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Construction of some classes of nonlinear PDE's admitting soliton solutions

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Contents

Abstract

We study a class of Lorentz invariant nonlinear field equations in multiple space dimensions. The main purpose is to obtain soliton-like solutions with variable exponent. The fields are characterized by a topological invariant, which we call the charge. We prove the existence of a static solution which minimizes the energy among the configurations with nontrivial charge.

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1. Introduction and motivation

Mathematics consists initially of a language, which makes it possible to transcribe problems of a quantitative nature: this is modeling. Once this transcription is made, tools are available to solve these problems, partially or completely. Then, one brings back the solution into its context of origin. Ordinary and partial differential equations (PDE's) are at the heart of mathematical modeling. They constitute the basic language in which most of the laws in physics or engineering can be written and one of the most important mathematical tools for modeling in the universe and socio-economical sciences. And they occur in many applications: in chemistry to model reactions, in economics to study market behavior, in finance to study financial derivatives, in image processing to study restoration in image damage, and so on.

We make some simple basic assumptions, actually they are the physical properties of the universe, which are shared today by every fundamental theory in physics and which will have very deep consequences (see [9]).

(α) The universe is variational; that is, we suppose that all the physical phenomena are governed by differential equations which admit a variational formulation. This variational principle is at once reasonable, if we think that all the fundamental equations of physics can be seen as the Euler–Lagrange equations of a suitable action functional.

Remark 1.0.1. The variational principle has a very long history and can be traced back even to the ideas of Aristotle. However, its very discovery has been attributed to P. L. M. de Maupertuis (1698–1759), after he was engaged in polemics with the followers of G. W. Leibniz; in his work "Examen philosophique de la preuve de l'existence de Dieu" (1756), he stated the principle of minimal action as an evidence of rationality in the divine creation. It is well known that these metaphysical ideas were then formalized by L. Euler and J. L. Lagrange in the eighteenth century, but the ultimate reason for which the variational principle holds true in nature is still today a mystery. On the subject, we refer to the essay "Le meilleur des mondes possibles. Mathématiques et destinée" by I. Ekeland [46].

(β) The universe is invariant for the Poincaré group; that is, we suppose that all the equations of the universe are invariant with respect to the group generated by the following transformations:

- time translations, i.e., transformations depending on one parameter having the form
$$\begin{cases} x \to x, \\ t \to t + t_0, \end{cases}$$
- space translations, i.e., transformations depending on three parameters having the form
$$\begin{cases} x \to x + x_0, \\ t \to t, \end{cases}$$
- space rotations, i.e., transformations depending on three parameters having the form
$$\begin{cases} x \to Rx, \ R \in O(3), \\ t \to t + t_0, \end{cases}$$
- Lorentz transformations, i.e., space-time rotations depending on one parameter ν having the form
$$\begin{cases} x_1 \to \gamma(x_1 - \nu t), \\ x_2 \to x_2, \\ x_3 \to x_3, \\ t \to \gamma\big(t - \frac{\nu}{c^2} x_1\big), \end{cases} \qquad \begin{cases} x_1 \to x_1, \\ x_2 \to \gamma(x_2 - \nu t), \\ x_3 \to x_3, \\ t \to \gamma\big(t - \frac{\nu}{c^2} x_2\big), \end{cases}$$
$$\begin{cases} x_1 \to x_1, \\ x_2 \to x_2, \\ x_3 \to \gamma(x_3 - \nu t), \\ t \to \gamma\big(t - \frac{\nu}{c^2} x_3\big), \text{ where } \gamma = \frac{c}{\sqrt{c^2 - \nu^2}}, \ |\nu| < c, \end{cases}$$

  and c is a constant (dimensionally a velocity). Indeed, the Poincaré group P is the ten-parameter Lie group generated by the above transformations together with the time and parity inversions $t \to -t$ and $x \to -x$. The assumptions of the first three invariances cannot be omitted if we want to make a physical theory, for they express the possibility of repeating experiments. More precisely, translational invariances ask for time and space to be homogeneous (i.e., whenever and wherever an experiment is performed, it gives the same results) and rotational invariance requires that the space be isotropic (i.e., there are no privileged directions in the universe). Finally, the Lorentz invariance is an empirical fact and we will see that it is the very cause of relativistic effects.

What is a soliton. Solitons are nonlinear waves. As a preliminary definition, a soliton is considered as a solitary, traveling wave pulse solution of a nonlinear PDE. The soliton phenomenon was first described in 1834 by John Scott Russell (1808–1882) who observed a solitary wave on the canal from Edinburgh to Glasgow in 1834. Reporting to the British Association, he wrote [75]: “I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that rare and beautiful phenomenon which I have called the Wave of Translation [...]."

A soliton is a solution of a field equation whose energy travels as a localized packet and which preserves its form under perturbations. The nonlinearity will play a significant role. For most dispersive evolution equations these solitary waves would scatter inelastically and lose "energy" due to the radiation. Not so for the solitons: after a fully nonlinear interaction, the solitary waves reemerge, retaining their identities with same speed and shape. It should have remarkable stability properties. In this respect solitons have a particle-like behavior. The soliton equations, in the mathematical sense, provide outstanding examples of completely integrable systems possessing an infinite number of degrees of freedom. That is why they so interest mathematicians.

History and problems with soliton solutions. Solitons also concern physicists and they even become indispensable to explain and describe many phenomena. They occur in many areas of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics and plasma physics. They also occur in many models in chemistry and biology (see [40,50,53,59,72,86]). Probably, the simplest equation which has soliton solutions is the sine-Gordon equation (see below in Chapter 1)

$$-\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial t^2}+\sin\psi=0,$$

where $\psi=\psi(x,t)$ is a scalar field, x and t are real numbers representing, respectively, the space and the time variables. In 1964, Derrick, in a celebrated paper [37], considers the more realistic three-space-dimension model (see below in Chapter 2)

$$-\Delta\psi+\frac{\partial^2\psi}{\partial t^2}+V'(\psi)=0,$$

Δ being the 3-dimensional Laplace operator and V' the gradient of a nonnegative C^1 real function V.

In [37] it is proved by a simple rescaling argument that the last equation in three space dimensions does not possess any nontrivial finite-energy static solutions. Derrick proposed some possible ways out of this difficulty. The first proposal was to consider models which are the Euler–Lagrange equations of the action functional relative to the functional

$$S=\iint\mathcal{L}\,dx\,dt.$$

The Lorentz invariant Lagrangian density proposed in [37] has the form

$$\mathcal{L}(\psi) = -(|\nabla\psi|^2 - |\psi_t|^2)^{p/2}, \qquad p > 3. \tag{1.0.1}$$

However, Derrick does not continue his analysis. He has been unable to demonstrate either the existence or nonexistence of stable solutions.

In this spirit, in recent years, a considerable amount of work has been done by V. Benci and collaborators (see [9, 11, 12, 14, 16–22]), which will constitute the core of this work.

1.1. Motivation. In the mathematical models (of solitons) studied in papers [14, 22], the space of the finite energy configurations (solution space) splits into infinitely many connected components according to the topological charge. Therein, the authors proved the existence of infinitely many solutions, which are constrained minima of the energy. More precisely, on every one-connected component characterized by a topological charge equal to $n \in \mathbb{N}$ there exists a solution of charge n. Since p is arbitrary in the static equation (see equation (4.1.10)), it is natural to consider $p = p(x)$ as a variable that depends on the connected component.

Our aim in this work is to carry out an existence analysis of the finite energy static solutions in more than one space dimension for a class of Lagrangian densities which include (1.0.1) and to generalize the results of Benci [14]. More precisely, we are concerned with "generalized Sobolev spaces with variable exponents".

The following models give a more concrete notion of processes that can be described by a soliton.

1.1.1. Equivalence between mass and energy (the celebrated Einstein equation). One of the main features of these soliton solutions is that they behave like relativistic particles. In fact, by using the Noether theorem, we can introduce the energy $E(\psi)$ and the mass $m(\psi)$ and it can be proved that the celebrated Einstein relation $E(\psi) = m(\psi)c^2$ holds true.

We shall consider Lagrangian densities of the form

$$\mathcal{L}_1(\psi, \rho) = -\tfrac{1}{2}\alpha(\rho) - V(\psi), \tag{1.1.1}$$

where V is a real function defined in an open subset $\Omega \subset \mathbb{R}^m$ and α is a real function defined by

$$\alpha(\rho) = \rho + \frac{\varepsilon}{3}|\rho|^3, \qquad \varepsilon > 0, \tag{1.1.2}$$

$$\rho = c^2|\nabla\psi|^2 - |\psi_t|^2.$$

The action functional related to (1.1.1) is

$$S_1(\psi) = \int_{\mathbb{R}^{3+1}} \mathcal{L}_1(\psi, \rho)\, dx\, dt = \int_{\mathbb{R}^{3+1}} \left(-\tfrac{1}{2}\alpha(\rho) - V(\psi)\right) dx\, dt.$$

The Euler–Lagrange equation is

$$\frac{\partial}{\partial t}((1 + \varepsilon|\rho|^2)\psi_t) - c^2\nabla((1 + \varepsilon|\rho|^2)\nabla\psi) + V'(\psi) = 0. \tag{1.1.3}$$

So the static solutions u solve the equation

$$-c^2\Delta u - c^6\varepsilon\Delta_6 u + V'(u) = 0. \tag{1.1.4}$$

Clearly, (1.1.4) is the Euler–Lagrange equation with respect to the energy functional

$$E(u) = \int_{\mathbb{R}^3} \left(\frac{c^2}{2} |\nabla u|^2 + \varepsilon \frac{c^6}{6} |\nabla u|^6 + V(u) \right) dx. \tag{1.1.5}$$

Equation (1.1.3) is probably the simplest Lorentz invariant equation which has static solitons. Nevertheless, these solitons have some interesting properties since they behave like relativistic bodies, namely:

- they experience a relativistic contraction in the direction of the motion;
- the rest mass is a scalar and not a tensor;
- the mass equals the energy;
- the mass increases with the velocity by the factor γ.

Our Lagrangian (1.1.1) is Lorentz invariant, thus it is reasonable to expect at least some of these features. However, it is somewhat surprising that they can be deduced from a single equation without extra assumptions. Moreover, this equation might be interpreted as the equation of an "elastic medium" in a Newtonian space-time. Thus, this model shows, from a purely formal point of view, that the main features of the special relativity can be deduced from a PDE in a Newtonian space-time.

1.1.2. Solitons and the electromagnetic field [19]. In Subsection 1.1.1, a Lorentz invariant equation was introduced in three space dimensions having soliton-like solutions, and the equation introduced is the Euler–Lagrange equation of the action functional

$$S_1(\psi) = \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \mathcal{L}_1 \, dx \, dt.$$

The soliton solutions behave like relativistic particles. Moreover, a topological invariant is associated to these solitons. If we interpret this invariant as the electric charge, it is natural to analyze the interaction between the soliton and the electromagnetic field and to try to construct a simple Lorentz invariant model for the electromagnetic theory, namely, a model describing particle-like matter interacting with the electromagnetic field through (deterministic) differential equations defined in a Newtonian space-time.

In the following, (A, ϕ) will denote the gauge potentials associated to the electromagnetic field $(E; H)$ by the relations

$$E = -(A_t + \nabla\phi), \tag{1.1.6}$$

$$H = \nabla \times A. \tag{1.1.7}$$

We need to define the Lagrangian density $\mathcal{L}_2$ of the electromagnetic field and the Lagrangian density $\mathcal{L}_3$ describing the interaction between them and the electromagnetic field:

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{8\pi}(|E|^2 - |H|^2) = \frac{1}{8\pi}(|A_t + \nabla\phi|^2 - |\nabla \times A|^2), \\ \mathcal{L}_3 &= (J(\psi, \nabla\psi, \psi_t) \mid A) - \varrho(\psi, \nabla\psi)\phi, \end{aligned}$$

where J is the electric current and ρ is the electric density (see [19, pp. 74, 78]).

The total action will be

$$S = S(\psi, A, \phi) = S_1(\psi) + S_2(A, \phi) + S_3(\psi, A, \phi)$$

with

$$S_i(\psi) = \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \mathcal{L}_i \, dx \, dt.$$

The model we introduce permits us to describe the interaction of a relativistic particle with an electromagnetic field by using only concepts of classical field theory. In this work we confine ourselves to analyzing some mathematical questions related to the existence of solutions for this model. More precisely, we prove the existence of static solutions (with nontrivial charge) of the Euler–Lagrange equations

$$dS = 0, \tag{1.1.8}$$

namely, solutions (u, A, ϕ) $(u = (u_1, \dots, u_4), A = (A_1, A_2, A_3))$ which do not depend on $t \in \mathbb{R}$. Let us point out that these solutions give rise to traveling solutions (ψ, A_v, ϕ_v) $(A_v = (A_{v,1}, A_{v,2}, A_{v,3}))$ with velocity $(v, 0, 0)$ where

$$\begin{aligned}
\psi(x,t) &= u\left(\frac{x_1 - vt}{\sqrt{1-(v/c)^2}}, x_2, x_3\right), \\
A_{v,1}(x,t) &= \frac{A_1\left(\frac{x_1-vt}{\sqrt{1-(v/c)^2}}, x_2, x_3\right) - \frac{v}{c}\phi\left(\frac{x_1-vt}{\sqrt{1-(v/c)^2}}, x_2, x_3\right)}{\sqrt{1-(v/c)^2}}, \\
A_{v,2}(x,t) &= A_2\left(\frac{x_1 - vt}{\sqrt{1-(v/c)^2}}, x_2, x_3\right), \\
A_{v,3}(x,t) &= A_3\left(\frac{x_1 - vt}{\sqrt{1-(v/c)^2}}, x_2, x_3\right), \\
\phi_v(x,t) &= \frac{\phi\left(\frac{x_1-vt}{\sqrt{1-(v/c)^2}}, x_2, x_3\right) - v/cA_1\left(\frac{x_1-vt}{\sqrt{1-(v/c)^2}}, x_2, x_3\right)}{\sqrt{1-(v/c)^2}},
\end{aligned}$$

ψ is a traveling soliton "surrounded" by the electromagnetic field (A_v, ϕ_v).

The aim of this work is to prove the existence of static solutions of the Euler–Lagrange equations relative to the action functional

$$S = S(\psi, A, \phi).$$

First we take the variation with respect to A. We get $dS[\delta A] = 0$ if and only if

$$\nabla \times (\nabla \times A) = 4\pi J(\psi, \nabla\psi, \psi_t) - \frac{\partial}{\partial t}(A_t + \nabla\phi). \tag{1.1.9}$$

Second we take the variation with respect to ϕ. We get $dS[\delta\phi] = 0$ if and only if

$$-\nabla(A_t + \nabla\phi) = 4\pi\varrho(\psi, \nabla\psi). \tag{1.1.10}$$

By (1.1.6) and (1.1.7), we get

$$\nabla \times H = 4\pi J(\psi, \nabla\psi, \psi_t) + E_t, \tag{1.1.11}$$

$$\nabla \cdot E = 4\pi\varrho(\psi, \nabla\psi), \tag{1.1.12}$$

which completes the Maxwell equations (1.1.6) and (1.1.7).

Now, if we want to take the variation with respect to the jth component of ψ, we notice that it has a complicated form. Anyway we can write the equation

$$dS[\delta\psi] = 0$$

in the form

$$\Box\psi^j - \varepsilon\Box_6\psi^j + \frac{\partial V}{\partial\xi_j}(\psi) = F_j \tag{1.1.13}$$

where the left-hand side derives from the variation of the action S_1 describing the matter field. The right-hand side F_j of (1.1.13), which derives from the interaction term S_3, depends on ψ (and its first and second derivatives) and on A and ϕ (and their first derivatives), such that

$$\begin{aligned}\Box_6\psi &= \frac{\partial}{\partial t}[(c^2|\nabla\psi|^2 - |\psi_t|^2)^2\psi_t)] - c^2\nabla[(c^2|\nabla\psi|^2 - |\psi_t|^2)^2\nabla\psi],\\ \Box\psi &= \psi_{tt} - c^2\Delta\psi.\end{aligned}$$

We confine ourselves to static solutions, that is, fields ψ, A, ϕ which do not depend on t. We immediately get

$$J(\psi, \nabla\psi, \psi_t) = 0,$$

and then (1.1.9), (1.1.10) and (1.1.13) give respectively

$$\Delta\phi = 4\pi\varrho(\psi, \nabla\psi), \tag{1.1.14}$$

$$\nabla\times(\nabla\times A) = 0, \tag{1.1.15}$$

$$\Delta\psi^j - \varepsilon\Delta_6\psi^j + \frac{\partial V}{\partial\xi_j}(\psi) = F_j, \tag{1.1.16}$$

where G_j depends on ψ (and its first and second derivatives) and ϕ (and its first derivatives). Clearly $A = 0$ (as well as $A = \nabla h$) solves (1.1.15), so the unknowns of our problem are (ψ, ϕ). In particular, since our field ψ does not depend on t, from now on, we rename it u. Finally we can state our main result.

THEOREM 1.1.1. *There exist fields*

$$u : \mathbb{R}^3 \to \mathbb{R}^4, \qquad \phi : \mathbb{R}^3 \to \mathbb{R}$$

such that $\mathrm{ch}(u) = 0$ *and* $(u, 0, \phi)$ *is a (weak) static solution of the Euler–Lagrange equation* (1.1.8).

For more information concerning this subsection refer to [22].

1.1.3. Solitons as a model for dislocations in a crystal. In 1939, J. Frenkel and T. Kontorova [54] introduced the SG equation (see Chapter 3.1) as a model for dislocations in a crystal. The displacement $\phi(x, t)$ of atoms connected by linear springs may propagate as a kink in the periodic crystal field. Around 1960, J. K. Perring and T. Skyrme [69] considered the SG equation, which is relativistic invariant, as a model for elementary particles (more rigorously, baryons). They examined collisions of kink-kink and kink-antikink and confirmed the particle-like stability of kinks (historically, A. Seeger, H. Donth and A. Kochendörfer [78] found kink-kink solutions and kink-antikink solutions in the study of the SG equation as a dislocation model).

1.1.4. Solitons in nonlinear optics. In 1967, S. L. McCall and E. L. Hahn [61] discovered an interesting phenomenon in nonlinear optics. Coherent light propagating in the system of 2-level atoms obeys the SG equation when the spectral widths are neglected (perfect resonance). The observed soliton-like behavior is called self-induced transparency (SIT). The 2π-pulse is the soliton and 0π-pulse is the breather. When the spectral widths ate not neglected, that is, the interaction between the media and the light wave is not resonant, the envelope of the electric field is described by the NLS equation. Further extension of research has been done in electromagnetically induced transparency (EIT) where two coherent lights propagate in the system of 3-level atoms. EIT and soliton propagations have attracted much attention [84].

1.2. Mathematical models with variable exponent. Now we illustrate some mathematical models with variable exponent. The nonlinear partial differential equations involving the $p(x)$-Laplacian are used in modeling many physical phenomena such as elasticity nonlinearity, electrorheological fluids (interaction between fluids and EMF) and thermorheological fluids, image restoration and propagation through porous medium.

1.2.1. Image restoration [33]. Image restoration is the adjustment of image, mesh or more generally of discrete data by variational methods. These have been successfully applied to solve problems in computer vision, computer graphics or further data analysis. The aim is to provide an approximation of the actual data from the observed data that suffered a deterioration from the environment (noise) or methods acquisition as quantification and discretization. We confine ourselves to the model [29] proposed by Blomgren, Chan, Mulet and Wong in 2000,

$$\min \int_\Omega |\nabla u|^{p(\nabla u)},$$

where $\lim_{s\to 0} p(s) = 2$, $\lim_{s\to\infty} p(s) = 1$, and p is decreasing. An image, ($u : \Omega \to \mathbb{R}^n$), is recovered from an observed, noisy image, $\Omega \subset \mathbb{R}^2$ being the domain of the image.

1.2.2. Electrorheological fluids [38]. An electrorheological fluid is composed of fine particles dispersed in a dielectric liquid. Under the action of an electric field, the particles are attracted to form fibers connecting the electrodes parallel to the direction of the electric field giving the following equations given by Rajagopal and Růžička in 2001 (see [71]):

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial E}{\partial x_i} + \operatorname{curl} E = 0, \\ \displaystyle\frac{\partial v}{\partial t} - \sum_{i=1}^{n} \frac{\partial S}{\partial x_i}(x, E, E(v)) + v|\nabla v| + \nabla\pi = g(x, E), \\ \displaystyle\sum_{i=1}^{n} \frac{\partial v}{\partial x_i} = 0, \end{cases}$$

where E is the electromagnetic field, $v : \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity of fields, π the pressure, v is the symmetric part of the gradient, S is a tensor and its expression is given by

$$S(x, E, z) = v(E)(1 + \|z\|^2)^{(p-2)/2} z, \qquad \forall z \in \mathbb{R}^3,$$

$p = p(\|E\|^2)$ and E depends on x.

Structure of this work. In particular, in this work, we will be concerned with generalizations of some of the results of V. Benci to the context of generalized Sobolev spaces with variable exponents.

This work is structured into seven chapters:

- In Chapter 1, we present the history and motivation for solitons of some classes of nonlinear PDE's.
- In Chapter 2, we introduce notation, definitions, lemmas and theorems which are used throughout this work.
- In Chapter 3, we present the simplest equation admitting soliton solutions (the sine–Gordon equation). In the second part, we present the more realistic $3+1$-dimensional model admitting soliton solutions given by the nonlinear Klein–Gordon equation.
- In Chapter 4, we introduce an existence result for an $n+1$-dimensional model generalizing the one suggested by Derrick in his first proposal (which is stated in Chapter 3).
- In Chapter 5, the main purpose is to obtain soliton-like solutions with variable exponent which generalize the results of Chapter 4.
- In Chapter 6, the main purpose is to obtain soliton-like solutions with twice variable exponent which generalize the results of Chapter 5.
- In Chapter 7, we gather together some auxiliary notions pertaining to Sobolev spaces, the inverse and continuity of multidimensional p-Lapacian and some notions of weak topology.

2. Preliminaries

In this chapter, we recall from the literature some notation, definitions, and auxiliary results which will be used throughout this work.

$\Omega \subset \mathbb{R}^n$: an open set in $\mathbb{R}^n$, $n \in \mathbb{N}^*$
$x \in \mathbb{R}^n$: $x = (x_1, \ldots, x_n)$
$\rightharpoonup$: weakly converges
$\sigma(E, E^*)$: weak topology on E
$\to$: strongly converges
$h' = \operatorname{grad} h = \left(\dfrac{\partial h}{\partial x_1}, \ldots, \dfrac{\partial h}{\partial x_n}\right)$
$L^p(\Omega) = \left\{h : \Omega \to \mathbb{R} : u \text{ is measurable and } \displaystyle\int_\Omega |h|^p < \infty\right\}$, $1 \le p < \infty$
$L^\infty(\Omega) = \{h : \Omega \to \mathbb{R} : h$ is measurable and $|h(x)| \le c$ a.e. in Ω
for some constant $c\}$
$\|h\|_{L^p} = \left(\displaystyle\int_\Omega |h|^p\right)^{1/p}$
$\|h\|_{L^\infty} = \inf\{c : |u(x)| \le c \text{ a.e. on } \Omega\}$
$W^{1,p}(\Omega) = \{h \in L^p(\Omega) : h' \in (L^p(\Omega))^n\}$
$W_0^{1,p}(\Omega)$: the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$
$\|h\|_{W^{1,p}} = \|h\|_{L^p} + \|h'\|_{L^p}$
$C_0^k(\Omega)$: the space of k times continuously differentiable functions with compact support in Ω
$C^\infty(\Omega)$: the space of infinitely differentiable functions on Ω
$C_0^\infty(\Omega)$: the space of C^∞ functions with compact support in Ω
(some authors write $\mathcal{D}(\Omega)$ or $C_c^\infty(\Omega)$ instead of $C_0^\infty(\Omega)$)
$\psi_t^j = \dfrac{\partial \psi^j}{\partial t}$
$\psi_i^j = \dfrac{\partial \psi^j}{\partial x_i}$
$u = (u_0, \tilde{u}) \in \mathbb{R} \times \mathbb{R}^n$
∇u : the Jacobian of u

$\|u\|_a = a\|\nabla u\|_{L^2} + \|\nabla u\|_{L^p} + \|u\|_{L^2},\ a > 0$

$E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a}$: the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with the norm $\|\cdot\|_a$

$\Delta_p u = \nabla(|\nabla u|^{p-2}\nabla u)$

$\nabla(|\nabla u|^{p-2}\nabla u)$: the vector whose jth component is given by $\operatorname{div}(|\nabla u|^{p-2}\nabla u^j)$

$\triangle u$: the vector whose jth component is given by $\operatorname{div}(\nabla u^j)$

$\Lambda_a = \{u \in E_a : u(x) \neq \eta \text{ for all } x \in \mathbb{R}^n\}$

$\partial\Lambda_a = \{u \in E_a : \text{there exists } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\}$

$\deg(h, \Omega, b) = \displaystyle\sum_{x \in h^{-1}(b)\cap\Omega} \operatorname{sgn} J_h(x)$

J_h : the determinant of the Jacobian matrix

$\operatorname{ch}(u) := \begin{cases} \deg(\tilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ 0 & \text{if } K_u = \emptyset \end{cases}$

$\Lambda_q^* = \{u \in \Lambda_a : \operatorname{ch}(u) \neq 0\}$

$\Lambda_a = \displaystyle\bigcup_{q\in\mathbb{Z}} \Lambda_q$

$O(n)$: the orthogonal group

$\mathcal{P}(\Omega) := \mathcal{P}(\Omega, \mu)$: the set of all μ-measurable functions $p : \Omega \to [1, \infty]$

$C_+(\mathbb{R}^n) = \{p \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) : p(x) > 1 \text{ for all } x \in \mathbb{R}^n\}$

$p^+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x)$

$p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x)$

$L^{p(\cdot)}(\Omega) = \left\{h : \Omega \to \mathbb{R} : h \text{ is measurable and } \displaystyle\int_\Omega |h(x)|^{p(x)}\,dx < \infty\right\}$

$\rho_{p(\cdot)}(h) = \displaystyle\int_\Omega |h(x)|^{p(x)}\,dx$

$\|h\|_{p(\cdot)} = \inf\left\{\sigma > 0 : \displaystyle\int_{\mathbb{R}^n} \left|\frac{h(x)}{\sigma}\right|^{p(x)} dx \leq 1\right\}$

$W^{1,p(\cdot)}(\Omega) = \{h \in L^{p(\cdot)}(\Omega) : h' \in (L^{p(\cdot)}(\Omega))^N\}$

$H_0^{1,p(\cdot)}(\Omega)$: the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$

$\|u\|_{a,p} = a\|\nabla u\|_{L^2} + \|\nabla u\|_{L^{p(\cdot)}} + \|u\|_{L^2},\ a > 0$

$E_{a,p} = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_{a,p}}$: the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ in the norm $\|\cdot\|_{a,p}$

$\Gamma_a = \{u \in E_{a,p} : u(x) \neq \eta \text{ for all } x \in \mathbb{R}^n\}$

$\partial\Gamma_a = \{u \in E_{a,p} : \text{there exists } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\}$

$\mathcal{L}(\mathbb{R}^n)$: the set of linear maps $\mathbb{R}^n \to \mathbb{R}^n$

2.1. Classical Sobolev spaces. For all information in this section, we refer to [30]. Let $\Omega \subset \mathbb{R}^n$ be an open set.

DEFINITION 2.1.1. Let $p \in \mathbb{R}$ with $1 \leq p < \infty$; we set

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^p \, dx < \infty \right\},$$

with the norm

$$\|u\|_{L^p} = \|u\|_p = \left(\int_\Omega |u|^p \right)^{1/p}.$$

DEFINITION 2.1.2. We set

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \begin{array}{l} u \text{ is measurable and there is a constant } c \\ \text{such that } |u(x)| \leq c \text{ a.e. on } \Omega \end{array} \right\}$$

with the norm

$$\|u\|_{L^\infty} = \inf\{c : |u(x)| \leq c \text{ a.e. on } \Omega\}.$$

- L^p is a separable Banach space; it is reflexive if $1 < p < \infty$ and the dual of L^p is isomorphic to $L^{p'}$ with
$$\frac{1}{p} + \frac{1}{p'} = 1.$$
- L^1 is a separable Banach space; it is never reflexive and the dual of L^1 is isomorphic to L^∞.
- L^∞ is Banach space; it is not separable, it is not reflexive and the dual of L^∞ is strictly bigger than L^1.

DEFINITION 2.1.3. For $1 \leq p < \infty$, the Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in (L^p(\Omega))^n\},$$

equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p}.$$

We set $W^{1,2}(\Omega) = H^1(\Omega)$.

REMARK 2.1.1.

$$\nabla u = \operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right)$$

such that $\frac{\partial u}{\partial x_i}$ is taken in the sense of distributions, i.e.,

$$\exists g_i \in L^p(\Omega), \qquad \int_\Omega u \frac{\partial \phi}{\partial x_i} = \int_\Omega g_i \phi, \qquad \forall \phi \in C_0^\infty(\Omega).$$

PROPOSITION 2.1.1 ([30, Proposition 8.1, p. 203]). *$W^{1,p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$. It is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$. H^1 is a separable Hilbert space.*

DEFINITION 2.1.4. Let $1 \leq p < \infty$. $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ *in* $W^{1,p}(\Omega)$. Set

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

Then $W_0^{1,p}$, equipped with the $W^{1,p}$ norm, is a separable Banach space; it is reflexive if $1 < p < \infty$. H_0^1, equipped with the H^1 scalar product, is a Hilbert space.

Corollary 2.1.2 ([30, Corollary 4.23, p. 109]). *$C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.*

Theorem 2.1.3 ([30, Theorem 9.2, p. 265]). *Let $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$. Then there exists a sequence $(u_k) \subset C_0^\infty(\mathbb{R}^n)$ such that*

$$\begin{aligned} u_{k|\Omega} &\to u && \text{in } L^p(\Omega), \\ \nabla u_{k|\omega} &\to \nabla u_{|\omega} && \text{in } (L^p(\Omega))^n \quad \text{for all } \omega \subset \Omega. \end{aligned}$$

In case $\Omega = \mathbb{R}^n$ and $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$, there exists a sequence $(u_k) \subset C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned} u_{k|\Omega} &\to u && \text{in } L^p(\mathbb{R}^n), \\ \nabla u_k &\to \nabla u && \text{in } (L^p(\mathbb{R}^n))^n. \end{aligned}$$

So, in particular, $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$.

Remark 2.1.2. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$, we have $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.

Corollary 2.1.4 ([30, Corollary 9.8 (density), p. 277]). *Assume that Ω is of class C^1, and let $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$. Then there exists a sequence $(u_k) \subset C_0^\infty(\mathbb{R}^n)$ such that $u_{k|\Omega} \to u$ in $W^{1,p}(\Omega)$. In other words, the restrictions to Ω of $C_0^\infty(\mathbb{R}^n)$ functions form a dense subspace of $W^{1,p}(\Omega)$.*

Theorem 2.1.5 ([30, Theorem 9.12, p. 282]). *Let $p > n$. Then $W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ with continuous injection. Furthermore, for all $u \in W^{1,p}(\mathbb{R}^n)$,*

$$|u(x) - u(y)| \leq C^\alpha |x - y| \, \|\nabla u\|_{L^p} \qquad \text{a.e. } x, y \in \mathbb{R}^n,$$

where $\alpha = 1 - n/p$ and C is a constant (depending only on p and N).

Furthermore $\lim_{|x|\to\infty} u(x) = 0$.

Corollary 2.1.6 ([30, Corollary 9.13, p. 283]). *Let $m \geq 1$ be an integer and let $p \in [1, \infty)$. Then*

$$\begin{aligned} &W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \ \frac{1}{q} = \frac{1}{p} - \frac{m}{n}, && \text{if } \frac{1}{p} - \frac{m}{n} > 0, \\ &W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \ \forall q \in [p, \infty), && \text{if } \frac{1}{p} - \frac{m}{n} = 0, \\ &W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n), && \text{if } \frac{1}{p} - \frac{m}{n} < 0, \end{aligned}$$

and all these injections are continuous.

Theorem 2.1.7 ([30, Theorem, p. 285]). *Suppose that Ω is bounded and of class C^1. Then we have the following compact injections:*

$$\begin{aligned} &W^{1,p}(\Omega) \subset L^q(\Omega), \ \forall q \in [1, p^*), \ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, && \text{if } p < n, \\ &W^{1,p}(\Omega) \subset L^q(\Omega), \ \forall q \in [p, \infty), && \text{if } p = n, \\ &W^{1,p}(\Omega) \subset C(\overline{\Omega}), && \text{if } p = n. \end{aligned}$$

In particular, $W^{1,p}(\Omega) \subset L^p(\Omega)$ with compact injection for all p (and all n).

THEOREM 2.1.8 ([30, Theorem, p. 278]). *Let* $1 \leq p < N$. *Then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n), \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n},$$

and there exists a constant $c = c(p, n)$ *such that*

$$\|u\|_{L^{p^\star}} \leq c\|\nabla u\|_{L^p} \qquad \forall u \in W^{1,p}(\mathbb{R}^n).$$

2.2. Lebesgue and Sobolev generalized spaces. Variable exponent Lebesgue spaces appeared in the literature for the first time in a 1931 article by Orlicz [67]. In that article the following question is considered: let (p_i) (with $p_i > 1$) and (x_i) be sequences of real numbers such that $\sum x^{p_i}$ converges. What are necessary and sufficient conditions on (y_i) for $\sum_i x_i y_i$ to converge? It turns out that the answer is that $\sum(\lambda y_i)^{q_i}$ should converge for some $\lambda > 0$ and $p_i = q_i/(q_i - 1)$. This is essentially Hölder's inequality in the space $\ell^{p()}$ for $p = (p_k)_{k\in\mathbb{N}}$ defined by

$$\ell^{p()} = \Big\{(x_k)_{k\in\mathbb{N}} \subset \mathbb{R} : \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty\Big\}.$$

Orlicz was also interested in the study of function spaces that contain all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega (\phi(\lambda|u(x)|)\, dx$$

for some $\lambda > 0$ and ϕ, and satisfying some natural assumptions, where Ω is an open set in $\mathbb{R}^N$. This space is denoted by $L^{\phi(\cdot)}$ and it is now called an Orlicz space.

However, we point out that in [67] the case of $|u(x)|^{p(x)}$ corresponding to variable exponents was not included. In the 1950's these problems were systematically studied by Nakano [64], who developed the theory of modular function spaces. Nakano explicitly mentioned variable exponent Lebesgue spaces as an example of more general spaces he considered [64, p. 284]. Later, some Polish mathematicians investigated the modular function spaces (see Musielak [63]). Variable exponent Lebesgue spaces on the real line have been independently developed by some Russian researchers. In that context, we refer to the work of Tsenov [83] and Sharapudinov [79].

In 1991, Kováčik and Rákosník [60] established several basic properties of the spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ with variable exponents. Their results were extended by Fan and Zhao [51] in the framework of Sobolev spaces $W^{m,p(\cdot)}$. Pioneering regularity results for functionals with nonstandard growth are due to Acerbi and Mingione [3]. Density of smooth functions in $W^{k,p(\cdot)}(\Omega)$ and related Sobolev embedding properties are due to Edmunds and Rákosník [42, 49].

The variable Lebesgue spaces, as their name implies, are generalizations of the classical Lebesgue spaces, replacing the constant exponent p with a variable exponent function $p(\cdot)$. The resulting Banach function spaces $L^{p(\cdot)}$ have many properties similar to the L^p spaces, but they also differ in surprising and subtle ways. For this reason the variable Lebesgue spaces have an intrinsic interest, but they are also important for their applications to partial differential equations and variational integrals with nonstandard growth conditions.

2.3. Generalized Lebesgue spaces. In this section we introduce generalized Lebesgue spaces and state some of their basic properties.

Definition 2.3.1. Let $(\mathcal{A}, \Sigma, \mu)$ be a σ-finite, complete measure space. Define $\mathcal{P}(A, \mu)$ to be the set of all μ-measurable functions $p : A \to [1, \infty]$. Functions $p \in \mathcal{P}(A, \mu)$ are called *variable exponents* on A.

In the special case that μ is the n-dimensional Lebesgue measure and Ω is an open subset of $\mathbb{R}^n$, we abbreviate $\mathcal{P}(\Omega) := \mathcal{P}(\Omega, \mu)$. For $p \in L^\infty(\mathbb{R}^n)$, with $1 < p^-$,

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{u : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} |u(x)|^{p(x)}\, dx < \infty\right\},$$

which is a Banach space when furnished with the *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf\left\{\sigma > 0 : \int_{\mathbb{R}^n} \left|\frac{u(x)}{\sigma}\right|^{p(x)} dx \le 1\right\}.$$

Remark 2.3.1. While this more technical definition is necessary when $p(\cdot)$ is unbounded, we can simplify it when $p^+ < \infty$.

Denote by $q(x)$ the *conjugate exponent* of $p(x)$, that is, the function satisfying

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \quad \text{pointwise in } \mathbb{R}.$$

Let $\Omega \subset \mathbb{R}^n$ be a measurable subset and meas $(\Omega) > 0$. The following Hölder-type inequality holds for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$:

$$\int_\Omega uv\, dx \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(\cdot)}|v|_{q(\cdot)} \le 2|u|_{p(\cdot)}|v|_{q(\cdot)}.$$

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the *modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_{\mathbb{R}^n} |u(x)|^{p(x)}\, dx,$$

which is convex and modular continuous (see [39, Theorem 3.4.1 (p. 87) and Theorem 3.4.9 (p. 89)]), i.e. $\rho_{p(\cdot)}$ satisfies the following properties:

- $\rho_{p(\cdot)}(u) = 0 \Leftrightarrow u = 0$,
- $\rho_{p(\cdot)}(u) = \rho_{p(\cdot)}(-u)$,
- $\rho_{p(\cdot)}(\alpha u + \beta v) \le \alpha\rho_{p(\cdot)}(u) + \beta\rho_{p(\cdot)}(v)$, $\forall u, v \in E$, $\forall \alpha, \beta \ge 0$, $\alpha + \beta = 1$, where

$$E = \{u : \Omega \to \mathbb{R} : u \text{ is a measurable function in } \Omega\}.$$

For $u \in L^{p(\cdot)}(\mathbb{R}^n)$, the relations between the modular and the Luxemburg norm are clarified by the following proposition.

Proposition 2.3.1. *Let* $u \in L^{p(\cdot)}(\mathbb{R}^n)$ *and* (u_m) *be a sequence in* $L^{p(\cdot)}(\mathbb{R}^n)$. *Then*

(1) $|u|_{p(\cdot)} < 1\ (= 1, > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1\ (= 1, > 1)$.
(2) $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} < \rho_{p(\cdot)}(u) < |u|_{p(\cdot)}^{p^+}$.
(3) $|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} < \rho_{p(\cdot)}(u) < |u|_{p(\cdot)}^{p^-}$.

(4) $|u-u_m|_{p(\cdot)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u-u_m) \to 0$.
(5) $|u|_{p(\cdot)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u) \to 0$, $|u|_{p(\cdot)} \to \infty \Leftrightarrow \rho_{p(\cdot)}(u) \to \infty$.

Theorem 2.3.2. *Let $p \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$. Then $C(\Omega) \cap L^{p(\cdot)}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$. If moreover Ω is open, then $C(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.*

Proposition 2.3.3. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $p^+ < \infty$, then $f \in L^{p(\cdot)}(\Omega)$ if and only if*

$$\rho_{p(\cdot)}(u) < \infty.$$

Lemma 2.3.4 ([39, Lemma 3.2.12, p. 78]). *Let $s \in \mathcal{P}(A, \mu)$. Then*

$$\tfrac{1}{2}\min\{\mu(A)^{1/s^+}, \mu(A)^{1/s^-}\} \le \|1\|_{L^{s(\cdot)}(A,\mu)} \le 2\max\{\mu(A)^{1/s^+}, \mu(A)^{1/s^-}\}.$$

2.4. Sobolev spaces with variable exponent. In this section, we define the variable exponent Sobolev space by

$$W^{1,p(\cdot)}(\mathbb{R}^n) = \{u \in L^{p(\cdot)}(\mathbb{R}^n) : \nabla u \in L^{p(\cdot)}(\mathbb{R}^n)\},$$

which is a Banach space equipped with the norm

$$|u|_{1,p} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)},$$

and it is reflexive for $1 < p^- \le p^+ < \infty$.

Definition 2.4.1 ([39, Definition 11.2.1, p. 346]). Let $p \in \mathcal{P}(\Omega)$ and $k \in \mathbb{N}$. The space $H_0^{k,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$.

Theorem 2.4.1. *Let $p \in \mathcal{P}(\Omega)$. Then $H_0^{k,p(\cdot)}(\Omega)$ is a Banach space. If $p(\cdot)$ is bounded, then $H_0^{k,p(\cdot)}(\Omega)$ is separable, and if $1 < p^- \le p^+ < \infty$, then it is reflexive and uniformly convex.*

Definition 2.4.2 ([39, Definition 4.1.1, p. 100]). We say that a function $\alpha : \Omega \to \mathbb{R}$ is *locally log-Hölder continuous* on Ω if there exists $c_1 > 0$ such that

$$|\alpha(x) - \alpha(y)| < \frac{c_1}{\log|e + 1/|y-x||}$$

for all $x, y \in \Omega$. We say that α satisfies the *log-Hölder decay condition* if there exist an $\alpha_\infty \in \mathbb{R}$ and a constant $c_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| < \frac{c_2}{\log|e + |x||}$$

for all $x \in \Omega$. We say that α is *globally log-Hölder continuous* on Ω if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.

The constants c_1 and c_2 are called the *local log-Hölder constant* and the *log-Hölder decay constant*, respectively. The maximum $\max\{c_1, c_2\}$ is just called the *log-Hölder constant* of α. The local log-Hölder condition was first used in the variable exponent context by Zhikov [87].

Definition 2.4.3 ([39, Definition 4.1.4]). We define the following class of variable exponents:

$$\mathcal{P}^{\log}(\Omega) := \{p \in \mathcal{P}(\Omega) : 1/p \text{ is globally log-Hölder continuous}\}.$$

REMARK 2.4.1 ([39, Remark 4.1.5, p. 101]). If $p \in \mathcal{P}(\Omega)$ with $p^+ < \infty$, then $p \in \mathcal{P}^{\log}(\Omega)$ if and only if p is globally Hölder continuous.

LEMMA 2.4.2 ([39, Lemma 4.1.6]). *Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be continuous and bounded, i.e., $-\infty < \alpha^- \leq \alpha^+ < \infty$. The following conditions are equivalent:*

(a) *α is locally* log-*Hölder continuous,*
(b) *for all balls B we have $|B|^{\alpha_B^- - \alpha_B^+} \leq c$,*
(c) *for all balls B and all $x \in B$ we have $|B|^{\alpha_B^- - \alpha(x)} \leq c$,*
(d) *for all balls B and all $x \in B$ we have $|B|^{\alpha(x) - \alpha_B^+} \leq c$.*

Instead of balls one can also use cubes.

We give two density results for generalized Lebesgue and Sobolev spaces.

COROLLARY 2.4.3. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ be a bounded exponent. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$.*

THEOREM 2.4.4. *Let $p \in \mathcal{P}(\Omega)$ be a bounded exponent. If $p \in E$ or $p \in \mathcal{P}^{\log}(\Omega)$, then $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$.*

2.5. Convergence theorems. The next theorems collect analogues of the classical Lebesgue convergence results.

THEOREM 2.5.1. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, let $(f_k) \subset L^{p(\cdot)}(\Omega)$ be a sequence of non-negative functions such that f_k increases to a function f pointwise a.e. Then either $f \in L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{1,p(\cdot)} \to \|f\|_{1,p(\cdot)}$, or $f \notin L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{1,p(\cdot)} \to \infty$.*

The next result is an analog of Fatou's lemma. It is proved in [58].

LEMMA 2.5.2 (Fatou's lemma). *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose the sequence $(f_k) \subset L^{p(\cdot)}(\Omega)$ is such that $f_k \to f$ pointwise a.e. If*

$$f(x) = \liminf_{k\to\infty} f_k(x) < \infty,$$

then $f \in L^{p(\cdot)}(\Omega)$ and

$$\int_\Omega f \leq \liminf_{k\to\infty} \int_\Omega f_k.$$

THEOREM 2.5.3 (Lebesgue dominated convergence theorem). *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose the sequence $(f_k) \subset L^{p(\cdot)}(\Omega)$ is such that $f_k \to f$ pointwise a.e., and there is a function $g \in L^{p(\cdot)}(\Omega, \mathbb{R}_+)$ such that for all k, $|f_k(x)| \leq g(x)$ a.e. on Ω. Then $f \in L^{p(\cdot)}$ and*

$$\|f_k - f\|_{L^{p(\cdot)}} \to 0 \qquad \text{as } k \to \infty.$$

The final convergence result shows that norm convergence yields pointwise convergence of subsequences. The proof depends on showing that norm convergence implies convergence in measure; see [58] for details.

THEOREM 2.5.4. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose the sequence $f_k \to f$ in norm in $L^{p(\cdot)}(\Omega)$. Then there exists a subsequence (f_{k_j}) that converges pointwise a.e. to f.*

2.6. Lower and upper semicontinuity

Definition 2.6.1. Let $f : X \to [-\infty, \infty]$ be a function. Define

$$\mathrm{dom}(f) := \{x \in X : f(x) < \infty\}.$$

We say that the function f is *proper* if $\mathrm{dom}(f) \neq \emptyset$.

Definition 2.6.2. Let X be a topological space. A function $f : X \to [-\infty, \infty]$ is called *lower* [*upper*] *semicontinuous at a point* $x_0 \in X$, abbreviated l.s.c. [u.s.c.], if for each $a \in \mathbb{R}$ with $f(x_0) > a$ [$f(x_0) < a$] there exists a neighbourhood U of x_0 such that

$$x \in U \Rightarrow f(x) > a \qquad [x \in U \Rightarrow f(x) < a].$$

We say that f is *l.s.c.* [*u.s.c.*] *on a set* $M \subseteq X$ if f is such at each point of M.

The next proposition gives a characterization of local semicontinuity of functions.

Proposition 2.6.1. *Let X be a Hausdorff space, let $f : X \to [-\infty, \infty]$, and let $x_0 \in X$. Then*

(i) *f is l.s.c. at x_0 if for each net $(x_\alpha)_{\alpha \in I}$ in X, whenever $x_\alpha \to x_0$ in X then*

$$\liminf_{\alpha \in I} f(x_\alpha) \geq f(x_0);$$

(ii) *f is u.s.c. at x_0 if whenever $x_\alpha \to x_0$ in X, then*

$$\limsup_{\alpha \in I} f(x_\alpha) \leq f(x_0);$$

(iii) *f is l.s.c. if and only if $-f$ is u.s.c.*

The following result is closely related to the characterization of local semicontinuity.

Proposition 2.6.2. *For any function $f : X \to [-\infty, \infty]$ and $x_0 \in X$:*

(i) *f is l.s.c. at x_0 if and only if $\liminf_{x \to x_0} f(x) = f(x_0)$,*
(ii) *f is u.s.c. at x_0 if and only if $\limsup_{x \to x_0} f(x) = f(x_0)$.*

The classical Weierstrass theorem states that a continuous function defined on a compact set achieves its minimum and its maximum on that set. The following refinement is a fundamental tool in proving the existence of solutions to minimization problems.

Theorem 2.6.3 (Weierstrass [10]). *Let X be a Hausdorff space, let $f : X \to [-\infty, \infty]$ be l.s.c., and let C be a compact subset of X. Suppose that $C \cap \mathrm{dom}(f) \neq \emptyset$. Then f achieves its infimum over C.*

2.7. Ekeland's variational principle. A variational principle which provides an approximate minimizer of a bounded below and lower semicontinuous function in a given neighborhood of a point was introduced by Ekeland [43] in 1972 (see also [44, 45]). It is known as Ekeland's variational principle (for short, E.V.P.). In 1981, Sullivan [82] established that the validity of E.V.P. on a metric space (X, d) is equivalent to the completeness of (X, d). In 1982, McLinden [62] showed how E.V.P., or more precisely an augmented form of it provided by Rockafellar [73], can be adapted to extremum problems of minimax type. In this section, we present several forms of Ekeland's variational principle.

From the Weierstrass theorem (Theorem 2.6.3), for each $C \subset X$ compact, if f is lower semicontinuous, then the constrained optimization problem

$$\inf_{x \in C} f(x) \tag{2.7.1}$$

has a solution. Note that the solution set of (2.7.1) is compact. Now the question is: "Can we achieve the infimum of the optimization problem

$$\inf_{x \in X} f(x) \tag{2.7.2}$$

without the compactness assumption?" The answer is "yes." But we need some kind of coercivity assumption as well as a convexity structure on C. But we can always obtain an approximate ϵ-solution, that is, a point x_ϵ for $\epsilon > 0$ satisfying

$$\inf_{x \in X} f(x) \leq f(x_\epsilon) \leq \inf_{x \in X} f(x) + \epsilon.$$

Ekeland's variational principle guarantees the existence of such an ϵ-solution where neither compactness nor convexity on the underlying space is needed.

THEOREM 2.7.1 (Strong form of Ekeland's variational principle [44]). *Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper, lower bounded and l.s.c. function. Let $\epsilon > 0$ and $x_* \in X$ be such that*

$$f(x_*) \leq \inf_{x \in X} f(x) + \epsilon.$$

Then for any given $\lambda > 0$, there exists $\bar{x} \in X$ such that

(a) $f(\bar{x}) \leq f(x_*)$,
(b) $d(x_*, \bar{x}) \leq \lambda$,
(c) $f(\bar{x}) < f(x) + \frac{\epsilon}{\lambda} d(x, \bar{x})$ *for all* $x \in X \setminus \{\bar{x}\}$.

Aubin and Frankowska [7] established the following form of Ekeland's variational principle which is equivalent to Theorem 2.7.1.

THEOREM 2.7.2. *Let (X, d) be a complete metric space and f a lower semicontinuous map from X to $\mathbb{R}$. Assume that f is lower bounded and set $c := \inf_{x \in X} f(x)$. Then for every $\epsilon > 0$, there exists u_ϵ such that*

$$c \leq f(u_\epsilon) \leq c + \epsilon, \qquad f(x) - f(u_\epsilon) + \epsilon d(x, u_\epsilon) > 0 \quad \text{for all } x \in X \setminus \{u_\epsilon\}.$$

We now present the so called weak formulation of Ekeland's variational principle.

COROLLARY 2.7.3 (Weak form of Ekeland's variational principle [5]). *Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be proper, lower bounded and l.s.c. Then for any given $\epsilon > 0$, there exists $\bar{x} \in X$ such that*

$$f(\bar{x}) \leq \inf_{x \in X} f(x) + \epsilon, \qquad f(\bar{x}) < f(x) + \epsilon d(x, x_*) \quad \text{for all } x \in X \setminus \{\bar{x}\}.$$

Ekeland's variational principle for proper, but extended real-valued lower semicontinuous and lower bounded functions on a metric space characterizes the completeness of the space.

Theorem 2.7.4 (Converse of E.V.P. [5]). *A metric space* (X, d) *is complete if for every function* $f : X \to \mathbb{R} \cup \{\infty\}$ *which is a proper, lower bounded and l.s.c., and for any given* $\epsilon > 0$*, there exists* $\bar{x} \in X$ *such that*

$$f(\bar{x}) \leq \inf_{x \in X} f(x) + \epsilon, \qquad f(\bar{x}) < f(x) + \epsilon d(x, x_*) \quad \textit{for all } x \in X \setminus \{\bar{x}\}.$$

2.8. Topological degree. For more information on this topic, see [65,66].

We recall the construction of Brouwer degree.

Definition 2.8.1 ([66, Definition 1.2.1, p. 4]). Let $\Omega \subset \mathbb{R}^N$ be open and bounded and $f \in C^1(\overline{\Omega})$. If $b \notin f(\partial\Omega)$ and $J_f(b) \neq 0$, then we define

$$\deg(f, \Omega, b) = \sum_{x \in f^{-1}(b)} \operatorname{sgn} J_f(x),$$

where $\deg(f, \Omega, b) = 0$ if $f^{-1}(b) = \emptyset$.

The definition of the degree can be extended to functions that are only continuous and also to nonregular values of b.

- Let $f \in C(\overline{\Omega})$. Then there exists a sequence $f_k \in C^1(\overline{\Omega})$, $k = 1, 2, \ldots$, such that
$$\|f_k - f\|_\infty := \sup_{x \in \overline{\Omega}} \|f_k(x) - f(x)\| \to 0,$$
and we can show that
$$\deg(f, \Omega, b) = \lim_{k \to \infty} \deg(f_k, \Omega, b).$$
- Let $f \in C(\overline{\Omega})$ and suppose $b \notin f(\partial\Omega)$ is not necessarily a regular value. Then there is a sequence b_k, $k = 1, 2, \ldots$, (of regular values of f) such that $b_k \to b$, and we can show that
$$\deg(f, \Omega, b) = \lim_{k \to \infty} \deg(f, \Omega, b_k),$$
and the limit is independent of the sequence b_k.

Theorem 2.8.1 ([66, Theorem 1.2.6, p. 7]). *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded subset and* $f : \overline{\Omega} \to \mathbb{R}^N$ *be a continuous mapping. If* $b \notin f(\partial\Omega)$*, then there exists an integer,* $\deg(f, \Omega, b)$*, satisfying the following properties:*

(1) (Normality) $\deg(I, \Omega, b) = 1$ *if and only if* $b \in \Omega$*, where* I *denotes the identity mapping.*
(2) (Solvability) *If* $\deg(f, \Omega, b) \neq 0$*, then* $f(x) = b$ *has a solution in* Ω*.*
(3) (Additivity) *Suppose* Ω_1, Ω_2 *are disjoint open subsets of* Ω *and* $b \notin f(\overline{\Omega} - (\Omega_1 \cup \Omega_2))$*. Then* $\deg(f, \Omega, b) = \deg(f, \Omega_1, b) + \deg(f, \Omega_2, b)$*.*
(4) (Homotopy) *If* $f_t(x) : [0,1] \times \overline{\Omega} \to \mathbb{R}^N$ *is continuous and* $b \notin \bigcup_{t \in [0,1]} f_t(\partial\Omega)$*, then* $\deg(f_t, \Omega, b)$ *does not depend on* $t \in [0,1]$*.*
(5) $\deg(f, \Omega, b)$ *is a constant on any connected component of* $\mathbb{R}^N \setminus f(\partial\Omega)$*.*

Other properties include (see [65]):

(6) (Excision) Let $A \subset \overline{\Omega}$ be a compact set and $b \notin f(A)$. Then
$$\deg(f, \Omega, b) = \deg(f, \Omega \setminus A, b).$$

(7) (Stability of topological degree with respect to uniform convergence) Let $f, g \in C(\overline{\Omega})$, $b \notin f(\partial\Omega) \cup g(\partial\Omega)$. If $\|g - f\|_\infty \leq \frac{1}{4} d(b, f(\partial\Omega) \cup g(\partial\Omega))$, then $\deg(f, \Omega, b) = \deg(g, \Omega, b)$.

(8) If $f = g$ on $\partial\Omega$, then $\deg(f, \Omega, b) = \deg(g, \Omega, b)$.

3. Solitons in one and three space dimensions

3.1. Solitons in one space dimension: sine-Gordon equation (SG). In this section, we consider the $1+1$-dimensional sine-Gordon equation

$$\psi_{tt} - \psi_{xx} + \sin\psi = 0, \tag{3.1.1}$$

where $\psi = \psi(x,t)$ is a scalar field, and x and t are real numbers.

This is probably the simplest equation admitting soliton solutions and can be used as a pattern for our study, to illustrate more clearly a soliton's properties and characteristics. Its name was coined by J. Rubinstein [74] as a pun on "Klein–Gordon" and it arises in the study of surfaces with constant negative Gaussian curvature in differential geometry, as well as in many physical applications, such as two-dimensional models of elementary particles, stability of fluid motions, propagation of crystal dislocations (see [1, 2, 8, 28, 34, 40, 41, 54, 55, 69, 72] and [77] for exhaustive discussions and references).

(3.1.1) is the Euler–Lagrange equation of the action functional

$$S(\psi) = \int_{\mathbb{R}\times\mathbb{R}} \left[\tfrac{1}{2}(\psi_t^2 - \psi_x^2) - V(\psi)\right] dx\, dt, \tag{3.1.2}$$

where we can choose $V(\psi) = 1 - \cos\psi$ so as to obtain $V \geq 0$, and then the related energy functional is given by

$$E(\psi) = \frac{1}{2}\int_{\mathbb{R}} \psi_t^2\, dx + \frac{1}{2}\int_{\mathbb{R}} \psi_x^2\, dx + \frac{1}{2}\int_{\mathbb{R}} V(\psi)\, dx. \tag{3.1.3}$$

Note that the potential V has a discrete infinite set $2\pi\mathbb{Z}$ of degenerate minima, where it vanishes. Obviously $\psi(x) = k\pi$ is a trivial solution of (3.1.2) for every $k \in \mathbb{Z}$, but of course we are interested in nontrivial solutions. In particular, we will concern ourselves with nonsingular finite-energy solutions (of which solitary waves are special cases).

So, let ψ be a classical solution with $E(\psi) < \infty$. By this we mean, of course, that all the integrals in (3.1.3) are finite, and this implies $1 - \cos\psi(\cdot,t) \in H^1(\mathbb{R})$ for all fixed t (because $[1-\cos\psi(\cdot,t)]^2 \leq 2[1-\cos\psi(\cdot,t)] \in L^1$ and $\left|\frac{d}{dx}\cos\psi(\cdot,t)\right| \leq |\psi_x(\cdot,t)| \in L^2$).

Hence $\lim_{x\to\pm\infty}[1-\cos\psi(x,t)] = 0$, and from this we deduce that every configuration of ψ satisfies the asymptotic conditions

$$\psi(\pm\infty,t) = \lim_{x\to\pm\infty}\psi(x,t) \in 2\pi\mathbb{Z}. \tag{3.1.4}$$

Moreover, if we assume that $\psi_t \in L^\infty(\mathbb{R}^2)$ (which is necessarily the case of solitary waves), then the functions of the variable t defined by the left-hand side of (3.1.4), being continuous and discrete-valued, must be constant,

$$\psi(\pm\infty,t) \equiv \psi(\pm\infty) \in 2\pi\mathbb{Z}; \tag{3.1.5}$$

that is, ψ preserves its asymptotic values as t varies. These facts suggest considering the sets

$$H_{(k_1,k_2)} = \left\{ f \in C^2(\mathbb{R};\mathbb{R}) : \lim_{x\to\infty} f(x,t) = 2k_1\pi,\ \lim_{x\to-\infty} f(x,t) = 2k_2\pi \right\}$$

with $k_1, k_2 \in \mathbb{Z}$, and the topological space

$$H = \bigcup_{(k_1,k_2)\in\mathbb{Z}^2} H_{(k_1,k_2)} \subset L^\infty(\mathbb{R}).$$

It is easy to see that $(k_1,k_2) \neq (h_1,h_2) \Rightarrow H_{(k_1,k_2)} \cap H_{(k_1,k_2)} = \emptyset$ and each $H_{(k_1,k_2)}$ is an open path-connected subset of H, called a sector. The property (3.1.5) implies that, for a solution ψ, the function $x \mapsto \psi(x,t)$ (which we call a *configuration* of ψ) stays always in the same connected component $H_{(k_1,k_2)}$ as time varies. This fact allows a topological classification of the finite-energy nonsingular solutions to equation (3.1.1) satisfying (3.1.5), each bearing thereby the pair of indices (k_1,k_2). Moreover, consistent with the invariance of equation (3.1.1) (and also of the action (3.1.2), under the change $\psi \mapsto \psi + 2\pi k$), we can fix k_1 and relate to such solutions a single integer index given by the difference $k_2 - k_1$, namely

$$Q(\psi) := \frac{1}{2\pi}\int_{\mathbb{R}} \psi_x(x,t)\,dx.$$

$Q(\psi)$ defines a topological index, called *topological charge.* Note that Q is essentially a boundary condition which is constant in time because of the finiteness of energy, in contrast with the other more familiar conserved quantities (see [9, Subsection 1.2.1]) coming from the symmetries of the action functional. Now, we turn to the particular case of static (i.e. t -independent) nonsingular finite-energy solutions. They solve the equation

$$-u'' + \sin u = 0 \qquad \text{for } u : \mathbb{R} \to \mathbb{R}, \tag{3.1.6}$$

which can be interpreted as a conservative system (or, by a mechanical analogy, as the equation of motion for a unit-mass point particle). The mechanical energy

$$E_M = \tfrac{1}{2}(u')^2 - V(u)$$

(in the analogy, kinetic energy plus potential energy) is constant with respect to x and must equal zero. Indeed $u(\pm\infty) \in 2\pi\mathbb{Z}$ implies $E_M = \lim_{x\to\pm\infty}(u')^2/2$. So $E_M < \infty$ implies $E_M = 0$. Hence in the phase plane we get the zero-energy orbits, that is, the solutions u for which

- $\forall x \in \mathbb{R}$, $u'(x) = \pm 2 \sin \frac{u(x)}{2}$,
- $\exists k \in \mathbb{Z}\ \forall x \in \mathbb{R}$, $2\pi k < u(x) < 2\pi(k+1)$,
- u is monotone, and either $\lim_{x\to-\infty} u(x) = 2k\pi$ and $\lim_{x\to\infty} u(x) = 2(k+1)\pi$, or $\lim_{x\to-\infty} u(x) = 2(k+1)\pi$ and $\lim_{x\to\infty} u(x) = 2k\pi$.

These imply that $Q(u) = \pm 1$ for these solutions.

Finally, upon integration,

$$x - x_0 = \pm \int_{x_0}^{x} \frac{du(x)}{u'(x)} \quad \text{and} \quad (u^{-1}(u(x)))' = \frac{d}{du} u^{-1}(u) \cdot u'(x) = 1,$$

so

$$x - x_0 = \pm \int_{u(x_0)}^{u(x)} \frac{d}{du} u^{-1}(u)\, du = \pm u^{-1}(u)|_{u(x_0)}^{u(x)}.$$

Thus we have for all $x \in \mathbb{R}$,

$$x - x_0 = \pm \int_{u(x_0)}^{u(x)} \frac{du}{2\sin(u/2)} = \pm \int_{u(x_0)}^{u(x)} \frac{d(\tan(u/4))}{\tan(u/4)} = \pm \ln \frac{\tan[u(x)/4]}{\tan[u(x_0)/4]}.$$

Using again the invariance $u \mapsto u + 2\pi k$, we impose $u(x_0) = \pi$, and we get the explicit solutions

$$u_K(x) = 4 \arctan e^{x - x_0} \quad \text{and} \quad u_A(x) = -4 \arctan e^{x - x_0}, \tag{3.1.7}$$

which are the so-called *kink* and *antikink*, respectively, and carry $Q(u_K) = 1$ and $Q(u_A) = -1$. Note that the translational invariance of (3.1.6) is reflected by the fact that a different choice of the constant x_0 only brings the solution to a shift in space.

Given on space-time $\mathbb{R} \times \mathbb{R}$ a nonlinear equation with associated energy functional (see [9, Section 1.2])

$$E(\psi) = \int_{\mathbb{R}} \varepsilon_\psi(x,t),$$

we define a *solitary wave* to be any nonsingular solution whose energy density has a space-time dependence of the form

$$\varepsilon_\psi(x,t) = \tilde{\varepsilon}_\psi(x - vt) \tag{3.1.8}$$

where $\tilde{\varepsilon}_\psi$ is a localized function and v is velocity in the direction of the motion. The energy density of both kink and antikink is given by the localized function

$$\tilde{\varepsilon}(x) = \frac{16 e^{2(x-x_0)}}{[1 + e^{2(x-x_0)}]^2},$$

and hence they are static solitary waves (i.e. corresponding to $v = 0$ in (3.1.8)).

By the Lorentz invariance of (3.1.2), traveling solitary waves can be trivially obtained by Lorentz-transforming (3.1.7), and their energy density turns out to be

$$\varepsilon(x,t) = \gamma^2 \tilde{\varepsilon}(\gamma[x - vt]),$$

which represents a single bump traveling undistorted with uniform velocity.

3.2. Solitons in three space dimensions: Derrick's Problem. In 1963, attempting to find a model for extended elementary particles in contrast with point particles, U. Enz [47] was led to study an equation like (3.1.1). He proved the existence of nonsingular time-independent solutions with energy density localized about a point on the x axis, and under a further requirement of stability, he found that the energy is bound to assume only certain discrete values, which can be seen as corresponding to the rest energies of elementary particles. Moving beyond this work, G. H. Derrick proposed, in a celebrated paper [37], the more realistic $3+1$-dimensional model given by the nonlinear Klein–Gordon equation

$$\Box\psi + W'(\psi) = 0, \tag{3.2.1}$$

where

$$\Box\psi = -\Delta\psi + \frac{\partial^2\psi}{\partial t^2}$$

(Δ being the 3-dimensional Laplace operator) and W' is the gradient of a nonnegative C^1 real function W.

Owing to the relativistic invariance of (3.2.1), moving waves can be trivially obtained from static solutions by boosting, i.e., turning to a moving coordinate frame by applying a Lorentz transformation. Thus, we are led to concern ourselves with finite-energy static solutions (of which solitary waves are a particular case) $u = u(x)$, $x \in \mathbb{R}^3$, with

$$E(u) = \int_{\mathbb{R}^N} \left[\tfrac{1}{2}|\nabla u|^2 + W(u)\right] dx < \infty, \tag{3.2.2}$$

where $N = 3$, that solve the equation

$$\Delta u + W'(u) = 0,$$

the latter equation also being the Euler–Lagrange equation of the energy functional (3.2.2).

In [37] Derrick showed that if the potential W is nonnegative, then any finite-energy static solution of (3.2.1) is necessarily trivial, namely, it takes a constant value which is a minimum point of W. On the other hand, if the nonnegativity of W is not required, no stable finite-energy static solution to equation (3.2.1) is permitted. In fact, the following theorem holds.

THEOREM 3.2.1. *Let $N \geq 3$. The energy functional* (3.2.2) *has no nontrivial local minima, i.e.,*

$$\delta^2 E(u) \geq 0 \text{ with } u \text{ nonconstant } \Rightarrow \delta E(u) \neq 0.$$

Moreover, if $W > 0$ then E does not have any nontrivial critical points at all, namely,

$$\delta E(u) = 0 \Rightarrow u \equiv u_0 \text{ with } W(u_0) = 0.$$

Proof. Using Derrick's simple rescaling argument, we set $u_\lambda(x) := u(\lambda x)$ and

$$E(u) = \frac{1}{2\lambda^{N-2}} \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{1}{\lambda^N} \int_{\mathbb{R}^N} W(u)\, dx =: \frac{1}{\lambda^{N-2}} I_1 + \frac{1}{\lambda^N} I_2.$$

If $\delta E(u)h = 0$ for any variation h, we have in particular

$$\left.\frac{d}{d\lambda} E(u_\lambda)\right|_{\lambda=1} = \frac{2-N}{2} I_1 + N I_2 = 0, \tag{3.2.3}$$

and therefore

$$\left.\frac{d^2}{d\lambda^2} E(u_\lambda)\right|_{\lambda=1} = \frac{(2-N)(1-N)}{2} I_1 + N(N+1) I_2 = (2-N) I_1.$$

Hence the second variation of E at any nonconstant critical point u is negative for a variation corresponding to a uniform stretching of u. Finally, if $W > 0$ then both I_1 and I_2 are nonnegative, and from (3.2.3) we deduce $I_1 = I_2 = 0$. ■

REMARK 3.2.1. According to Enz's results, as well as to our previous discussion on equation (3.1.1), the above argument is not applicable for the 1 + 1-dimensional case: if

$N = 1$, we obtain $E(u_\lambda) = \lambda I_1 + I_2/\lambda$, yielding on differentiation $I_1 = 2I_2$, which gives no contradiction.

On the other hand, if we consider a nonpositive potential, we are forced to seek saddle points, instead of minima, and for these static solutions we have lack of stability. As an example we recall that if we take

$$W(\xi) = \tfrac{1}{2}\xi^2 - \tfrac{1}{4}\xi^4,$$

then critical points of the energy functional

$$E(u) = \int_{\mathbb{R}^3} \left[\tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{2}u^2 - \tfrac{1}{4}u^4\right] dx$$

have been found in [27] and [70], and for more general potentials in [25, 81]; but in [4] and [24] it has been proved that these static solutions are not stable.

In [37], these facts led Derrick to say, "We are thus faced with the disconcerting fact that no equation of type (3.2.1) has any time-independent solutions which could reasonably be interpreted as elementary particles."

Derrick proposed some possible ways out of this difficulty. The first proposal was to consider models which are the Euler–Lagrange equations of the action functional relative to the functional

$$S = \int\int \mathcal{L}\, dx\, dt.$$

The Lorentz invariant Lagrangian density proposed in [37] has the form

$$\mathcal{L}(\psi) = -(|\nabla\psi|^2 - |\psi_t|^2)^{p/2}. \tag{3.2.4}$$

For $p = 2$, the Euler–Lagrange equations reduce to (3.2.1). For every integer $p > 3$, the nonexistence proof in [37] for the finite energy static solutions fails. However, Derrick does not continue his analysis and he concludes that a Lagrangian density of type (3.2.4) leads to a very complicated differential equation. He has been unable to demonstrate either the existence or nonexistence of stable solutions.

In this spirit, a considerable amount of work has been done by V. Benci and collaborators, and a model equation proposed in [22] will be the topic of the next chapter.

4. Solitons in several space dimensions

We introduce here an existence result for a $3+1$-dimensional model generalizing the one suggested by Derrick in his first proposal. A first existence result is stated in [22], which also gives a topological classification of static solutions by means of a topological invariant: the topological charge. In order to prove the existence of static solutions with nontrivial charge, a study of the behaviour of sequences of bounded energy is needed, in the spirit of the concentration-compactness principle. A further generalization is carried out in [14], which develops an existence analysis of the finite-energy static solutions in higher spatial dimension and for a larger class of Lorentz invariant Lagrangian densities.

4.1. Statement of the problem. The class of Lagrangian densities we consider generalizes the problem studied in [22], so as to include the Derrick proposal. First we introduce some notation. If n and m are positive integers, $\mathbb{R}^{n+1}$ and $\mathbb{R}^m$ will denote respectively the physical space-time (typically $n=3$) and the internal parameters space. We are interested in the multidimensional case, so we assume that $n \geq 2$. A point in $\mathbb{R}^{n+1}$ will be denoted by (x,t), where $x \in \mathbb{R}^n$ denotes the space variable and $t \in \mathbb{R}$ denotes the time variable. The fields we are interested in are maps $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^m$, $\psi = (\psi_1, \ldots, \psi_m)$. We set

$$\rho = |\nabla \psi|^2 - |\psi_t|^2,$$

$\nabla\psi$ and ψ_t denoting, respectively, the Jacobian with respect to x and the derivative with respect to t, so that

$$|\nabla\psi|^2 = \sum_{1\leq j\leq m,\, 1\leq i\leq n} |\psi_i^j|^2, \qquad |\psi_t|^2 = \sum_{1\leq j\leq m} |\psi_t^j|^2,$$

where $\psi_i^j = \frac{\partial \psi^j}{\partial x_i}$, $1 \leq i \leq n$, $1 \leq j \leq m$.

We shall consider Lagrangian densities of the form

$$\mathcal{L}(\psi,\rho) = -\tfrac{1}{2}\alpha(\rho) - V(\psi), \tag{4.1.1}$$

where V is a real function defined in an open subset $\Omega \subset \mathbb{R}^m$ and α is a real function defined by

$$\alpha(\rho) = a\rho + b|\rho|^{p/2}, \qquad p > n, \tag{4.1.2}$$

where $a \geq 0$ and $b > 0$.

The results of [22] dealt with the case $a=1$, $n=3$ and $p=6$. If $a=0$ and $n=3$, then (4.1.1) is equivalent to the Lagrangian density (3.2.4) proposed by Derrick [37], when we look for static solutions.

The action functional related to (4.1.1) is

$$S(\psi) = \int_{\mathbb{R}^{n+1}} \mathcal{L}(\psi, \rho)\, dx\, dt = \int_{\mathbb{R}^{n+1}} \big(-\tfrac{1}{2}\alpha(\rho) - V(\psi)\big)\, dx\, dt.$$

So the Euler–Lagrange equations are (a system of m scalar equations in $n+1$ dimensions, see [32])

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \psi_t^j}\right) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{L}}{\partial \psi_i^j}\right) - \frac{\partial \mathcal{L}}{\partial \psi^j} = 0 \tag{4.1.3}$$

where $\psi_i^j = \frac{\partial \psi^j}{\partial x_i}$, $1 \le i \le n$ and $1 \le j \le m$, and

$$\frac{\partial \mathcal{L}}{\partial \psi^j} = -\frac{1}{2}\alpha'(\rho) \underbrace{\frac{\partial \rho}{\partial \psi^j}}_{=0} - \frac{\partial V}{\partial \xi_j}(\psi), \tag{4.1.4}$$

$$\frac{\partial \mathcal{L}}{\partial \psi_i^j} = -\frac{1}{2}\alpha'(\rho)\frac{\partial \rho}{\partial \psi_i^j} = -\alpha'(\rho)\psi_i^j, \tag{4.1.5}$$

$$\frac{\partial \mathcal{L}}{\partial \psi_t^j} = -\frac{1}{2}\alpha'(\rho)\frac{\partial \rho}{\partial \psi_t^j} = \alpha'(\rho)\psi_t^j. \tag{4.1.6}$$

Substituting (4.1.4)–(4.1.6) into (4.1.3), we get

$$\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t^j) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\alpha'(\rho)\psi_i^j) + \frac{\partial V}{\partial \xi_j}(\psi) = 0, \quad 1 \le i \le n,\ 1 \le j \le m. \tag{4.1.7}$$

So we have

$$\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t^j) - \operatorname{div}(\alpha'(\rho)\nabla\psi^j) + \frac{\partial V}{\partial \xi_j}(\psi) = 0, \quad 1 \le j \le m.$$

Thus

$$\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t) - \nabla(\alpha'(\rho)\nabla\psi) + V'(\psi) = 0 \quad \text{in } \mathbb{R}^m \tag{4.1.8}$$

where $\nabla(\alpha'(\rho)\nabla\psi)$ denotes the vector whose jth component is $\operatorname{div}(\alpha'(\rho)\nabla\psi^j)$, and V' denotes the gradient of V.

Remark 4.1.1. Lorentz transformations, i.e., space-time rotations depending on one parameter v have the form

$$\begin{cases} x_1 \mapsto \gamma(x_1 - vt), \\ x_2 \mapsto x_2, \\ x_3 \mapsto x_3, \\ t \mapsto \gamma\big(t - \frac{v}{c^2}x_1\big), \end{cases}$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$, $|v| < c$ and c is a constant (dimensionally a velocity); for simplicity, we assume $c = 1$.

Equation (4.1.8) is Lorentz invariant (see Subsection 4.1.1 below). The static solutions $\psi(x,t) = u(x)$ of (4.1.8) solve the equation

$$-\nabla(\alpha'(\rho)\nabla u) + V'(u) = 0. \tag{4.1.9}$$

If we use (4.1.2), then (4.1.9) becomes

$$-a\Delta u - b\frac{p}{2}\Delta_p + V'(u) = 0, \quad \text{where} \quad \Delta_p u = \nabla(|\nabla u|^{p-2}\nabla u). \tag{4.1.10}$$

It is easy to verify that if $u = u(x)$ is a solution of (4.1.10) and $\mathrm{v} = (v, 0, \dots, 0)$ with $|\mathrm{v}| < 1$, then the field

$$\psi_{\mathrm{v}}(x,t) = u(\gamma(x_1 - vt), x_2, \dots, x_n) \tag{4.1.11}$$

is a solution of (4.1.8) (see Subsection 4.1.2). Notice that the function ψ_{v} experiences a contraction by a factor of

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

in the direction of motion; this is a consequence of the fact that (4.1.8) is Lorentz invariant. Clearly, (4.1.10) are the Euler–Lagrange equations with respect to the energy functional

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^2 + \frac{b}{2}|\nabla u|^p + V(u) \right) dx, \tag{4.1.12}$$

where $m = n + 1$, so the time independent fields u are maps $u : \mathbb{R}^n \to \mathbb{R}^m$. For every $\xi \in \mathbb{R}^{n+1}$, we write

$$\xi = (\xi_0, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^n.$$

As to the function V, we assume that $V : \Omega \to \mathbb{R}$ where $\Omega = \mathbb{R}^{n+1} \setminus \{\eta\}$, $\eta = (1, 0)$, and V is positive and singular at η. More precisely we assume:

(V1) $V \in C^1(\Omega, \mathbb{R})$.
(V2) $V(\xi) \geq V(0) = 0$.
(V3) V is twice differentiable at 0 and the Hessian matrix $V''(0)$ is nondegenerate.
(V4) There exist $c, \rho > 0$ such that if $|\xi| < \rho$, then

$$V(\eta + \xi) \geq c|\xi|^{-q}, \quad \text{where} \quad \frac{1}{q} = \frac{1}{n} - \frac{1}{p}. \tag{4.1.13}$$

(V5) For every $\xi \in \Omega \setminus \{0\}$ we have

$$V(\xi) > 0, \quad \text{and} \quad \liminf_{|\xi| \to \infty} V(\xi) = v > 0.$$

Taking $a = 1$, we observe that for $j = 1, \dots, n + 1$, $\alpha(0) = a = 1$, and since 0 is a minimum for V, we can choose a base in $\mathbb{R}^{n+1}$ which diagonalizes $V''(0)$ as

$$\begin{pmatrix} m_1^2 & & 0 \\ & \ddots & \\ 0 & & m_{n+1}^2 \end{pmatrix}$$

such that

$$V'(\xi) = V''(0)\xi + o(\xi) \simeq V''(0)\xi$$

in a neighborhood of 0.

Then, linearizing (4.1.8) at 0 and taking $a = 1$, we get a system of Klein–Gordon equations

$$\Box\psi^j + m_j^2 \psi^j = 0, \qquad 1 \leq j \leq n + 1,$$

where the m_j^2 denote the eigenvalues of $V''(0)$ and $\Box\psi = \frac{\partial \psi}{\partial t^2} - \Delta\psi$.

Example 4.1.1. A potential satisfying assumptions (V1)–(V5) is

$$V(\xi) = \omega^2\left(|\xi|^2 + \frac{|\xi|^4}{|\xi-\eta|^q}\right), \quad \text{where} \quad q = \frac{np}{p-n}.$$

Definition 4.1.1. A *soliton* is a solution of (4.1.8) having the form of (4.1.11), where u is a local minimum of the energy functional (4.1.12).

4.1.1. Lorentz invariant field equations. For simplicity we take $n = 1$ and $m = 1$, so equation (4.1.8) in $\mathbb{R}^2$ becomes

$$\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t) - \frac{\partial}{\partial x}(\alpha'(\rho)\psi_x) + V'(\psi) = 0, \tag{4.1.14}$$

where

$$\begin{aligned}\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t) &= \alpha''(\rho)\rho_t\psi_t + \alpha'(\rho)\psi_{tt},\\ \frac{\partial}{\partial x}(\alpha'(\rho)\psi_x) &= \alpha''(\rho)\rho_x\psi_x + \alpha'(\rho)\psi_{xx}.\end{aligned}$$

Then equation (4.1.14) becomes

$$\alpha''(\rho)\underbrace{(\rho_t\psi_t - \rho_x\psi_x)}_{\text{A}} + \alpha'(\rho)\underbrace{(\psi_{tt} - \psi_{xx})}_{\text{B}} + V'(\psi) = 0.$$

To prove that (4.1.14) is Lorentz invariant, it is sufficient to prove that parts A and B are invariant under the Lorentz transformations

$$X = \gamma(x - vt), \qquad T = \gamma(t - vx),$$

and

$$\psi(X, T) = \psi(\gamma(x - vt), \gamma(t - vx)).$$

First we show that part B is Lorentz invariant. We have

$$\psi_x = \gamma\psi_X - v\gamma\psi_T, \qquad \psi_t = -v\gamma\psi_X + \gamma\psi_T.$$

Then

$$\begin{aligned}\psi_{xx} &= (\gamma\psi_X - v\gamma\psi_T)_x = \gamma(\psi_X)_x - v\gamma(\psi_T)_x,\\ \psi_{tt} &= (-v\gamma\psi_X + \gamma\psi_T)_t = -v\gamma(\psi_X)_t + \gamma(\psi_T)_t,\end{aligned}$$

which implies

$$\begin{aligned}\psi_{xx} &= \gamma^2\psi_{XX} + (v\gamma)^2\psi_{TT} - 2v\gamma^2\psi_{TX},\\ \psi_{tt} &= (v\gamma)^2\psi_{XX} + \gamma^2\psi_{TT} - 2v\gamma^2\psi_{TX}.\end{aligned}$$

So,

$$\psi_{tt} - \psi_{xx} = \psi_{TT} - \psi_{XX}.$$

The proof that part A is invariant uses the same arguments.

4.1.2. Static solution and stability. For simplicity and with no loss of generality, we take $n = m = 1$. Then equation (4.1.10) in $\mathbb{R}^2$ becomes

$$-au_{xx} - b\frac{p}{2}\frac{\partial}{\partial x}(|u_x|^{p-2}u_x) + V'(u) = 0. \tag{4.1.15}$$

Let u be a solution of (4.1.15). We show that $\psi(x,t) = u(\gamma(x_1 - vt))$ is a solution of (4.1.14).

By easy calculation, we have

$$\begin{cases} \rho = |\psi_x|^2 - |\psi_t|^2 = u_x^2, \\ \psi_x = \gamma u_x, \qquad \psi_t = (\gamma v) u_x, \\ \psi_{xx} = \gamma^2 u_{xx}, \qquad \psi_{tt} = (\gamma v)^2 u_{xx}, \\ \rho_x = \gamma (u_x^2)_x, \qquad \rho_t = (\gamma v)(u_x^2)_x. \end{cases} \tag{4.1.16}$$

Recall that $\gamma = 1/\sqrt{1-v^2}$. We have

$$\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t) - \frac{\partial}{\partial x}(\alpha'(\rho)\psi_x) = \alpha''(\rho)(\rho_t \psi_t - \rho_x \psi_x) + \alpha'(\rho)(\psi_{tt} - \psi_{xx}),$$

and using (4.1.16) we get

$$\begin{aligned} \frac{\partial}{\partial t}(\alpha'(\rho)\psi_t) - \frac{\partial}{\partial x}(\alpha'(\rho)\psi_x) &= -\alpha''(u_x^2)((u_x^2)_x u_x) - \alpha'(u_x^2)(u_{xx}) \\ &= -\frac{\partial}{\partial x}(\alpha'(u_x^2)u_x) = -\frac{\partial}{\partial x}\left(a u_x + b\frac{p}{2}|u_x|^{p-2}u_x\right). \end{aligned}$$

So, $\psi(x,t) = u(\gamma(x_1 - vt))$ is a solution of (4.1.14).

4.2. Functional setting. Let $p > n \geq 2$. With no loss of generality, we can consider the functional (4.1.12) with $b = 1$. It will be convenient to write

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^2 + \frac{1}{2}|\nabla u|^p + V(u) \right) dx.$$

We define E_a to be the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_a = a\|\nabla u\|_{L^2} + \|\nabla u\|_{L^p} + \|u\|_{L^2}, \qquad p > n \geq 2,\ a \geq 0,$$

i.e.,

$$E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a},$$

$$\|u\|_{L^2} = \Big(\sum_{j=1}^{n+1} \|u_j\|_{L^2}^2\Big)^{1/2}, \quad \|\nabla u\|_{L^2} = \Big(\sum_{j=1}^{n+1} \|\nabla u_j\|_{L^2}^2\Big)^{1/2}, \quad \|\nabla u\|_{L^p} = \Big(\sum_{j=1}^{n+1} \|\nabla u_j\|_{L^p}^p\Big)^{1/p}.$$

For every $a > 0$, the norms $\|\cdot\|_a$ are equivalent, so we have to study only the two cases: $a = 0$ and $a > 0$.

Proposition 4.2.1. *The Banach space E_0 is continuously embedded in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$ for every $s \in [2, \infty]$.*

Proof. Since E_0 is continuously embedded in $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, it is sufficient to show that E_0 is also embedded in $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Since $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in E_0, and also in L^s (see Corollary 2.1.2), it is sufficient to prove that there exists $c > 0$ such that, for every $u \in C_0(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\|u\|_{L^\infty} \leq c\|u\|_0.$$

We fix $u \in C_0(\mathbb{R}^n, \mathbb{R}^{n+1})$ and consider a family of cubes $Q_k \subset \mathbb{R}^n$ such that

$$\operatorname{meas}(Q_k) = 1, \qquad \bigcup_{k \in \mathbb{N}} Q_k = \mathbb{R}^n.$$

Then, by a well-known inequality (see [30, p. 283]), for every $k \in \mathbb{N}$,

$$|u(x)| \leq \left| \int_{Q_k} u\, dy \right| + M\|\nabla u\|_{L^p(Q_k)}, \tag{4.2.1}$$

with $M \geq 0$ independent of u. Thus

$$|u(x)| \leq \operatorname{meas}(Q_k)\|u\|_{L^2} + M\|\nabla u\|_{L^p(Q_k)} \leq \|u\|_{L^2(\mathbb{R}^n)} + M\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq (1+M)\|u\|_0.$$

Hence

$$\|u\|_{L^\infty} \leq c\|u\|_0 \quad \text{for } c = 1 + M. \ ■$$

COROLLARY 4.2.2. *The Banach space* E_0 *is continuously embedded in* $W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1})$.

Proof. By definition of E_0, we have

$$\|u\|_0 > \|\nabla u\|_{L^p} \quad \text{for every } u \in E_0.$$

From Proposition 4.2.1 there exists $c_1 > 0$ such that

$$c_1\|u\|_0 > \|u\|_{L^p},$$

and so $\|u\|_0 > c\|u\|_{W^{1,p}}$. ■

COROLLARY 4.2.3. *For every* $a > 0$, *the space* E_a *can be identified with the Banach space*

$$W = W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1}),$$

equipped with the usual norm

$$\|u\|_W = \|u\|_{W^{1,2}} + \|u\|_{W^{1,p}}.$$

Proof. $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1})$ and also in $W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see Theorem 2.1.8). For any $u \in E_a$ we have

$$\|u\|_a \leq \sup(1, a)\|u\|_W.$$

From Corollary 4.2.2, there exists $c > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$,

$$\|u\|_a \geq c(\|u\|_{W^{1,2}} + \|u\|_{W^{1,p}}). \ ■$$

By Proposition 4.2.1 and well-known Sobolev embeddings we have the following:

REMARK 4.2.1. Since $p > n$, by the preceding corollaries and well-known Sobolev embeddings (see Theorem 2.1.5), we easily get some useful properties of the Banach space E_a:

(1) We have

$$E_a \subset W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1}) \subset L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}), \tag{4.2.2}$$

and so if $\{u_k\}$ converges weakly in E_a to u, then it converges uniformly on every compact subset of $\mathbb{R}^n$.

(2) Furthermore, all functions in E_a are Hölder continuous of order $(p-n)/p$:

$$|u(x) - u(y)| \leq c|x - y|^{(p-n)/p}\|\nabla u\|_{L^p}, \tag{4.2.3}$$

i.e. $E_a \subset C^{0,(p-n)/p}(\mathbb{R}^n, \mathbb{R}^{n+1})$ is a locally compact injection.

(3) For every $a \geq 0$, the functions in E_a are bounded and decay to zero at infinity,

$$\lim_{|x|\to\infty} u(x) = 0. \tag{4.2.4}$$

The presence of Δ_p in (4.1.10) implies that the functions u on which the energy f_a is finite are continuous and decay to 0 at infinity; the presence of the singular term $V'(u)$ implies that such maps u take values in Ω. So the nontrivial topological properties of Ω (namely $\pi_n(\Omega) = \mathbb{Z}$) permit one, as in the sine-Gordon equation (Chapter 2), to give a topological classification of the static configurations. This classification is carried out by means of a topological invariant, the topological charge (see Definition 4.3.1), which depends only on the region where the function is concentrated, namely the support. We point out that in other models (see [23, 48, 49, 80]), the topological classification follows from the fact that the field u takes values in suitable manifolds.

Recall that η is a singular point of the potential V, so it is reasonable to consider in E_a the open subset

$$\Lambda_a = \{u \in E_a : u(x) \neq \eta \text{ for all } x \in \mathbb{R}^n\}.$$

In fact, if $u \in \Lambda_a$, by Remark 4.2.1 we have

$$\inf_{x\in\mathbb{R}^n} |u(x) - \eta| = d > 0.$$

Then, by using Proposition 4.2.1 (E_0 is continuously embedded in L^∞), we deduce that there exists a small neighborhood of u contained in Λ_a.

The boundary of Λ_a is given by

$$\partial\Lambda_a = \{u \in E_a : \text{there exists } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\}.$$

We can show that Λ_a has a rich topological structure, more precisely it consists of infinitely many connected components. These components are identified by the topological charge we are going to introduce.

4.3. Topological charge and connected components of Λ_a. For simplicity, we consider the function space

$$C = \left\{u : \mathbb{R}^n \to \mathbb{R}^{n+1} \setminus \{\eta\} : u \text{ is continuous and } \lim_{|x|\to\infty} u(x) = 0\right\}$$

where $\eta = (1, 0)$. Every function $u \in C$ can be written in the form $u(x) = (u_0(x), \tilde{u}(x)) \in \mathbb{R}^{n+1}$ where $u_0 : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{u} : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 4.3.1. For every $u \in C$ we define

$$K_u = \{x \in \mathbb{R}^n : u_0(x) > 1\}.$$

Then we define the *topological charge* of u by

$$\operatorname{ch}(u) := \begin{cases} \deg(\tilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ 0 & \text{if } K_u = \emptyset, \end{cases}$$

where

$$\deg(\tilde{u}, K_u, 0) = \sum_{x\in\tilde{u}^{-1}(0)} \operatorname{sgn} J_{\tilde{u}}(x)$$

(Brouwer degree) with $J_{\tilde{u}}$ denoting the determinant of the Jacobian matrix.

For more information on this subject, see Section 2.8.

We notice that the above definition is well posed. Indeed, since

$$\lim_{|x|\to\infty} u(x) = 0,$$

we see that K_u is an open, bounded set; moreover, for every $x \in \partial K_u$, we have $u(x) \neq \eta$ and $\tilde{u}(x) \neq 0$.

PROPOSITION 4.3.1. *For every* $u = (u_0, \tilde{u}) \in \Lambda_a$ *there exist* $l \in \mathbb{N}$ *and* $x^1, \dots, x^l \in \mathbb{R}^n$, $R_1, \dots, R_l > 0$, *for which we set* $B^i = B(x^i, R_i)$, *such that*

$$B^i \cap B^j = \emptyset, \qquad i \neq j;$$

$$\forall x \in \mathbb{R}^n \setminus \bigcup_{i=1}^{l} B^i, \qquad u_0(x) < 1;$$

$$K_u \subset \bigcup_{i=1}^{l} B^i; \tag{4.3.1}$$

$$\operatorname{ch}(u) = \deg\Big(\tilde{u}, \bigcup_{i=1}^{l} B^i, 0\Big) = \sum_{i=1}^{l} \deg(\tilde{u}, B^i, 0). \tag{4.3.2}$$

Proof. Let $u \in \Lambda_a$. If $K_u = \emptyset$, we have $\deg(\tilde{u}, K_u, 0) = 0$. Now assume $K_u \neq \emptyset$.

Recall that E_a is a reflexive Banach space continuously embedded in $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Let $x^1 \in \mathbb{R}^n$ be a maximum point for u_0. By Remark 4.2.1, x^1 always exists. Since $K_u \neq \emptyset$, we have $u_0(x^1) > 1$.

We take R_1 such that $u_0(x) > 1$ for all $x \in B(x^1, R_1)$. For simplicity we set $B^1 = B(x^1, R_1)$. Now we distinguish two cases:

(A1) $u_0(x) \leq 1$ for all $x \in \mathbb{R}^n \setminus B^1$,
(B1) $u_0(x) > 1$ for some $x \in \mathbb{R}^n \setminus B^1$.

In case (A1) the proposition holds with $l = 1$; indeed, by additivity we have

$$\deg(\tilde{u}, B^1, 0) = \deg(\tilde{u}, K_u, 0) + \deg(\tilde{u}, B^1 \setminus \overline{K}_u, 0).$$

Since $B^1 \setminus \overline{K}_u = \emptyset$, we obtain $\deg(\tilde{u}, B^1 \setminus \overline{K}_u, 0) = 0$, and then

$$\operatorname{ch}(u) = \deg(\tilde{u}, B^1, 0). \tag{4.3.3}$$

Let us consider case (B1). Let x^2 be a maximum point for u_0 in $\mathbb{R}^n \setminus B^1$; we have $u_0(x^2) > 1$. We set $B^2 = B(x^2, R_2)$ and we take R_1 such that

$$u_0(x) > 1 \quad \forall x \in B^2.$$

Also in this second step we have an alternative: either

(A2) $u_0(x) \leq 1$ for all $x \in \mathbb{R}^n \setminus (B^1 \cup B^2)$, or
(B2) $u_0(x) > 1$ for some $x \in \mathbb{R}^n \setminus (B^1 \cup B^2)$.

If case (A2) holds true, the proposition holds with $l = 2$; indeed, the spheres B^1 and B^2 are disjoint for R_1, R_2 sufficiently small, and with the same arguments as in (4.3.3),

$$\operatorname{ch}(u) = \deg(\tilde{u}, B^1, 0) + \deg(\tilde{u}, B^2, 0),$$

so that

$$K_u \subset B^1 \cup B^2.$$

In case (B2) we consider a maximum point of u_0 in $\mathbb{R}^n \setminus (B^1 \cup B^2)$ and we repeat the argument used in case (B1).

By Remark 4.2.1 this alternative process terminates in a finite number of steps. ■

The open set $K_u := \{x \in \mathbb{R}^n : u_0(x) > 1\}$ is the support of u; by Proposition 4.3.1 there exists $R = R(u)$ such that $K_u \subset B(0, R)$.

THEOREM 4.3.2. *For every $u \in \Lambda_a$ there exists $r = r(u) > 0$ such that, for every $v \in \Lambda_a$,*

$$\|u - v\|_\infty \le r \;\Rightarrow\; \mathrm{ch}(u) = \mathrm{ch}(v). \tag{4.3.4}$$

Proof. Let $u^k = (u_0^k, \widetilde{u^k}) \in \Lambda_a$ be uniformly convergent to $u = (u_0, \tilde{u})$. By Proposition 4.3.1 there exists $l \in \mathbb{N}$ such that $\mathrm{ch}(u) = \deg(\tilde{u}, \bigcup_{i=1}^l B^i, 0) = \sum_{i=1}^l \deg(\tilde{u}, B^i, 0)$. We shall show that, for k sufficiently large,

$$\mathrm{ch}(u^k) = \mathrm{ch}(u).$$

First we show that

$$K_{u^k} \subset \bigcup_{i=1}^l B^i.$$

Let $x \notin \bigcup_{i=1}^l B^i$. Then $u_0(x) < 1$ since u_0^k is uniformly convergent to u_0; for n sufficiently large, $u_0^k(x) < 1$, and so

$$x \notin K_{u^k}.$$

Using the excision property of the degree, we have

$$\deg\Big(\widetilde{u^k}, \bigcup_{i=1}^l B^i, 0\Big) = \underbrace{\deg(\widetilde{u^k}, K_{u^k}, 0)}_{=\mathrm{ch}(u^k)} + \underbrace{\deg\Big(\widetilde{u^k}, \bigcup_{i=1}^l B^i \setminus \overline{K}_{u^k}, 0\Big)}_{=0} \tag{4.3.5}$$

where the second term on the right-hand side of (4.3.5) is 0. Indeed, let $x \in \bigcup_{i=1}^l B^i \setminus \overline{K}_{u^k}$. Then $u_0^k(x) < 1$ and $u_0(x) > 1$ for n sufficiently large. We get a contradiction: $u_0(x) < 1$ and $u_0(x) > 1$. So,

$$\bigcup_{i=1}^l B^i \setminus \overline{K}_{u^k} = \emptyset,$$

and thus

$$\deg\Big(\widetilde{u^k}, \bigcup_{i=1}^l B^i \setminus \overline{K}_{u^k}, 0\Big) = 0. \tag{4.3.6}$$

From (4.3.5), (4.3.6) and by additivity, we have

$$\mathrm{ch}(u^k) = \deg\Big(\widetilde{u^k}, \bigcup_{i=1}^l B^i, 0\Big) = \sum_{i=1}^l \deg(\widetilde{u^k}, B^i, 0).$$

Now, using the previous proposition and the continuity of topological degree with respect to uniform convergence, we get, for k sufficiently large,

$$\mathrm{ch}(u^k) = \sum_{i=1}^{l} \deg(\tilde{u}, B^i, 0) = \mathrm{ch}(u), \qquad \mathrm{ch}(u) = \deg(\tilde{u}, B(0,R), 0). \ \blacksquare \tag{4.3.7}$$

So, by Theorem 4.3.2, the topological charge is continuous with respect to uniform convergence. Now, for every $q \in \mathbb{Z}$, we set

$$\Lambda_q = \{u \in \Lambda_a : \mathrm{ch}(u) = q\}.$$

The continuity of the topological charge with respect to uniform convergence and the continuity of the embeddings of E_a in L^∞ (see Proposition 4.2.1) ensure that the topological charge is continuous on Λ_a, and it follows that Λ_q is open in E_a, since we also have

- $\Lambda_a = \bigcup_{q\in\mathbb{Z}} \Lambda_q$,
- $\Lambda_q \cap \Lambda_p = \emptyset$ if $p \neq q$.

We conclude that every Λ_q is a connected component of Λ_a. We assume that the space dimension is odd, and hence for every $q \in \mathbb{Z}$ the component Λ_q is homeomorphic to Λ_{-q}.

So for every $u \in \Lambda_a$ we can define the charge $\mathrm{ch}(u) \in \mathbb{Z}$. Now, we consider the minimizer set of f_a in the open set

$$\Lambda_q^* = \{u \in \Lambda_a : \mathrm{ch}(u) \neq 0\}.$$

REMARK 4.3.1. We can easily see that $\mathrm{ch}(u) \neq 0$ implies $\|u\|_{L^\infty} > 1$.

4.4. Properties of the energy functional. Firstly we are going to study the behavior of the energy f_a when u approaches the boundary of Λ_a, in the spirit of a well-known result of Gordon [57] concerning strongly attractive potentials. We remark that $\partial\Lambda_a = E_a \setminus \Lambda_a$.

LEMMA 4.4.1. *Let $(u_k) \subset \Lambda_a$ be a weakly converging sequence. If the weak limit belongs to $\partial\Lambda_a$, then*

$$f_a(u_k) \to \infty \qquad \text{as } k \to \infty.$$

Proof. Suppose $u_k \rightharpoonup u \in \partial\Lambda_a$ as $k \to \infty$. Since $u \in \partial\Lambda_a$, there exists $x_* \in \mathbb{R}^n$ such that $u(x_*) = \eta$.

Since V is nonnegative, it is sufficient to show that there exists a small ball centered at x_* such that

$$\lim_{k\to\infty} \int_{B(x_*,R)} V(u_k(x))\, dx = \infty.$$

Using the fact that (u_k) is bounded in E_a, and uniform convergence on compact sets, we have

$$u_k(x_*) \to u(x_*) \qquad \text{as } k \to \infty. \tag{4.4.1}$$

Since (u_k) is bounded in E_a, (∇u_k) is bounded in L^p. Then from (4.2.3) we obtain

$$|u_k(x) - u_k(x_*)| \le c|x - x_*|^{(p-n)/p}. \tag{4.4.2}$$

Thus

$$|u_k(x) - \eta| \le |u_k(x) - u_k(x_*)| + |u_k(x_*) - \eta|. \tag{4.4.3}$$

By (4.4.1) and (4.4.2), there exists $\epsilon_k > 0$ such that

$$|u_k(x) - \eta| \le c|x - x_*|^{(p-n)/p} + \epsilon_k$$

where $\epsilon_k \to 0$ as $k \to \infty$. Let $x \in B(0, r)$. For r sufficiently small, there is $\rho > 0$ such that

$$|u_k(x) - \eta| \le cr^{(p-n)/p} + \epsilon_k < \rho. \tag{4.4.4}$$

From (4.4.4) and (V4), we have

$$V(u_k(x)) \ge c|u_k(x) - \eta|^{np/(p-n)}. \tag{4.4.5}$$

Then from (4.4.4) and (4.4.5), we obtain

$$V(u_k(x)) \ge \frac{c}{|x - x_*| + \epsilon_k}.$$

Restricting our attention to $B(x_*, r)$, we have

$$\lim_{k\to\infty} \int_{B(x_*,r)} V(u_k(x))\, dx \ge \int_{B(x_*,r)} \lim_{k\to\infty} V(u_k(x))\, dx \ge c \int_{B(x_*,r)} \frac{1}{|x - x_*|^n}\, dx = \infty. \blacksquare$$

Lemma 4.4.2. *The functional f_a takes real values and is continuous on Λ_a.*

Proof. We have

$$f_a(u) = \underbrace{\int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^2 + \frac{b}{2}|\nabla u|^p\right) dx}_{} + \underbrace{\int_{\mathbb{R}^n} V(u)\, dx}_{}.$$

First we show that $f_a(u) < \infty$. The first term on the left-hand side above is finite and continuous. Let us prove that the second term is finite and continuous.

We have

$$V(\xi) = V(0) + V'(0)\xi + V''(0)\xi \cdot \xi + o(\xi^2);$$

by (V1) and (V2), then

$$V(\xi) = V''(0)\xi \cdot \xi + o(\xi^2).$$

By (V3), there exist a small neighborhood of $0 \in \mathbb{R}^{n+1}$ and $M > 0$ such that, for every ξ in that neighbourhood, we have

$$V(\xi) \le M|\xi|^2. \tag{4.4.6}$$

Since every $u \in E_a$ decays to zero at infinity (see (4.2.4)), there exists a ball B_u such that, for every $x \in \mathbb{R}^n \setminus B_u$, we have $|u(x)| < \epsilon$; then by (4.4.6) and for ϵ sufficiently small,

$$V(u(x)) \le M|u(x)|^2. \tag{4.4.7}$$

From $u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, we deduce

$$\int_{\mathbb{R}^n \setminus B_u} V(u)\, dx < \infty.$$

On the other hand, since u is continuous (see (4.2.3)), we also have

$$\int_{B_u} V(u)\, dx < \infty.$$

Let $(u_k) \subset \Lambda_a$ be a sequence such that $f_a(u_k) < \infty$ and $u_k \to u$ in E_a. We show that

$$\int_{\mathbb{R}^n} V(u_k) \to \int_{\mathbb{R}^n} V(u).$$

Since $f_a(u_k) < \infty$, Lemma 4.4.1 shows that u belongs to Λ_a.

We have $u_k \to u$ in $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see (4.2.2)). We deduce that $V(u_k) \to V(u)$ uniformly on $\mathbb{R}$, so

$$\int_{B_u} V(u_k)\,dx \to \int_{B_u} V(u)\,dx. \tag{4.4.8}$$

On the other hand, by (4.4.7) we have

$$\int_{\mathbb{R}^n \setminus B_u} V(u(x))\,dx \le \int_{\mathbb{R}^n \setminus B_u} |u(x)|^2\,dx,$$

and since $u_k \to u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, using the dominated convergence (see Theorem 2.5.3) we conclude that

$$\int_{\mathbb{R}^n \setminus B_u} V(u_k)\,dx \to \int_{\mathbb{R}^n \setminus B_u} V(u)\,dx. \ \blacksquare \tag{4.4.9}$$

LEMMA 4.4.3. *The map $f' : E_a \to E_a'$ defined by*

$$\begin{aligned}\langle f_a'(u), v\rangle &= \langle -a\Delta u - b\Delta_p u + V'(u), v\rangle \\ &= \int_{\mathbb{R}^n} \left(a(\nabla u|\nabla v) + b\frac{p}{2}|\nabla u|^{p-2}(\nabla u|\nabla v) + V'(u)\cdot v \right) dx\end{aligned}$$

is continuous.

Proof. We have

$$f_a'(u) = \underbrace{-a\Delta u - b\frac{p}{2}\Delta_p u}_{} + \underbrace{V'(u)}_{}.$$

The first term is dealt with in Appendix B. Let us prove that the second term is continuous.

Let $(u_k) \subset \Lambda_a$ be a sequence such that $f_a(u_k) < \infty$ and $u_k \to u$. We show that

$$V'(u_k) \to V'(u_k) \quad \text{in } E_a'.$$

Since $f_a(u_k) < \infty$, Lemma 4.4.1 shows that u belongs to Λ_a. Recall that E_a is continuously embedded in L^∞ (see (4.2.2)). We have

$$\|V'(u_k) - V'(u)\|_{E_a'} = \sup_{\|h\|_{E_a} \le 1} \langle V'(u_k) - V'(u), h\rangle_{E_a' \times E_a}$$

with

$$\begin{aligned}\langle V'(u_k) - V'(u), h\rangle_{E_a' \times E_a} &= \int_{\mathbb{R}^n} (V'(u_k) - V'(u))h\,dx \\ &= \underbrace{\int_{B_u} (V'(u_k) - V'(u))h\,dx}_{(1)} + \underbrace{\int_{\mathbb{R}^n \setminus B_u} (V'(u_k) - V'(u))h\,dx}_{(2)}.\end{aligned}$$

In (1), since $\|h\|_{L^\infty} \le \|h\|_{E_a} \le 1$, with the same reasoning as in (4.4.8), we have

$$\int_{B_u} (V'(u_k) - V'(u))h\,dx < \epsilon/2,$$

with the same choice of B_u as in the proof of Lemma 4.4.2.

In (2), we have $V'(\xi) = (V''(0)\xi + o(\xi)$. Then by (V3),

$$\int_{\mathbb{R}^n \setminus B_u} (V'(u_k))h\,dx = M \int_{\mathbb{R}^n \setminus B_u} |u_k|\,|h|\,dx \le \|u_k\|_{L^2}\|h\|_{L^2} \le \|u_k\|_{L^2}. \tag{4.4.10}$$

From (4.4.10) with the same reasoning as in (4.4.9), we have

$$\int_{\mathbb{R}^n/B_u} (V'(u_k) - V'(u))h\,dx < \epsilon/2. \blacksquare$$

LEMMA 4.4.4. *The functional f_a is coercive in Λ_a; that is, for every sequence $(u_k) \subset \Lambda_a$ such that $\|u_k\|_a \to \infty$, we have $f_a(u_k) \to \infty$.*

Proof. In the case $a > 0$ and $n > 2$, we have

$$\|u\|_a = a\|\nabla u\|_{L^2} + \|\nabla u\|_{L^p} + \|u\|_{L^2}.$$

Let $u_k \in \Lambda_a$ be such that $\|u_k\|_a \to \infty$ as $k \to \infty$. It is clear that if

$$a\|\nabla u_k\|_{L^2} + \|\nabla u_k\|_{L^p} \to \infty \quad \text{as } k \to \infty, \tag{4.4.11}$$

then

$$f_a(u_k) \to \infty \quad \text{as } k \to \infty.$$

Assume now that there exists $c_* > 0$ such that

$$a\|\nabla u_k\|_{L^2} + \|\nabla u_k\|_{L^p} < c_*, \tag{4.4.12}$$

$$\|u_k\|_{L^2} \to \infty \quad \text{as } k \to \infty. \tag{4.4.13}$$

We shall prove that

$$\int_{\mathbb{R}^n} V(u_k)\,dx \to \infty \quad \text{as } k \to \infty.$$

From (V3), for every $r > 0$ there exists $\omega_r > 0$ such that

$$|\xi| \le r \;\Rightarrow\; V(\xi) \ge \omega_r |\xi|^2. \tag{4.4.14}$$

For every $k \in \mathbb{N}$, we set

$$A_k = \{x \in \mathbb{R}^n : |u_k(x)| \le r\},$$

where $u_k \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$. By the Sobolev inequality (see Theorem 2.1.8),

$$\|u_k\|_{L^{2^*}} \le c\|\nabla u_k\|_{L^2}, \qquad 2^* = \frac{2n}{n-2},\ n > 2. \tag{4.4.15}$$

From (4.4.12), we obtain

$$\|u_k\|_{L^{2^*}} < c_*. \tag{4.4.16}$$

Moreover, from (4.2.1), there exists $M \ge 0$ independent of u_k such that

$$\begin{aligned} |u_k(x)| &\le \left|\int_{Q_k} u\,dy\right| + M\|\nabla u_k\|_{L^p(Q_k)} \quad (\operatorname{meas}(Q_k) = 1) \\ &\le \|u\|_{L^{2^*}(Q_k)} + M\|\nabla u_k\|_{L^p(Q_k)}. \end{aligned}$$

By (4.4.11) and (4.4.16), for any $x \in \mathbb{R}^n$, we have

$$|u_k(x)| < c_* + Mc_*. \tag{4.4.17}$$

Then there exists $c > 0$ such that

$$\operatorname{meas}(\mathbb{R}^n \setminus A_k) < c. \tag{4.4.18}$$

From (4.4.17) and (4.4.18), we deduce that there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}^n \setminus A_k} |u_k|^2\,dx < c_1. \tag{4.4.19}$$

By (4.4.16), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} V(u_k)\,dx &\geq \int_{A_k} V(u_k)\,dx \geq \omega_r \int_{A_k} \|u_k\|^2\,dx \\ &\geq \omega_r \biggl(\|u_k\|_{L^2}^2 - \int_{\mathbb{R}^n \setminus A_k} |u_k|^2\,dx \biggr).\end{aligned}$$

From (4.4.19) and (4.4.15), we have

$$\int_{\mathbb{R}^n} V(u_k)\,dx \geq \omega_r(\|u_k\|_{L^2}^2 - c_1) \to \infty \quad \text{as } k \to \infty.$$

If $a = 0$ or $n = 2$, by (V5), there exists $r_* > 0$ such that, for every $\xi \in \mathbb{R}^n$ with $|\xi| \geq r_*$, we have

$$V(\xi) \geq \nu/2. \tag{4.4.20}$$

Let $(u_k) \subset \Lambda_a$ be a sequence such that $\|u_k\|_0 \to \infty$ as $k \to \infty$. Since the functional f_a is invariant with respect to translations in $\mathbb{R}^n$, we can assume

$$\|u_k\|_{L^\infty} = |u_k(0)|. \tag{4.4.21}$$

Now, we consider the case

$$\|\nabla u_k\|_{L^p} \leq M_* \quad \text{and} \quad \|u_k\|_{L^2} \to \infty \quad \text{as } k \to \infty.$$

Here we have two subcases:

$$\text{(a) } \|u_k\|_{L^\infty} \to \infty \text{ as } k \to \infty, \text{ or} \tag{4.4.22}$$

$$\text{(b) } \|u_k\|_{L^\infty} \text{ is bounded.} \tag{4.4.23}$$

In subcase (a), by (4.4.22), we can choose a sequence $(r_k) \subset (0, \infty)$ such that

$$r_* \leq \|u_k\|_{L^\infty} - K r_k^{(p-n)/p} \quad \text{and} \quad r_k \to \infty, \tag{4.4.24}$$

where $K = cM_*$ and c is the constant of (4.2.3). For every $y \in \mathbb{R}^n$, we have

$$|u_k(0)| - |u_k(y)| \leq |u_k(0) - u_k(y)|.$$

Hence by (4.2.3), we obtain

$$|u_k(0)| - |u_k(y)| \leq K|y|^{(p-n)/p}.$$

From (4.4.21), we get

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K|y|^{(p-n)/p}.$$

For $|y| \leq r_k$ and (4.4.24), we have

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K r_k^{(p-n)/p} \geq r_*. \tag{4.4.25}$$

From (4.4.20) and (4.4.25), we get

$$\int_{\mathbb{R}^n} V(u_k)\,dx \geq \int_{B(0,r_k)} V(u_k)\,dx \geq \frac{\nu}{2}\,\mathrm{meas}(B(0,r_k)).$$

This implies that $\int_{\mathbb{R}^n} V(u_k)\,dx \to \infty$ as $r_k \to \infty$.

In subcase (b), we assume there exists $\bar{M} > 0$ such that $\|u_k\|_{L^\infty} \leq \bar{M}$. From (4.4.16), we obtain

$$\int_{\mathbb{R}^n} V(u_k)\,dx \geq \omega_{\bar{M}} \|u_k\|_{L^2} \to \infty \quad \text{as } k \to \infty. \ \blacksquare$$

Corollary 4.4.5. *For every $b > 0$, there exists $d = d(b)$ such that, for every $u \in \Lambda_a$,*

$$f_a(u) \le b \;\Rightarrow\; \min_{x\in\mathbb{R}^n} |u(x) - \eta| \ge d.$$

Proof. Towards a contradiction, assume that there exist $b > 0$ and a sequence $(u_k) \subset \Lambda_a$ such that $f_a(u_k) \le b$ and

$$\min_{x\in\mathbb{R}^n} |u_k(x) - \eta| < 1/k. \tag{4.4.26}$$

For every $k \in \mathbb{N}$, by Remark 4.2.1, there exists $x_k \in \mathbb{R}^n$ such that

$$|u(x_k) - \eta| = \min_{x\in\mathbb{R}^n} |u(x) - \eta|.$$

Then we can consider the sequence $\psi_k = u(\cdot + x_k)$. Since

$$f_a(\psi_k) = f_a(u_k) \le b, \tag{4.4.27}$$

we see that $(f_a\psi_k)$ is bounded in E_a. Then, up to a subsequence, $(f_a\psi_k)$ weakly converges to some ψ.

Now, from the definition of ψ_k and (4.4.26), we obtain

$$\psi(0) = \lim_{k\to\infty} \psi_k(0) = \eta.$$

Thus $\psi \in \partial\Lambda_a$. Taking into account (4.4.27) we have a contradiction with Proposition 4.4.1. ■

Lemma 4.4.6. *The functional f_a is weakly lower semicontinuous in Λ_a.*

Proof. Let $u \in \Lambda_a$ and let a sequence $(u_k) \subset \Lambda_a$ weakly converge to u. We show that

$$\liminf_{k\to\infty} f_a(u_k) \ge f_a(u).$$

The result is obvious when $\liminf_{k\to\infty} f_a(u_k) = \infty$, and so we consider

$$f_a(u_k) = \underbrace{\frac{a}{2}\|\nabla u_k\|_{L^2}^2 + \frac{b}{2}\|\nabla u_k\|_{L^p}^p}_{\text{A}} + \underbrace{\int_{\mathbb{R}^n} V(u_k)\,dx}_{\text{B}}.$$

Part A is convex and strongly continuous, hence is weakly lower semicontinuous (see [30, Remark 6, p. 61]).

Now we have to study part B. Since (u_k) converges to u uniformly on every compact set, for a fixed sphere $B_R(0)$ we have

$$\lim_{k\to\infty} \int_{B_R(0)} V(u_k)\,dx = \int_{B_R(0)} V(u)\,dx.$$

On the other hand, since V is nonnegative, we have

$$\liminf_{k\to\infty} \int_{\mathbb{R}^n} V(u_k)\,dx \ge \liminf_{k\to\infty} \int_{B_R(0)} V(u_k)\,dx = \int_{B_R(0)} V(u)\,dx,$$

and taking the limit as $R \to \infty$, we obtain

$$\liminf_{k\to\infty} \int_{\mathbb{R}^n} V(u_k)\,dx \ge \int_{\mathbb{R}^n} V(u)\,dx. \;\blacksquare$$

Proposition 4.4.7. *There exists $\Delta_a > 0$ such that, for every $u \in \Lambda_a$ satisfying $\|u\|_{L^\infty} \ge 1$,*

$$f_a(u) \ge \Delta_a.$$

Proof. By continuous injection in Proposition 4.2.1,

$$\|u\|_a \geq \|u\|_{L^\infty} \geq 1,$$

and by the coercivity of f_a, we get

$$\|u\|_a \geq 1 \;\Rightarrow\; \exists \Delta_a > 0,\ f_a(u) \geq \Delta_a. \;\blacksquare$$

4.5. Nontrivial solution. We have $\Lambda_a = \bigcup_{q\in\mathbb{Z}} \Lambda_a^q$, and the natural idea is to minimize E_a on each Λ_a^q. Unfortunately, in this approach a number of problems arise: Λ_a^q is not weakly closed, the operator Δ_p is not weakly continuous, and the concentration-compactness methods cannot be applied directly. These difficulties can be overcome if we are able to “localize” the charge, so that every bump has its own charge, and the charge of any configuration equals the sum of the bumps’ charges.

4.5.1. The splitting lemma. The proof of our main result is based on the following proposition, in the spirit of the concentration-compactness principle for unbounded domains (see [13, 59]).

PROPOSITION 4.5.1 (Splitting Proposition). *Let* $(u_k) \subset \Lambda_a^*$ *be a sequence and* M *be a positive real number such that*

$$f_a(u_k) \leq M. \tag{4.5.1}$$

Then there exists $l \in \mathbb{N}$ *such that*

$$1 \leq l \leq M/\Delta_a, \tag{4.5.2}$$

where Δ_a *was introduced in Proposition* 4.4.7, *and there exist* $\bar{u}_1, \dots, \bar{u}_l \in \Lambda_a$, $(x_k^1), \dots, (x_k^l) \subset \mathbb{R}^n$, *and* $R_1, \dots, R_l$ *such that, up to a subsequence,*

$$u_k(\cdot + x_k^i) \rightharpoonup \bar{u}_i; \tag{4.5.3}$$

$$\|\bar{u}_i\|_{L^\infty} \geq 1; \tag{4.5.4}$$

$$|x_k^i - x_k^j| \to \infty, \qquad i \neq j; \tag{4.5.5}$$

$$\sum_{i=1}^{l} f_a(\bar{u}_i) \leq \liminf_{k\to\infty} f_a(u_k); \tag{4.5.6}$$

$$\forall x \in \mathbb{R}^n \setminus \bigcup_{i=1}^{l} B^i(u_k), \qquad |u_k(x)| \leq 1. \tag{4.5.7}$$

Moreover,

$$\operatorname{ch}(u_k) = \sum_{i=1}^{l} \operatorname{ch}(\bar{u}_i), \tag{4.5.8}$$

$$\limsup_{k\to\infty} \Big\| u_k - \sum_{i=1}^{l} \bar{u}_i(\cdot - x_k^i) \Big\|_{L^\infty} \leq 1. \tag{4.5.9}$$

Proof. The proof is divided into two parts. First, by an iterative procedure, we prove the existence of $l \in \mathbb{N}$, $\bar{u}_1, \dots, \bar{u}_l \in \Lambda_a$, $(x_k^1), \dots, (x_k^l) \subset \mathbb{R}^n$, and $R_1, \dots, R_l$ such that (4.5.2)–(4.5.7) are satisfied. Then from these properties we shall easily deduce (4.5.8) and (4.5.9).

For simplicity, whenever necessary, we shall tacitly consider a subsequence of u_k. First of all, we arbitrarily choose $\gamma \in \,]0,1[$. Let $x_k^1 \in \mathbb{R}^n$ be a maximum point for $|u_k|$ (see Remark 4.3.1). We have $|u_k(x_k^1)| > 1$. We set

$$u_k^1 = u_k(\cdot + x_k^1)$$

and we obtain

$$\|u_k^1\|_\infty = |u_k^1(0)| > 1. \tag{4.5.10}$$

Since $f_a(u_k^1) = f_a(u_k)$ and the functional f_a is coercive, the sequence (u_k^1) is bounded in E_a, and we have

$$u_k^1 \rightharpoonup \bar{u}_1 \in E_a. \tag{4.5.11}$$

From (4.5.10) and (4.2.2), it follows that

$$\|\bar{u}_1\|_\infty \geq 1. \tag{4.5.12}$$

Since $u_k^1 \subset \Lambda_a$ and $f_a(u_k^1)$ is bounded, by (4.5.11) and Lemma 4.4.1 we get $\bar{u}_1 \in \Lambda_a$.

Since f_a is weakly lower semicontinuous, we have

$$f_a(\bar{u}_1) \leq \liminf_{k\to\infty} f_a(u_k^1) = \liminf_{k\to\infty} f_a(u_k). \tag{4.5.13}$$

We set $\bar{u}_1 = (\bar{u}_{01}, \tilde{\bar{u}}_1) \in \mathbb{R} \times \mathbb{R}^n$. Now, using (4.2.4), we consider $R_1 > 0$ such that

$$\forall x \in \mathbb{R}^n \setminus B(0, R_1), \qquad |\bar{u}_1(x)| \leq \gamma; \tag{4.5.14}$$

for simplicity we set $B_k^1 = B(x_k^1, R_1)$.

Now we distinguish two cases: either

(A1) for k sufficiently large

$$\forall x \in \mathbb{R}^n \setminus B_k^1, \qquad |u_k(x)| \leq 1;$$

(B1) possibly after passing to a subsequence,

$$\exists x \in \mathbb{R}^n \setminus B_k^1, \qquad |u_k(x)| > 1.$$

In case (A1) the first part of the proposition is proved with $l = 1$. Now, let us consider case (B1). Let x_k^2 be a maximum point for $|u_k|$ in $\mathbb{R}^n \setminus B_k^1$. We have $u_k(x_k^2) > 1$. We set

$$u_k^2 = u_k(\cdot + x_k^2)$$

and we obtain

$$\|u_k^2\|_\infty = |u_k^2(0)| > 1.$$

Just as for (u_k^1), we have

$$u_k^2 \rightharpoonup \bar{u}_2 \in \Lambda_a, \tag{4.5.15}$$

with

$$\|\bar{u}_2\|_\infty \geq 1. \tag{4.5.16}$$

Now we have to show that

$$|x_k^1 - x_k^2| \to \infty. \tag{4.5.17}$$

We set

$$y_k = x_k^1 - x_k^2,$$

and towards a contradiction we assume that the sequence (y_k) is bounded in $\mathbb{R}^n$. Then, up to subsequence, we have $y_k \to \tilde{y}$.

Since $|y_k| = |x_k^1 - x_k^2| \geq R_1$, we get $|\tilde{y}| \geq R_1$; then, using (4.5.14),

$$|\bar{u}_1(\tilde{y})| \leq \gamma < 1. \tag{4.5.18}$$

On the other hand,

$$1 \leq |u_k(x_k^2)| = |u_k(y_k + x_k^1)| = |u_k^1(y_k)|.$$

Then, by (4.5.18),

$$\begin{aligned} 0 < 1 - |\bar{u}_1(\tilde{y})| &\leq |u_k^1(y_k)| - |\bar{u}_1(\tilde{y})| \leq |u_k^1(y_k) - \bar{u}_1(\tilde{y})| \\ &\leq |u_k^1(y_k) - \bar{u}_1(y_k)| + |\bar{u}_1(y_k) - \bar{u}_1(\tilde{y})| \\ &\leq \Big(\sup_{|y-\tilde{y}|\leq 1} |u_k^1(y) - \bar{u}_1(y)| \Big) + |\bar{u}_1(y_k) - \bar{u}_1(\tilde{y})|. \end{aligned}$$

By (4.2.3), $\bar{u}_1$ is continuous and by (4.2.2) we have locally a compact injection $E_a \subset L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Then letting $k \to \infty$ we get a contradiction, $0 < 1 - |\bar{u}_1(\tilde{y})| \leq 0$.

Now we show that

$$f_a(\bar{u}_1) + f_a(\bar{u}_2) \leq \lim_{k\to\infty} f_a(u_k). \tag{4.5.19}$$

For simplicity we set, for all $u \in \Lambda_a$ and $A \subset \mathbb{R}^n$,

$$f_{a/A}(u) = \int_A \Big(\frac{a}{2}|\nabla u|^2 + \frac{1}{2}|\nabla u|^p + V(u) \Big)\, dx.$$

Since f_a is continuous and $f_a(0) = 0$, and by (4.2.4), for a fixed $\eta > 0$ there exists $\rho > 0$ such that

$$f_{a/C_{B_\rho(0)}}(\bar{u}_1) < \eta/2 \quad \text{and} \quad f_{a/C_{B_\rho(0)}}(\bar{u}_2) < \eta/2, \qquad C_{B_\rho(0)} = \mathbb{R}^{n+1} \setminus B(0,\rho).$$

From (4.5.17) it follows that the spheres $B_\rho(x_k^1)$ and $B_\rho(x_k^2)$ are disjoint for k sufficiently large. Then we obtain

$$\begin{aligned} \liminf_{k\to\infty} f_a(u_k) &\geq \liminf_{k\to\infty} \big(f_{a/C_{B_\rho(x_k^1)}}(u_k) + f_{a/C_{B_\rho(x_k^2)}}(u_k) \big) \\ &\geq \liminf_{k\to\infty} f_{a/C_{B_\rho(x_k^1)}}(u_k) + \liminf_{k\to\infty} f_{a/C_{B_\rho(x_k^2)}}(u_k) \\ &= \liminf_{k\to\infty} f_{a/C_{B_\rho(0)}}(u_k^1) + \liminf_{k\to\infty} f_{a/C_{B_\rho(0)}}(u_k^2) \\ &\geq f_{a/C_{B_\rho(0)}}(\bar{u}_1) + f_{a/C_{B_\rho(0)}}(\bar{u}_2) > f_a(\bar{u}_1) + f_a(\bar{u}_2) - \eta. \end{aligned}$$

From the arbitrariness of η, we get (4.5.19).

Finally, just as for $\bar{u}_1$, from (4.2.4) we get $R_2 > 0$ such that

$$\forall x \in C_{B_{R_2}(0)}, \qquad |\bar{u}_2(x)| \leq \gamma,$$

and we set $B_k^2 = B_{R_2}(x_k^2)$. Also in this second step we have an alternative: either

(A2) for k sufficiently large

$$\forall x \in C_{B_k^1 \cup B_k^2}, \qquad |u_k(x)| \leq 1,$$

or

(B2) possibly after passing to a subsequence,

$$\exists x \in C_{B_k^1 \cup B_k^2}, \qquad |u_k(x)| > 1.$$

If case (A2) holds true, the first part of the proposition is proved with $l = 2$; in case (B2) we consider a maximum point of $|u_k|$ in $C_{B_k^1 \cup B_k^2}$ and we repeat the same argument used in case (B1). This alternative process terminates after a finite number of steps. Indeed, now we prove (4.5.2).

From (4.5.12) and (4.5.16),

$$\|\bar{u}_i\|_\infty \geq 1 \quad i = 1, \ldots, l,$$

and with Proposition 4.4.7, we get $f_a(\bar{u}_i) \geq \Delta_a > 0$. Then from (4.5.1) and (4.5.6),

$$l \cdot \Delta_a \leq \sum_{i=1}^{l} f_a(\bar{u}_i) \leq \liminf_{k \to \infty} f_a(u_k) \leq M.$$

So, we get (4.5.2). We notice that this estimate is independent of the sequence (u_k).

Now we prove (4.5.8). We consider k sufficiently large such that (4.5.7) holds and

$$B_k^i \cap B_k^j = \emptyset \quad \text{for } i \neq j. \tag{4.5.20}$$

By using the same arguments used in Proposition 4.3.1 we have, $K_{u_k} \subset \bigcup_{i=1}^{l} B_k^i$, and

$$\begin{aligned} \mathrm{ch}(u_k) &= \deg\Big(\tilde{u}_k, \bigcup_{i=1}^{l} B_k^i, 0\Big) = \sum_{i=1}^{l} \deg(\tilde{u}_k, B_k^i, 0) \\ &= \sum_{i=1}^{l} \deg(\tilde{u}_k^i, B_{R_i}(0), 0). \end{aligned} \tag{4.5.21}$$

On the other hand, for every $i \in \{1, \ldots, l\}$, since (u_k^i) converges uniformly to $\bar{u}_i$ on $B_{R_i}(0)$, we obtain

$$\deg(\tilde{u}_k^i, B_{R_i}(0), 0) = \deg(\bar{\tilde{u}}_i, B_{R_i}(0), 0); \tag{4.5.22}$$

recall that $u_k^i \rightharpoonup \bar{u}_i \in \Lambda_a$, $\bar{u}_i = (\bar{u}_{0i}, \bar{\tilde{u}}_i) \in \mathbb{R} \times \mathbb{R}^n$, $u_k^i = (u_{0k}^i, \tilde{u}_k^i) \in \mathbb{R} \times \mathbb{R}^n$.

Since $|\bar{u}_i(x)| \leq \gamma < 1$ for all $x \in C_{B_{R_i}(0)}$, by the excision property of the topological degree we have

$$\deg(\bar{\tilde{u}}_i, B_{R_i}(0), 0) = \deg(\bar{\tilde{u}}_i, K_{u_i}, 0) + \deg(\bar{\tilde{u}}_i, B_{R_i}(0) \setminus \bar{K}_{u_i}, 0).$$

Let $x \in B_{R_i}(0) \setminus \bar{K}_{\bar{u}_i}$. Then $\gamma < u_{0i}(x) \leq 1$. From the arbitrariness of γ, we get $u_{0i}(x) = 1$, and since $\{\bar{u}_i\} \subset \Lambda_a$ we have $\bar{\tilde{u}}_i \neq 0$, so by the solvability property of the topological degree,

$$\deg(\bar{\tilde{u}}_i, B_{R_i}(0) \setminus \bar{K}_{u_i}, 0) = 0;$$

thus

$$\mathrm{ch}(\bar{u}_i) = \deg(\bar{\tilde{u}}_i, B_{R_i}(0), 0). \tag{4.5.23}$$

From (4.5.21)–(4.5.23),

$$\mathrm{ch}(u_k) = \sum_{i=1}^{l} \mathrm{ch}(\bar{u}_i).$$

Finally, in order to prove (4.5.9), since u_k^i converges uniformly to $\bar{u}_i$ in $B_{R_i}(0)$ for every $i \in \{1, \ldots, l\}$, we assume that

$$\forall x \in B_i^k, \quad |u_k(x) - \bar{u}_i(x - x_i^k)| < \gamma. \tag{4.5.24}$$

We shall prove that, for k large enough,

$$\forall x \in \mathbb{R}^n, \qquad \Big|u_k(x) - \sum_{i=1}^{l} \bar{u}_i(x - x_i^k)\Big| < 1 + l\gamma. \tag{4.5.25}$$

Indeed, if $x \in \bigcup_{i=1}^{l} B_i^k$, then, by (4.5.20), there exists a unique index $j \in \{1, \dots, l\}$ such that $x \in B_j^k$, and then

$$\begin{aligned}\Big|u_k(x) - \sum_{i=1}^{l} \bar{u}_i(x - x_i^k)\Big| &\le |u_k(x) - \bar{u}_j(x - x_j^k)| + \sum_{i \ne j} |\bar{u}_i(x - x_i^k)| \\ &< \gamma + (l-1)\gamma = l\gamma < 1 + l\gamma.\end{aligned} \tag{4.5.26}$$

On the other hand, if $x \notin \bigcup_{i=1}^{l} B_i^k$, then, by (4.5.7),

$$\Big|u_k(x) - \sum_{i=1}^{l} \bar{u}_i(x - x_i^k)\Big| \le |u_k(x)| + \sum_{i=1}^{l} |\bar{u}_i(x - x_i^k)| \le 1 + l\gamma.$$

Now fix $\eta > 1$. Choosing γ sufficiently small, we have

$$1 + l\gamma < \eta. \tag{4.5.27}$$

Substituting (4.5.27) into (4.5.25), we get

$$\forall x \in \mathbb{R}^n, \qquad \Big|u_k(x) - \sum_{i=1}^{l} \bar{u}_i(x - x_i^k)\Big| < \eta,$$

and, by the arbitrariness of $\eta > 1$, we obtain (4.5.9). ■

4.5.2. Existence of minima in connected components of Λ_a. The minimum is attained on the set Λ_a, and it is easy to see that $u \equiv 0$ is a trivial solution of the problem. But of course we are interested in nontrivial solutions. Now, we consider the following problem:

$$I_* = \inf_{u \in \Lambda_a^*} f_a(u), \qquad \Lambda_a^* = \{u \in E_a : \mathrm{ch}(u) \ne 0\}.$$

The functional is bounded below and the set Λ_a^* is not empty. We consider fields u having the form

$$u(x) = \left(\frac{2}{1 + |x|^m}, \frac{1}{1 + |x|^m} x\right). \tag{4.5.28}$$

LEMMA 4.5.2. *There exists $m \ge 1$ such that the field u defined in* (4.5.28) *belongs to Λ_a^*.*

Proof. Clearly, if m is sufficiently large, then $u \in E_a$. For the sake of contradiction, suppose that there exists $\bar{x} \in \mathbb{R}^n$ such that $u(\bar{x}) = \eta = (1, 0)$. We deduce that

$$\frac{2}{1 + |\bar{x}|^m} = 1, \qquad \frac{1}{1 + |\bar{x}|^m} \bar{x} = 0.$$

We get a contradiction: $|\bar{x}| = 1$ and $\bar{x} = 0$. So, $u \in \Lambda_a$.

We show that $\mathrm{ch}(u) \ne 0$. Set $g(x) = \frac{1}{2}x$. Then

$$K_u = \left\{x \in \mathbb{R}^n : \frac{2}{1 + |x|^m} > 1\right\} = B(0, 1),$$

and if $|x| = 1$ then $g(x) = \frac{1}{1+|x|^m}x$. By the properties of the topological degree (see Theorem 2.8.1), we get

$$\deg\Big(\frac{1}{1+|x|^m}x, B(0,1), 0\Big) = \deg\big(g(x), B(0,1), 0\big) \neq 0. \ \blacksquare$$

Moreover, the set Λ_a^* is open in E_a; indeed, the topological charge is continuous with respect to uniform convergence (see Theorem 4.3.2), and the continuity of the embedding of E_a in L^∞ (see Proposition 4.2.1) ensures that the topological charge is continuous on Λ_a.

THEOREM 4.5.3. *Let $a \geq 0$, $b > 0$, and $p > n > 2$. If V satisfies* (V1)–(V5)*, then there exists a weak solution of* (4.1.10) (*i.e., a static solution of* (4.1.8)) *which is a minimizer of the energy functional* (4.1.12) *in the class of maps whose topological charge is nonzero.*

Proof. By Lemma 4.4.2 and Proposition 4.4.7, we have

$$\forall u \in \Lambda_a^*, \qquad 0 < \Delta_a \leq f_a(u) < \infty.$$

We consider a minimizing sequence $(u_k) \subset \Lambda_a^*$. It obviously has bounded energy; then we can apply Proposition 4.5.1. There exist $l \in \mathbb{N}$ and $\bar{u}_1, \ldots, \bar{u}_l \in \Lambda_a$ such that, up to a subsequence, (4.5.6) and (4.5.8) hold true. Since $\mathrm{ch}(u_k) \neq 0$, from (4.5.8) we deduce that there exists $\bar{i} \in \{1, \ldots, l\}$ such that $\mathrm{ch}(\bar{u}_{\bar{i}}) \neq 0$. Then, from (4.5.6),

$$I_* \leq f_a(\bar{u}_{\bar{i}}) \leq \sum_{i=1}^{l} f_a(\bar{u}_i) \leq \liminf_{k\to\infty} f_a(u_k) = I_*.$$

So we conclude that $f_a(\bar{u}_{\bar{i}}) = I_*$. Moreover, since Λ_a^* is an open set, there exists a weak solution of (4.1.10) (i.e., a static solution of (4.1.8)). Hence we deduce a solution of (4.1.8) having the form (4.1.11). ■

REMARK 4.5.1. The functional exhibits an invariance under rotations and translations; indeed, for every function u and $g \in O(n)$, if we set $u_g(x) = u(gx)$, we immediately get

$$f_a(u_g) = f_a(u).$$

Then our theorem gives the existence of an orbit of minimum solutions. This orbit consists of two connected components, which are identified, respectively, by $\bar{u}$ and

$$\bar{u} \circ \mathcal{P}(x) = \bar{u}(-x).$$

Since typically $n = 3$ is odd, $\bar{u} \circ \mathcal{P}$ and $\bar{u}$ have opposite topological charge.

4.5.3. Resolution of the static equation. In this subsection we prove the existence of a solution for the static equation.

THEOREM 4.5.4. *The minimum points $u \in \Lambda_a$ for the functional f_a are weak solutions of the system* (4.1.10)*.*

Proof. Let u be a minimum point of f_a and $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Let e_j denote the jth vector of the canonical basis in $\mathbb{R}^n$. If ϵ is sufficiently small, then $u + \epsilon e_j h \in \Lambda_a$ and $f_a(u + \epsilon e_j h) < \infty$. Since u is a minimum point of f_a, we have

$$0 = \frac{df(u+\epsilon e_j h)}{d\epsilon}\bigg|_{\epsilon=0} = \int_{\mathbb{R}^n} \Big(a\nabla u_j \nabla h + b\frac{p}{2}(|\nabla u|^{p-2}\nabla u_j \nabla h) + \frac{\partial V(\xi)}{\partial \xi_j} h\Big)\, dx.$$

By Green's formula,

$$\int_{\mathbb{R}^n} b\frac{p}{2}(|\nabla u|^{p-2}\nabla u_j \nabla h)\,dx = -\int_{\mathbb{R}^n} b\frac{p}{2}\,\mathrm{div}(|\nabla \cdot u|^{p-2}\nabla u_j)h\,dx.$$

So

$$\int_{\mathbb{R}^n}\left(-a\Delta u_j - b\frac{p}{2}\,\mathrm{div}(|\nabla \cdot u|^{p-2}\nabla u_j) + \frac{\partial V(\xi)}{\partial \xi_j}\right)h\,dx = 0$$

for $1 \le j \le n+1$, and for any $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Then

$$\int_{\mathbb{R}^n}\left[-a\Delta u - b\frac{p}{2}\Delta_p u + V'(u)\right]\phi\,dx = 0 \quad \text{for every } \phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

This implies by density

$$-a\Delta u - \frac{b}{2}\Delta_p u + V'(u) = 0. \blacksquare$$

4.6. Compactness properties related to symmetry. We fix $a \in \mathbb{R}_+ \setminus \{0\}$; for the sake of simplicity we assume $a = 1$; so hereafter in the notation we omit the subscript a. We consider the completion E of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_E = \|\nabla u\|_{L^2} + \|\nabla u\|_{L^p} + \|u\|_{L^2}.$$

By Corollary 4.2.3, the space E coincides with

$$W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

In the space E, we can consider the following $O(n)$-action: for every $u \in E$ and $g \in O(n)$,

$$T_g u(x) = (u_0(gx), g^{-1}\tilde{u}(gx)). \tag{4.6.1}$$

We see below that if the potential V has a suitable symmetry property, it is possible to prove the existence of infinitely many finite energy solutions. More precisely, we assume:

(V6) There exist ρ_1 and $r > 1$ such that

$$|V'(\xi) - V''(0)\xi| \le c_0|\xi|^r \quad \text{whenever } |\xi| \le \rho_1.$$

(V7) For every $\xi = (\xi_0, \tilde{\xi})$ and every $g \in O(n)$,

$$V(\xi_0, g\tilde{\xi}) = V(\xi_0, \tilde{\xi}).$$

An easy calculation and assumption (V7) give the following lemma.

Lemma 4.6.1. *The open set Λ and the functional f are invariant under the action* (4.6.1)*; that is, for every $g \in O(n)$ and $u \in \Lambda$, we have*

$$T_g(u) \in \Lambda, \qquad f(T_g(u)) = f(u).$$

Now, let F denote the subspace of fixed points,

$$F = \{u \in E : T_g u = u,\ \forall g \in O(n)\}.$$

We shall show that

$$\Lambda_F = \Lambda \cap F$$

is a natural constraint to finding the critical points of f. This means that any $u \in \Lambda_F$ such that

$$\langle f'(u), v\rangle = 0 \quad \text{for any } v \in F$$

gives $f'(u) = 0$ (see Lemma 4.6.2). This fact is usual in Hilbert spaces, but unfortunately E is only a Banach space. Moreover, we shall need a continuous projection $\mathbf{P} : E \to F$. We can define $\mathbf{P}$ by using the $O(n)$-action. For every $u \in E$, we set

$$\mathbf{P}u = \int_{O(n)} T_g u \, dg, \tag{4.6.2}$$

dg being the Haar measure on the group $O(n)$. This map $\mathbf{P}$ is continuous and takes its values in F; moreover we have $\mathbf{P} \circ \mathbf{P} = \mathbf{P}$. So we conclude that $\mathbf{P}$ is a projection of E onto F and F is a closed subspace.

Lemma 4.6.2. *For all $u \in \Lambda_F$ and $v \in E$ we have*

$$\langle f'(u), v \rangle = \langle f'(u), \mathbf{P}v \rangle,$$

$\mathbf{P}$ being the projection of E onto F.

Proof. Since the functional f is invariant, the map $f' : E \to E'$ is "equivariant," that is,

$$\langle f'(u), T_g v \rangle = \langle f'(T_{g^{-1}} u), v \rangle. \tag{4.6.3}$$

Now we recall that the integral commutes with continuous linear forms, and so

$$\begin{aligned} \langle f'(u), \mathbf{P}v \rangle &= \left\langle f'(u), \int_{O(n)} T_g v \, dg \right\rangle = \int_{O(n)} \langle f'(u), T_g v \rangle \, dg \\ &= \int_{O(n)} \langle f'(T_{g^{-1}} u), v \rangle \, dg \quad \text{by (4.6.3)} \\ &= \int_{O(n)} \langle f'(u), v \rangle \, dg \quad \text{since } u \in F \\ &= \langle f'(u), v \rangle \int_{O(n)} dg = \langle f'(u), v \rangle, \end{aligned}$$

where the last equality follows from the fact that $\int_{O(n)} dg = 1$.

From this lemma we deduce that every local minimum of f restricted to Λ_F is a critical point of f. ■

Proposition 4.6.3. *The space F, equipped with the norm $\| \cdot \|_E$, is compactly embedded in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$ for every $s \in]2, 2^*[$, where*

$$2^* = \begin{cases} \infty & \text{if } n = 2, \\ 2n/(n-2) & \text{if } n > 2. \end{cases} \tag{4.6.4}$$

This is an easy consequence of the following theorem, which is proved in Appendix A.

Theorem 4.6.4. *If $\mathcal{W}$ is a bounded subset of $W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$, then*

$$\mathcal{W}_R = \{u \in \mathcal{W} : |u| \text{ is a radial function}\}$$

is relatively compact in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$ for every $s \in]2, 2^*[$.

Proof of Proposition 4.6.3. We have to prove that every bounded set in F is relatively compact in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$. To prove this, we can employ Theorem 4.6.4; we have only to notice that, for every $u \in F$, $|u|$ is a radial function. If $u = (u_0, \tilde{u}) \in F$, the function

u_0 is $O(n)$-invariant, and the field u is $O(n)$-equivariant; that is, for every $x \in \mathbb{R}^n$ and $g \in O(n)$,

$$u_0(gx) = u_0(x), \qquad \tilde{u}(gx) = g\tilde{u}(x).$$

So we have

$$|u(gx)|^2 = |u_0(gx)|^2 + |\tilde{u}(gx)|^2 = |u_0(x)|^2 + |g\tilde{u}(x)|^2 = |u_0(x)|^2 + |\tilde{u}(x)|^2 = |u(x)|^2.$$

This means that $|u(x)|^2$ depends only on $|x|$. ∎

PROPOSITION 4.6.5. *The functional* $f|_{\Lambda_F}$ *satisfies the Palais–Smale condition, i.e., every sequence* $(u_k) \subset \Lambda_F$ *such that*

(a) $(f(u_k))$ *is bounded, and*
(b) $(f'|_{\Lambda_F}(u_k))$ *converges to* 0 *in* F'

contains a convergent subsequence. We remark that (b) *means that*

$$\langle f'(u_k), v\rangle \le \varepsilon_k \|v\| \quad \textit{for every } v \in F,$$

where $\varepsilon_k \to 0$.

In the proof of the proposition, we need the following lemmas.

LEMMA 4.6.6. *The map* $\mathcal{A} : E \to E'$ *defined by*

$$\begin{aligned}\langle \mathcal{A}u, v\rangle &= \langle -\Delta u - \Delta_p u + V''(0)u, v\rangle \\ &= \int_{\mathbb{R}^n} \big((\nabla u|\nabla v) + |\nabla u|^{p-2}(\nabla u|\nabla v) + V''(0)u.v\big)\, dx\end{aligned}$$

is invertible with continuous inverse.

The proof is given in Appendix B.

LEMMA 4.6.7. *For all* $u \in F$ *and* $v \in E$,

$$\langle \mathcal{A}u, v\rangle = \langle \mathcal{A}u, \mathbf{P}v\rangle. \tag{4.6.5}$$

Proof. First we notice that if $u \in F$, then for every $g \in O(n)$,

$$\langle \mathcal{A}u, T_g v\rangle = \langle \mathcal{A}u, v\rangle.$$

Now, since the integral commutes with continuous linear forms, we have

$$\langle \mathcal{A}u, \mathbf{P}v\rangle = \left\langle \mathcal{A}u, \int_{O(n)} T_g v\, dg \right\rangle = \int_{O(n)} \langle \mathcal{A}u, v\rangle\, dg = \langle \mathcal{A}u, v\rangle. \ ∎$$

Proof of Proposition 4.6.5. Let (u_k) be a sequence in Λ_F such that

$$f(u_k) \text{ is bounded,} \tag{4.6.6}$$

$$f'(u_k) = -\Delta u_k - \Delta_p u_k + V'(u_k) \to 0 \quad \text{in } F'. \tag{4.6.7}$$

By (4.6.6), since the functional f is coercive, the sequence (u_k) is bounded in E. First, we shall prove that, up to a subsequence, (u_k) is strongly convergent to $u \in E$. Using the operator $\mathcal{A}$ we can write

$$\mathcal{A}(u_k) = f'(u_k) - U'(u_k), \tag{4.6.8}$$

where

$$U(\xi) = V(\xi) - V''(0)\xi \cdot \xi.$$

Using (4.6.6) and Corollary 4.4.5, we deduce that there exists $d > 0$ such that, for every $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$|u_k(x) - \eta| \geq d. \tag{4.6.9}$$

From (4.6.9), since (u_k) is bounded in L^∞, we deduce that, for a suitable $M > 0$,

$$|U'(u_k)| \leq M. \tag{4.6.10}$$

Now we set

$$A_k = \{x \in \mathbb{R}^n : |u_k| \geq \rho_1\},$$

where ρ_1 is introduced in (V6). Since (u_k) is bounded in L^2, the measure of A_k is uniformly bounded.

Since $r > 1$ (see (V6)), we can find s such that

$$(2^*)' < s' < 2 \leq rs',$$

where

$$(2^*)' = \begin{cases} 1 & \text{if } 2^* = \infty, \\ 2^*/(2^* - 1) & \text{if } 2^* < \infty. \end{cases}$$

Now, by (V6), we have, for every $x \in \mathbb{R}^n \setminus A_k$,

$$U'(u_k(x)) \leq C_0 |u_k(x)|^r,$$

and using (4.6.10), we get

$$\begin{aligned} \int_{\mathbb{R}^n} |U'(u_k)|^{s'} \, dx &= \int_{\mathbb{R}^n \setminus A_k} |U'(u_k)|^{s'} \, dx + \int_{A_k} |U'(u_k)|^{s'} \, dx \\ &\leq \int_{\mathbb{R}^n \setminus A_k} C_0^{s'} |u_k|^{rs'} + M^{s'} \operatorname{meas}(A_k) \\ &\leq M_1 \|u_k\|_{L^{rs'}}^{rs'} + M_2. \end{aligned} \tag{4.6.11}$$

We know that (u_k) is bounded in E; then Corollary 4.2.3 implies that (u_k) is bounded in $L^{rs'}(\mathbb{R}^n, \mathbb{R}^{n+1})$. So, from (4.6.11), we deduce the boundedness of $U'(u_k)$ in $L^{s'}(\mathbb{R}^n, \mathbb{R}^{n+1})$. Since $(2^*)' < s' < 2$, we have

$$2 < s = \frac{s'}{s' - 1} < 2^*,$$

and from Proposition 4.6.3, $U'(u_k)$ is compactly embedded into L^s. Then, since $U'(u_k)$ is bounded in $L^{s'}$ and by Theorem 7.3.2 (Schauder's theorem), $(U'(u_k))$ is strongly convergent in F' (up to a subsequence). So,

$$f'(u_k) - U'(u_k) \to \chi \in F'. \tag{4.6.12}$$

Now we can define $\mathcal{P}\chi \in E'$ by setting

$$\langle \mathcal{P}\chi, \omega \rangle = \langle \chi, \mathbf{P}\omega \rangle \qquad \text{for every } \omega \in E, \tag{4.6.13}$$

where $\mathbf{P}$ is defined in (4.6.2). Now we want to prove that

$$\mathcal{A}u_k \to \mathcal{P}\chi$$

in E'; indeed, using (4.6.5), (4.6.13) and (4.6.8), for every $\omega \in E$ we have

$$\langle \mathcal{A}u_k - \mathcal{P}\chi, \omega \rangle = \langle \mathcal{A}u_k - \chi, \mathbf{P}\omega \rangle = \langle f'(u_k) - U'(u_k) - \chi, \mathbf{P}\omega \rangle \to 0.$$

Using Lemma 4.6.6, we deduce

$$u_k = \mathcal{A}^{-1}\mathcal{A}u_k \to \mathcal{A}^{-1}\mathcal{P}\chi = u.$$

Finally we have $u \in \Lambda_F$: in fact, $u \in F$ since F is closed, and $u \in \Lambda$ by Lemma 4.4.1 and (4.6.6). ■

4.7. Infinitely many solutions. In this section, we prove under some symmetry assumptions the existence of infinitely many solutions, which are constrained minima of the energy. More precisely, for every $N \in \mathbb{N}$ there exists a solution of charge N.

THEOREM 4.7.1. *Assume* $p > n \geq 2$ *and* $a > 0$. *Assume that* V *satisfies* (V1)–(V7). *Then, for any* $N \in \mathbb{N}$, *there exists a solution* u_N *of* (4.1.10) *such that* $\mathrm{ch}(u_N) = N$. *Moreover,*

$$\lim_N f(u_N) = \infty.$$

For the proof of Theorem 4.7.1 we shall prove the following statements:

(A) for every $N \geq 1$, the connected component

$$\Lambda_F^N = \{u \in \Lambda_F : \mathrm{ch}(u) = N\}$$

is not empty;

(B) for every $N \geq 1$, the energy functional attains its minimum in Λ_F^N;

(C) if we denote by u_N a minimizer of the energy in Λ_F^N, then

$$\lim_N f(u_N) = \infty.$$

4.7.1. Symmetric fields with arbitrary charge. This subsection is devoted to the proof of statement (A). We shall give a complete proof of

$$\Lambda_F^N \neq \emptyset \tag{4.7.1}$$

in the case of N odd; the case of N even is analogous.

To this end, we shall study suitable fields in F having the form

$$u(x) = \big(A(|x|), B(|x|)x\big), \tag{4.7.2}$$

A and B being two scalar fields such that $u \in \Lambda$. Indeed, an easy calculation shows that fields having the form (4.7.2) are fixed points for the action (4.6.1), so they belong to Λ_F. More precisely, we consider fields u having the form (4.7.2) with

$$A(|x|) = \frac{a}{1 + (|x|/2\pi)^m} \cos|x|, \tag{4.7.3}$$

$$B(|x|) = \frac{1}{1 + (|x|/2\pi)^m} \sin|x|, \tag{4.7.4}$$

and we show that $u \in \Lambda_F$.

LEMMA 4.7.2. *There exists an* $m \geq 1$ *such that, for every* $a \in \mathbb{R} \setminus \mathbb{Q}$, *the field* u *defined by* (4.7.2)–(4.7.4) *belongs to* Λ (*by the above remark, it belongs to* Λ_F).

Proof. By an easy calculation in polar coordinates, and for m sufficiently large, we find that the field u defined by (4.7.2)–(4.7.4) belongs to E. Now we assume $a \in \mathbb{R} \setminus \mathbb{Q}$ and we prove that $u \in \Lambda$.

We have to show that, for every $x \in \mathbb{R}^n$,

$$u(x) \neq \eta = (1, 0).$$

For the sake of contradiction, suppose that there exists $\bar{x} \in \mathbb{R}^n$ such that $u(\bar{x}) = (1, 0)$. Using the definition of u, we deduce that

$$A(|\bar{x}|) = \frac{a}{1 + (|\bar{x}|/2\pi)^m} \cos |\bar{x}| = 1, \tag{4.7.5}$$

$$B(|\bar{x}|) = \frac{a}{1 + (|\bar{x}|/2\pi)^m} \sin |\bar{x}| = 0. \tag{4.7.6}$$

From (4.7.6) we deduce that $|\bar{x}| = k\pi$, $k \in \mathbb{N}$; then from (4.7.5) we get

$$\pm \frac{a}{1 + (k/2)^m} = 1,$$

and this contradicts $a \in \mathbb{R} \setminus \mathbb{Q}$. So $\Lambda \neq \emptyset$. ∎

PROPOSITION 4.7.3. *Let $N = 2L + 1$ ($L \in \mathbb{N}$). Then any field u of type* (4.7.2)–(4.7.4), *with m as in Lemma* 4.7.2 *and $a \in \mathbb{R} \setminus \mathbb{Q}$ such that*

$$1 + L^m < a < 1 + (L + 1)^m \tag{4.7.7}$$

has charge equal to N.

Proof. By the definition of charge (see Definition 4.3.1), we have to prove that

$$\deg(\tilde{u}, K_u, 0) = 2L + 1,$$

where

$$\tilde{u}(x) = \frac{\sin |x|}{1 + (|x|/2\pi)^m} x \quad \text{and} \quad K_u = \left\{ x \in \mathbb{R}^n : \frac{a \cos |x|}{1 + (|x|/2\pi)^m} > 1 \right\}.$$

We fix $\epsilon \in\,]0, \pi/2[$ and set

$$K^0 = \{|x| < \epsilon\}.$$

Moreover, for every $j = 1, \dots, L$, we set

$$K^j = \{2j\pi - \epsilon < |x| < 2j\pi + \epsilon\}.$$

The open subsets $\{K^j\}_{1 \leq j \leq L}$ are disjoint.

We show that

$$K^0 \cup K^1 \cup \cdots \cup K^L \subset K_u.$$

Let $x \in K^j$. Then $2j\pi - \epsilon < |x| < 2j\pi + \epsilon$. We develop $\cos |x|$ on the ball $B(0, \epsilon)$ for ϵ small enough. We set $t = |x| \geq 0$ and write

$$\cos t = 1 - \frac{t^2}{2} + \frac{t^3}{9} R(t),$$

so that $R(t) = \sin \theta t$ with $0 < \theta < 1$. We deduce that

$$\frac{a \cos t}{1 + (t/2\pi)^m} > \frac{a(1 - t^2/2)}{1 + (j + t/2\pi)^m}.$$

So it is enough to prove that

$$\frac{a(1 - t^2/2)}{1 + (j + t/2\pi)^m} > 1.$$

We consider the function

$$s(t)=\frac{(1+t^m)(1-t^2/2)}{1+(j+t/2\pi)^m}, \tag{4.7.8}$$

where $0\le j\le L$. The function s is continuous and strictly decreasing, and so s is locally bijective with $s(0)=1$, and for t small enough we have

$$1-s(t)<\epsilon_0. \tag{4.7.9}$$

From (4.7.7) we can choose $\epsilon_0=1-(1+t^m)/a$. Then from (4.7.9) and (4.7.8),

$$\frac{a(1-t^2/2)}{1+(j+t/2\pi)^m}>1.$$

So we have

$$K^0\cup K^1\cup\cdots\cup K^L\subset K_u.$$

Moreover, using the right-hand inequality of (4.7.7), we can prove

$$K_u=\{x\in K_u:\tilde{u}(x)=0\}\subset K^0\cup K^1\cup\cdots\cup K^L.$$

So, by the excision and additivity properties of the topological degree, we conclude

$$\deg(\tilde{u},K_u,0)=\deg\Big(\tilde{u},\bigcup_{j=1}^{L}K^j,0\Big)+\underbrace{\deg\Big(\tilde{u},K_u\setminus\bigcup_{j=1}^{L}K^j,0\Big)}_{=0}=\sum_{j=1}^{L}\deg(\tilde{u},K^j,0).$$

Clearly, the conclusion will follow if we prove that

$$\deg(\tilde{u},K^0,0)=1, \tag{4.7.10}$$

$$\deg(\tilde{u},K^j,0)=2 \quad \text{for } j=1,\dots,L. \tag{4.7.11}$$

First we prove (4.7.10). Consider the function

$$v_0=\frac{\sin\epsilon}{1+(\epsilon/2\pi)^m}x.$$

We notice that for every $x\in\partial K^0$ (that is, such that $|x|=\epsilon$), we have $\tilde{u}=v_0\neq 0$. We obtain

$$\deg(v_0,K^0,0)=\sum_{x\in {v_0}^{-1}(0)}\operatorname{sgn} J_{v_0(x)}=1,$$

where the equality follows from the fact that v_0 is the identity up to a multiplicative constant.

Since the degree depends only on the values on the boundary, we conclude that

$$\deg(\tilde{u},K^0,0)=\deg(v_0,K^0,0)=1.$$

Now, for every $j\in\{1,\dots,L\}$, we set

$$B_+^j=\{|x|<j\pi+\epsilon\},\qquad B_-^j=\{|x|<j\pi-\epsilon\}.$$

Since $\overline{K^j}=\overline{B_+^j\setminus B_-^j}$, by the additivity of degree, we have

$$\deg(\tilde{u},K^j,0)=\deg(\tilde{u},B_+^j,0)-\deg(\tilde{u},B_-^j,0). \tag{4.7.12}$$

Then we consider the function

$$v_j^+(x) = \frac{\sin \epsilon}{1 + (|2j\pi + \epsilon|/2\pi)^m} x.$$

For every $x \in \partial B_j^+$, we have $\tilde{u} = v_j^+ \neq 0$. So, by the boundary dependence of the degree, we conclude, as before, that

$$\deg(\tilde{u}, B_+^j, 0) = \deg(v_j^+, B_+^j, 0) = 1. \tag{4.7.13}$$

Analogously we have

$$\deg(\tilde{u}, B_-^j, 0) = \deg(v_j^-, B_-^j, 0) = 1 \tag{4.7.14}$$

with

$$v_j^-(x) = \frac{\sin \epsilon}{1 + (|2j\pi - \epsilon|/2\pi)^m} x.$$

Substituting (4.7.13) and (4.7.14) into (4.7.12), we get (4.7.11). ■

By the preceding proposition, for every $N \geq 1$ odd, we can construct a field $u \in \Lambda_F$ having the form (4.7.2) such that $\mathrm{ch}(u) = N$.

The case of N even is analogous: We can consider again a field $u \in \Lambda_F$ having the form (4.7.2) with coefficients

$$A(|\bar{x}|) = \frac{a}{1 + (|\bar{x}|/2\pi)^m} \sin |\bar{x}| = 1, \tag{4.7.15}$$

$$B(|\bar{x}|) = -\frac{a}{1 + (|\bar{x}|/2\pi)^m} \cos |\bar{x}| = 0. \tag{4.7.16}$$

With the same choice of m as in Lemma 4.7.2, for every $L \geq 1$, we can find $a \in \mathbb{R} \setminus \mathbb{Q}$ such that the field defined by (4.7.2), (4.7.15) and (4.7.16) has charge $2L$.

4.7.2. Minimizers in Λ_F^N. We recall that for

$$\Lambda_F^N = \{u \in \Lambda_F : \mathrm{ch}(u) = N\} = \Lambda^N \cap F \neq \emptyset,$$

Λ_F^N is a connected component of Λ_F.

Fix $N \geq 1$ and consider

$$c_N = \inf_{\Lambda_F^N} f.$$

The proof of our main result is based on Theorem 2.7.2 (Ekeland's lemma), Proposition 4.6.5, Lemma 4.6.2 and Proposition 4.5.1 (splitting lemma).

We recall that, for every $u \in \Lambda$ with $\mathrm{ch}(u) \neq 0$, we have $\|u\|_{L^\infty} \geq 1$ (see Remark 4.3.1), which from Proposition 4.4.7 implies

$$f(u) \geq \Delta^* > 0. \tag{4.7.17}$$

So we conclude that

$$c_N \geq \Delta^* > 0.$$

We want to prove that the value c_N is attained in Λ_F^N.

For every $c \in \mathbb{R}$, the sublevels of f are given by

$$f^c = \{u \in \Lambda_F : f(u) \leq c\}.$$

Taking into account Lemma 4.4.1, and since F is a closed subspace, it is easy to prove that the f^c are complete in E, as well as in F. By (4.7.17), f is lower bounded on Λ_F^N, so from Theorem 2.7.2 (Ekeland's lemma) there exists a sequence $(u_k^N) \subset \Lambda_F^N$ such that

$$c_N \leq f(u_k^N) \leq c_N + \frac{1}{k}, \tag{4.7.18}$$

and

$$\forall v \in \Lambda_F^N, \qquad f(v) + \frac{1}{k}\|v - u_k^N\|_E \geq f(u_k^N). \tag{4.7.19}$$

Since f is C^1, we have

$$f(v) = f(u_k^N) + \langle f'(u_k^N), v - u_k^N \rangle + o(v - u_k^N).$$

Then by (4.7.19) we have

$$\forall v \in \Lambda_F^N, \qquad \langle f'(u_k^N), u_k^N - v \rangle \leq \frac{1}{k}\|v - u_k^N\|_E + o(v - u_k^N). \tag{4.7.20}$$

We take $v = u_k^N - \epsilon h$ such that $h \in F$ (for ϵ small enough). Then $v \in \Lambda_F^N = \Lambda^N \cap F$: indeed, F is a subspace and Λ^N is open in Λ.

So for all $h \in F$ we have

$$\frac{\langle f'(u_k^N), h \rangle}{\|h\|_E} \leq \frac{1}{k} + \frac{o(\epsilon h)}{\|\epsilon h\|_E}.$$

Then

$$f'(u_k^N) \to 0 \quad \text{ in } F', \tag{4.7.21}$$

and from (4.7.18),

$$f(u_k^N) \to c_N. \tag{4.7.22}$$

Moreover, the functional f restricted to Λ_F satisfies the Palais–Smale condition (see Proposition 4.6.5). Then by (4.7.21) and (4.7.22), up to a subsequence,

$$u_k^N \to u^N \quad \text{ in } \Lambda_F^N,$$

and since f is C^1, we have $f(u^N) = c_N$, and

$$f'(u^N) = 0 \quad \text{ in } F'. \tag{4.7.23}$$

From (4.7.23) and Lemma 4.6.2 we deduce that

$$f'(u^N) = 0 \quad \text{ in } E',$$

so u^N is a critical point of f. Then u^N is a weak solution of (4.1.10) (i.e., a static solution of (4.1.8)).

We want to show that $f(u^N) \to \infty$ as $N \to \infty$. For the sake of contradiction, assume that, up to a subsequence, $f(u^N) \leq M$. Then, by Proposition 4.5.1 (see (4.5.8)), there exists $Q \in \mathbb{N}$ such that (up to a subsequence)

$$\mathrm{ch}(u^N) = Q,$$

and this contradicts $\mathrm{ch}(u^N) = N \to \infty$.

5. Solitons in generalized Sobolev spaces

The aim of this chapter is to carry out an existence analysis of the finite-energy static solutions in more than one space dimension for a class of Lagrangian densities L which include (4.1.1) with variable exponents. We study a class of Lorentz invariant nonlinear field equations in several space dimensions. The main purpose is to obtain soliton-like solutions with variable exponents. The fields are characterized by a topological invariant, which we call the charge. We prove the existence of a static solution which minimizes the energy among the configurations with nontrivial charge. The study of PDE's with $p(x)$-growth condition has received more and more attention in recent years. The specific attention accorded to such kinds problems is due to applications in mathematical physics. More precisely, such an equation is used in electrorheological fluids [76] and in elastic mechanics [87]. They also have wide applications in different research fields (see [6,33,51] and the reference therein).

5.1. Statement of the problem. The class of Lagrangian densities is more general than in [14], with Lagrangian density with variable exponent, in such a way as to include the Lorentz invariant Lagrangian density proposed in [14]. First we introduce some notation. For n and m positive integers, we will consider, respectively, the physical space-time (typically $n = 3$) and the internal parameters space. We are interested in the multi-dimensional case, so we assume that $n \geq 2$. A point in $\mathbb{R}^{n+1}$ will be denoted by $X = (x, t)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The fields we are interested in are maps $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^m$, $\psi = (\psi_1, \ldots, \psi_m)$. We set

$$\rho = |\nabla\psi|^2 - |\psi_t|^2,$$

with $\nabla\psi$ and ψ_t denoting, respectively, the Jacobian with respect to x and the derivative with respect to t. Let

$$s : \mathbb{R}^{n+1} \to \mathbb{R}.$$

We shall consider Lagrangian densities of the form

$$\mathcal{L}(\psi, \rho) = -\tfrac{1}{2}\alpha(\rho, s) - V(\psi), \tag{5.1.1}$$

where V is a real function defined in an open subset $\Omega \subset \mathbb{R}^m$ and α is a real function defined by

$$\alpha(\rho, s) = a\rho + b|\rho|^{s(\cdot)/2}, \qquad a \geq 0,\, b > 0,\, s(0) > n. \tag{5.1.2}$$

The results of Chapter 4 were concerned with the case $s(\cdot) \equiv p$ (i.e., we fixed the variable exponent). The action functional related to (5.1.1) is

$$S(\psi) = \int_{\mathbb{R}^{n+1}} \mathcal{L}(\psi, \rho)\, dx\, dt = \int_{\mathbb{R}^{n+1}} \big(-\tfrac{1}{2}\alpha(\rho, s) - V(\psi)\big)\, dx\, dt.$$

So the Euler–Lagrange equations are

$$\frac{\partial}{\partial t}(\alpha' \psi_t) - \nabla\big(\alpha' \nabla \psi\big) + V'(\psi) = 0, \tag{5.1.3}$$

where $\nabla(\alpha'\nabla\psi)$ denotes the vector whose jth component is given by $\operatorname{div}(\alpha'\nabla\psi^j)$, and V' denotes the gradient of V. Equation (5.1.3) is Lorentz invariant. Static solutions $\psi(x,t) = u(x)$ of (5.1.3) solve the equation

$$-\nabla\big(\alpha'\nabla u\big) + V'(u) = 0. \tag{5.1.4}$$

Set $s(x,t) = p(x)$ on $\mathbb{R}^n$ (the restrictions of s on $\mathbb{R}^n$). Using (5.1.2) and (5.1.4) we obtain

$$-a\Delta u - \frac{b}{2}\Delta_{p(\cdot)} + V'(u) = 0, \tag{5.1.5}$$

where

$$\Delta_{p(\cdot)} u = \nabla\big(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u\big).$$

We introduce the following notation and function spaces:

$$C_+(\mathbb{R}^n) = \{p \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) : p(x) > 1 \text{ for all } x \in \mathbb{R}^n\}$$

and

$$p^+ = \sup_{x\in\mathbb{R}^n} p(x), \qquad p^- = \inf_{x\in\mathbb{R}^n} p(x).$$

We assume that

(p$_1$) $S(x,t) = p\big(\frac{x_1 - t\nu}{\sqrt{1-\nu^2}}, \dots, x_n\big)$, where ν is a parameter used in the Lorentz transformation,

(p$_2$) $\lim_{x\to\infty} p(x) = p_\infty = p^- > n$.

Recall that the results of Chapter 4 were concerned with the case

$$p(\cdot) \equiv p^- > n.$$

Under (p$_1$), it is easy to verify that, if $u = u(x)$ is a solution of (5.1.3) and $v = (\nu, 0, \dots, 0)$ with $|\nu| < 1$, then the field

$$\psi_\nu(x,t) = u\left(\frac{x_1 - \nu t}{\sqrt{1-\nu^2}}, x_2, \dots, x_n\right) \tag{5.1.6}$$

is a solution of (5.1.3). Notice that the function undergoes a contraction by a factor of

$$\gamma = \frac{1}{\sqrt{1-\nu^2}}$$

in the direction of motion; this is a consequence of the fact that (5.1.3) is Lorentz invariant. Clearly, (5.1.5) are the Euler–Lagrange equations with respect to the energy functional

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^2 + \frac{b}{2}|\nabla u|^{p(x)} + V(u)\right) dx, \tag{5.1.7}$$

where $m = n + 1$, so the time independent fields u are maps

$$u : \mathbb{R}^n \to \mathbb{R}^m.$$

For every $\xi \in \mathbb{R}^{n+1}$, we write $\xi = (\xi_0, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^n$. $V : \Omega \to \mathbb{R}$ where $\Omega = \mathbb{R}^{n+1} \setminus \{\eta\}$, $\eta = (1, 0)$, and V is positive and singular in η. More precisely, we assume:

(V_1) $V \in C^1(\Omega, \mathbb{R})$.
(V_2) $V(\xi) \geq V(0) = 0$.
(V_3) V is twice differentiable in 0 and the Hessian matrix $V''(0)$ is nondegenerate.
(V_4) There exist $c, \rho > 0$ such that if $|\xi| < \rho$ then

$$V(\eta + \xi) \geq c(|\xi|^{-q^+} + |\xi|^{-q^-}) \quad \text{where} \quad \frac{1}{q^-} = \frac{1}{n} - \frac{1}{p^-}, \ \frac{1}{q^+} = \frac{1}{n} - \frac{1}{p^+}.$$

(V_5) For every $\xi \in \Omega \setminus \{0\}$ we have

$$V(\xi) > 0, \qquad \liminf_{|\xi| \to \infty} V(\xi) = v > 0.$$

Example 5.1.1. A potential satisfying assumptions (V_1)–(V_5) is

$$V(\xi) = \omega_0^2 \left(|\xi|^2 + \frac{|\xi|^4}{|\xi - \eta|^{q^+} + |\xi - \eta|^{q^-}} \right).$$

Definition 5.1.1. We define a $p(\cdot)$-*soliton* to be a solution of equation (5.1.3) having the form (5.1.6), where u is a local minimum of the energy functional.

5.2. Solution space. Let $p^- > n \geq 2$; with no loss of generality, we can consider the functional (5.1.7) with $b = 1$. It will be convenient to set

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2} |\nabla u|^2 + \frac{1}{2} |\nabla u|^{p(x)} + V(u) \right) dx,$$

and define E_a to be the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_a = a \|\nabla u\|_{L^2} + \|\nabla u\|_{L^{p(\cdot)}} + \|u\|_{L^2}, \qquad p^- > n \geq 2,\ a \geq 0,$$

i.e.,

$$E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a}, \qquad \|u\|_{L^2} = \Big(\sum_{j=1}^{n+1} \|u_j\|_{L^2}^2 \Big)^{1/2}, \qquad \|\nabla u\|_{L^2} = \Big(\sum_{j=1}^{n+1} \|\nabla u_j\|_{L^2}^2 \Big)^{1/2},$$

and

$$\|\nabla u\|_{p(\cdot)} = \inf \left\{ \sigma > 0 : \int_{\mathbb{R}^n} \left| \frac{\nabla u(x)}{\sigma} \right|^{p(x)} dx \leq 1 \right\}.$$

For every $a > 0$, the norms $\| \cdot \|_a$ are equivalent, so we have to study only two cases: $a = 0$, $a > 0$.

Proposition 5.2.1. *The Banach space E_0 is continuously embedded in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$ for every $s \in [2, \infty]$.*

Proof. Since E_0 is continuously embedded in $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, it is sufficient to show that it is also embedded in $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Since $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in E_0, it is sufficient to prove that there exists $c > 0$ such that, for every $u \in C_0(\mathbb{R}^n, \mathbb{R}^{n+1})$,

$$\|u\|_{L^\infty} \leq c \|u\|_0.$$

We fix $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ and consider a family of cubes $Q_k \subset \mathbb{R}^n$ such that $\operatorname{meas}(Q_k) = 1$, $\bigcup_{k\in\mathbb{N}} Q_k = \mathbb{R}^n$. Then, by a well-known inequality (see below (5.2.4) in Proposition 5.2.5), for every $k \in \mathbb{N}$ and $Q_k \subset \mathbb{R}^n$,

$$|u(x)| \le \left| \int_{Q_k} u\,dy \right| + M\|\nabla u\|_{L^{p(\cdot)}(Q_k)}, \tag{5.2.1}$$

where $M \ge 0$ is independent of u. Thus

$$|u(x)| \le \operatorname{meas}(Q_k)\|u\|_{L^2} + M\|\nabla u\|_{L^{p(\cdot)}(Q_k)} \le \|u\|_{L^2(\mathbb{R}^n)} + M\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le (1+M)\|u\|_0.$$

Hence

$$\|u\|_{L^\infty} \le c\|u\|_0, \qquad c = 1 + M. \ \blacksquare$$

Corollary 5.2.2. *The Banach space E_0 is continuously embedded in $L^{p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$.*

Proof. Since $2 \le n < p_0 \le p^- < p^+ < \infty$ and $E_0 = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_0}$, it follows that $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in E_0 and also in $L^{p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see [39, Theorem 3.4.12]). It is sufficient to prove that there exists $c > 0$ such that

$$\|u\|_{L^{p(\cdot)}} \le c\|u\|_0 \quad \text{for all } u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

Let B be the support of u. Then

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|u\|_{L^{p(\cdot)}(B)}.$$

From [39, Theorem 3.3.1, p. 82], we have

$$\|u\|_{L^{p(\cdot)}(B)} \le \|u\|_{L^{p^+}(B)}.$$

It is clear that

$$\|u\|_{L^{p^+}(B)} \le \|u\|_{L^{p^+}(\mathbb{R}^n)}.$$

From Proposition 5.2.1, we deduce that there exists $c > 0$ such that

$$\|u\|_{L^{p^+}(\mathbb{R}^n)} \le c\|u\|_0.$$

This implies that

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|u\|_{L^{p(\cdot)}(B)} \le \|u\|_{L^{p^+}(B)} \le \|u\|_{L^{p^+}(\mathbb{R}^n)} \le c\|u\|_0. \ \blacksquare$$

Corollary 5.2.3. *The Banach space E_0 is continuously embedded in $H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$.*

Proof. By definition of E_0, for every $u \in E_0$ we have

$$\|u\|_0 \ge \|\nabla u\|_{L^{p(\cdot)}}.$$

From Corollary 5.2.2 there exists $c_1 > 0$ such that

$$c_1\|u\|_0 \ge \|u\|_{L^{p(\cdot)}}, \quad \text{and so} \quad \|u\|_0 \ge c\|u\|_{H_0^{1,p(\cdot)}}. \ \blacksquare$$

Corollary 5.2.4. *For every $a > 0$, the space E_a can be identified with the Banach space*

$$W = H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1}),$$

equipped with the usual norm

$$\|u\|_W = \|u\|_{W^{1,2}} + \|u\|_{W^{1,p(\cdot)}}.$$

Proof. $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in $H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see Definition 2.4.1), and $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in $W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see [30]). For any $u \in E_a$ we have

$$\|u\|_a \leq \sup(1, a)\|u\|_W.$$

From Corollary 5.2.2, there exists $c > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$,

$$\|u\|_a \geq c(\|u\|_{W^{1,2}} + \|u\|_{W^{1,p(\cdot)}}). \blacksquare$$

PROPOSITION 5.2.5. *Since* $p > n$, *for every value* $a \geq 0$, *all functions in* E_a *are bounded, continuous, and decay to zero at infinity. Furthermore,*

$$|u(x)-u(y)| \leq c \sup(|x-y|^{1-n/p^-}, |x-y|^{1-n/p^+})\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \quad \text{for all } x, y \in \mathbb{R}. \tag{5.2.2}$$

Proof. By Proposition 5.2.1 we have

$$E_a \subset E_0 \subset L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}), \tag{5.2.3}$$

and since $E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a}$, it is easy to conclude that all functions in E_a are bounded, and decay to zero at infinity. Now we show the inequality.

Fix $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ and consider a family of cubes $Q_k \subset \mathbb{R}^n$ such that

$$\operatorname{meas}(Q_k) = 1, \qquad \bigcup_{k\in\mathbb{N}} Q_k = \mathbb{R}^n,$$

with each Q_k an open cube, containing 0, whose sides—of length r—are parallel to the coordinate axes. For $x \in Q_k$ we have

$$u(x) - u(0) = \int_0^1 du(tx) = \int_0^1 \frac{du(tx)}{dt}\, dt,$$

where

$$\frac{du(tx)}{dt} = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(tx) \cdot \frac{\partial(tx_i)}{\partial t} = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(tx) \cdot x_i.$$

Then

$$|u(x) - u(0)| \leq \int_0^1 \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}(tx)\right| |x_i|\, dt, \qquad x \in Q_k,\ |x_i| < r.$$

Hence

$$|u(x) - u(0)| \leq r \int_0^1 \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}(tx)\right| dt.$$

Let

$$\overline{u} = \frac{1}{|Q_k|} \int_{Q_k} u\, dx.$$

Integrating the last inequality on Q_k we obtain, for every $k \in \mathbb{N}$ and $Q_k \subset \mathbb{R}^n$,

$$\int_{Q_k} |u(x) - u(0)|\, dx \geq \left|\int_{Q_k} (u(x) - u(0))\, dx\right| = |Q_k|\, |\overline{u} - u(0)|$$

and

$$|Q_k|\, |\overline{u} - u(0)| \leq r \int_{Q_k} dx \int_0^1 \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}(tx)\right| dt, \qquad |Q_k| = r^n.$$

Then

$$\begin{aligned}|\overline{u}-u(0)| &\le \frac{1}{r^{n-1}}\int_0^1 dt\int_{Q_k}\sum_{i=1}^n\left|\frac{\partial u}{\partial x_i}(tx)\right|dx\\ &\le r^{n-1}\int_0^1 dt\int_{tQ_k}\sum_{i=1}^n\left|\frac{\partial u}{\partial x_i}(y)\right|\frac{dy}{t^n}, \qquad tQ_k\subset Q_k,\ t\in(0,1).\end{aligned}$$

From Hölder's inequality and Lemma 2.3.4, we have

$$\begin{aligned}\int_{tQ_k}\left|\frac{\partial u}{\partial x_i}(y)\right|dy &\le 2\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(\cdot)}(Q_k)} + \|1\|_{L^{q(\cdot)}(tQ_k)}\\ &\le 2\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(\cdot)}(Q_k)}(|tQ_k|^{n/q^-}+|tQ_k|^{n/q^+}),\end{aligned}$$

where $1/p(x)+1/q(x)=1$. Then we have

$$|\overline{u}-u(0)|\le\frac{2}{r^{n-1}}\int_0^1 dt\,\|\nabla u\|_{L^{p(\cdot)}(Q_k)}\frac{1}{t^n}((tr)^{n/q^-}+(tr)^{n/q^+}).$$

We can easily show that

$$2\frac{r^{n/q^-}}{r^{n-1}}\|\nabla u\|_{L^{p(\cdot)}(Q_k)}\int_0^1\frac{t^{n/q^-}}{t^n}\,dt = 2\frac{r^{1-n/p^-}}{1-n/p^-}\|\nabla u\|_{L^{p(\cdot)}(Q_k)}$$

and

$$2\frac{r^{n/q^+}}{r^{n-1}}\|\nabla u\|_{L^{p(\cdot)}(Q_k)}\int_0^1\frac{t^{n/q^+}}{t^n}\,dt = 2\frac{r^{1-n/p^+}}{1-n/p^+}\|\nabla u\|_{L^{p(\cdot)}(Q_k)}.$$

We deduce from this that

$$|\overline{u}-u(0)|\le c\max(r^{1-n/p^-},r^{1-n/p^+})\|\nabla u\|_{L^{p(\cdot)}(Q_k)}.$$

By translation, this inequality remains true for all cubes Q_k whose sides—of length r—are parallel to the coordinate axes. Thus we have

$$|\overline{u}-u(x)|\le c\max(r^{1-n/p^-},r^{1-n/p^+})\|\nabla u\|_{L^{p(\cdot)}(Q_k)}. \tag{5.2.4}$$

By adding these (and using the triangle inequality) we obtain

$$|u(x)-u(y)|\le c'\max(r^{1-n/p^-},r^{1-n/p^+})\|\nabla u\|_{L^{p(\cdot)}(Q_k)}.$$

Given any $x,y\in\mathbb{R}^n$, there exists a cube with side $r=2|x-y|$ such that

$$\begin{aligned}|u(x)-u(y)| &\le c\max(|x-y|^{1-n/p^-},|x-y|^{1-n/p^+})\|\nabla u\|_{L^{p(\cdot)}(Q_k)}\\ &\le c\max(|x-y|^{1-n/p^-},|x-y|^{1-n/p^+})\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \ \blacksquare\end{aligned}$$

REMARK 5.2.1. We deduce from Proposition 5.2.5 that if $u\in E_a$ with $n<p^-<\infty$, then u is bounded and $\lim_{|x|\to\infty}u(x)=0$.

Recall that η is a singular point of the potential V, so it is reasonable to consider in E_a the subset

$$\Gamma_a=\{u\in E_a : u(x)\neq\eta \text{ for all } x\in\mathbb{R}^n\}.$$

The subset Γ_a is open in E_a. Indeed, by Remark 5.2.1, we have

$$\inf_{x\in\mathbb{R}^n}|u(x)-\eta|=d>0.$$

Then, by (5.2.3) (E_a is continuously embedded in L^∞), we deduce that for all $u \in \Gamma_a$, there exists a small neighborhood of u contained in Γ_a. The boundary of Γ_a is given by

$$\partial\Gamma_a = \{u \in E_a : \text{there exist } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\} = E_a \setminus \Gamma_a.$$

5.3. Topological charge and connected components of Γ_a. Recall that the topological charge ch was defined in Section 4.3.

Now, for every $q \in \mathbb{Z}$ we set

$$\Gamma_a^q = \{u \in \Gamma_a : \mathrm{ch}(u) = q\}.$$

Since the topological charge is continuous with respect to uniform convergence and the continuity of the embeddings of E_a in L^∞ ensures that the topological charge is continuous on Γ_a, it follows that Γ_a^q is open in E_a, and we also have

$$\Gamma_a = \bigcup_{q\in\mathbb{Z}} \Gamma_a^q, \qquad \Gamma_a^q \cap \Gamma_a^p = \emptyset, \quad p \neq q.$$

We conclude that every Γ_a^q is a connected component of Γ_a. We observe that for every $q \in \mathbb{Z}$ the component Γ_a^q is homeomorphic to Γ_a^{-q}. So for every $u \in \Gamma_a$ we can define the charge $\mathrm{ch}(u) \in \mathbb{Z}$. Now, we consider the minimizer set of f_a in the open set

$$\Gamma_a^* = \{u \in \Gamma_a : \mathrm{ch}(u) \neq 0\}.$$

5.4. Properties of the energy functional

Lemma 5.4.1. *The functional f_a takes real values and is continuous on Γ_a.*

Proof. We have

$$\begin{aligned} f_a(u) &= \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^2 + \frac{b}{2}|\nabla u|^{p(x)}\right) dx + \int_{\mathbb{R}^n} V(u)\,dx \\ &= \underbrace{\frac{a}{2}\|\nabla u\|_{L^2}^2 + \frac{b}{2}\rho_{p(\cdot)}(u)} + \underbrace{\int_{\mathbb{R}^n} V(u)\,dx}\,. \end{aligned}$$

The first term on the right-hand side is finite and continuous. Let us prove that the second term is finite and continuous.

We have $V(\xi) = (V''(0)\xi \cdot \xi + o(\xi^2)$. By ($\mathrm{V}_3$) there exist a small neighborhood of $0 \in \mathbb{R}^{n+1}$ and $M > 0$ such that, for every ξ in that neighborhood, we have

$$V(\xi) \leq M|\xi|^2. \tag{5.4.1}$$

Since every $u \in E_a$ decays to zero at infinity (see Proposition 5.2.5 below), there exists a ball B_u such that, for every $x \in \mathbb{R}^n \setminus B_u$ we have $|u(x)| < \epsilon$; then by (5.4.1) and for ϵ sufficiently small,

$$V(u(x)) \leq M|u(x)|^2. \tag{5.4.2}$$

From $u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, we deduce

$$\int_{\mathbb{R}^n\setminus B_u} V(u)\,dx < \infty.$$

On the other hand, since u is continuous (see Proposition 5.2.5), we also have

$$\int_{B_u} V(u)\,dx < \infty.$$

Let $(u_k) \subset \Lambda_a$ be a sequence such that $f_a(u_k) < \infty$ and $u_k \to u$ in E_a. We show that

$$\int_{\mathbb{R}^n} V(u_k) \to \int_{\mathbb{R}^n} V(u).$$

Since $f_a(u_k) < \infty$, Lemma 5.4.3 shows that u belongs to Λ_a.

By (5.2.3) we have $u_k \to u$ on $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Then $V(u_k) \to V(u)$ uniformly on $\mathbb{R}$. We deduce that

$$\int_{B_u} V(u_k)\,dx \to \int_{B_u} V(u)\,dx. \tag{5.4.3}$$

By (5.4.2),

$$\int_{\mathbb{R}^n \setminus B_u} V(u(x))\,dx \le \int_{\mathbb{R}^n \setminus B_u} |u(x)|^2\,dx,$$

and since $u_k \to u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, using the dominated convergence theorem we get

$$\int_{\mathbb{R}^n \setminus B_u} V(u_k)\,dx \to \int_{\mathbb{R}^n \setminus B_u} V(u)\,dx. \ \blacksquare \tag{5.4.4}$$

LEMMA 5.4.2. *The functional f_a is coercive in Γ_a, that is, for every sequence $(u_k) \subset \Gamma_a$ such that $\|u_k\|_a \to \infty$, we have $f_a(u_k) \to \infty$.*

Proof. For $a > 0$, $n > 2$, we have

$$\|u\|_a = a\|\nabla u\|_{L^2} + \|\nabla u\|_{L^{p(\cdot)}} + \|u\|_{L^2}.$$

Let $u_k \in \Gamma_a$ be such that $\|u_k\|_a \to \infty$ as $k \to \infty$. It is clear that if

$$a\|\nabla u_k\|_{L^2} + \|\nabla u_k\|_{L^{p(\cdot)}} \to \infty \quad \text{as } k \to \infty, \tag{5.4.5}$$

then $f_a(u_k) \to \infty$ as $k \to \infty$.

Assume now that there exists $c_* > 0$ such that

$$a\|\nabla u_k\|_{L^2} + \|\nabla u_k\|_{L^{p(\cdot)}} < c_* \tag{5.4.6}$$

and

$$\|u_k\|_{L^2} \to \infty \quad \text{as } k \to \infty. \tag{5.4.7}$$

We shall prove that

$$\int_{\mathbb{R}^n} V(u_k)\,dx \to \infty \quad \text{as } k \to \infty.$$

From (V_3), we know that for every $r > 0$ there exists $\omega_r > 0$ such that

$$|\xi| \le r \ \Rightarrow\ V(\xi) \ge \omega_r |\xi|^2. \tag{5.4.8}$$

For every $k \in \mathbb{N}$, we set

$$A_k = \{x \in \mathbb{R}^n : |u_k(x)| \le r\},$$

where $u_k \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$. By the Sobolev inequality,

$$\|u_k\|_{L^{2^*}} \le c\|\nabla u_k\|_{L^2}, \qquad 2^* = \frac{2n}{n-2},\ n > 2. \tag{5.4.9}$$

From (5.4.6), we obtain

$$\|u_k\|_{L^{2^*}} < c_*. \tag{5.4.10}$$

Moreover, from (5.2.4), there exists $M \geq 0$ independent of u_k such that

$$\begin{aligned} |u_k(x)| &\leq \left| \int_{Q_k} u_k \, dy \right| + M \|\nabla u_k\|_{L^{p(\cdot)}(Q_k)} \qquad (\mathrm{meas}(Q_k) = 1) \\ &\leq \|u_k\|_{L^{2^*}(Q_k)} + M \|\nabla u_k\|_{L^{p(\cdot)}(Q_k)}. \end{aligned}$$

By (5.4.6) and (5.4.10), for any $x \in \mathbb{R}^n$, we have

$$|u_k(x)| < c_* + M c_*. \tag{5.4.11}$$

Then there exists $c > 0$ such that

$$\mathrm{meas}(\mathbb{R}^n \setminus A_k) < c. \tag{5.4.12}$$

From (5.4.11) and (5.4.12), we deduce that there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}^n \setminus A_k} |u_k|^2 \, dx < c_1. \tag{5.4.13}$$

By (5.4.8), we obtain

$$\int_{\mathbb{R}^n} V(u_k) \, dx \geq \int_{A_k} V(u_k) \, dx \geq \omega_r \int_{A_k} \|u_k\|^2 \, dx \geq \omega_r \left(\|u_k\|_{L^2}^2 - \int_{\mathbb{R}^n \setminus A_k} |u_k|^2 \, dx \right).$$

From (5.4.13) and (5.4.7), we have

$$\lim_k \int_{\mathbb{R}^n} V(u_k) \, dx \geq \omega_r (\|u_k\|_{L^2}^2 - c_1) \to \infty \qquad \text{as } k \to \infty.$$

In the case $a = 0$ or $n = 2$, by (V_5), there exists $r_* > 0$ such that, for every $\xi \in \mathbb{R}^n$ with $|\xi| \geq r_*$,

$$V(\xi) \geq \nu/2. \tag{5.4.14}$$

Let $(u_k) \subset \Gamma_a$ be a sequence such that $\|u_k\|_0 \to \infty$ as $k \to \infty$. Since the functional f_a is invariant with respect to translations in $\mathbb{R}^n$, we can assume

$$\|u_k\|_{L^\infty} = |u_k(0)|. \tag{5.4.15}$$

Now, we consider the case

$$\|\nabla u_k\|_{L^{p(\cdot)}} \leq M_* \quad \text{and} \quad \|u_k\|_{L^2} \to \infty \qquad \text{as } k \to \infty.$$

Here we have two subcases:

(a) $\|u_k\|_{L^\infty} \to \infty$ as $k \to \infty$. (5.4.16)

(b) $\|u_k\|_{L^\infty}$ is bounded. (5.4.17)

In subcase (a), by (5.4.16), we can choose a sequence $(r_k) \subset (0, \infty)$ such that

$$r_* \leq \|u_k\|_{L^\infty} - K(r_k^{(p^+ - n)/p^+} + r_k^{(p^- - n)/p^-}) \quad \text{and} \quad r_k \to \infty, \tag{5.4.18}$$

where $K = cM_*$ and c is the same constant as in (5.2.2). For every $y \in \mathbb{R}^n$, we have

$$|u_k(0)| - |u_k(y)| \leq |u_k(0) - u_k(y)|.$$

Hence by (5.2.2), we obtain

$$|u_k(0)| - |u_k(y)| \leq K(|y|^{(p^+ - n)/p^+} + |y|^{(p^- - n)/p^-}).$$

From (5.4.15), we get

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K(|y|^{(p^+-n)/p^+} + |y|^{(p^--n)/p^-}).$$

For $|y| \leq r_k$, from (5.4.18) we have

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K(r_k^{(p^+-n)/p^+} + r_k^{(p^--n)/p^-}) \geq r_*. \tag{5.4.19}$$

From (5.4.14) and (5.4.19), we get

$$\int_{\mathbb{R}^n} V(u_k)\,dx \geq \int_{B(0,r_k)} V(u_k)\,dx \geq \frac{\nu}{2}\operatorname{meas}(B(0,r_k)).$$

This implies that

$$\int_{\mathbb{R}^n} V(u_k)\,dx \to \infty \quad \text{as } r_k \to \infty.$$

In subcase (b), we assume there exists $\bar{M} > 0$ such that $\|u_k\|_{L^\infty} \leq \bar{M}$. From (5.4.8), we obtain

$$\int_{\mathbb{R}^n} V(u_k)\,dx \geq \omega_{\bar{M}}\|u_k\|_{L^2} \to \infty \quad \text{as } k \to \infty. \ \blacksquare$$

We are going to study the behavior of the energy f_a when u approaches the boundary of Γ_a; we remark that $\partial\Gamma_a = E_a \setminus \Gamma_a$.

LEMMA 5.4.3. *Let $(u_k) \subset \Gamma_a$ be a weakly converging sequence. If the weak limit belongs to $\partial\Gamma_a$, then $f_a(u_k) \to \infty$ as $k \to \infty$.*

Proof. Let $(u_k) \subset \Gamma_a$ be such that $u_k \rightharpoonup u \in \partial\Gamma_a$ as $k \to \infty$. Since $u \in \partial\Gamma_a$, there exists $x_* \in \mathbb{R}^n$ such that $u(x_*) = \eta$. Using the fact that (u_k) is bounded in E_a, by the uniform convergence on compact sets, we have

$$u_k(x_*) \to u(x_*) \quad \text{as } k \to \infty. \tag{5.4.20}$$

Since (u_k) is bounded in E_a, (∇u_k) is bounded in $L^{p(\cdot)}$. Then from (5.2.2), we obtain

$$|u_k(x) - u_k(x_*)| \leq c\max(|x-x_*|^{(p^+-n)/p^+}, |x-x_*|^{(p^--n)/p^-}). \tag{5.4.21}$$

Thus

$$|u_k(x) - \eta| \leq |u_k(x) - u_k(x_*)| + |u_k(x_*) - \eta|. \tag{5.4.22}$$

By (5.4.20) and (5.4.21), there exists $\epsilon_k > 0$ such that

$$|u_k(x) - \eta| \leq c\max(|x-x_*|^{(p^+-n)/p^+}, |x-x_*|^{(p^--n)/p^-}) + \epsilon_k \tag{5.4.23}$$

where $\epsilon_k \to 0$ as $k \to \infty$. Let $x \in B(0,r)$. For r sufficiently small, there exists $\rho > 0$ such that

$$|u_k(x) - \eta| \leq c\max(r^{(p^+-n)/p^+}, r^{(p^--n)/p^-}) + \epsilon_k < \rho. \tag{5.4.24}$$

From (5.4.24) and (V_4), we have

$$V(u_k(x)) \geq c\big(|u_k(x)-\eta|^{-np^+/(p^+-n)} + |u_k(x)-\eta|^{-np^-/(p^--n)}\big). \tag{5.4.25}$$

Then from (5.4.23) and (5.4.25), we obtain

$$V(u_k(x)) \geq \frac{c}{|x-x_*|^n + \epsilon_k}.$$

Restricting our attention to $B(x_*, r)$, we have

$$\lim_{k\to\infty}\int_{B(x_*,r)} V(u_k(x))\,dx \geq \int_{B(x_*,r)} \lim_{k\to\infty} V(u_k(x))\,dx \geq c\int_{B(x_*,r)} \frac{1}{|x-x_*|^n}\,dx = \infty. \ ■$$

Corollary 5.4.4. *For every $b > 0$, there exist $d_* = d(b)$ such that, for every $u \in \Gamma_a$,*

$$f_a(u) \leq b \ \Rightarrow\ \min_x |u(x) - \eta| \geq d_*.$$

Proof. The proof is the same as that of Corollary 4.4.5. ■

Lemma 5.4.5. *The functional f_a is weakly lower semicontinuous in Γ_a.*

Proof. The proof is the same as that of Lemma 4.4.6. ■

Proposition 5.4.6. *There exists $\Delta_a > 0$ such that for every $u \in \Gamma_a$ satisfying $\|u\|_{L^\infty} \geq 1$,*

$$f_a(u) \geq \Delta_a.$$

It is easy to see that $\mathrm{ch}(u) \neq 0$ *implies* $\|u\|_{L^\infty} > 1$.

Proof. By continuous injection in Proposition 5.2.1,

$$\|u\|_a \geq \|u\|_0 \geq \|u\|_{L^\infty} \geq 1,$$

and by the coercivity of f_a, we get

$$\|u\|_a \geq 1 \ \Rightarrow\ \exists \Delta_a > 0 \text{ such that } f_a(u) \geq \Delta_a. \ ■$$

5.5. Existence result

Theorem 5.5.1. *The minimum points $u \in \Gamma_a$ for the functional f_a are weak solutions of the system* (5.1.5).

Proof. Let u be a minimum point of f_a and $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Let e_j denote the jth vector of the canonical basis in $\mathbb{R}^n$. If ϵ is sufficiently small, then $u + \epsilon e_j h \in \Gamma_a$ and $f_a(u + \epsilon e_j h) < \infty$. Since u is a minimum point of f_a, we have

$$0 = \frac{df(u+\epsilon e_j h)}{d\epsilon}\bigg|_{\epsilon=0} = \int_{\mathbb{R}^n} \left(a\nabla u_j \nabla h + \frac{b}{2}(p(\cdot)|\nabla u|^{p-2}\nabla u_j \nabla h) + \frac{\partial V(\xi)}{\partial \xi_j} h\right) dx.$$

By Green's formula,

$$\int_{\mathbb{R}^n} \frac{b}{2}(p(\cdot)|\nabla u|^{p-2}\nabla u_j \nabla h)\,dx = \int_{\mathbb{R}^n} -\frac{b}{2}\,\mathrm{div}(p(\cdot)|\nabla\cdot u|^{p-2}\nabla u_j) h\,dx,$$

$$\int_{\mathbb{R}^n} a\nabla u_j \nabla h = \int_{\mathbb{R}^n} -a\Delta u_j h.$$

So

$$\int_{\mathbb{R}^n} \left(-a\Delta u_j - \frac{b}{2}\,\mathrm{div}(p(\cdot)|\nabla\cdot u|^{p-2}\nabla u_j) + \frac{\partial V(\xi)}{\partial \xi_j}\right) h\,dx = 0$$

for $1 \leq j \leq n+1$, and for any $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Then

$$\int_{\mathbb{R}^n} \left[-a\Delta u - \frac{b}{2}\Delta_{p(\cdot)} u + V'(u)\right] \phi\,dx = 0 \quad \text{for every } \phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

This implies by density

$$-a\Delta u - \frac{b}{2}\Delta_{p(\cdot)} u + V'(u) = 0. \ ■$$

PROPOSITION 5.5.2 (Splitting Proposition). *Let* $(u_k) \subset \Gamma_a^*$ *be a sequence and* M *be a positive real number such that* $f_a(u_k) \leq M$. *Then there exists* $l \in \mathbb{N}$ *such that* $1 \leq l \leq M/\Delta_a$ *where* Δ_a *was introduced in Proposition* 5.4.6 *and there exist* $\bar{u}_1, \ldots, \bar{u}_l \in \Gamma_a$ *and* $(x_k^1), \ldots, (x_k^l) \subset \mathbb{R}^n$ *such that, up to a subsequence,*

$$u_k(\cdot + x_k^i) \rightharpoonup \bar{u}_i, \qquad |x_k^i - x_k^j| \to \infty, \qquad i \neq j,$$
$$\sum_{i=1}^{l} f_a(\bar{u}_i) \leq \liminf_{k\to\infty} f_a(u_k), \qquad \mathrm{ch}(u_k) = \sum_{i=1}^{l} \mathrm{ch}(\bar{u}_i).$$

Proof. Use Lemmas 5.4.2, 5.4.3 and 5.4.5, and the same method as in Proposition 4.5.1. ∎

The minimum is attained on the set Γ_a, and it is easy to see that $u \equiv 0$ is a trivial solution of the problem. But of course we are interested in nontrivial solutions, We consider the following problem:

$$I_* = \inf_{u\in\Gamma_a^*} f_a(u), \qquad \Gamma_a^* = \{u \in E_a : \mathrm{ch}(u) \neq 0\}.$$

The functional is bounded below and the set Γ_a^* is not empty. We consider fields u having the form

$$u(x) = \left(\frac{2}{1+|x|^m}, \frac{1}{1+|x|^m}x\right). \tag{5.5.1}$$

LEMMA 5.5.3. *There exists an* $m \geq 1$ *such that the field* u *defined in* (5.5.1) *belongs to* Γ_a^*.

Proof. Clearly, if m is sufficiently large, then the field u defined in (5.5.1) belongs to E_a. For the sake of contradiction, suppose that there exists $\bar{x} \in \mathbb{R}^n$ such that $u(\bar{x}) = \eta = (1, 0)$. We deduce that

$$\frac{2}{1+|\bar{x}|^m} = 1, \qquad \frac{1}{1+|\bar{x}|^m}\bar{x} = 0.$$

We get a contradiction: $|\bar{x}| = 1$ and $\bar{x} = 0$. So, $u \in \Gamma_a$.

We show that $\mathrm{ch}(u) \neq 0$. Set $g(x) = \frac{1}{2}x$. Then we have

$$K_u = \left\{x \in \mathbb{R}^n : \frac{2}{1+|x|^m} > 1\right\} = B(0,1), \qquad \text{if } |x| = 1 \text{ then } g(x) = \frac{1}{1+|x|^m}x,$$

and by the properties of the topological degree (see Section 2.8), we get

$$\deg\left(\frac{1}{1+|x|^m}x, B(0,1), 0\right) = \deg\big(g(x), B(0,1), 0\big) \neq 0. \ ∎$$

Moreover, the set Γ_a^* is open in the space E_a; indeed,

$$\Gamma_a^* = \bigcup_{q\in\mathbb{N}^*} \Gamma_a^q, \qquad \Gamma_a^q \cap \Gamma_a^p = \emptyset,\ p \neq q,$$

where Γ_a^q is a connected component.

THEOREM 5.5.4. *Let* $a \geq 0$, $b > 0$, $p^- > n > 2$. *If* V *satisfies* (V_1)–(V_5) *and* p *satisfies* (p_1)–(p_2), *then there exists a weak solution of* (5.1.5) (*i.e., a static solution of* (5.1.3)) *which is a minimizer of the energy functional* (5.1.7) *in the class of maps whose topological charge is different from* 0.

Proof. By the Splitting Proposition and the same technique used in Chapter 4 (Theorem 4.5.3), we show that there exists a weak solution to (5.1.5). And with suitable change of variable in (5.1.6) we deduce a solution of equation (5.1.3). ■

Remark 5.5.1. The functional exibits an invariance for the symmetry group of rotations and translations; indeed, for every function u and $g \in O(n)$, if we set $u_g(x) = u(gx)$, we have immediately

$$f_a(u_g) = f_a(u).$$

Then our theorem gives the existence of an orbit of minimum solutions. This orbit consists of two connected components, which are identified, respectively, by $\bar{u}$ and

$$\bar{u} \circ \mathcal{P}(x) = \bar{u}(-x).$$

Since typically $n = 3$ is odd, $\bar{u} \circ \mathcal{P}$ and $\bar{u}$ have opposite topological charge.

6. Derrick's Problem with twice variable exponent

In the mathematical models (solitons) studied in Chapter 4 the space of finite energy configurations (solution space) splits into infinitely many connected components according to the topological charge. In that chapter we proved the existence of infinitely many solutions, which are constrained minima of the energy. More precisely, on each connected component characterized by a topological charge equal to $N \in \mathbb{N}$ there exists a solution of charge N. Since p is arbitrary in the static equation, it is natural to consider $p = p(x)$ as a variable that depends on the connected component.

The aim of this chapter is to carry out an existence analysis of the finite-energy static solutions in more than one space dimension ($n \geq 2$) for a class of Lagrangian densities L which include (5.1.1) in Chapter 5, generalizing the results of Chapter 4. More precisely, we are concerned with generalized Sobolev spaces with twice variable exponent $r(\cdot) \leq n$ and $p(\cdot) > n$.

6.1. Statement of the problem. The class of Lagrangian densities we consider generalizes the problem studied in Chapter 5, in such a way as to include the Derrick proposal.

First we introduce some notation. Positive integers n and m will correspond, respectively, to the physical space-time (typically $n = 3$) and the internal parameter space. We are interested in the multi-dimensional case, so we assume that $n \geq 2$. A point in $\mathbb{R}^{n+1}$ will be denoted by $X = (x, t)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The fields we are interested in are maps $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^m$, $\psi = (\psi_1, \dots, \psi_m)$. We set

$$\rho = |\nabla\psi|^2 - |\psi_t|^2,$$

$\nabla\psi$ and ψ_t denoting, respectively, the Jacobian with respect to x and the derivative with respect to t. Let

$$s : \mathbb{R}^{n+1} \to \mathbb{R}.$$

We shall consider Lagrangian densities of the form

$$\mathcal{L}(\psi, \rho) = -\tfrac{1}{2}\alpha(\rho, k, s) - V(\psi), \tag{6.1.1}$$

where V is a real function defined in an open subset $\Omega \subset \mathbb{R}^m$ and α is a real function defined by

$$\alpha(\rho, s, k) = a\rho|\rho|^{k(\cdot)/2-1} + b|\rho|^{s(\cdot)/2}, \qquad a \geq 0,\ b > 0,\ s(0) > n,\ 2 \leq k(\cdot) < n. \tag{6.1.2}$$

The results of [14] were concerned with the case $s(\cdot) \equiv p$ and $k(\cdot) \equiv 2$ (we fixed the variable exponent). The action functional related to (6.1.1) is

$$S(\psi) = \int_{\mathbb{R}^{n+1}} \mathcal{L}(\psi, \rho)\, dx\, dt = \int_{\mathbb{R}^{n+1}} \bigl(-\tfrac{1}{2}\alpha(\rho, k, s) - V(\psi)\bigr)\, dx\, dt.$$

So the Euler–Lagrange equations are

$$\frac{\partial}{\partial t}(\alpha' \psi_t) - \nabla\bigl(\alpha' \nabla \psi\bigr) + V'(\psi) = 0, \tag{6.1.3}$$

where $\nabla(\alpha' \nabla \psi)$ denotes the vector whose jth component is given by $\operatorname{div}(\alpha' \nabla \psi^j)$, and V' denotes the gradient of V. Equation (6.1.3) is Lorentz invariant. Static solutions $\psi(x,t) = u(x)$ of (6.1.3) solve the equation

$$-\nabla(\alpha' \nabla u) + V'(u) = 0. \tag{6.1.4}$$

Set $k(x,t) = r(x)$ and $s(x,t) = p(x)$ on $\mathbb{R}^n$ (the restrictions of s to $\mathbb{R}^n$). Using (6.1.2) and (6.1.4) we obtain

$$-\frac{a}{2}\Delta_{r(\cdot)} - \frac{b}{2}\Delta_{p(\cdot)} + V'(u) = 0, \tag{6.1.5}$$

where

$$\Delta_{r(\cdot)} u = \nabla(r(\cdot)|\nabla u|^{r(\cdot)-2}\nabla u), \qquad \Delta_{p(\cdot)} u = \nabla(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u).$$

We introduce the following notation and function spaces:

$$C_+(\mathbb{R}^n) = \{p \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) : p(x) > 1 \text{ for all } x \in \mathbb{R}^n\}$$

and

$$p^+ = \sup_{x \in \mathbb{R}^n} p(x), \qquad p^- = \inf_{x \in \mathbb{R}^n} p(x).$$

We assume that:

(p$_1$) $S(x,t) = p\bigl(\frac{x_1 - t\nu}{\sqrt{1-\nu^2}}, \dots, x_n\bigr)$, where ν is a parameter used in the Lorentz transformation,
(p$_2$) $\lim_{x\to\infty} p(x) = p_\infty = p^- > n$,
(r$_1$) $k(x,t) = r\bigl(\frac{x_1 - t\nu}{\sqrt{1-\nu^2}}, \dots, x_n\bigr)$, where ν is a parameter used in the Lorentz transformation,
(r$_2$) $\lim_{x\to\infty} r(x) = r_\infty = 2 = r^- \le r^+ \le n$,
(r$_3$) there exists $c > 0$ such that for all balls B and all $x \in B$ we have $|B|^{r_B^- - r(x)} < c$,
(r$_4$) for all $x \in \mathbb{R}^n$, $|r(x) - 2| < 1/(\log|e + |x||)$.

Recall that the results of [14] were concerned with the case

$$r(\cdot) \equiv 2 \quad \text{and} \quad p(\cdot) \equiv p^- > n.$$

Under (p$_1$), it is easy to verify that if $u = u(x)$ is a solution of (6.1.3) and $v = (\nu, 0, \dots, 0)$ with $|\nu| < 1$, then the field

$$\psi_\nu(x,t) = u\biggl(\frac{x_1 - \nu t}{\sqrt{1-\nu^2}}, x_2, \dots, x_n\biggr) \tag{6.1.6}$$

is a solution of (6.1.3). Notice that the function undergoes a contraction by a factor of

$$\gamma = \frac{1}{\sqrt{1-\nu^2}}$$

in the direction of the motion; this is a consequence of the fact that (6.1.3) is Lorentz invariant. Clearly, (6.1.5) are the Euler–Lagrange equations with respect to the energy functional

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2} |\nabla u|^{r(x)} + \frac{b}{2} |\nabla u|^{p(x)} + V(u) \right) dx, \tag{6.1.7}$$

where $m = n + 1$, so the time independent fields u are maps

$$u : \mathbb{R}^n \to \mathbb{R}^m.$$

For every $\xi \in \mathbb{R}^{n+1}$, we write $\xi = (\xi_0, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^n$. Let $V : \Omega \to \mathbb{R}$, where $\Omega = \mathbb{R}^{n+1} \setminus \{\eta\}$, $\eta = (1, 0)$, and V is positive and singular in η. More precisely, we assume:

(V_1) $V \in C^1(\Omega, \mathbb{R})$.
(V_2) $V(\xi) \geq V(0) = 0$.
(V_3) V is twice differentiable at 0 and the Hessian matrix $V''(0)$ is nondegenerate.
(V_4) There exist $c, \rho > 0$ such that if $|\xi| < \rho$ then

$$V(\eta + \xi) \geq c(|\xi|^{-q^+} + |\xi|^{-q^-})$$

where

$$\frac{1}{q^-} = \frac{1}{n} - \frac{1}{p^-}, \qquad \frac{1}{q^+} = \frac{1}{n} - \frac{1}{p^+}.$$

(V_5) For every $\xi \in \Omega \setminus \{0\}$ we have

$$V(\xi) > 0, \qquad \liminf_{|\xi| \to \infty} V(\xi) = v > 0.$$

(V_6) There exists $R > 0$ such that $|\xi| < R \Rightarrow V(\xi) \geq \omega_R |\xi|^{r^+}$ for some $\omega_R > 0$.

EXAMPLE 6.1.1. A potential satisfying assumptions (V_1)–(V_6) is

$$V(\xi) = \omega_0^2 \left(|\xi|^{r^+} + \frac{|\xi|^4}{|\xi - \eta|^{q^+} + |\xi - \eta|^{q^-}} \right).$$

6.2. Solution space. Let $p^- > n \geq 2$, $2 = r^- \leq r^+ \leq n$, and with no loss of generality, consider the functional (6.1.7) with $b = 1$. It will be convenient to set

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2} |\nabla u|^{r(x)} + \frac{1}{2} |\nabla u|^{p(x)} + V(u) \right) dx,$$

and define E_a to be the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_a = a\|\nabla u\|_{L^{r(\cdot)}} + \|\nabla u\|_{L^{p(\cdot)}} + \|u\|_{L^{r(\cdot)}}, \qquad a \geq 0,$$

i.e.,

$$E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a},$$

$$\|u\|_{r(\cdot)} = \inf\left\{ \sigma > 0 : \int_{\mathbb{R}^n} \left| \frac{u(x)}{\sigma} \right|^{r(x)} dx \leq 1 \right\},$$

$$\|\nabla u\|_{r(\cdot)} = \inf\left\{ \sigma > 0 : \int_{\mathbb{R}^n} \left| \frac{\nabla u(x)}{\sigma} \right|^{r(x)} dx \leq 1 \right\},$$

$$\|\nabla u\|_{p(\cdot)} = \inf\left\{ \sigma > 0 : \int_{\mathbb{R}^n} \left| \frac{\nabla u(x)}{\sigma} \right|^{p(x)} dx \leq 1 \right\}.$$

For every $a > 0$, the norms $\|\cdot\|_a$ are equivalent, so we have to study only two cases: $a = 0$, $a > 0$.

Remark 6.2.1. We have $\lim_{x\to\infty} r(x) = r_\infty = 2 = r^- \le r^+ \le n$.

- From (r_3), (r_4), Lemma 2.4.2 and Definition 2.4.2, it is easy to see that $r(\cdot)$ is globally Hölder continuous.
- From Definition 2.4.3 and Remark 2.4.1 it is easy to see that $r \in \mathcal{P}^{\log}(\mathbb{R}^n)$.
- $L^{r(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ is continuously embedded in $L^{r^-}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see [39, Proposition 4.1.8, p. 103]).

Proposition 6.2.1. *The Banach space E_0 is continuously embedded in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$ for every $s \in [r^-, \infty]$, $r^- = 2$.*

Proof. From Remark 6.2.1, E_0 is continuously embedded in $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, so it is sufficient to show that E_0 is also embedded in $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Since $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in E_0, it is sufficient to prove that there exists $c > 0$ such that, for every $u \in C_0(\mathbb{R}^n, \mathbb{R}^{n+1})$,

$$\|u\|_{L^\infty} \le c\|u\|_0.$$

We fix $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ and consider a family of cubes $Q_k \subset \mathbb{R}^n$ such that

$$\operatorname{meas}(Q_k) = 1, \qquad \bigcup_{k\in\mathbb{N}} Q_k = \mathbb{R}^n.$$

Then, by (5.2.4),

$$|u(x)| \le \left|\int_{Q_k} u\,dy\right| + M\|\nabla u\|_{L^{p(\cdot)}(Q_k)}, \tag{6.2.1}$$

where $M \ge 0$ is independent of u. Thus

$$\begin{aligned}|u(x)| &\le \operatorname{meas}(Q_k)\|u\|_{L^{r(\cdot)}} + M\|\nabla u\|_{L^{p(\cdot)}(Q_k)} \le \|u\|_{L^{r(\cdot)}(\mathbb{R}^n)} + M\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le (1+M)\|u\|_0.\end{aligned}$$

Hence

$$\|u\|_{L^\infty} \le c\|u\|_0, \qquad c = 1 + M. \ \blacksquare$$

Corollary 6.2.2. *The Banach space E_0 is continuously embedded in $L^{p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$.*

Proof. Since $2 \le n < p_0 \le p^- < p^+ < \infty$ and $E_0 = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_0}$, $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in E_0 and hence also in $L^{p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see [39, Theorem 3.4.12]). So it is sufficient to prove that there exists $c > 0$ such that

$$\|u\|_{L^{p(\cdot)}} \le c\|u\|_0 \quad \text{for all } u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

Let B be the support of u. Then

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|u\|_{L^{p(\cdot)}(B)}.$$

From [39, Theorem 3.3.1, p. 82], we have

$$\|u\|_{L^{p(\cdot)}(B)} \le \|u\|_{L^{p^+}(B)}.$$

It is clear that

$$\|u\|_{L^{p^+}(B)} \le \|u\|_{L^{p^+}(\mathbb{R}^n)}.$$

From Proposition 6.2.1, we deduce that there exists $c > 0$ such that

$$\|u\|_{L^{p^+}(\mathbb{R}^n)} \le c\|u\|_0.$$

This implies that

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|u\|_{L^{p(\cdot)}(B)} \leq \|u\|_{L^{p^+}(B)} \leq \|u\|_{L^{p^+}(\mathbb{R}^n)} \leq c\|u\|_0. \ \blacksquare$$

COROLLARY 6.2.3. *The Banach space* E_0 *is continuously embedded in* $H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$.

Proof. By definition of the space E_0, for every $u \in E_0$ we have

$$\|u\|_0 \geq \|\nabla u\|_{L^{p(\cdot)}}.$$

From Corollary 6.2.2 there exists $c_1 > 0$ such that $c_1\|u\|_0 \geq \|u\|_{L^{p(\cdot)}}$, and so

$$\|u\|_0 \geq c\|u\|_{H_0^{1,p(\cdot)}}. \ \blacksquare$$

REMARK 6.2.2. We have $r_\infty = 2 = r^- \leq r^+ \leq n$.

- From Remark 6.2.1 we have $r \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then $H_0^{1,r(\cdot)}(\mathbb{R}^n) = W_0^{1,r(\cdot)}(\mathbb{R}^n)$ (see [39, Corollary 11.2.4, p. 347]).
- Since $r \in \mathcal{P}(\mathbb{R}^n)$ is bounded, we have $W_0^{1,r(\cdot)}(\mathbb{R}^n) = W^{1,r(\cdot)}(\mathbb{R}^n)$ (see [39, Corollary 9.1.3, p. 291]).

COROLLARY 6.2.4. *For every* $a > 0$*, the space* E_a *can be identified with the Banach space*

$$W = H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap W^{1,r(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1}),$$

equipped with the usual norm

$$\|u\|_W = \|u\|_{W^{1,r(\cdot)}} + \|u\|_{W^{1,p(\cdot)}}.$$

Proof. $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in $H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see Definition 2.4.1), and $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in $W^{1,r(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see [39, Theorem 9.1.6, p. 291]). For any $u \in E_a$ we have

$$\|u\|_a \leq \max(1, a)\|u\|_W.$$

From Corollary 6.2.2, there exists $c > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\|u\|_a \geq c(\|u\|_{W^{1,2}} + \|u\|_{W^{1,p(\cdot)}}). \ \blacksquare$$

PROPOSITION 6.2.5. *For* $p > n$*, for every value* $a \geq 0$*, the functions in* E_a *are bounded, continuous, and decay to zero at infinity. Furthermore,*

$$|u(x) - u(y)| \leq c\max(|x-y|^{1-n/p^-}, |x-y|^{1-n/p^+})\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \text{ for all } x, y \in \mathbb{R}. \quad (6.2.2)$$

Proof. The proof is the same as in Proposition 5.2.5. ∎

REMARK 6.2.3. By Proposition 6.2.1 we have

$$E_a \subset E_0 \subset L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}). \quad (6.2.3)$$

We deduce from Proposition 6.2.5 that if $u \in E_a$ with $n < p^- < \infty$, then u is bounded and

$$\lim_{|x|\to\infty} u(x) = 0.$$

Recall that η is a singular point of the potential V, so it is reasonable to consider in E_a the open subset

$$\Gamma_a = \{u \in E_a : u(x) \neq \eta \text{ for all } x \in \mathbb{R}^n\}.$$

The subset Γ_a is open in E_a. Indeed, by Remark 6.2.3, we have

$$\inf_{x\in\mathbb{R}^n} |u(x)-\eta| = d > 0.$$

Then, by (6.2.3), E_a is continuously embedded in L^∞, and we deduce that for all $u \in \Gamma_a$, there exists a small neighborhood of u contained in Γ_a.

The boundary of Γ_a is given by

$$\partial\Gamma_a = \{u \in E_a : \text{there exists } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\} = E_a \setminus \Gamma_a.$$

6.3. Topological charge and connected components of Γ_a. Recall that the topological charge ch was defined in Section 4.3.

Now, for every $q \in \mathbb{Z}$ we set

$$\Gamma_a^q = \{u \in \Gamma_a : \mathrm{ch}(u) = q\}.$$

Since the topological charge is continuous with respect to uniform convergence and the continuity of the embeddings E_a in L^∞ ensures that the topological charge is continuous on Γ_a, it follows that Γ_a^q is open in E_a, since we also have $\Gamma_a = \bigcup_{q\in\mathbb{Z}} \Gamma_a^q$ and $\Gamma_a^q \cap \Gamma_a^p = \emptyset$ for $p \neq q$. We conclude that every Γ_a^q is a connected component of Γ_a. We observe that for every $q \in \mathbb{Z}$ the component Γ_a^q is homeomorphic to the component Γ_a^{-q} contained in the space C, which we have considered in the preceding section. So for every $u \in \Gamma_a$ we can define the charge $\mathrm{ch}(u) \in \mathbb{Z}$. Now, we consider the minimizer set of f_a in the open set

$$\Gamma_a^* = \{u \in \Gamma_a : \mathrm{ch}(u) \neq 0\}.$$

6.4. Properties of the energy functional

Lemma 6.4.1. *The functional f_a takes real values and is continuous on Γ_a.*

Proof. We have

$$\begin{aligned} f_a(u) &= \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^{r(x)} + \frac{b}{2}|\nabla u|^{p(x)}\right) dx + \int_{\mathbb{R}^n} V(u)\,dx \\ &= \underbrace{\frac{a}{2}\rho_{r(x)}(u) + \frac{b}{2}\rho_{p(\cdot)}(u)}_{} + \underbrace{\int_{\mathbb{R}^n} V(u)\,dx}_{}\,. \end{aligned}$$

The first term on the right-hand side is finite and continuous. Let us prove that the second term is finite and continuous.

We have $V(\xi) = V''(0)\xi\cdot\xi + o(\xi^2)$. By ($\mathrm{V}_3$) there exist a small neighborhood of $0 \in \mathbb{R}^{n+1}$ and $M > 0$ such that, for every ξ in that neighborhood, we have

$$V(\xi) \leq M|\xi|^2. \tag{6.4.1}$$

Since every $u \in E_a$ decays to zero at infinity (see Proposition 6.2.5), there exists a ball B_u such that, for every $x \in \mathbb{R}^n \setminus B_u$, $|u(x)| < \epsilon$, so by (6.4.1), for ϵ sufficiently small,

$$V(u(x)) \leq M|u(x)|^2. \tag{6.4.2}$$

Since $u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, we deduce

$$\int_{\mathbb{R}^n\setminus B_u} V(u)\,dx < \infty.$$

On the other hand, since u is continuous (see Proposition 6.2.5), we also have

$$\int_{B_u} V(u)\,dx < \infty.$$

Let $(u_k) \subset \Lambda_a$ be a sequence such that $f_a(u_k) < \infty$ and $u_k \to u \in E_a$. We show that

$$\int_{\mathbb{R}^n} V(u_k) \to \int_{\mathbb{R}^n} V(u).$$

Since $f_a(u_k) < \infty$ and by Lemma 6.4.3, u belongs to Λ_a. By (6.2.3) we have $u_k \to u$ in $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Then

$$\int_{B_u} V(u_k)\,dx \to \int_{B_u} V(u)\,dx. \tag{6.4.3}$$

By (6.4.2),

$$\int_{\mathbb{R}^n\setminus B_u} V(u(x))\,dx \le \int_{\mathbb{R}^n\setminus B_u} |u(x)|^2\,dx,$$

and since $u_k \to u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, the dominated convergence theorem gives

$$\int_{\mathbb{R}^n\setminus B_u} V(u_k)\,dx \to \int_{\mathbb{R}^n\setminus B_u} V(u)\,dx. \ \blacksquare \tag{6.4.4}$$

Lemma 6.4.2. *The functional f_a is coercive in Γ_a, that is, for every sequence $(u_k) \subset \Gamma_a$ such that $\|u_k\|_a \to \infty$, we have $f_a(u_k) \to \infty$.*

Proof. In the case $a > 0$, $n > r(x) \ge 2$, we have

$$\|u\|_a = a\|\nabla u\|_{L^{r(\cdot)}} + \|\nabla u\|_{L^{p(\cdot)}} + \|u\|_{L^{r(\cdot)}}.$$

Let $(u_k) \subset \Gamma_a$ be such that $\|u_k\|_a \to \infty$ as $k \to \infty$. It is clear that if

$$a\|\nabla u_k\|_{L^{r(\cdot)}} + \|\nabla u_k\|_{L^{p(\cdot)}} \to \infty \quad \text{as } k \to \infty, \tag{6.4.5}$$

then $f_a(u_k) \to \infty$ as $k \to \infty$.

Assume now that there exists $c_* > 0$ such that

$$a\|\nabla u_k\|_{L^{r(\cdot)}} + \|\nabla u_k\|_{L^{p(\cdot)}} < c_* \tag{6.4.6}$$

and

$$\|u_k\|_{L^{r(\cdot)}} \to \infty \quad \text{as } k \to \infty. \tag{6.4.7}$$

We shall prove that

$$\int_{\mathbb{R}^n} V(u_k)\,dx \to \infty \quad \text{as } k \to \infty.$$

By (V_3), for every $R > 0$ there exists $\omega_R > 0$ such that

$$|\xi| \le R \ \Rightarrow\ V(\xi) \ge \omega_R|\xi|^2,\ r^- = 2. \tag{6.4.8}$$

By (V_6) and (6.4.8), there exists $R > 0$ such that

$$|\xi| \le R \ \Rightarrow\ V(\xi) \ge \omega_R|\xi|^{r(\cdot)},\ \omega_R > 0. \tag{6.4.9}$$

For every $k \in \mathbb{N}$, we set

$$A_k = \{x \in \mathbb{R}^n : |u_k(x)| \le R\},$$

where $u_k \in W^{1,r(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$. By the inequality (see [39, Theorem 8.3.1, p. 265])

$$\|u_k\|_{L^{r^*(\cdot)}} \leq c\|\nabla u_k\|_{L^{r(\cdot)}}, \qquad r^*(x) = \frac{r(x)\cdot n}{n - r(x)},\ n > r(x) \geq 2, \tag{6.4.10}$$

from (6.4.6), we obtain

$$\|u_k\|_{L^{r^*(\cdot)}} < c_*. \tag{6.4.11}$$

Moreover, from (5.2.4), there exists $M \geq 0$ independent of u_k such that

$$\begin{aligned} |u_k(x)| &\leq \left|\int_{Q_k} u_k\, dy\right| + M\|\nabla u_k\|_{L^{p(\cdot)}(Q_k)} \qquad (\mathrm{meas}(Q_k) = 1) \\ &\leq \|u_k\|_{L^{r^*(\cdot)}(Q_k)} + M\|\nabla u_k\|_{L^{p(\cdot)}(Q_k)}. \end{aligned}$$

By (6.4.5) and (6.4.11), for any $x \in \mathbb{R}^n$, we have

$$|u_k(x)| < c_* + Mc_*. \tag{6.4.12}$$

Now, there exists $c > 0$ such that

$$\mathrm{meas}(\mathbb{R}^n \setminus A_k) < c. \tag{6.4.13}$$

From (6.4.12) and (6.4.13), we deduce that there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}^n \setminus A_k} |u_k|^{r(x)}\, dx < c_1. \tag{6.4.14}$$

By (6.4.11), we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} V(u_k)\, dx &\geq \int_{A_k} V(u_k)\, dx \\ &\geq \omega_R \int_{A_k} \|u_k\|^{r(x)}\, dx \\ &\geq \omega_R\Big(\rho_{r(x)}(u) - \int_{\mathbb{R}^n \setminus A_k} |u_k|^{r(x)}\, dx\Big). \end{aligned}$$

From (6.4.14) and (6.4.7), we have

$$\lim_k \int_{\mathbb{R}^n} V(u_k)\, dx \geq \omega_R(\rho_{r(x)}(u) - c_1) \to \infty \qquad \text{as } k \to \infty.$$

In the case $a = 0$ or $n = 2 \equiv r(\cdot)$, by (V_5), there exists $r_* > 0$ such that, for every $\xi \in \mathbb{R}^n$ with $|\xi| \geq r_*$, we have

$$V(\xi) \geq \nu/2. \tag{6.4.15}$$

Let $(u_k) \subset \Gamma_a$ be a sequence such that $\|u_k\|_0 \to \infty$ as $k \to \infty$. Since the functional f_a is invariant with respect to translations in $\mathbb{R}^n$, we can assume

$$\begin{aligned} &\|u_k\|_{L^\infty} = |u_k(0)|, \\ &\|\nabla u_k\|_{L^{p(\cdot)}} \leq M_* \quad \text{and} \quad \|u_k\|_{L^2} \to \infty \text{ as } k \to \infty. \end{aligned} \tag{6.4.16}$$

Here we have two subcases:

$$\text{(a) } \|u_k\|_{L^\infty} \to \infty \text{ as } k \to \infty. \tag{6.4.17}$$

$$\text{(b) } \|u_k\|_{L^\infty} \text{ is bounded.} \tag{6.4.18}$$

In subcase (a), by (6.4.17), we can choose a sequence $(R_k) \subset (0,\infty)$ such that

$$R_* \leq \|u_k\|_{L^\infty} - K(R_k^{(p^+-n)/p^+} + R_k^{(p^--n)/p^-}) \quad \text{and} \quad R_k \to \infty, \tag{6.4.19}$$

where $K = cM_*$ and c is the same constant as in (6.2.2). For every $y \in \mathbb{R}^n$, we have

$$|u_k(0)| - |u_k(y)| \leq |u_k(0) - u_k(y)|.$$

Hence by (6.2.2), we obtain

$$|u_k(0)| - |u_k(y)| \leq K(|y|^{(p^+-n)/p^+} + |y|^{(p^--n)/p^-}).$$

From (6.4.16), we get

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K(|y|^{(p^+-n)/p^+} + |y|^{(p^--n)/p^-}).$$

For $|y| \leq R_k$ and (6.4.19), we have

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K(R_k^{(p^+-n)/p^+} + R_k^{(p^--n)/p^-}) \geq R_*. \tag{6.4.20}$$

From (6.4.15) and (6.4.20), we get

$$\int_{\mathbb{R}^n} V(u_k)\,dx \geq \int_{B(0,r_k)} V(u_k)\,dx \geq \frac{\nu}{2}\operatorname{meas}(B(0,R_k)).$$

This implies that $\int_{\mathbb{R}^n} V(u_k)\,dx \to \infty$ as $R_k \to \infty$.

In subcase (b), we assume there exists $\bar{M} > 0$ such that $\|u_k\|_{L^\infty} \leq \bar{M}$. From (6.4.8), we obtain $\int_{\mathbb{R}^n} V(u_k)\,dx \geq \omega_{\bar{M}}\|u_k\|_{L^2} \to \infty$ as $k \to \infty$. ■

We are going to study the behaviour of f_a when u approaches the boundary of Γ_a. We remark that $\partial\Gamma_a = E_a \setminus \Gamma_a$.

LEMMA 6.4.3. *Let* $(u_k) \subset \Gamma_a$ *be a weakly converging sequence. If the weak limit belongs to* $\partial\Gamma_a$, *then* $f_a(u_k) \to \infty$ *as* $k \to \infty$.

Proof. The proof is the same as that of Lemma 5.4.3. ■

COROLLARY 6.4.4. *For every* $b > 0$, *there exists* $d_* = d(b)$ *such that, for every* $u \in \Gamma_a$,

$$f_a(u) \leq b \ \Rightarrow\ \min_x |u(x) - \eta| \geq d_*.$$

Proof. The proof is the same as that of Corollary 4.4.5. ■

LEMMA 6.4.5. *The functional* f_a *is weakly lower semicontinuous in* Γ_a.

Proof. The proof is the same as that of Lemma 4.4.6. ■

PROPOSITION 6.4.6. *There exists* $\Delta_a > 0$ *such that, for every* $u \in \Gamma_a$ *with* $\|u\|_{L^\infty} \geq 1$, *we have* $f_a(u) \geq \Delta_a$. *It is easy to see that* $\operatorname{ch}(u) \neq 0$ *implies* $\|u\|_{L^\infty} > 1$.

Proof. The proof is the same as that of Proposition 5.4.6. ■

6.5. Existence result

THEOREM 6.5.1. *The minimum points* $u \in \Gamma_a$ *for the functional* f_a *are weak solutions of the system* (6.1.5).

Proof. Let u be a minimum point of f_a and $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Let e_j denote the jth vector of the canonical basis in $\mathbb{R}^n$. If ϵ is sufficiently small, then $u + \epsilon e_j h \in \Gamma_a$ and $f_a(u + \epsilon e_j h) < \infty$. Since u is a minimum point of f_a, for $1 \le j \le n+1$ we have

$$\begin{aligned} 0 &= \frac{df(u+\epsilon e_j h)}{d\epsilon}\bigg|_{\epsilon=0} \\ &= \int_{\mathbb{R}^n} \left(\frac{a}{2}(r(x)|\nabla u|^{r(x)-2}\nabla u_j \nabla h) + \frac{b}{2}(p(x)|\nabla u|^{p(x)-2}\nabla u_j \nabla h) + \frac{\partial V(\xi)}{\partial \xi_j} h \right) dx. \end{aligned}$$

By Green's formula,

$$\int_{\mathbb{R}^n} \frac{b}{2}(p(x)|\nabla u|^{p-2}\nabla u_j \nabla h)\, dx = \int_{\mathbb{R}^n} -\frac{b}{2} \operatorname{div}(p(x)|\nabla \cdot u|^{p-2}\nabla u_j) h\, dx.$$

So

$$\int_{\mathbb{R}^n} \left(-\frac{a}{2} \operatorname{div}(r(x)|\nabla \cdot u|^{r-2}\nabla u_j) - \frac{b}{2} \operatorname{div}(p(x)|\nabla \cdot u|^{p-2}\nabla u_j) + \frac{\partial V(\xi)}{\partial \xi_j} \right) \cdot h\, dx = 0$$

for $1 \le j \le n+1$ and for any $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Then

$$\int_{\mathbb{R}^n} \left[-\frac{a}{2}\Delta_{r(\cdot)}u - \frac{b}{2}\Delta_{p(\cdot)}u + V'(u) \right] \phi\, dx = 0 \quad \text{for every } \phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

This implies by density

$$-\frac{a}{2}\Delta_{r(\cdot)}u - \frac{b}{2}\Delta_{p(\cdot)}u + V'(u) = 0. \ ■$$

PROPOSITION 6.5.2 (Splitting Proposition). *Let $(u_k) \subset \Gamma_a^*$ be a sequence and M be a positive real number such that $f_a(u_k) \le M$. Then there exists $l \in \mathbb{N}$ such that*

$$1 \le l \le M/\Delta_a,$$

where Δ_a was introduced in Proposition 6.4.6, and there exist $\bar{u}_1, \dots, \bar{u}_l \in \Gamma_a$ and $(x_k^1), \dots, (x_k^l) \subset \mathbb{R}^n$ such that, up to a subsequence,

$$\begin{gathered} u_k(\cdot + x_k^i) \rightharpoonup \bar{u}_i, \quad |x_k^i - x_k^j| \to \infty, \quad i \ne j, \\ \sum_{i=1}^{l} f_a(\bar{u}_i) \le \liminf_{k\to\infty} f_a(u_k), \quad \operatorname{ch}(u_k) = \sum_{i=1}^{l} \operatorname{ch}(\bar{u}_i). \end{gathered}$$

Proof. From Lemmas 6.4.2, 6.4.3 and 6.4.5, and by the same method as used for Proposition 4.5.1. ■

The minimum is attained on the set Γ_a, and it is easy to see that $u \equiv 0$ is a trivial solution. But of course we are interested in nontrivial solutions. We consider the following problem:

$$I_* = \inf_{u\in\Gamma_a^*} f_a(u), \quad \Gamma_a^* = \{u \in E_a : \operatorname{ch}(u) \ne 0\}.$$

The functional is bounded below and the set Γ_a^* is not empty. We consider fields u having the form

$$u(x) = \left(\frac{2}{1+|x|^m}, \frac{1}{1+|x|^m}x \right). \tag{6.5.1}$$

LEMMA 6.5.3. *There exists an* $m \geq 1$ *such that the field* u *defined in* (6.5.1) *belongs to* Γ_a^*.

Proof. The proof is the same as that of Lemma 5.5.3. ■

Moreover, the set Γ_a^* is open in the space E_a; indeed, $\Gamma_a^* = \bigcup_{q \in \mathbb{N}^*} \Gamma_a^q$ and $\Gamma_a^q \cap \Gamma_a^p = \emptyset$ for $p \neq q$, where Γ_a^q is a connected component.

THEOREM 6.5.4. *Let* $a \geq 0$, $b > 0$, $p^- > n > 2$ *and* $2 = r^- \leq r^+ \leq n$. *If* V *satisfies* (V_1)–(V_6) *and* p *satisfies* (p_1)–(p_2), *and if* r *satisfies* (r_1)–(r_4), *then there exists a weak solution of* (6.1.5) (*i.e., a static solution of* (6.1.3)), *which is a minimizer of the energy functional* (6.1.7) *in the class of maps whose topological charge is different from* 0.

Proof. By the Splitting Proposition and the same technique used in the proof of Theorem 4.5.3 we conclude that there exists a weak solution (static) of (6.1.5). And with a suitable change of variable (6.1.6), we deduce a solution of (6.1.3). ■

REMARK 6.5.1. The functional exibits an invariance for the symmetry group of rotations and translations; indeed, for every function u and $g \in O(n)$, if we set $u_g(x) = u(gx)$, we have immediately

$$f_a(u_g) = f_a(u).$$

Thus our theorem gives the existence of an orbit of minimum solutions. This orbit consists of two connected components, which are identified, respectively, by $\bar{u}$ and

$$\bar{u} \circ \mathcal{P}(x) = \bar{u}(-x).$$

Since typically $n = 3$ is odd, $\bar{u} \circ \mathcal{P}$ and $\bar{u}$ have opposite topological charge.

7. Appendix

7.1. Appendix A. Compact embeddings. In this appendix we first prove a result which slightly extends a compactness theorem of [26, 81]. We set

$$W^{1,2}_R(\mathbb{R}^n,\mathbb{R}) = \{u \in W^{1,2}(\mathbb{R}^n,\mathbb{R}) : u \text{ radial}\}.$$

PROPOSITION 7.1.1. *Let* $n \geq 2$. *Then* $W^{1,2}_R(\mathbb{R}^n,\mathbb{R})$ *is compactly embedded in* $L^s(\mathbb{R}^n,\mathbb{R})$ *for every* $s \in \,]2, 2^*[$, *where*

$$2^* = \begin{cases} \infty & \text{if } n = 2, \\ 2n/(n-2) & \text{if } n > 2. \end{cases}$$

Proof. For $n > 2$, the proof is given in [26, Theorem A.I′]. We give the proof for $n = 2$. First we recall that, for every $m \in [2, \infty[$,

$$W^{1,2}(\mathbb{R}^2,\mathbb{R}) \subset L^m(\mathbb{R}^2,\mathbb{R}). \tag{7.1.1}$$

Fix $s \in \,]2, \infty[$ and consider a bounded sequence $(u_k) \subset W^{1,2}_R(\mathbb{R}^2,\mathbb{R})$; by (7.1.1), (u_k) is bounded in $L^s(\mathbb{R}^2,\mathbb{R})$. So, up to a subsequence,

$$u_k \rightharpoonup u \quad \text{in } L^s(\mathbb{R}^2,\mathbb{R}). \tag{7.1.2}$$

Now we have to prove that the convergence is strong. Let $m \in \,]s, \infty[$. Clearly, (u_k) is bounded in $L^2(\mathbb{R}^2,\mathbb{R}) \cap L^m(\mathbb{R}^2,\mathbb{R})$. Now we apply the compactness Lemma A.I of [26] with

$$P(t) = t^s, \qquad Q(t) = t^2 + t^m.$$

We conclude that

$$\|u_k\|_{L^s} \to \|u\|_{L^s}. \tag{7.1.3}$$

From (7.1.2) and (7.1.3) we have $u_k \to u$ in $L^s(\mathbb{R}^2,\mathbb{R})$. ■

THEOREM 7.1.2. *If* $\mathcal{W}$ *is a bounded subset of* $W^{1,2}(\mathbb{R}^n,\mathbb{R}^{n+1})$, *then*

$$\mathcal{W}_R = \{u \in \mathcal{W} : u \text{ radial}\}$$

is relatively compact in $L^s(\mathbb{R}^n,\mathbb{R}^{n+1})$ *for every* $s \in \,]2, 2^*[$.

Proof. Fix $s \in \,]2, 2^*[$ and consider a sequence $(u_k) \subset \mathcal{W}_R$. We have to show that there exists a subsequence that is strongly convergent in $L^s(\mathbb{R}^n,\mathbb{R}^{n+1})$. Since (u_k) is bounded in $W^{1,2}(\mathbb{R}^n,\mathbb{R}^{n+1})$, there exists $u \in W^{1,2}(\mathbb{R}^n,\mathbb{R}^{n+1})$ such that, up to a subsequence,

$$u_k \rightharpoonup u \quad \text{in } W^{1,2}(\mathbb{R}^n,\mathbb{R}^{n+1}). \tag{7.1.4}$$

From the continuous embedding $W^{1,2}(\mathbb{R}^n,\mathbb{R}^{n+1}) \hookrightarrow L^s(\mathbb{R}^n,\mathbb{R}^{n+1})$ we deduce that

$$u_k \rightharpoonup u \quad \text{in } L^s(\mathbb{R}^n,\mathbb{R}^{n+1}). \tag{7.1.5}$$

On the other hand, $(|u_k|)$ is bounded in $W^{1,2}(\mathbb{R}^n, \mathbb{R})$. Indeed,

$$\int \big|\nabla |u_k|\big|^2 \, dx \leq \int |\nabla u_k|^2 \, dx.$$

Then, by Proposition 7.1.1, we get

$$|u_k| \to \chi \quad \text{in } L^s(\mathbb{R}^n, \mathbb{R}), \tag{7.1.6}$$

and, up to a subsequence, $|u_k| \to \chi$ a.e. in $\mathbb{R}^n$. Moreover, from (7.1.4) we deduce $u_k \rightharpoonup u$ in $L^s_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^{n+1})$, and therefore, by a Cantor diagonal process, we can select a subsequence such that $u_k \to u$ a.e. in $\mathbb{R}^n$. So we conclude that

$$\chi = |u|. \tag{7.1.7}$$

From (7.1.6) and (7.1.7) we deduce

$$\|u_k\|_{L^s} \to \|u\|_{L^s}. \tag{7.1.8}$$

Then (7.1.5) and (7.1.8) allow us to conclude that $u_k \to u$ in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$. ∎

7.2. Appendix B. Continuity and invertibility of Δ_p

Lemma 7.2.1. *The map* $\Delta_p : E \to E'$ *defined by*

$$\langle -\Delta_p u, v\rangle_{E'_a \times E_a} = \int_{\mathbb{R}^n} |\nabla u|^{p-2} (\nabla u \mid \nabla v) \, dx, \qquad p > 2,$$

is continuous.

Proof. Recall that E is the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Let $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Then

$$\begin{aligned}
\langle \Delta_p u - \Delta_p v, h\rangle &= \int_{\mathbb{R}^n} \big(|\nabla v|^{p-2}(\nabla u \mid \nabla h) - |\nabla u|^{p-2}(\nabla v \mid \nabla h)\big)\, dx \\
&= \int_{\mathbb{R}^n} \big(|\nabla v|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla v \mid \nabla h\big)\, dx \\
&\leq \int_{\mathbb{R}^n} \big||\nabla v|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla v\big| \cdot |\nabla h| \, dx \\
&\leq \beta \int_{\mathbb{R}^n} \big||\nabla v|^{p-2} + |\nabla u|^{p-2}\big| \cdot |\nabla u - \nabla v| \cdot |\nabla h| \, dx \quad \text{(from Lemma 7.6.2)} \\
&\leq \beta\big(\|\nabla v\|_{L^p}^{p-2} + \|\nabla u\|_{L^p}^{p-2}\big) \cdot \|\nabla u - \nabla v\|_{L^p} \cdot \|\nabla h\|_{L^p} \quad \text{(from Hölder's inequality).}
\end{aligned}$$

∎

Lemma 4.6.6 follows from the next result.

Theorem 7.2.2. *If* H *is a positive definite matrix of order* $N+1$, *then the map* $\mathcal{A} : E \to E'$ *defined by*

$$\langle \mathcal{A}u, v\rangle = \langle -\Delta u - \Delta_p u + Hu, v\rangle = \int_{\mathbb{R}^n} \big((\nabla u \mid \nabla v) + |\nabla u|^{p-2}(\nabla u \mid \nabla v) + Hu \cdot v\big)\, dx$$

is invertible with continuous inverse.

For the proof we need some preliminary results. The first deals with the monotonicity of $-\Delta_p u$; for the convenience of the reader we give a simple proof (see also [68] and [15] for the scalar case).

LEMMA 7.2.3. *There exists a constant $c > 0$ such that, for every $u, v \in E$,*

$$\langle \Delta_p u - \Delta_p v, u - v\rangle \geq c\|\nabla u - \nabla v\|_{L^p}^p, \qquad p > 2. \tag{7.2.1}$$

Proof. We prove (7.2.1) for $u, v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Then our statement follows by density.

For all $u, v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\langle \Delta_p u - \Delta_p v, u - v\rangle = \int_{\mathbb{R}^n} \left[|\nabla u|^p + |\nabla v|^p - (\nabla u \mid \nabla v)(|\nabla u|^{p-2} + |\nabla v|^{p-2})\right] dx$$

and

$$\|\nabla u - \nabla v\|_{L^p}^p = \int_{\mathbb{R}^n} |\nabla u - \nabla v|^p \, dx$$

where

$$|\nabla u| = \sqrt{\sum_{j,i}\left(\frac{\partial u^j}{\partial x_i}\right)^2}, \qquad (\nabla u | \nabla v) = \sum_{j,i} \frac{\partial u^j}{\partial x_i}\frac{\partial v^j}{\partial x_i}.$$

So it is enough to prove that there exists $c > 0$ such that, for all $X, Y \in \mathbb{R}^{n(n+1)}$,

$$|X|^p + |Y|^p - (X \mid Y)(|X|^{p-2} + |Y|^{p-2}) \geq c|X - Y|^p. \tag{7.2.2}$$

Substituting

$$-(X \mid Y) = \tfrac{1}{2}(|X - Y|^2 - |X|^2 - |Y|^2)$$

into (7.2.2), we get

$$\begin{aligned} &\tfrac{1}{2}(|X|^p + |Y|^p) + \tfrac{1}{2}|X - Y|^2(|X|^{p-2} + |Y|^{p-2}) \\ &\qquad \geq c|X - Y|^p + \tfrac{1}{2}\big(|Y|^2|X|^{p-2} + |X|^2|Y|^{p-2}\big). \end{aligned} \tag{7.2.3}$$

We notice that (7.2.3) can be obtained from the inequalities

$$\begin{aligned} &\tfrac{1}{2}(|X|^p + |Y|^p) \geq \tfrac{1}{2}(|Y|^2|X|^{p-2} + |X|^2|Y|^{p-2}), \\ &\tfrac{1}{2}|X - Y|^2(|X|^{p-2} + |Y|^{p-2}) \geq c|X - Y|^p. \end{aligned}$$

The first holds true for any vectors X, Y by Lemma 7.6.5; the second is also true by Lemma 7.6.4. ∎

Proof of Theorem 7.2.2. First we prove that $\mathcal{A}$ is invertible. For every $h \in E'$, the solution of $\mathcal{A}u = h$ can be obtained as the critical points of the functional

$$\mathcal{J}(u) = \frac{1}{p}\int |\nabla u|^p \, dx + \frac{1}{2}\int |\nabla u|^2 \, dx + \frac{1}{2}\int Hu \cdot u \, dx - \langle h, u\rangle.$$

Since the matrix H is positive definite, we have

$$\frac{1}{p}\int |\nabla u|^p \, dx + \frac{1}{2}\int |\nabla u|^2 \, dx + \frac{1}{2}\int Hu \cdot u \, dx \geq \frac{1}{p}\|\nabla u\|_{L^p}^p + m\|u\|_{W^{1,2}}^2. \tag{7.2.4}$$

On the other hand, for every $\lambda > 0$,

$$\langle h, u\rangle \leq \|h\|_{E'}\|u\|_E \leq \frac{1}{2\lambda}\|h\|_{E'} + \frac{\lambda}{2}\|u\|_E, \tag{7.2.5}$$

where

$$\|u\|_E = \|\nabla u\|_{L^p} + \|u\|_{W^{1,2}}.$$

Taking into account (7.2.4), (7.2.5), we conclude that the functional $\mathcal{J}$ is lower bounded. Moreover, it is strictly convex, so it has a unique critical point. Now, let (h_k) be a sequence of elements of E' and $h \in E'$ such that $h_k \to h$ in E'. Then, we can consider (u_k) and u in E such that

$$\mathcal{A}(u_k) = h_k, \qquad \forall k \in \mathbb{N}, \qquad \mathcal{A}(u) = h.$$

We want to prove that $u_k \to u$ in E. By (7.2.1), using again the fact that H is positive definite, we get $c_1 > 0$ such that

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle \geq c_1(\|\nabla u - \nabla v\|_{L^p}^p + \|u - v\|_{W^{1,2}}^2).$$

Then

$$\begin{aligned} c_1\big(\|\nabla u_k - \nabla u\|_{L^p}^p + \|u_k - u\|_{W^{1,2}}^2\big) &\leq \langle \mathcal{A}(u_k) - \mathcal{A}(u), u_k - u\rangle = \langle h_k - h, u_k - u\rangle \\ &\leq \|h_k - h\|_{E'}\|u_k - u\|_E, \end{aligned}$$

that is,

$$\frac{1}{c_1}\|h_k - h\|_{E'} \geq \frac{\|\nabla u_k - \nabla u\|_{L^p}^p + \|u_k - u\|_{W^{1,2}}^2}{\|\nabla u_k - \nabla u\|_{L^p} + \|u_k - u\|_{W^{1,2}}}.$$

By applying Lemma 7.6.3 to $a_k = \|\nabla u_k - \nabla u\|_{L^p}$ and using $b_k = \|u_k - u\|_{W^{1,2}}$, we deduce that

$$\lim_k \|\nabla u_k - \nabla u\|_{L^p} = 0, \qquad \lim_k \|\nabla u_k - \nabla u\|_{W^{1,2}} = 0.$$

So $u_k \to u$ in E. ■

7.3. Appendix C. Linear operators

PROPOSITION 7.3.1 ([30, Proposition 3.5, p. 58]). *Let* (x_n) *be a sequence in* E. *Then*

(i) $[x_k \rightharpoonup x$ *weakly in* $\sigma(E, E^*)] \Leftrightarrow [\langle f, x_k\rangle \to \langle f, x\rangle\ \forall f \in E^*]$.
(ii) *If* $x_k \to x$ *strongly, then* $x_k \rightharpoonup x$ *weakly in* $\sigma(E, E^*)$.
(iii) *If* $x_k \rightharpoonup x$ *weakly in* $\sigma(E, E^*)$, *then* $(\|x_k\|)$ *is bounded and* $\|x\| \leq \liminf \|x_k\|$.
(iv) *If* $x_k \rightharpoonup x$ *weakly in* $\sigma(E, E^*)$ *and* $f_k \to f$ *strongly* E^* (*i.e.*, $\|f_k - f\|_{E^*} \to 0$), *then* $\langle f_k, x_k\rangle \to \langle f, x\rangle$.

THEOREM 7.3.2 ([30, Theorem 6.4 (Schauder), p. 159]). $T \in \mathcal{K}(E, F)$ *if and only if* $T^* \in \mathcal{K}(E^*, F^*)$, *where* $\mathcal{K}(E, F)$ *denotes the set of all compact operators from* E *to* F, E^* *denotes the dual space of* E, *and* T^* *denotes the adjoint of* T.

DEFINITION 7.3.1. Let E be a reflexive and separable Banach space, and $\mathcal{A}$ a mapping from E to E'. Then $\mathcal{A}$ is *monotone* if

$$\forall u, v \in E, \quad \langle \mathcal{A}u - \mathcal{A}v, u - v\rangle \geq 0.$$

$\mathcal{A}$ is *coercive* if

$$\lim_{\|u\|_E \to \infty} \frac{\langle \mathcal{A}u, v\rangle}{\|u\|_E} = \infty.$$

7.4. Appendix D. Orthogonal group

DEFINITION 7.4.1. $g \in \mathcal{L}(\mathbb{R}^n)$ is said to be *orthogonal* when it preserves the scalar product: for all $x, y \in \mathbb{R}^n$,

$$\langle g(x), g(y)\rangle = \langle x, y\rangle.$$

We denote by $O(n)$ the set of orthogonal maps.

Properties:

1. g is orthogonal if and only if g preserves the norm.
2. If g is orthogonal, then it is bijective.
3. If $g \in O(n)$ then its inverse is also in $O(n)$. Furthermore, if $f \in O(n)$ then $g \circ f \in O(n)$.

We conclude that $O(n)$ is a group. Furthermore, it is a compact group.

Proposition 7.4.1. *The following are equivalent:*

(i) *g is orthogonal.*
(ii) *There exists an o.n.b. in which the matrix of g is orthogonal.*
(iii) *In all o.n.b., the matrix of g is orthogonal.*

Here o.n.b. means orthonormal base.

Definition 7.4.2. The *orthogonal matrices* are matrices $M \in \mathcal{M}_n(\mathbb{R})$ satisfying

$$^tM \cdot M = I_n.$$

We denote by $O_n(\mathbb{R})$ the set of orthogonal matrices.

Remark 7.4.1. For all $g \in O(n)$ there exists $M \in O_n(\mathbb{R})$ such that

$$g(x) = M \cdot x, \qquad |g(x)| = |M \cdot x| = |x|,$$

with $\det M = J_g = 1$, J being denoting the Jacobian matrix.

Proposition 7.4.2. *For all $M \in O_n(\mathbb{R})$ and $A \in \mathcal{M}_n(\mathbb{R})$,*

$$|A \cdot M| = |M \cdot A| = |A|.$$

7.5. Appendix E. Haar measure. Let G a locally compact group.

Theorem 7.5.1 ([31, 85]).

- *Existence: There exists on G a Radon measure invariant under left translations. Such a measure is called a left Haar measure on G.*
- *Unicity: All left Haar measures on G are proportional.*
- *Convention: If G is compact, there is a canonical choice of Haar measure on G, namely the left measure Haar that is a probability measure on G (i.e. the measure of G is 1). In general, we choose a left Haar measure on G, which one calls (wrongly) the Haar measure to G and denotes λG or more simply λ. Other notation: $d\lambda(x) = dx$.*

We notice that λ is left translation invariant, meaning that $\lambda(gB) = \lambda(B)$ for any Borel subset B of G and any $g \in G$.

7.6. Appendix F. Elementary calculus

Lemma 7.6.1 ([68, Lemma A.0.5, p. 80]). *Let $x, y \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ be the standard scalar product on $\mathbb{R}^n$. Then*

$$\langle x|x|^{p-2} - y|y|^{p-2}, x - y\rangle \geq \begin{cases} c_p|x-y|^p & \text{if } p \geq 2, \\ c_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1 < p < 2. \end{cases}$$

Lemma 7.6.2 (see [56]).

(i) *If* $p \in [2, \infty)$, *then*

$$\big| z|z|^{p-2} - y|y|^{p-2} \big| \le \beta |z-y|(|z|+|y|)^{p-2} \quad \textit{for all } z, y \in \mathbb{R}^n$$

with β *independent of* y *and* z.

(ii) *If* $p \in (1, 2]$, *then*

$$\big| z|z|^{p-2} - y|y|^{p-2} \big| \le \beta (|z|+|y|)^{p-1} \quad \textit{for all } z, y \in \mathbb{R}^n$$

with β *independent of* y *and* z.

Lemma 7.6.3. *Let* (a_k) *and* (b_k) *be sequences of nonnegative numbers such that*

$$\lim_k \frac{a_k^p + b_k^2}{a_k + b_k} = 0. \tag{7.6.1}$$

Then

$$\lim_k a_k = \lim_k b_k = 0.$$

Proof. Since $a_k, b_k \ge 0$, from (7.6.1) we immediately deduce

$$\lim_k \frac{a_k^p}{a_k + b_k} = 0, \tag{7.6.2}$$

$$\lim_k \frac{b_k^2}{a_k + b_k} = 0. \tag{7.6.3}$$

For contradiction, assume that, up to a subsequence,

$$a_k \ge \delta > 0. \tag{7.6.4}$$

From (7.6.2) and (7.6.4) we deduce $\lim_k (a_k + b_k) = \infty$. Then, up to a subsequence, either

$$\lim_k a_k = \infty, \tag{7.6.5}$$

or $\lim_k b_k = \infty$. Suppose that (7.6.5) holds true. Then we write (7.6.2) as

$$\lim_k \frac{a_k^{p-1}}{1 + (b_k/a_k)} = 0,$$

from which we deduce $\lim_k b_k/a_k = \infty$. So, for k sufficiently large,

$$a_k \le b_k. \tag{7.6.6}$$

Then it is easy to deduce

$$\frac{1}{2} b_k \le \frac{b_k^2}{a_k + b_k}. \tag{7.6.7}$$

Now, from (7.6.7) and (7.6.3) we deduce $\lim_k b_k = 0$. On the other hand, (7.6.5) and (7.6.6) imply $\lim_k b_k = \infty$, a contradiction. The proof for the other case is analogous. ■

Lemma 7.6.4. *Let* $a, b \ge 0$ *and* $1 \le p < \infty$. *Then*

$$(a+b)^p \le 2^{p-1}(a^p + b^p).$$

Proof. The case $a = 0$ is trivial. If $a > 0$, the inequality is equivalent to

$$(1+x)^p \le 2^{p-1}(1 + x^p).$$

We set

$$f(x) = \frac{(1+x)^p}{1+x^p}$$

which satisfies

$$f(0) = 1 = \lim_{x\to\infty} f(x)$$

and $f(x) > 0$ for all $0 < x < \infty$.

So, for $x \geq 0$, f attains a maximum only at the point $x = 1$, $f'(1) = 0$. Now $f(1) = 2^{p-1}$ immediately gives the result. ■

LEMMA 7.6.5. *Let* $a, b \geq 0$ *and* $p \geq 2$. *Then*

$$a^2b^{p-2} + b^2a^{p-2} \leq a^p + b^p.$$

Proof. By homogeneity we can assume that $a = 1$ and $b < 1$. The inequality is equivalent to

$$b^p - b^{p-2} - b^2 + 1 \geq 0.$$

Indeed, $b^p - b^{p-2} - b^2 + 1 = (b^2 - 1)(b^{p-2} - 1) \geq 0$. ■

REMARK 7.6.1. From Lemma 7.6.4, it is easy to see that under the assumption $1 \leq p^- \leq p^+ < \infty$, we have

$$(a+b)^{p(x)} \leq 2^{p^+-1}(a^{p(x)} + b^{p(x)}).$$

From Lemma 7.6.5, and under the assumption $2 \leq p^- \leq p^+$, we have

$$a^2b^{p(x)-2} + b^2a^{p(x)-2} \leq a^{p(x)} + b^{p(x)}.$$

Conclusions and perspectives

In this work, we study a class of Lorentz invariant nonlinear field equations in several space dimensions with classical Sobolev spaces and generalized Sobolev spaces (with variable exponents) as a functional setting. The main purpose is to obtain soliton-like solutions. The fields are characterized by a topological invariant, which we call the charge. We prove the existence of a static solution which minimizes the energy among the configurations with nontrivial charge. And with a suitable change of variable we deduce the solution to the dynamic equation (soliton solution). Moreover, under some symmetry assumptions, we prove the existence of infinitely many solutions, which are constrained minima of the energy. More precisely, for every $N \in \mathbb{N}$ there exists a solution of charge N. We notice that the nature of the convergence of energy is the same as that of the topological charge; when the charge explodes, the energy explodes too. That gives important information characterizing the solution.

We plan to look for the total generalization of the problem in general Sobolev spaces with variable exponents as a functional setting.

As another perspective, we propose some possible ways:

- Generalization of the problem with an inclusion approach.
- Numerical treatment of soliton solutions with finite element methods.
- Weakening the assumptions of the problems in Chapters 4 and 5.

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