^{-m_2}.$$

Set $\delta_1=2+\sqrt{3}$ and $\delta_2=1+\sqrt{2}$. Taking the continued fraction expansion of $\log\delta_i/\log\alpha$ $(i=1,2)$ such that the suitable denominator exceeds $6\cdot 10^{16}$, we found that

$$q_{1,31}=156827205418169727\approx 1.56\cdot 10^{17}$$

and

$$q_{2,28}=98827474195551603\approx 9.88\cdot 10^{16}$$

are satisfactory for $i=1$ and $i=2$, respectively. We now apply Lemma 4.1 with $m=n_2$, $n=m_2$, $k=m_2$, $A=14.8$, $B=2.4$, $M=10^{16}$, $\kappa=\log\delta_i/\log\alpha$ and $\mu=\chi$. Further, in the two cases $Q=q_{1,31}$ and $Q=q_{2,28}$, we get $\xi_1>0.039$ and $\xi_2>0.071$. Consequently, $m_2<49.9$, $n_2<23.1$ in the first case, and $m_2<48.7$, $n_2<33.7$ in the second case. However, since we have assumed that $m_2>100$, we get a contradiction, so $m_2\le 100$, leading to $n_2\le 69.2$. Checking the last range we only obtained the possibilities

$$X_1=2=T_3\quad\text{and}\quad X_2=7=T_5,$$

and

$$X_1=1=T_1=T_2\quad\text{and}\quad X_3=7=T_5,$$

respectively. But these are known from the investigation of trivial cases. Hence the proof is complete.

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