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Almost prime triples and Chen's theorem

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1. Introduction. One of the central problems in the theory of prime distribution, namely the twin prime conjecture, states that there exist infinitely many primes p such that $p+2$ is also prime. Although the conjecture has resisted all attacks, there have been spectacular partial achievements. One well known result is due to Chen [2, 3], who proved that there exist infinitely many primes p such that $p+2$ has at most 2 prime factors. Another result belongs to Zhang [13], building on the work of Goldston, Pintz and Yıldırım [5], who showed that there exist infinitely many prime pairs whose differences are less than 70000000. The bound 70000000 has been improved to 600 by Maynard [10], and the bounded gaps result for prime tuples was also obtained by Maynard [10] and Tao (unpublished).

The twin prime conjecture is the simplest case of the Hardy–Littlewood conjecture [7], which postulates an asymptotic for prime tuples in general. Another special case of the Hardy–Littlewood conjecture states that the number of primes $p \le x$ such that $p+2$ and $p+6$ are prime simultaneously should be asymptotic to $\frac{Cx}{\log^3 x}$ for a positive constant C (see (4.6)). In this direction, it has been proved by Porter [12] that there exist infinitely many natural numbers n such that $n(n+2)(n+6)$ has at most eight prime factors, and Maynard [11] was able to replace 8 by 7.

Very recently, Heath-Brown and Xiannan Li [8] showed that there exist infinitely many primes p such that $p+2$ and $p+6$ have at most 2 and at most 76 prime factors respectively. This result improves upon those of Chen [3] and Maynard [11].

Let P_r denote an almost-prime with at most r prime factors, counted according to multiplicity. In this paper we shall show, by a delicate sieve process, the following result.

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Theorem 1. *Let* $\pi_{1,2,r}(x)$ *denote the number of primes* $p \le x$ *such that* $p+2$ *and* $p+6$ *are* P_2 *and* P_r *respectively. Then*

$$\pi_{1,2,14}(x) \gg \frac{x}{\log^3 x}.$$

By the same method, we can prove the following result.

Theorem 2. *For even integer* N, *let* $S(N)$ *denote the number of solutions to the equation* $N = p + P_2$ *with* $p + 6 = P_{14}$. *Then, for sufficiently large* N,

$$S(N) \gg \frac{N}{\log^3 N}.$$

2. Notation and preliminary lemmas. In this paper, $\varepsilon \in (0, 10^{-10})$ and x denotes a sufficiently large real number in terms of ε. By $A \asymp B$ we mean that $A \ll B$ and $B \ll A$. The letter p, with or without subscript, is reserved for prime numbers. We denote by (m, n) the greatest common divisor of m and n, and by $\{m, n\}$ a 2-dimensional vector. As usual, $\varphi(n)$ denotes Euler's function. By $\tau(n)$ we denote the divisor function.

Lemma 1 ([8, Proposition 1]). *Let* $\mathscr{W}$ *be a finite subset of* $\mathbb{N}^2$. *Suppose that* $z_1, z_2 \ge 2$ *with* $\log z_1 \asymp \log z_2$ *and write* $\mathbf{z} = \{z_1, z_2\}$. *For* $\mathbf{d} = \{d_1, d_2\}$ *and* $\mathbf{n} = \{n_1, n_2\}$, *we write* $\mathbf{d} \,|\, \mathbf{n}$ *to mean that* $d_1 \,|\, n_1$ *and* $d_2 \,|\, n_2$. *Set*

$$\mathscr{W}_{\mathbf{d}} = \{\mathbf{n} \in \mathscr{W} : \mathbf{d} \,|\, \mathbf{n}\}, \qquad S(\mathscr{W}, \mathbf{z}) = \sum_{\substack{\{n_1, n_2\} \in \mathscr{W} \\ p|n_1 \Rightarrow p \ge z_1 \\ p|n_2 \Rightarrow p \ge z_2}} 1.$$

Suppose that

$$|\mathscr{W}_{\mathbf{d}}| = h(\mathbf{d})X + R(\mathbf{d})$$

for some $X > 0$ *independent of* $\mathbf{d}$ *and some multiplicative function* $h(\mathbf{d}) \in [0, 1)$ *such that* $h(p, 1) + h(1, p) - 1 < h(p, p) \le h(p, 1) + h(1, p)$ *for all primes* p, *and*

$$h(p, 1), h(1, p) \le cp^{-1}, \qquad h(p, p) \le cp^{-2}$$

for some constant $c \ge 2$.

Let $h_1(d) = h(d, 1)$ *and* $h_2(d) = h(1, d)$. *Suppose that*

$$\prod_{w \le p < z} (1 - h_j(p))^{-1} \le \frac{\log z}{\log w}\left(1 + \frac{L}{\log w}\right) \qquad (j = 1, 2)$$

for $z \ge w \ge 2$ *and some positive constant* L. *Then:*

$$\text{(i)}\quad S(\mathscr{W},\mathbf{z}) \le XV(z_0,h^*)V_1V_2F(s_1)F(s_2)\{1+O((\log D_1D_2)^{-1/18})\} + O_\varepsilon\Big(\sum_{d_1d_2\le(D_1D_2)^{1+\varepsilon}} \tau^4(d_1d_2)|R(\{d_1,d_2\})|\Big),$$

$$\text{(ii)}\quad S(\mathscr{W},\mathbf{z}) \ge XV(z_0,h^*)V_1V_2\{f(s_1)F(s_2)+F(s_1)f(s_2)-F(s_1)F(s_2)\} \times \big(1+O((\log D_1D_2)^{-1/18})\big) + O_\varepsilon\Big(\sum_{d_1d_2\le(D_1D_2)^{1+\varepsilon}} \tau^4(d_1d_2)|R(\{d_1,d_2\})|\Big)$$

for any $\varepsilon > 0$, *where* $f(s)$ *and* $F(s)$ *denote the standard functions in the linear sieve theory, and*

$$z_0 = \exp(\sqrt[3]{\log z_1z_2}), \qquad s_j = \frac{\log D_j}{\log z_j} \quad (j=1,2)$$

$$V(z_0,h^*) = \prod_{p<z_0}(1-h^*(p)), \qquad h^*(p) = h(p,1)+h(1,p)-h(p,p),$$

$$V_j = \prod_{z_0\le p<z_j}(1-h_j(p)) \quad (j=1,2).$$

Lemma 2 ([6, (3.11) and (3.12)]). *For the functions* $f(s)$ *and* $F(s)$ *in the linear sieve,*

$$F(s) = \begin{cases} \dfrac{2e^\gamma}{s}, & 0<s\le 3,\\[2mm] \dfrac{2e^\gamma}{s}\Big(1+\displaystyle\int_2^{s-1}\frac{\log(t-1)}{t}\,dt\Big), & 3\le s\le 5,\\[2mm] \dfrac{2e^\gamma}{s}\Big(1+\displaystyle\int_2^{s-1}\frac{\log(t-1)}{t}\,dt \\ \quad +\displaystyle\int_2^{s-3}\frac{\log(t-1)}{t}\,dt\int_{t+2}^{s-1}\frac{1}{u}\log\frac{u-1}{t+1}\,du\Big), & 5\le s\le 7; \end{cases}$$

$$f(s) = \begin{cases} \dfrac{2e^\gamma}{s}\log(s-1), & 2\le s\le 4,\\[2mm] \dfrac{2e^\gamma}{s}\Big(\log(s-1)+\displaystyle\int_3^{s-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(u-1)}{u}\,du\Big), & 4\le s\le 6,\\[2mm] \dfrac{2e^\gamma}{s}\Big(\log(s-1)+\displaystyle\int_3^{s-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(u-1)}{u}\,du \\ \quad +\displaystyle\int_2^{s-4}\frac{\log(t-1)}{t}\,dt\int_{t+2}^{s-2}\frac{1}{u}\log\frac{u-1}{t+1}\log\frac{s}{u+2}\,du\Big), & 6\le s\le 8, \end{cases}$$

where $\gamma = 0.577\ldots$ *denotes the Euler constant.*

Lemma 3 ([8, Lemma 2], deduced from [4, Theorem 22.3]). *Let*

$$P(y_1,\dots,y_k;z_1,\dots,z_k)=\{m=p_1\cdots p_k \mid y_1\le p_1\le z_1,\dots,y_k\le p_k\le z_k\},$$

$$\pi_k(x;q,a)=\sum_{\substack{m\in P(y_1,\dots,y_k;z_1,\dots,z_k)\\ m\le x,\, m\equiv a\,(\mathrm{mod}\, q)}} 1,$$

$$\pi_k(x;q)=\sum_{\substack{m\in P(y_1,\dots,y_k;z_1,\dots,z_k)\\ m\le x,\, (m,q)=1}} 1.$$

Then for any $A>0$ *and* $l\ge 1$ *there exists* $B=B(A,l)>0$ *such that*

$$\sum_{q\le x^{1/2}\log^{-B}x}\tau^l(q)\max_{(a,q)=1}\left|\pi_k(x;q,a)-\frac{\pi_k(x;q)}{\varphi(q)}\right|\ll\frac{x}{\log^A x},$$

where the implied constant depends only on k, l *and* A.

Lemma 4 ([9, Lemmas 20 and 21]). *Let*

$$z=x^{1/u},\qquad Q(z)=\prod_{p<z}p.$$

Then for $1<u_0\le u\le 100$,

$$\sum_{\substack{n\le x\\ (n,Q(z))=1}}1=w(u)\frac{x}{\log z}+O\left(\frac{x}{\log^2 z}\right),$$

where $w(u)$ *is determined by the differential-difference equation*

$$\begin{cases} w(u)=1/u, & 1\le u\le 2,\\ (uw(u))'=w(u-1), & u\ge 2.\end{cases}$$

Moreover,

$$w(u)\le\begin{cases}0.5617, & u\ge 4,\\ 0.5644, & u\ge 3.\end{cases}$$

3. Weighted sieve. Let $\mathscr{A}$ denote a finite set of positive integers and $\mathscr{P}$ an infinite set of primes, and denote

$$\mathscr{P}(q)=\{p\in\mathscr{P}:(p,q)=1\},\qquad \mathscr{A}_d=\{a\in\mathscr{A}:a\equiv 0\ (\mathrm{mod}\ d)\}.$$

For sufficiently large x, set

$$D_1=x^{0.475-\varepsilon},\qquad D_2=x^{0.025},\qquad z=D_2^{1/5}.$$

For the rest of this paper we always assume that

$$\mathscr{P}=\{p:p>2\},\qquad P(w)=\prod_{\substack{p<w\\ p\in\mathscr{P}}}p,$$

$$\mathscr{A}=\{p+2:7<p\le x,\ (p+6,P(z))=1\},$$

$$S(\mathscr{A};\mathscr{P},w)=\sum_{\substack{a\in\mathscr{A}\\(a,P(w))=1}}1.$$

LEMMA 5. *For* $0<\alpha<\beta<1/3$,

$$\begin{aligned}2\pi_{1,2,r}(x)\ge 2S(\mathscr{A};\mathscr{P},x^{\alpha})&-\sum_{x^{\alpha}\le p\le x^{\beta}}S(\mathscr{A}_p;\mathscr{P},x^{\alpha})\\&-\sum_{x^{\alpha}\le p_1\le x^{\beta}\le p_2\le (x/p_1)^{1/2}}S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)\\&-2\sum_{x^{\beta}\le p_1\le p_2\le (x/p_1)^{1/2}}S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)\\&+\sum_{x^{\alpha}\le p_1\le p_2\le p_3\le x^{\beta}}S(\mathscr{A}_{p_1p_2p_3};\mathscr{P}(p_1),p_2)\\&-\sum_{k=r+1}^{199}S(\mathscr{A}^{(k)};\mathscr{P},x^{\alpha})-\sum_{k=r+1}^{199}S(\mathscr{A}^{(k)};\mathscr{P},x^{\beta})+O(x^{1-\alpha}),\end{aligned}$$

where

$$\begin{aligned}\mathscr{A}^{(k)}&=\{p+2:7<p\le x,p+6\in\mathscr{M}_k\},\\\mathscr{M}_k&=\{m=p_1\cdots p_k:13<p_1\cdots p_k\le x+6,\ z\le p_1\le\cdots\le p_k\}.\end{aligned}$$

Proof. It is similar to that of [1, Lemma 5]. By the trivial inequality

$$\begin{aligned}(3.1)\quad \pi_{1,2,r}(x)&\ge\sum_{\substack{a\in\mathscr{A},a=P_2\\(a,P(x^{\beta}))=1}}1-\sum_{k=r+1}^{199}\sum_{\substack{a\in\mathscr{A},\,a=P_2\\(a,P(x^{\beta}))=1,\,a+4\in\mathscr{M}_k}}1\\&\ge S(\mathscr{A};\mathscr{P},x^{\beta})-\sum_{x^{\beta}\le p_1\le p_2\le(x/p_1)^{1/2}}S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)\\&\quad-\sum_{k=r+1}^{199}S(\mathscr{A}^{(k)};\mathscr{P},x^{\beta})\end{aligned}$$

and Buchstab's identity, we have

$$\begin{aligned}(3.2)\quad \pi_{1,2,r}(x)&\ge S(\mathscr{A};\mathscr{P},x^{\alpha})-\sum_{x^{\alpha}\le p\le x^{\beta}}S(\mathscr{A}_p;\mathscr{P},x^{\alpha})\\&+\sum_{x^{\alpha}\le p_1\le p_2\le x^{\beta}}S(\mathscr{A}_{p_1p_2};\mathscr{P},p_1)-\sum_{x^{\beta}\le p_1\le p_2\le(x/p_1)^{1/2}}S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)\\&-\sum_{k=r+1}^{199}S(\mathscr{A}^{(k)};\mathscr{P},x^{\beta})+O(x^{1-\alpha}),\end{aligned}$$

where we have used the trivial bound

$$\sum_{x^\alpha \le p \le x^\beta} S(\mathscr{A}_{p^2}; \mathscr{P}, p) \ll x^{1-\alpha}. \tag{3.3}$$

Similar to (3.1), we have the obvious inequality

$$\begin{aligned}
\pi_{1,2,r}(x) &\ge S(\mathscr{A}; \mathscr{P}, x^\alpha) - \sum_{x^\alpha \le p_1 \le p_2 \le (x/p_1)^{1/2}} S(\mathscr{A}_{p_1p_2}; \mathscr{P}(p_1), p_2) \\
&\quad - \sum_{k=r+1}^{199} S(\mathscr{A}^{(k)}; \mathscr{P}, x^\alpha) \\
&= S(\mathscr{A}; \mathscr{P}, x^\alpha) - \sum_{x^\alpha \le p_1 \le p_2 \le x^\beta} S(\mathscr{A}_{p_1p_2}; \mathscr{P}(p_1), p_2) \\
&\quad - \sum_{x^\alpha \le p_1 \le x^\beta \le p_2 \le (x/p_1)^{1/2}} S(\mathscr{A}_{p_1p_2}; \mathscr{P}(p_1), p_2) \\
&\quad - \sum_{x^\beta \le p_1 \le p_2 \le (x/p_1)^{1/2}} S(\mathscr{A}_{p_1p_2}; \mathscr{P}(p_1), p_2) \\
&\quad - \sum_{k=r+1}^{199} S(\mathscr{A}^{(k)}; \mathscr{P}, x^\alpha).
\end{aligned} \tag{3.4}$$

It follows from Buchstab's identity that

$$\begin{aligned}
&\sum_{x^\alpha \le p_1 \le p_2 \le x^\beta} S(\mathscr{A}_{p_1p_2}; \mathscr{P}, p_1) - \sum_{x^\alpha \le p_1 \le p_2 \le x^\beta} S(\mathscr{A}_{p_1p_2}; \mathscr{P}(p_1), p_2) \\
&\qquad = \sum_{x^\alpha \le p_1 \le p_2 \le p_3 \le x^\beta} S(\mathscr{A}_{p_1p_2p_3}; \mathscr{P}(p_1), p_2) + O(x^{1-\alpha}),
\end{aligned} \tag{3.5}$$

where a bound similar to (3.3) is applied. Now Lemma 5 follows from (3.2) and (3.4)–(3.5).

LEMMA 6. *We have*

$$\begin{aligned}
4\pi_{1,2,14}(x) &\ge 3S(\mathscr{A}; \mathscr{P}, x^{1/13}) + S(\mathscr{A}; \mathscr{P}, x^{1/8.4}) \\
&\quad + \sum_{x^{1/13} \le p_1 \le p_2 \le x^{1/8.4}} S(\mathscr{A}_{p_1p_2}; \mathscr{P}, x^{1/13}) \\
&\quad + \sum_{x^{1/13} \le p_1 \le x^{1/8.4} \le p_2 \le D_1 x^{-2/13} p_1^{-1}} S(\mathscr{A}_{p_1p_2}; \mathscr{P}, x^{1/13}) \\
&\quad - \sum_{x^{1/13} \le p \le x^{1/3.145}} S(\mathscr{A}_p; \mathscr{P}, x^{1/13}) - \sum_{x^{1/13} \le p \le x^{1/3.81}} S(\mathscr{A}_p; \mathscr{P}, x^{1/13}) \\
&\quad - \sum_{x^{1/13} \le p_1 \le x^{1/3.145} \le p_2 \le (x/p_1)^{1/2}} S(\mathscr{A}_{p_1p_2}; \mathscr{P}(p_1), p_2)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{x^{1/8.4}\le p_1\le x^{1/3.81}\le p_2\le (x/p_1)^{1/2}} S\Bigl(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),\Bigl(\frac{x}{p_1p_2}\Bigr)^{1/2}\Bigr)\\
&- 2\sum_{x^{1/3.145}\le p_1\le p_2\le (x/p_1)^{1/2}} S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)\\
&- 2\sum_{x^{1/3.81}\le p_1\le p_2\le (x/p_1)^{1/2}} S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)\\
&- \sum_{x^{1/13}\le p_1\le p_2\le p_3\le p_4\le x^{1/8.4}} S(\mathscr{A}_{p_1p_2p_3p_4};\mathscr{P}(p_1),p_2)\\
&- \sum_{x^{1/13}\le p_1\le p_2\le p_3\le x^{1/8.4}\le p_4\le D_1x^{-2/13}p_3^{-1}} S(\mathscr{A}_{p_1p_2p_3p_4};\mathscr{P}(p_1),p_2)\\
&- \sum_{k=15}^{199} S(\mathscr{A}^{(k)};\mathscr{P},x^{1/13}) - \sum_{k=15}^{199} S(\mathscr{A}^{(k)};\mathscr{P},x^{1/8.4})\\
&- \sum_{k=15}^{199} S(\mathscr{A}^{(k)};\mathscr{P},x^{1/3.145}) - \sum_{k=15}^{199} S(\mathscr{A}^{(k)};\mathscr{P},x^{1/3.81}) + O(x^{12/13})\\
={}& 3S_1+S_2+S_3+S_4-S_5-S_6-S_7-S_8\\
&- 2S_9-2S_{10}-S_{11}-S_{12}-\varSigma_1-\varSigma_2-\varSigma_3-\varSigma_4+O(x^{12/13}).
\end{aligned}
$$

Proof. The argument is similar to that for [1, Lemma 6]. By Buchstab's identity, we have

$$
\begin{aligned}
S(\mathscr{A};\mathscr{P},x^{1/8.4}) ={}& S(\mathscr{A};\mathscr{P},x^{1/13}) - \sum_{x^{1/13}\le p\le x^{1/8.4}} S(\mathscr{A}_p;\mathscr{P},x^{1/13})\\
&+ \sum_{x^{1/13}\le p_1\le p_2\le x^{1/8.4}} S(\mathscr{A}_{p_1p_2};\mathscr{P},x^{1/13})\\
&- \sum_{x^{1/13}\le p_1\le p_2\le p_3\le x^{1/8.4}} S(\mathscr{A}_{p_1p_2p_3};\mathscr{P},p_1) + O(x^{12/13}),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&\sum_{x^{1/8.4}\le p\le x^{1/3.81}} S(\mathscr{A}_p;\mathscr{P},x^{1/8.4})\\
&\quad\le \sum_{x^{1/8.4}\le p\le x^{1/3.81}} S(\mathscr{A}_p;\mathscr{P},x^{1/13})\\
&\qquad - \sum_{x^{1/13}\le p_1\le x^{1/8.4}\le p_2\le D_1x^{-2/13}p_1^{-1}} S(\mathscr{A}_{p_1p_2};\mathscr{P},x^{1/13})\\
&\qquad + \sum_{x^{1/13}\le p_1\le p_2\le x^{1/8.4}\le p_3\le D_1x^{-2/13}p_2^{-1}} S(\mathscr{A}_{p_1p_2p_3};\mathscr{P},p_1) + O(x^{12/13})
\end{aligned}
\tag{3.7}
$$

and

$$\begin{aligned}\sum_{x^{1/8.4}\le p_1\le x^{1/3.81}\le p_2\le (x/p_1)^{1/2}} & S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)\\ &= \sum_{x^{1/8.4}\le p_1\le x^{1/3.81}\le p_2\le (x/p_1)^{1/3}} S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)\\ &\quad + \sum_{\substack{x^{1/8.4}\le p_1\le x^{1/3.81}\\ (x/p_1)^{1/3}\le p_2\le (x/p_1)^{1/2}}} S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2),\end{aligned}\tag{3.8}$$

where a bound similar to (3.3) is employed. If $p_2 \le (x/p_1)^{1/3}$, then $p_2 \le (x/p_1p_2)^{1/2}$, and it follows from Buchstab's identity that

$$\begin{aligned}\sum_{x^{1/8.4}\le p_1\le x^{1/3.81}\le p_2\le (x/p_1)^{1/3}} & S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)\\ &= \sum_{x^{1/8.4}\le p_1\le x^{1/3.81}\le p_2\le (x/p_1)^{1/3}} S\left(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),\left(\frac{x}{p_1p_2}\right)^{1/2}\right)\\ &\quad + \sum_{x^{1/8.4}\le p_1\le x^{1/3.81}\le p_2\le p_3\le \left(\frac{x}{p_1p_2}\right)^{1/3}} S(\mathscr{A}_{p_1p_2p_3};\mathscr{P}(p_1p_2),p_3).\end{aligned}\tag{3.9}$$

On the other hand, If $p_2 \ge (x/p_1)^{1/3}$, then $p_2 \ge \left(\frac{x}{p_1p_2}\right)^{1/2}$, and we have

$$\begin{aligned}\sum_{\substack{x^{1/8.4}\le p_1\le x^{1/3.81}\\ (x/p_1)^{1/3}\le p_2\le (x/p_1)^{1/2}}} & S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)\\ &\le \sum_{\substack{x^{1/8.4}\le p_1\le x^{1/3.81}\\ (x/p_1)^{1/3}\le p_2\le (x/p_1)^{1/2}}} S\left(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),\left(\frac{x}{p_1p_2}\right)^{1/2}\right).\end{aligned}\tag{3.10}$$

From (3.8)–(3.10), we get

$$\begin{aligned}\sum_{x^{1/8.4}\le p_1\le x^{1/3.81}\le p_2\le (x/p_1)^{1/2}} & S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)\\ &\le \sum_{x^{1/8.4}\le p_1\le x^{1/3.81}\le p_2\le (x/p_1)^{1/2}} S\left(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),\left(\frac{x}{p_1p_2}\right)^{1/2}\right)\\ &\quad + \sum_{x^{1/8.4}\le p_1\le x^{1/3.81}\le p_2\le p_3\le \left(\frac{x}{p_1p_2}\right)^{1/3}} S(\mathscr{A}_{p_1p_2p_3};\mathscr{P}(p_1p_2),p_3).\end{aligned}\tag{3.11}$$

By Buchstab's identity, we obtain

$$
\begin{aligned}
(3.12)\qquad & \sum_{x^{1/13}\le p_1\le p_2\le p_3\le x^{1/3.145}} S(\mathscr{A}_{p_1p_2p_3};\mathscr{P}(p_1),p_2)\\
&- \sum_{x^{1/13}\le p_1\le p_2\le p_3\le x^{1/8.4}} S(\mathscr{A}_{p_1p_2p_3};\mathscr{P},p_1)\\
&- \sum_{x^{1/13}\le p_1\le p_2\le x^{1/8.4}\le p_3\le D_1x^{-2/13}p_2^{-1}} S(\mathscr{A}_{p_1p_2p_3};\mathscr{P},p_1)\\
&- \sum_{x^{1/8.4}\le p_1\le x^{1/3.81}\le p_2\le p_3\le \left(\frac{x}{p_1p_2}\right)^{1/3}} S(\mathscr{A}_{p_1p_2p_3};\mathscr{P}(p_1p_2),p_3)\\
\ge &- \sum_{x^{1/13}\le p_1\le p_2\le p_3\le p_4\le x^{1/8.4}} S(\mathscr{A}_{p_1p_2p_3p_4};\mathscr{P}(p_1),p_2)\\
&- \sum_{x^{1/13}\le p_1\le p_2\le p_3\le x^{1/8.4}\le p_4\le D_1x^{-2/13}p_2^{-1}} S(\mathscr{A}_{p_1p_2p_3p_4};\mathscr{P}(p_1),p_2)+O(x^{12/13}),
\end{aligned}
$$

where a bound similar to (3.3) is used. Now by Lemma 5 with $(\alpha,\beta)=(1/13,1/3.145)$ and $(\alpha,\beta)=(1/8.4,1/3.81)$, Lemma 6 follows from (3.6)–(3.7) and (3.11)–(3.12).

4. Proof of Theorem 1. In this section we assume the notation in Section 3, in particular,

$$D_1=x^{0.475-\varepsilon},\qquad D_2=x^{0.025},\qquad z=D_2^{1/5}.$$

4.1. Lower bounds of S_j $(j=1,2,3,4)$. Let

$$\mathscr{W}=\bigl\{\{p+2,p+6\}:7<p\le x\bigr\}.$$

We will apply Lemma 1 to $\mathscr{W}$. We note that for prime $p>7$, $d_1\mid(p+2)$ and $d_2\mid(p+6)$ imply that

$$(d_1,d_2)=(d_1,2)=(d_2,6)=1, \tag{4.1}$$

therefore we can take

$$|\mathscr{W}_{\mathbf{d}}|=h(\mathbf{d})X+R(\mathbf{d}),\qquad X=\pi(x),$$

$$h(\mathbf{d})=\begin{cases}\frac{1}{\varphi(d_1d_2)}, & (d_1,d_2)=(d_1,2)=(d_2,6)=1,\\ 0, & \text{otherwise.}\end{cases}$$

By the Chinese remainder theorem, we have

$$|R(\mathbf{d})|\le|r(d_1d_2)|, \tag{4.2}$$

where

$$|r(d)|=\max_{(a,d)=1}\Biggl|\sum_{\substack{p\le x\\ p\equiv a\,(\mathrm{mod}\,d)}}1-\frac{\pi(x)}{\varphi(d)}\Biggr|+O(1). \tag{4.3}$$

Hence by the Bombieri–Vinogradov theorem, we obtain

$$\sum_{d_1 d_2 \le (D_1 D_2)^{1+\varepsilon}} \tau^4(d_1 d_2)|R(\mathbf{d})| \ll \frac{x}{\log^{10} x}. \tag{4.4}$$

It is easy to show that

$$h_1(p) = \begin{cases} 0, & p = 2, \\ \frac{1}{p-1}, & p \ge 3; \end{cases} \qquad h_2(p) = \begin{cases} 0, & p = 2, 3, \\ \frac{1}{p-1}, & p \ge 5; \end{cases}$$

$$h^*(p) = \begin{cases} 0, & p = 2, \\ \frac{1}{2}, & p = 3, \\ \frac{2}{p-1}, & p \ge 5. \end{cases}$$

Therefore

$$\begin{aligned} V(z_0, h^*)V_1V_2 &= \frac{1}{2}\prod_{3<p\le z_0}\left(1-\frac{2}{p-1}\right)\prod_{z_0<p\le x^{1/13}}\left(1-\frac{1}{p-1}\right)\prod_{z_0<p\le z}\left(1-\frac{1}{p-1}\right) \\ &= \bigl(1+O(z_0^{-1})\bigr)CV(x^{1/13})V(z), \end{aligned} \tag{4.5}$$

where

$$C = \frac{9}{2}\prod_{p>3}\left(1-\frac{3p-1}{(p-1)^3}\right), \qquad V(w) = \prod_{p<w}\left(1-\frac{1}{p}\right). \tag{4.6}$$

By Lemma 1(ii) and (4.4)–(4.5),

$$\begin{aligned} S_1 &= S(\mathscr{W}, \{x^{1/13}, z\}) \\ &\ge (1+O(\varepsilon))C\pi(x)V(x^{1/13})V(z) \\ &\quad \times \{f(6.175)F(5) + F(6.175)f(5) - F(6.175)F(5)\} \\ &\quad + O\left(\frac{x}{\log^{10} x}\right) \\ &= (1+O(\varepsilon))\frac{4C\pi(x)}{(\log D_1)(\log D_2)} \\ &\quad \times \{f_0(6.175)F_0(5) + F_0(6.175)f_0(5) - F_0(6.175)F_0(5)\} \\ &\quad + O\left(\frac{x}{\log^{10} x}\right) \\ &\ge 2.42877\frac{4C\pi(x)}{(\log D_1)(\log D_2)}, \end{aligned} \tag{4.7}$$

where Mertens' prime formula

$$V(w) = \left(1 + O\left(\frac{1}{\log w}\right)\right)\frac{e^{-\gamma}}{\log w}$$

and numerical integration are used, and

$$f_0(s) = \frac{s}{2e^\gamma} f(s), \qquad F_0(s) = \frac{s}{2e^\gamma} F(s).$$

In the same manner, we get

$$\begin{aligned}(4.8) \qquad S_2 &\geq (1+O(\varepsilon))\frac{4C\pi(x)}{(\log D_1)(\log D_2)} \\ &\quad \times \{f_0(3.99)F_0(5) + F_0(3.99)f_0(5) - F_0(3.99)F_0(5)\} \\ &\quad + O\biggl(\frac{x}{\log^{10} x}\biggr) \\ &\geq 1.53433\frac{4C\pi(x)}{(\log D_1)(\log D_2)}.\end{aligned}$$

By Lemma 1(ii),

$$\begin{aligned}(4.9) \qquad S(\mathscr{A}_{p_1p_2}; \mathscr{P}, x^{1/13}) &= S(\mathscr{W}_{\{p_1p_2,1\}}, \{x^{1/13}, z\}) \\ &\geq (1+O(\varepsilon))\frac{4C\pi(x)}{p_1p_2\bigl(\log\frac{D_1}{p_1p_2}\bigr)(\log D_2)}\biggl\{f_0\biggl(\frac{13\log\frac{D_1}{p_1p_2}}{\log x}\biggr)F_0(5) \\ &\quad + F_0\biggl(\frac{13\log\frac{D_1}{p_1p_2}}{\log x}\biggr)f_0(5) - F_0\biggl(\frac{13\log\frac{D_1}{p_1p_2}}{\log x}\biggr)F_0(5)\biggr\} \\ &\quad + O_\varepsilon\biggl(\sum_{d_1d_2 \leq \frac{D^{1+\varepsilon}}{p_1p_2}} \tau^4(d_1d_2)|R(\{p_1p_2d_1, d_2\})|\biggr).\end{aligned}$$

It follows from (4.2)–(4.3), (4.5) and (4.9) that

$$\begin{aligned}(4.10) \qquad S_3 &\geq (1+O(\varepsilon))\frac{4C\pi(x)}{(\log D_1)(\log D_2)} \\ &\quad \times 0.475\{f_0(5)G_0 + F_0(5)g_0 - G_0F_0(5)\} + O\biggl(\frac{x}{\log^{10} x}\biggr) \\ &\geq 0.21948\frac{4C\pi(x)}{(\log D_1)(\log D_2)},\end{aligned}$$

where the Bombieri–Vinogradov theorem and summation by parts are used, and

$$\begin{aligned}G_0 &= \int_{1/13}^{1/8.4}\frac{dt_1}{t_1}\int_{t_1}^{1/8.4}\frac{dt_2}{t_2(0.475-t_1-t_2)} \\ &\quad + \int_{1/13}^{1/8.4}\frac{dt_1}{t_1}\int_{t_1}^{1/8.4}\frac{dt_2}{t_2(0.475-t_1-t_2)}\int_2^{5.175-13(t_1+t_2)}\frac{\log(t_3-1)}{t_3}\,dt_3,\end{aligned}$$

$$
\begin{aligned}
g_0 = &\int_{1/13}^{1/8.4} \frac{dt_1}{t_1} \int_{t_1}^{1/8.4} \frac{\log(5.175 - 13(t_1 + t_2))}{t_2(0.475 - t_1 - t_2)}\, dt_2 \\
&+ \int_{1/13}^{2.675/26} \frac{dt_1}{t_1} \int_{t_1}^{2.175/13 - t_1} \frac{dt_2}{t_2(0.475 - t_1 - t_2)} \\
&\times \int_{3}^{5.175 - 13(t_1 + t_2)} \frac{dt_3}{t_3} \int_{2}^{t_3 - 1} \frac{\log(t_4 - 1)}{t_4}\, dt_4.
\end{aligned}
$$

By similar arguments,

$$
\begin{aligned}
S_4 &\geq (1 + O(\varepsilon)) \frac{4C\pi(x)}{(\log D_1)(\log D_2)} \\
&\quad \times 0.475\{f_0(5)H_0 + F_0(5)h_0 - H_0F_0(5)\} + O\left(\frac{x}{\log^{10} x}\right) \\
&\geq 0.44275 \frac{4C\pi(x)}{(\log D_1)(\log D_2)},
\end{aligned}
\tag{4.11}
$$

where

$$
\begin{aligned}
H_0 = &\int_{1/13}^{1/8.4} \frac{dt_1}{t_1} \int_{1/8.4}^{0.475 - 2/13 - t_1} \frac{dt_2}{t_2(0.475 - t_1 - t_2)} \\
&+ \int_{1/13}^{1/8.4} \frac{dt_1}{t_1} \int_{1/8.4}^{0.475 - 2/13 - t_1} \frac{dt_2}{t_2(0.475 - t_1 - t_2)} \int_{2}^{5.175 - 13(t_1 + t_2)} \frac{\log(t_3 - 1)}{t_3}\, dt_3, \\
h_0 = &\int_{1/13}^{1/8.4} \frac{dt_1}{t_1} \int_{1/8.4}^{0.475 - 2/13 - t_1} \frac{\log(5.175 - 13(t_1 + t_2))}{t_2(0.475 - t_1 - t_2)}\, dt_2.
\end{aligned}
$$

4.2. Upper bounds of S_5 and S_6. By Lemma 1(i) and (4.5),

$$
\begin{aligned}
S(\mathscr{A}_p; \mathscr{P}, x^{1/13}) &= S(\mathscr{W}_{\{p,1\}}, \{x^{1/13}, z\}) \\
&\leq (1 + O(\varepsilon)) \frac{4C\pi(x)}{(p-1)\left(\log \frac{D_1}{p}\right)(\log D_2)} F_0\left(\frac{13 \log \frac{D_1}{p}}{\log x}\right) F_0(5) \\
&\quad + O_\varepsilon\Bigl(\sum_{d_1 d_2 \leq D^{1+\varepsilon}/p} \tau^4(d_1 d_2) |R(\{pd_1, d_2\})|\Bigr).
\end{aligned}
\tag{4.12}
$$

From (4.2)–(4.3) and (4.12), we get

$$
\begin{aligned}
S_5 &\leq (1 + O(\varepsilon)) \frac{4C\pi(x)}{(\log D_1)(\log D_2)} 0.475 J_0 F_0(5) + O\left(\frac{x}{\log^{10} x}\right) \\
&\leq 3.80708 \frac{4C\pi(x)}{(\log D_1)(\log D_2)},
\end{aligned}
\tag{4.13}
$$

where the Bombieri–Vinogradov theorem and summation by parts are applied, and

$$
\begin{aligned}
J_0 = & \int_{1/13}^{1/3.145} \frac{dt}{t(0.475-t)} + \int_{1/13}^{0.475-3/13} \frac{dt_1}{t_1(0.475-t_1)} \int_{2}^{5.175-13t_1} \frac{\log(t_2-1)}{t_2}\, dt_2 \\
& + \int_{1/13}^{0.475-5/13} \frac{dt_1}{t_1(0.475-t_1)} \int_{2}^{3.175-13t} \frac{\log(t_2-1)}{t_2}\, dt_2 \int_{t_2+2}^{5.175} \frac{1}{t_3} \log\frac{t_3-1}{t_2+1}\, dt_3.
\end{aligned}
$$

Similar to S_5, we have

$$
\begin{aligned}
S_6 & \le (1+O(\varepsilon))\frac{4C\pi(x)}{(\log D_1)(\log D_2)} 0.475 K_0 F_0(5) + O\left(\frac{x}{\log^{10} x}\right) \\
& \le 3.11185 \frac{4C\pi(x)}{(\log D_1)(\log D_2)},
\end{aligned}
\tag{4.14}
$$

where

$$
\begin{aligned}
K_0 = & \int_{1/13}^{1/3.81} \frac{dt}{t(0.475-t)} + \int_{1/13}^{0.475-3/13} \frac{dt_1}{t_1(0.475-t_1)} \int_{2}^{5.175-13t_1} \frac{\log(t_2-1)}{t_2}\, dt_2 \\
& + \int_{1/13}^{0.475-5/13} \frac{dt_1}{t_1(0.475-t_1)} \int_{2}^{3.175-13t} \frac{\log(t_2-1)}{t_2}\, dt_2 \int_{t_2+2}^{5.175} \frac{1}{t_3} \log\frac{t_3-1}{t_2+1}\, dt_3.
\end{aligned}
$$

4.3. Upper bounds of S_j $(7 \le j \le 12)$. Let

$$
\begin{aligned}
\mathscr{N} = \{ & m = p_1p_2p_3p_4n : p_1p_2p_3p_4n \le x+2, \\
& x^{1/13} \le p_1 \le p_2 \le p_3 \le p_4 \le x^{1/8.4},\ p \mid n \Rightarrow p \ge p_2\}, \\
\mathscr{M} = \{ & m = n-2 : n \in \mathscr{N}\}.
\end{aligned}
$$

Then

$$
S_{11} = \sum_{\substack{m\in\mathscr{M} \\ m=p \\ (m+6,P(z))=1}} 1 \le \sum_{\substack{m\in\mathscr{M} \\ (m,P(D_1^{1/2}))=1 \\ (m+6,P(z))=1}} 1 + O(x^{1/2}).
\tag{4.15}
$$

Set

$$
\mathscr{W}^{(1)} = \big\{\{m, m+6\} : m \in \mathscr{M}\big\}.
$$

If $\mathbf{d} \mid \{m, m+6\}$, then $\mathbf{d} = \{d_1, d_2\}$ satisfies (4.1), and we may write

$$
|\mathscr{W}^{(1)}_{\mathbf{d}}| = h(\mathbf{d})|\mathscr{N}| + R^{(1)}(\mathbf{d}),
$$

where

$$
\begin{aligned}
(4.16)\qquad |R^{(1)}(\mathbf{d})| &\le \max_{(a,d_1d_2)=1}\Bigg|\sum_{\substack{n\in\mathscr{N}\\ n\equiv a\,(\mathrm{mod}\,d_1d_2)}} 1-\frac{1}{\varphi(d_1d_2)}\sum_{\substack{n\in\mathscr{N}\\ (n,d_1d_2)=1}} 1\Bigg|\\
&\quad+\frac{1}{\varphi(d_1d_2)}\sum_{\substack{n\in\mathscr{N}\\ (n,d_1d_2)>1}} 1\\
&= R_1^{(1)}(\mathbf{d})+R_2^{(1)}(\mathbf{d}).
\end{aligned}
$$

By a trivial argument,

$$
\begin{aligned}
(4.17)\qquad &\sum_{d_1d_2\le (D_1D_2)^{1+\varepsilon}} \tau^4(d_1d_2)R_2^{(1)}(\mathbf{d})\\
&= \sum_{d_1d_2\le (D_1D_2)^{1+\varepsilon}} \frac{\tau^4(d_1d_2)}{\varphi(d_1d_2)}\sum_{d\mid d_1d_2}\sum_{\substack{n\in\mathscr{N}\\ (n,d_1d_2)=d}} 1\\
&\ll \sum_{d_1d_2\le (D_1D_2)^{1+\varepsilon}} \tau^4(d_1d_2)\frac{x}{z\varphi(d_1d_2)}\tau(d_1d_2)\\
&\ll \frac{x}{\log^{10}x}.
\end{aligned}
$$

By a splitting process to remove the dependence between the factors of the elements of $\mathscr{N}$, and by Lemma 3, we obtain

$$
(4.18)\qquad \sum_{d_1d_2\le (D_1D_2)^{1+\varepsilon}} \tau^4(d_1d_2)R_1^{(1)}(\mathbf{d}) \ll \frac{x}{\log^{10}x}.
$$

By Lemma 1(i), (4.5) and (4.15)–(4.18),

$$
\begin{aligned}
(4.19)\qquad S_{11} &= S(\mathscr{W}^{(1)},\{D_1^{1/2},z\})\\
&\le (1+O(\varepsilon))\frac{4C|\mathscr{N}|}{(\log D_1)(\log D_2)}F_0(2)F_0(5)+O\bigg(\frac{x}{\log^{10}x}\bigg).
\end{aligned}
$$

It follows from Lemma 4 and summation by parts that

$$
\begin{aligned}
(4.20)\qquad |\mathscr{N}| &= \sum_{x^{1/13}\le p_1\le p_2\le p_3\le p_4\le x^{1/8.4}}\ \sum_{\substack{n\le \frac{x+2}{p_1p_2p_3p_4}\\ (n,Q(p_2))=1}} 1\\
&= \bigg(1+O\bigg(\frac{1}{\log x}\bigg)\bigg)\pi(x)\int_{1/13}^{1/8.4}\frac{dt_1}{t_1}\int_{t_1}^{1/8.4}\frac{dt_2}{t_2^2}\\
&\qquad\times\int_{t_2}^{1/8.4}\frac{dt_3}{t_3}\int_{t_3}^{1/8.4}\frac{w(\frac{1-t_1-t_2-t_3-t_4}{t_2})}{t_4}\,dt_4
\end{aligned}
$$

$$\leq 0.5618\pi(x)\int\limits_{1/13}^{1/8.4}\frac{dt_1}{t_1}\int\limits_{t_1}^{1/8.4}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\frac{1}{8.4t_2}\,dt_2$$
$$\leq 0.00934\pi(x).$$

From (4.19)–(4.20), we get

$$S_{11}\leq 0.01312\frac{4C\pi(x)}{(\log D_1)(\log D_2)}. \tag{4.21}$$

In a similar way,

$$S_{12}\leq(1+O(\varepsilon))\frac{4C\pi(x)}{(\log D_1)(\log D_2)}F_0(2)F_0(5)Q_0+O\left(\frac{x}{\log^{10}x}\right) \tag{4.22}$$
$$\leq 0.06803\frac{4C\pi(x)}{(\log D_1)(\log D_2)},$$

where

$$Q_0=\int\limits_{1/13}^{1/8.4}\frac{dt_1}{t_1}\int\limits_{t_1}^{1/8.4}\frac{dt_2}{t_2^2}\int\limits_{t_2}^{1/8.4}\frac{dt_3}{t_3}\int\limits_{1/8.4}^{0.475-2/13-t_3}\frac{w(\frac{1-t_1-t_2-t_3-t_4}{t_2})}{t_4}\,dt_4$$
$$\leq 0.5645\int\limits_{1/13}^{1/8.4}\frac{dt_1}{t_1}\int\limits_{t_1}^{1/8.4}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log 8.4\left(0.475-\frac{2}{13}-t_3\right)dt_2$$
$$\leq 0.04839;$$

$$S_7\leq(1+O(\varepsilon))\frac{4C\pi(x)}{(\log D_1)(\log D_2)}F_0(2)F_0(5)L_0+O\left(\frac{x}{\log^{10}x}\right) \tag{4.23}$$
$$\leq 1.10174\frac{4C\pi(x)}{(\log D_1)(\log D_2)},$$

where

$$L_0=\int\limits_{2.145}^{12}\frac{\log\left(2.145-\frac{3.145}{t+1}\right)}{t}\,dt;$$

$$S_8\leq(1+O(\varepsilon))\frac{4C\pi(x)}{(\log D_1)(\log D_2)}F_0(2)F_0(5)M_0+O\left(\frac{x}{\log^{10}x}\right) \tag{4.24}$$
$$\leq 1.01329\frac{4C\pi(x)}{(\log D_1)(\log D_2)},$$

where

$$M_0=\int\limits_{2.81}^{7.4}\frac{\log\left(2.81-\frac{3.81}{t+1}\right)}{t}\,dt;$$

$$\begin{aligned}(4.25)\qquad S_9 &\le (1+O(\varepsilon))\frac{4C\pi(x)}{(\log D_1)(\log D_2)}F_0(2)F_0(5)N_0 + O\bigg(\frac{x}{\log^{10} x}\bigg)\\ &\le 0.00674\frac{4C\pi(x)}{(\log D_1)(\log D_2)},\end{aligned}$$

where

$$N_0 = \int_2^{2.145} \frac{\log(t-1)}{t}\,dt;$$

and

$$\begin{aligned}(4.26)\qquad S_{10} &\le (1+O(\varepsilon))\frac{4C\pi(x)}{(\log D_1)(\log D_2)}F_0(2)F_0(5)P_0 + O\bigg(\frac{x}{\log^{10} x}\bigg)\\ &\le 0.14781\frac{4C\pi(x)}{(\log D_1)(\log D_2)},\end{aligned}$$

where

$$P_0 = \int_2^{2.81} \frac{\log(t-1)}{t}\,dt.$$

4.4. Upper bounds of Σ_j $(1 \le j \le 4)$. Let $\mathscr{M}_k$ be defined in Lemma 5. Then

$$\begin{aligned}(4.27)\qquad S(\mathscr{A}^{(k)};\mathscr{P},x^{1/13}) &= \sum_{\substack{m\in\mathscr{M}_k,\, m-6=p\\ (m-4,P(x^{1/13}))=1}} 1\\ &\le \sum_{\substack{m\in\mathscr{M}_k,\,(m-6,P(D_2^{1/2}))=1\\ (m-4,P(x^{1/13}))=1}} 1 + O(x^{1/2}).\end{aligned}$$

By arguments similar to those for S_{11}, we get

$$\begin{aligned}(4.28)\qquad &\sum_{\substack{m\in\mathscr{M}_k,\,(m-6,P(D_2^{1/2}))=1\\ (m-4,P(x^{1/13}))=1}} 1\\ &\le (1+O(\varepsilon))\frac{4C|\mathscr{M}_k|}{(\log D_1)(\log D_2)}F_0(2)F_0(6.175) + O\bigg(\frac{x}{\log^{10} x}\bigg).\end{aligned}$$

By the prime number theorem and summation by parts,

$$\begin{aligned}(4.29)\qquad |\mathscr{M}_k| &\le \sum_{z\le p_1\le\cdots\le p_{k-1}\le\left(\frac{x+6}{p_1\cdots p_{k-2}}\right)^{1/2}} \frac{x}{p_1\cdots p_{k-1}\log\frac{x}{p_1\cdots p_{k-1}}}\\ &= \bigg(1+O\bigg(\frac{1}{\log x}\bigg)\bigg)c_k\pi(x),\end{aligned}$$

where

$$c_k = \int_{k-1}^{199} \frac{dt_1}{t_1} \int_{k-2}^{t_1-1} \frac{dt_2}{t_2} \cdots \int_{3}^{t_{k-4}-1} \frac{dt_{k-3}}{t_{k-3}} \int_{2}^{t_{k-3}-1} \frac{\log(t_{k-2}-1)\,dt_{k-2}}{t_{k-2}}.$$

By numerical integration,

$$\begin{aligned}(4.30)\qquad & c_{15} < 0.003088,\ \ c_{16} < 0.000646,\ \ c_{17} < 0.000124,\ \ c_{18} < 0.000011,\\ & c_k < 0.000001 \quad \text{for } 19 \le k \le 199,\end{aligned}$$

and

$$C_0 = \sum_{k=15}^{199} c_k < 0.00408.$$

From (4.27)–(4.30), we get

$$\begin{aligned}(4.31)\qquad \varSigma_1 &\le (1+O(\varepsilon))\frac{4C_0C\pi(x)}{(\log D_1)(\log D_2)}F_0(2)F_0(6.175) + O\biggl(\frac{x}{\log^{10} x}\biggr)\\ &\le 0.00708\frac{4C\pi(x)}{(\log D_1)(\log D_2)}.\end{aligned}$$

By arguments similar to those leading to (4.31), we have

$$\begin{aligned}(4.32)\qquad \varSigma_2 &\le (1+O(\varepsilon))\frac{4C_0C\pi(x)}{(\log D_1)(\log D_2)}F_0(2)F_0(3.99) + O\biggl(\frac{x}{\log^{10} x}\biggr)\\ &\le 0.00468\frac{4C\pi(x)}{(\log D_1)(\log D_2)},\end{aligned}$$

$$\begin{aligned}(4.33)\qquad \varSigma_3 &\le (1+O(\varepsilon))\frac{4C_0C\pi(x)}{(\log D_1)(\log D_2)}F_0(2)F_0(2) + O\biggl(\frac{x}{\log^{10} x}\biggr)\\ &\le 0.00408\frac{4C\pi(x)}{(\log D_1)(\log D_2)}\end{aligned}$$

and

$$\begin{aligned}(4.34)\qquad \varSigma_4 &\le (1+O(\varepsilon))\frac{4C_0C\pi(x)}{(\log D_1)(\log D_2)}F_0(2)F_0(2) + O\biggl(\frac{x}{\log^{10} x}\biggr)\\ &\le 0.00408\frac{4C\pi(x)}{(\log D_1)(\log D_2)}.\end{aligned}$$

4.5. Proof of Theorem 1. By Lemma 6, (4.7)–(4.8), (4.10)–(4.11), (4.13)–(4.14), (4.21)–(4.26) and (4.31)–(4.34), we get

$$\begin{aligned}4\pi_{1,2,14} \ge \bigl(&3\times 2.42877 + 1.53443 + 0.21948 + 0.44275 - 3.80708 - 3.11185\\ &- 1.10174 - 1.01328 - 2\times 0.00674 - 2\times 0.14781 - 0.01312\\ &- 0.06803 - 0.00708 - 0.00468 - 2\times 0.00408\bigr)\frac{4C\pi(x)}{(\log D_1)(\log D_2)}\\ \ge{}& 0.03883\frac{4C\pi(x)}{(\log D_1)(\log D_2)}\end{aligned}$$

and

$$\pi_{1,2,14} \geq 0.0097 \frac{4C\pi(x)}{(\log D_1)(\log D_2)}. \tag{4.35}$$

Now Theorem 1 follows from (4.35).

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