

SOME HOMOLOGICAL PROPERTIES OF IDEALS WITH  
COHOMOLOGICAL DIMENSION ONE

BY

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**Abstract.** Let  $R$  denote a commutative Noetherian ring and let  $I$  be an ideal of  $R$  such that  $H_i^I(R) = 0$  for all  $i \geq 2$ . We prove some results concerning the homological properties of  $I$ .

**1. Introduction.** Let  $R$  denote a commutative Noetherian ring and let  $I$  be an ideal of  $R$ . Hartshorne [H1] defined an  $R$ -module  $L$  to be  *$I$ -cofinite* if  $\text{Supp } L \subseteq V(I)$  and  $\text{Ext}_R^i(R/I, L)$  is a finitely generated module for all  $i$ . He posed the following question:

*Is the category  $\mathcal{C}(R, I)_{\text{cof}}$  of  $I$ -cofinite modules an Abelian subcategory of the category of all  $R$ -modules? That is, if  $f : M \rightarrow N$  is an  $R$ -homomorphism of  $I$ -cofinite modules, are  $\ker f$  and  $\text{coker } f$   $I$ -cofinite?*

Hartshorne gave a counterexample to show that the answer is not affirmative in general (see [H1, §3]). On the positive side, he proved that if  $I$  is a prime ideal of dimension one in a complete regular local ring  $R$ , then the answer is “yes”. Delfino and Marley [DM] extended this result to arbitrary complete local rings. Kawasaki [K2] generalized Delfino and Marley’s result to an arbitrary ideal  $I$  of dimension one in a local ring  $R$ . Finally, Melkersson [Me1] generalized Kawasaki’s result to all ideals of dimension one of any arbitrary Noetherian ring  $R$ . More recently, in [BNS], as a generalization of Melkersson’s result it is shown that for any ideals  $I$  in a commutative Noetherian ring  $R$ , the category of all  $I$ -cofinite  $R$ -modules  $M$  with  $\dim M \leq 1$  is an Abelian subcategory of the category of all  $R$ -modules.

Recall that, for an  $R$ -module  $M$ , the *cohomological dimension of  $M$  with respect to  $I$* , denoted by  $\text{cd}(I, M)$ , is the smallest integer  $n \geq 0$  such that  $H_i^I(M) = 0$  for all  $i > n$ .

2010 *Mathematics Subject Classification*: Primary 13D45; Secondary 14B15, 13E05.

*Key words and phrases*: Abelian category, cofinite modules, cohomological dimension, local cohomology, Noetherian ring.

Received 5 April 2016; revised 16 November 2016.

Published online 8 June 2017.

The cohomological dimension has been studied by several authors, including Faltings [F], Hartshorne [H2], Huneke–Lyubeznik [HL], Divaani-Aazar et al. [DNT], Hellus [He], Hellus–Stückrad [HS1], Mehrvarz et al. [MBN], and Ghasemi et al. [GBA].

Recall that, for any proper ideal  $I$  of  $R$ , the *arithmetic rank* of  $I$ , denoted by  $\text{ara}(I)$ , is the least number of elements of  $I$  required to generate an ideal which has the same radical as  $I$ .

Now, let  $I$  be an ideal of an arbitrary Noetherian ring  $R$ . Kawasaki [K1, Theorem 2.1] proved that if  $\text{ara}(I) = 1$  then the category  $\mathcal{C}(R, I)_{\text{cof}}$  of  $I$ -cofinite modules is an Abelian subcategory of the category of all  $R$ -modules.

It is well known that for any proper ideal  $I$  of a Noetherian ring  $R$  we have  $\text{cd}(I, R) \leq \text{ara}(I)$ . (See [BS, Corollary 3.3.3].) In particular, for any ideal  $I$  with  $\text{ara}(I) = 1$  we have  $\text{cd}(I, R) \leq 1$ . So, as a generalization of Kawasaki’s interesting result, it is natural to ask the following question:

**QUESTION 1.** *Let  $R$  be a Noetherian ring and  $I$  be an ideal of  $R$  with  $\text{cd}(I, R) \leq 1$ . Is the category  $\mathcal{C}(R, I)_{\text{cof}}$  of  $I$ -cofinite modules an Abelian subcategory of the category of all  $R$ -modules?*

In Section 2 we give an affirmative answer to Question 1 whenever  $R$  is a local Noetherian ring. In Section 3 we present some equivalent conditions for the exactness of ideal transform functors. Using these results, for any ideal  $I$  generated by two elements, we provide some necessary and sufficient conditions for the non-vanishing of the local cohomology module  $H_I^2(R)$ . Finally, in Section 4 we give some conditions for the vanishing of the Bass and Betti numbers of some special local cohomology modules. Also, we give a formula for the cohomological dimension of some special ideals in Noetherian domains.

Throughout this paper,  $R$  is a commutative Noetherian ring and  $I$  an ideal of  $R$ . Also, for an  $R$ -module  $M$ ,  $\Gamma_I(M)$  denotes the submodule of  $M$  consisting of all elements annihilated by some power of  $I$ , i.e.,  $\bigcup_{n=1}^{\infty} (0 :_M I^n)$ . For every  $R$ -module  $L$ , we denote by  $\text{mAss}_R L$  the set of minimal elements of  $\text{Ass}_R L$  with respect to inclusion. Also, for any ideal  $\mathfrak{a}$  of  $R$ , we denote  $\{\mathfrak{p} \in \text{Spec } R : \mathfrak{p} \supseteq \mathfrak{a}\}$  by  $V(\mathfrak{a})$ . Finally, for any ideal  $\mathfrak{b}$  of  $R$ , the *radical* of  $\mathfrak{b}$ , denoted by  $\text{Rad}(\mathfrak{b})$ , is defined to be the set  $\{x \in R : x^n \in \mathfrak{b} \text{ for some } n \in \mathbb{N}\}$ . For any unexplained notation and terminology we refer the reader to [BS] and [Mat].

**2. A category of modules which is Abelian.** The following well known lemma plays a key role in the proof of Proposition 2.2.

**LEMMA 2.1.** *Let  $R$  be a Noetherian ring,  $I$  be an ideal of  $R$  and  $T$  be an  $R$ -module. Then the following statements are equivalent:*

- (i) *The  $R$ -module  $H_I^i(T)$  is Artinian for  $i \geq 0$ .*

- (ii) The  $R$ -module  $\text{Ext}_R^i(R/I, T)$  is Artinian for  $i \geq 0$ .
- (iii) The  $R$ -module  $\text{Ext}_R^i(R/I, T)$  is Artinian for  $0 \leq i \leq \text{cd}(I, R)$ .

*Proof.* See [Me2, Theorem 5.5] and [AM, Theorem 2.9]. ■

PROPOSITION 2.2. *Let  $(R, \mathfrak{m})$  be a Noetherian local ring,  $I$  be an ideal of  $R$  and  $M$  be an  $R$ -module. Then the following conditions are equivalent:*

- (i) The  $R$ -modules  $\text{Tor}_i^R(R/I, M)$  are finitely generated for all  $i \geq 0$ .
- (ii) The  $R$ -modules  $\text{Ext}_i^R(R/I, M)$  are finitely generated for all  $0 \leq i \leq \text{cd}(I, R)$ .

*Proof.* (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i). Let  $T := D(M)$ , where  $D(-) := \text{Hom}_R(-, E)$  denotes the Matlis dual functor and  $E := E_R(R/\mathfrak{m})$  is the injective hull of the residue field  $R/\mathfrak{m}$ . Then by adjointness we have  $\text{Ext}_R^i(R/I, T) \simeq D(\text{Tor}_i^R(R/I, M))$  for all  $0 \leq i \leq \text{cd}(I, R)$ . In particular, the  $R$ -modules  $\text{Ext}_R^i(R/I, T)$  are Artinian for all  $0 \leq i \leq \text{cd}(I, R)$ . So, it follows from Lemma 2.1 and adjointness that the  $R$ -modules  $\text{Ext}_R^i(R/I, T) \simeq D(\text{Tor}_i^R(R/I, M))$  are Artinian for all  $i \geq 0$ . Now, [KLW, Lemma 1.15(a)] implies that the  $R$ -modules  $\text{Tor}_i^R(R/I, M)$  are finitely generated for all  $i \geq 0$ , as required. ■

The following easy consequence of Proposition 2.2 plays a key role in the proof of Theorem 2.4.

COROLLARY 2.3. *Let  $(R, \mathfrak{m})$  be a Noetherian local ring,  $I$  be an ideal of  $R$  with  $\text{cd}(I, R) = n$  and let  $M$  be an  $R$ -module with  $\text{Supp } M \subseteq V(I)$ . Then the  $R$ -module  $M$  is  $I$ -cofinite if and only if the  $R$ -modules  $\text{Tor}_i^R(R/I, M)$  are finitely generated for all  $0 \leq i \leq n$ .*

*Proof.* Apply Proposition 2.2 and [Me2, Theorem 2.1]. ■

The following result gives an affirmative answer to Question 1 for the local case.

THEOREM 2.4. *Let  $I$  be an ideal of a Noetherian local ring  $(R, \mathfrak{m})$  such that  $\text{cd}(I, R) \leq 1$ . Let  $\mathcal{C}(R, I)_{\text{cof}}$  denote the category of  $I$ -cofinite  $R$ -modules. Then  $\mathcal{C}(R, I)_{\text{cof}}$  is an Abelian subcategory of the category of all  $R$ -modules.*

*Proof.* Let  $M, N \in \mathcal{C}(R, I)_{\text{cof}}$  and let  $f : M \rightarrow N$  be an  $R$ -homomorphism. It is enough to show that the  $R$ -modules  $\ker f$  and  $\text{coker } f$  are  $I$ -cofinite.

To this end, observe that the exact sequence

$$0 \rightarrow \ker f \rightarrow M \rightarrow \text{im } f \rightarrow 0$$

induces an exact sequence

$$\text{Tor}_0^R(R/I, M) \rightarrow \text{Tor}_0^R(R/I, \text{im } f) \rightarrow 0,$$

which by [Me2, Theorem 2.1] implies that the  $R$ -module  $\text{Tor}_0^R(R/I, \text{im } f)$  is finitely generated. Now, the exact sequence

$$0 \rightarrow \text{im } f \rightarrow N \rightarrow \text{coker } f \rightarrow 0$$

induces an exact sequence

$$\begin{aligned} \text{Tor}_1^R(R/I, N) \rightarrow \text{Tor}_1^R(R/I, \text{coker } f) \rightarrow \text{Tor}_0^R(R/I, \text{im } f) \\ \rightarrow \text{Tor}_0^R(R/I, N) \rightarrow \text{Tor}_0^R(R/I, \text{coker } f) \rightarrow 0. \end{aligned}$$

By [Me2, Theorem 2.1] the  $R$ -modules  $\text{Tor}_1^R(R/I, N)$  and  $\text{Tor}_0^R(R/I, N)$  are finitely generated, which implies that so are

$$\text{Tor}_0^R(R/I, \text{coker } f) \quad \text{and} \quad \text{Tor}_1^R(R/I, \text{coker } f).$$

Therefore, Corollary 2.3 shows that the  $R$ -module  $\text{coker } f$  is  $I$ -cofinite. Now, the assertion follows from the exact sequences

$$0 \rightarrow \text{im } f \rightarrow N \rightarrow \text{coker } f \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \ker f \rightarrow M \rightarrow \text{im } f \rightarrow 0. \blacksquare$$

**COROLLARY 2.5.** *Let  $I$  be an ideal of a Noetherian local ring  $(R, \mathfrak{m})$  such that  $\text{cd}(I, R) \leq 1$ . Let*

$$X^\bullet : \dots \rightarrow X^i \xrightarrow{f^i} X^{i+1} \xrightarrow{f^{i+1}} X^{i+2} \rightarrow \dots$$

*be a complex such that  $X^i \in \mathcal{C}(R, I)_{\text{cof}}$  for all  $i \in \mathbb{Z}$ . Then for each  $i \in \mathbb{Z}$  the  $i$ th cohomology module  $H^i(X^\bullet)$  is in  $\mathcal{C}(R, I)_{\text{cof}}$ .*

*Proof.* The assertion follows from Theorem 2.4.  $\blacksquare$

**COROLLARY 2.6.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring,  $I$  be an ideal of  $R$  with  $\text{cd}(I, R) \leq 1$ , and  $M$  be an  $I$ -cofinite  $R$ -module. Then the  $R$ -modules  $\text{Tor}_i^R(N, M)$  and  $\text{Ext}_R^i(N, M)$  are  $I$ -cofinite for all finitely generated  $R$ -modules  $N$  and all integers  $i \geq 0$ .*

*Proof.* Since  $N$  is finitely generated,  $N$  has a free resolution with finitely generated free  $R$ -modules. Now the assertion follows from Corollary 2.5 by computing the  $R$ -modules  $\text{Tor}_i^R(N, M)$  and  $\text{Ext}_R^i(N, M)$ , using this free resolution.  $\blacksquare$

**3. Vanishing of the extension and torsion functors.** In this section we present some equivalent conditions for the exactness of ideal transform functors. Using these results, for any ideal  $I$  generated by two elements, we provide some necessary and sufficient conditions for the non-vanishing of the local cohomology module  $H_I^2(R)$ .

The following lemma is needed in the proof of Proposition 3.2.

**LEMMA 3.1.** *Let  $R$  be a Noetherian ring and  $I$  be an ideal of  $R$ . Let  $M$  be an  $R$ -module such that  $\text{Tor}_i^R(R/I, M) = 0$  for all  $i \geq 0$ . Then  $\text{Hom}_R(R/I, M) = 0$ .*

*Proof.* Let  $I = (x_1, \dots, x_n)$  and let

$$K_\bullet(\underline{x}, M) : 0 \rightarrow M \xrightarrow{f_0} \bigoplus_{k=1}^{C_n^1} M \xrightarrow{f_1} \bigoplus_{k=1}^{C_n^2} M \rightarrow \dots \rightarrow \bigoplus_{k=1}^{C_n^{n-1}} M \xrightarrow{f_{n-1}} M \rightarrow 0$$

be the Koszul complex of  $M$  with respect to  $\underline{x} = x_1, \dots, x_n$ .

We prove that  $H_i(\underline{x}; M) = 0$  for all  $0 \leq i \leq n$ . By definition,

$$H_0(\underline{x}; M) = \text{coker } f_{n-1} = M/IM \simeq R/I \otimes_R M = 0.$$

So,  $\text{im } f_{n-1} = M$ . Therefore, using the hypothesis we find that

$$\text{Tor}_i^R(R/I, \text{im } f_{n-1}) = 0 \quad \text{for each } i \geq 0.$$

Hence, the exact sequence

$$0 \rightarrow \ker f_{n-1} \rightarrow \bigoplus_{k=1}^{C_n^{n-1}} M \rightarrow \text{im } f_{n-1} \rightarrow 0$$

implies that  $\text{Tor}_i^R(R/I, \ker f_{n-1}) = 0$  for each  $i \geq 0$ . The exact sequence

$$(3.1.1) \quad 0 \rightarrow \text{im } f_{n-2} \rightarrow \ker f_{n-1} \rightarrow H_1(\underline{x}; M) \rightarrow 0$$

induces the exact sequence

$$(3.1.2) \quad R/I \otimes_R \ker f_{n-1} \rightarrow R/I \otimes_R H_1(\underline{x}; M) \rightarrow 0.$$

Now as  $R/I \otimes_R \ker f_{n-1} = 0$ , it follows from (3.1.2) that

$$R/I \otimes_R H_1(\underline{x}; M) = 0.$$

By the definition of the Koszul complex we have  $I H_1(\underline{x}; M) = 0$ . Therefore,  $H_1(\underline{x}; M) \simeq R/I \otimes_R H_1(\underline{x}; M) = 0$ . Now it follows from the exact sequence (3.1.1) that  $\text{Tor}_i^R(R/I, \text{im } f_{n-2}) = 0$  for each  $i \geq 0$ . Moreover, the exact sequence

$$0 \rightarrow \ker f_{n-2} \rightarrow \bigoplus_{k=1}^{C_n^{n-2}} M \rightarrow \text{im } f_{n-2} \rightarrow 0$$

implies that  $\text{Tor}_i^R(R/I, \ker f_{n-2}) = 0$  for each  $i \geq 0$ . Proceeding in the same way we can see that  $H_i(\underline{x}; M) = 0$  for all  $0 \leq i \leq n$ .

Now, since  $H_i(\underline{x}; M) \simeq H^{n-i}(\underline{x}; M)$  for all  $0 \leq i \leq n$ , we conclude that

$$\text{Hom}_R(R/I, M) \simeq (0 :_M I) \simeq H^0(\underline{x}; M) \simeq H_n(\underline{x}; M) = 0. \quad \blacksquare$$

The following proposition plays a key role in the proof of our main results.

**PROPOSITION 3.2.** *Let  $R$  be a Noetherian ring,  $I$  be an ideal of  $R$ , and  $M$  be an  $R$ -module. Then the following conditions are equivalent:*

- (i)  $\text{Tor}_n^R(R/I, M) = 0$  for all  $n \geq 0$ .
- (ii)  $\text{Ext}_R^n(R/I, M) = 0$  for all  $n \geq 0$ .

*Proof.* (i) $\Rightarrow$ (ii). We argue by induction on  $n$ . For  $n = 0$ , the assertion holds by Lemma 3.1. Assume, inductively, that  $n > 0$  and the result has been proved for smaller values of  $n$ . Then there is an exact sequence

$$(3.2.1) \quad 0 \rightarrow M \rightarrow E_R(M) \rightarrow E_R(M)/M \rightarrow 0.$$

Since by hypothesis we have  $(0 :_M I) = 0$ , it follows that  $(0 :_{E_R(M)} I) = 0$ , and hence  $\Gamma_I(E_R(M)) = 0$ . By [Mat, Theorem 18.5] the  $R$ -module  $E_R(M)$  is isomorphic to the direct sum of a family of indecomposable injective  $R$ -modules. Let  $\mathfrak{p}$  be a prime ideal of  $R$  such that  $E_R(R/\mathfrak{p})$  is a direct summand of  $E_R(M)$ . Then  $\Gamma_I(E_R(R/\mathfrak{p})) = 0$ . Therefore, as  $\text{Ass}_R E_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$ , it follows that  $I \not\subseteq \mathfrak{p}$ . So there exists  $a \in I$  such that  $a \notin \mathfrak{p}$ . Then by [Mat, Theorem 18.4(iii)], multiplication by  $a$  is an automorphism on  $E_R(R/\mathfrak{p})$ . Therefore, it is also an automorphism on  $\text{Tor}_i^R(R/I, E_R(R/\mathfrak{p}))$  for all  $i \geq 0$ . But, since  $a \in I$ , multiplication by  $a$  on  $\text{Tor}_i^R(R/I, E_R(R/\mathfrak{p}))$  is the zero map for all  $i \geq 0$ . Thus,  $\text{Tor}_i^R(R/I, E_R(R/\mathfrak{p})) = 0$  for all  $i \geq 0$ . Since for each  $i \geq 0$  the torsion functor  $\text{Tor}_i^R(R/I, -)$  commutes with the direct sums, it follows that  $\text{Tor}_i^R(R/I, E_R(M)) = 0$ , for all  $i \geq 0$ . So, the exact sequence (3.2.1) implies that  $\text{Tor}_i^R(R/I, E_R(M)/M) = 0$  for all  $i \geq 0$ . Hence, by applying the inductive hypothesis to the  $R$ -module  $E_R(M)/M$  we have

$$\text{Ext}_R^{i+1}(R/I, M) \simeq \text{Ext}_R^i(R/I, E_R(M)/M) = 0 \quad \text{for } 0 \leq i \leq n-1.$$

So,  $\text{Ext}_R^n(R/I, M) = 0$ . This completes the inductive step.

(ii) $\Rightarrow$ (i). Assume the opposite. Then there is an integer  $j \geq 0$  such that  $\text{Tor}_j^R(R/I, M) \neq 0$ . Let  $\mathfrak{p} \in \text{Supp Tor}_j^R(R/I, M)$ . Then localizing at  $\mathfrak{p}$ , without loss of generality, we may assume that  $(R, \mathfrak{m})$  is a Noetherian local ring. Let  $T := D(M)$ , where  $D(-)$  denotes the Matlis dual functor. Then by adjointness,

$$\text{Tor}_i^R(R/I, T) \simeq D(\text{Ext}_R^i(R/I, M)) = 0$$

for all  $i \geq 0$ . Therefore, by the previous part of the proof,  $\text{Ext}_R^i(R/I, T) = 0$  for each  $i \geq 0$ . So, again by adjointness,

$$D(\text{Tor}_j^R(R/I, M)) \simeq \text{Ext}_R^j(R/I, T) = 0,$$

which implies that  $\text{Tor}_j^R(R/I, M) = 0$ , a contradiction. ■

LEMMA 3.3. *Let  $R$  be a Noetherian ring and  $I$  be an ideal of  $R$ . Let  $E$  be an injective  $R$ -module and  $K$  be a submodule of  $E$ . Then*

$$\frac{L}{\Gamma_I(L)} \otimes_R \frac{R}{I} = 0, \quad \text{where } L := \frac{E}{K}.$$

*Proof.* In view of [BS, Proposition 2.1.4] the  $R$ -module  $E_1 := \Gamma_I(E)$  is injective. Therefore, there exists an injective submodule  $E_2$  of  $E$  such that  $E_1 + E_2 = E$  and  $E_1 \cap E_2 = 0$ . Since  $(E_1 + K)/K$  is a submodule of  $\Gamma_I(L)$ , the  $R$ -module  $L/\Gamma_I(L)$  is a homomorphic image of the  $R$ -module  $E_2$ . So, it

is enough to prove that  $E_2 \otimes_R R/I = 0$ . Thus, we must prove that  $E_2 = IE_2$ . Since  $\Gamma_I(R)E_2 \subseteq \Gamma_I(E_2) = 0$ , it follows that  $\Gamma_I(R)E_2 = 0$ . Therefore,

$$E_2 = (0 :_{E_2} \Gamma_I(R)) \simeq \text{Hom}_R(R/\Gamma_I(R), E_2).$$

On the other hand, by [BS, Lemma 2.1.1] there is an exact sequence

$$0 \rightarrow R/\Gamma_I(R) \xrightarrow{a} R/\Gamma_I(R)$$

for some  $a \in I$ ; applying the  $R$ -linear exact functor  $\text{Hom}_R(-, E_2)$  induces the exact sequence

$$\text{Hom}_R(R/\Gamma_I(R), E_2) \xrightarrow{a} \text{Hom}_R(R/\Gamma_I(R), E_2) \rightarrow 0.$$

Therefore,  $\text{Hom}_R(R/\Gamma_I(R), E_2) = a \text{Hom}_R(R/\Gamma_I(R), E_2)$ . Hence,

$$\begin{aligned} E_2/aE_2 &\simeq E_2 \otimes_R R/Ra \simeq \text{Hom}_R(R/\Gamma_I(R), E_2) \otimes_R R/Ra \\ &\simeq \text{Hom}_R(R/\Gamma_I(R), E_2)/a \text{Hom}_R(R/\Gamma_I(R), E_2) \simeq 0. \end{aligned}$$

So,  $E_2 = aE_2$ , and hence  $E_2 = IE_2$ . ■

LEMMA 3.4. *Let  $R$  be a Noetherian ring and let  $I \subseteq J$  be two ideals of  $R$ . Let  $M$  be an  $R$ -module and  $t \geq 2$  be an integer such that  $\text{Tor}_j^R(R/J, H_i^1(M)) = 0$  for all  $i > t$  and all  $j \geq 0$ . Then  $\text{Tor}_j^R(R/J, H_i^t(M)) = 0$  for  $j = 0, 1$ .*

*Proof.* Let

$$0 \rightarrow M \xrightarrow{\varepsilon} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} \dots$$

be a minimal injective resolution for  $M$ . Set  $K_i := \ker f_i$  for  $i \geq 0$ . By splitting this minimal injective resolution into short exact sequences we get the isomorphisms

$$\begin{aligned} H_I^t(M) &\simeq H_I^{t-1}(K_1) \simeq H_I^{t-2}(K_2) \simeq \dots \simeq H_I^2(K_{t-2}) \simeq H_I^1(K_{t-1}) \\ &\simeq H_I^1(K_{t-1}/\Gamma_I(K_{t-1})). \end{aligned}$$

Set  $N := K_{t-1}/\Gamma_I(K_{t-1})$ . Then  $\Gamma_I(N) = 0$  and

$$K_{t-1} = \ker f_{t-1} = \text{im } f_{t-2} \simeq E_{t-2}/\ker f_{t-2} = E_{t-2}/K_{t-2}.$$

So, there exists a submodule  $K$  of  $E := E_{t-2}$  such that  $N \simeq E/K$ . By [BS, Remark 2.2.7] there is an exact sequence

$$(3.4.1) \quad 0 \rightarrow N \rightarrow D_I(N) \rightarrow H_I^1(N) \rightarrow 0.$$

For each  $i \geq 2$  we have

$$H_I^i(N) \simeq H_I^i(K_{t-1}) \simeq H_I^{t+i-1}(M).$$

Therefore, by hypothesis,

$$\text{Tor}_j^R(R/J, H_I^i(N)) \simeq \text{Tor}_j^R(R/J, H_I^{t+i-1}(M)) = 0$$

for all  $i \geq 2$  and all  $j \geq 0$ . Thus, by Proposition 3.2,  $\text{Ext}_R^j(R/J, H_I^i(N)) = 0$  for all  $i \geq 2$  and all  $j \geq 0$ . The exact sequence (3.4.1) yields

$$H_I^i(D_I(N)) \simeq H_I^i(N)$$

for  $i \geq 2$ . So,

$$\text{Ext}_R^j(R/J, H_I^i(D_I(N))) \simeq \text{Ext}_R^j(R/J, H_I^i(N)) = 0$$

for all  $i \geq 2$  and all  $j \geq 0$ . Also, by [BS, Corollary 2.2.8] we have  $H_I^i(D_I(N)) = 0$  for  $i = 0, 1$ . So  $\text{Ext}_R^j(R/J, H_I^i(D_I(N))) = 0$  for all  $i, j \geq 0$ . Therefore, by [GBA, Lemma 2.1] we have  $\text{Ext}_R^j(R/J, D_I(N)) = 0$  for  $j \geq 0$ . Thus by Proposition 3.2,  $\text{Tor}_i^R(R/J, D_I(N)) = 0$  for all  $i \geq 0$ . On the other hand, in view of Lemma 3.3 we have  $N \otimes_R R/I = 0$ , and hence the hypothesis  $I \subseteq J$  implies that  $N \otimes_R R/J = 0$ . Hence, from the long exact sequence induced by (3.4.1) it follows that  $\text{Tor}_j^R(R/J, H_I^1(N)) = 0$  for  $j = 0, 1$ . Therefore,  $\text{Tor}_j^R(R/J, H_I^t(M)) \simeq \text{Tor}_j^R(R/J, H_I^1(N)) = 0$  for  $j = 0, 1$ . ■

**COROLLARY 3.5.** *Let  $R$  be a Noetherian ring and  $I \subseteq J$  be ideals of  $R$ . Let  $K$  be an  $R$ -module such that  $2 \leq \text{cd}(I, K) = t$ . Then  $\text{Tor}_i^R(R/J, H_I^t(K)) = 0$  for  $i = 0, 1$ .*

*Proof.* This follows from Lemma 3.4. ■

**PROPOSITION 3.6.** *Let  $R$  be a Noetherian ring,  $I$  be an ideal of  $R$ , and  $M$  be an  $R$ -module. Then the following conditions are equivalent:*

- (i)  $\text{Tor}_i^R(R/I, M) = 0$  for all  $i \geq 0$ .
- (ii)  $\text{Tor}_i^R(R/I, M) = 0$  for all  $0 \leq i \leq \text{cd}(I, R)$ .

*Proof.* (i)  $\Rightarrow$  (ii) is clear.

(ii)  $\Rightarrow$  (i). Assume the opposite. Then there is an integer  $j > \text{cd}(I, R)$  such that  $\text{Tor}_j^R(R/I, M) \neq 0$ . Let  $\mathfrak{p} \in \text{Supp Tor}_j^R(R/I, M)$ . Localizing at  $\mathfrak{p}$ , we may assume that  $(R, \mathfrak{m})$  is a Noetherian local ring. Let  $T := D(M)$ , where  $D(-)$  denotes the Matlis dual functor. Then by adjointness,

$$\text{Ext}_R^i(R/I, T) \simeq D(\text{Tor}_i^R(R/I, M)) = 0$$

for all  $0 \leq i \leq \text{cd}(I, R)$ . Now, by [AM, Theorem 2.9] we have  $H_I^i(T) = 0$  for all  $0 \leq i \leq \text{cd}(I, R)$ . Hence,  $H_I^i(T) = 0$  for all  $i \geq 0$ . So, in view of [AM, Theorem 2.9] we have  $\text{Ext}_R^i(R/I, T) = 0$  for all  $i \geq 0$ . Consequently, by adjointness,

$$D(\text{Tor}_i^R(R/I, M)) \simeq \text{Ext}_R^i(R/I, T) = 0$$

for all  $i \geq 0$ . Thus,  $\text{Tor}_i^R(R/I, M) = 0$  for all  $i \geq 0$ , a contradiction. ■

**THEOREM 3.7.** *Let  $R$  be a Noetherian ring and  $I$  be an ideal of  $R$ . Then the following conditions are equivalent:*

- (i)  $\text{cd}(I, R) \leq 1$ .
- (ii) *The ideal transform functor  $D_I(-)$  is exact.*
- (iii) *For every  $R$ -module  $M$ , if  $\text{Tor}_i^R(R/I, M) = 0$  for  $i = 0, 1$ , then  $\text{Tor}_i^R(R/I, M) = 0$  for all  $i \geq 0$ .*

(iv) For every  $R$ -module  $M$ , if  $\text{Ext}_R^i(R/I, M) = 0$  for  $i = 0, 1$ , then  $\text{Ext}_R^i(R/I, M) = 0$  for all  $i \geq 0$ .

*Proof.* (i) $\Leftrightarrow$ (ii). See [BS, Lemma 6.3.1].

(i) $\Rightarrow$ (iii) follows from Proposition 3.6.

(iii) $\Rightarrow$ (i). Assume the opposite. Then  $\text{cd}(I, R) \geq 2$ . Let  $t = \text{cd}(I, R)$ . Then by Corollary 3.5 for the  $R$ -module  $M := H_I^t(R)$  we have  $\text{Tor}_i^R(R/I, M) = 0$  for  $i = 0, 1$ . So, by hypothesis  $\text{Tor}_i^R(R/I, M) = 0$  for all  $i \geq 0$ . Thus, by Proposition 3.2 we have  $\text{Ext}_R^i(R/I, M) = 0$  for all  $i \geq 0$ . Therefore  $(0 :_{H_I^t(R)} I) \simeq \text{Hom}_R(R/I, H_I^t(R)) = 0$ . Hence  $H_I^t(R) = 0$ , a contradiction.

(i) $\Rightarrow$ (iv). Let  $M$  be an  $R$ -module such that  $\text{Ext}_R^i(R/I, M) = 0$  for  $i = 0, 1$ . Then by [AM, Theorem 2.9] we have  $H_I^i(M) = 0$  for  $i = 0, 1$ . On the other hand, by [BS, Lemma 6.3.1] we have  $H_I^i(M) = 0$  for all  $i \geq 2$ . Therefore,  $H_I^i(M) = 0$  for all  $i \geq 0$ . Hence, [AM, Theorem 2.9] yields  $\text{Ext}_R^i(R/I, M) = 0$  for all  $i \geq 0$ .

(iv) $\Rightarrow$ (i). By [BS, Corollary 2.2.8] we have  $H_I^i(D_I(R)) = 0$  for  $i = 0, 1$ . Hence,  $\text{Ext}_R^i(R/I, D_I(R)) = 0$  for  $i = 0, 1$  by [AM, Theorem 2.9]. So, by hypothesis,  $\text{Ext}_R^i(R/I, D_I(R)) = 0$  for all  $i \geq 0$ . Hence, Proposition 3.2 yields  $R/I \otimes_R D_I(R) = 0$ , and so  $D_I(R) = ID_I(R)$ . Now the assertion follows from [BS, Lemma 6.3.1 and Proposition 6.3.5]. ■

**THEOREM 3.8.** *Let  $R$  be a Noetherian ring and let  $I = Ra_1 + Ra_2$  be an ideal of  $R$ . Then the following conditions are equivalent:*

- (i)  $\text{cd}(I, R) = 2$ .
- (ii) There exists an  $R$ -module  $M$  such that  $\text{Tor}_i^R(R/I, M) = 0$  for  $i = 0, 1$  and  $\text{Tor}_2^R(R/I, M) \neq 0$ .
- (iii) There exists an  $R$ -module  $M$  such that  $\text{Ext}_R^i(R/I, M) = 0$  for  $i = 0, 1$  and  $\text{Ext}_R^2(R/I, M) \neq 0$ .

*Proof.* (i) $\Rightarrow$ (ii). If  $\text{cd}(I, R) = 2$ , then for the  $R$ -module  $M := H_I^2(R)$  by Corollary 3.5 we have  $\text{Tor}_i^R(R/I, M) = 0$  for  $i = 0, 1$ . Now, we claim that  $\text{Tor}_2^R(R/I, M) \neq 0$ . Assume the opposite. Then it follows from Proposition 3.6 that  $\text{Tor}_i^R(R/I, M) = 0$  for all  $i \geq 0$ . Proposition 3.2 shows that  $\text{Ext}_R^i(R/I, M) = 0$  for all  $i \geq 0$ . Therefore,  $\text{Hom}_R(R/I, H_I^2(R)) = 0$ , which implies that  $H_I^2(R) = 0$ , a contradiction.

(ii) $\Rightarrow$ (i). Under the given hypothesis it follows from Theorem 3.7 that  $\text{cd}(I, R) \geq 2$ . On the other hand, by [BS, Theorem 3.3.1] we have  $\text{cd}(I, R) \leq 2$ . So,  $\text{cd}(I, R) = 2$ , as required.

(i) $\Rightarrow$ (iii). If  $\text{cd}(I, R) = 2$ , then for the  $R$ -module  $M := D_I(R)$ , by [BS, Corollary 2.2.8] we have  $H_I^i(M) = 0$  for  $i = 0, 1$ . Hence, by [AM, Theorem 2.9] we have  $\text{Ext}_R^i(R/I, M) = 0$  for  $i = 0, 1$ . Moreover, by [BS, Remark 2.2.7] there is an exact sequence

$$0 \rightarrow R/\Gamma_I(R) \rightarrow D_I(R) \rightarrow H_I^1(R) \rightarrow 0,$$

which induces the isomorphisms

$$H_I^2(M) \simeq H_I^2(R/\Gamma_I(R)) \simeq H_I^2(R) \neq 0.$$

Now, if  $\text{Ext}_R^2(R/I, M) = 0$ , then  $\text{Ext}_R^i(R/I, M) = 0$  for  $i = 0, 1, 2$ . So, it follows from [AM, Theorem 2.9] that  $H_I^i(M) = 0$  for  $i = 0, 1, 2$ , which is a contradiction, because  $H_I^2(M) \neq 0$ . Hence, for the  $R$ -module  $M := D_I(R)$  we have  $\text{Ext}_R^i(R/I, M) = 0$  for  $i = 0, 1$  and  $\text{Ext}_R^2(R/I, M) \neq 0$ .

(iii) $\Rightarrow$ (i). Repeat the argument for (ii) $\Rightarrow$ (i). ■

REMARK. For a given proper ideal  $I$  of a Noetherian ring  $R$ , there are some other well known equivalent conditions for  $\text{cd}(I, R) \leq 1$ . For instance, see [BS, Lemma 6.3.1 and Proposition 6.3.5], and for more properties of such ideals see [B].

**4. Vanishing of the Bass and Betti numbers of local cohomology modules.** In this section we give some conditions for the vanishing of some Bass and Betti numbers of special local cohomology modules. Also, we give a formula for the cohomological dimension of special ideals over Noetherian domains.

THEOREM 4.1. *Let  $R$  be a Noetherian ring and let  $I \subseteq J$  be ideals of  $R$  such that  $\text{cd}(J, R) = 1$ . Then:*

- (i) *For every  $R$ -module  $M$  we have  $\text{Ext}_R^j(R/J, H_I^i(M)) = 0$  for all  $i \geq 2$  and  $j \geq 0$ .*
- (ii) *For every  $R$ -module  $M$  and every finitely generated  $R$ -module  $N$  with  $\text{Supp } N \subseteq V(J)$  we have  $\text{Ext}_R^j(N, H_I^i(M)) = 0$  for all  $i \geq 2$  and  $j \geq 0$ .*
- (iii) *For every  $R$ -module  $M$  and each prime ideal  $\mathfrak{p} \in V(J)$  we have*

$$\mu^j(\mathfrak{p}, H_I^i(M)) = 0 \quad \text{for all } i \geq 2 \text{ and } j \geq 0.$$

*(Here  $\mu^j(\mathfrak{p}, H_I^i(M))$  denotes the  $j$ th Bass number of the  $R$ -module  $H_I^i(M)$  with respect to  $\mathfrak{p}$ .)*

- (iv) *For every  $R$ -module  $M$  and each prime ideal  $\mathfrak{p} \in V(J)$  we have*

$$\beta_j(\mathfrak{p}, H_I^i(M)) = 0 \quad \text{for all } i \geq 2 \text{ and } j \geq 0.$$

*(Here  $\beta_j(\mathfrak{p}, H_I^i(M))$  denotes the  $j$ th Betti number of the  $R$ -module  $H_I^i(M)$  with respect to  $\mathfrak{p}$ .)*

*Proof.* (i) Let  $M$  be an arbitrary  $R$ -module. We may assume  $\text{cd}(I, M) = t \geq 2$ . By Corollary 3.5 we have  $\text{Tor}_j^R(R/J, H_I^t(M)) = 0$  for  $j = 0, 1$ . Now, it follows from Proposition 3.6 that  $\text{Tor}_i^R(R/J, H_I^t(M)) = 0$  for all  $i \geq 0$ . Therefore, by Proposition 3.2 we have  $\text{Ext}_R^i(R/J, H_I^t(M)) = 0$  for all  $i \geq 0$ . Now, if  $t \geq 3$  then by Lemma 3.4 we have  $\text{Tor}_j^R(R/J, H_I^{t-1}(M)) = 0$  for  $j = 0, 1$ . So, it follows from Proposition 3.6 that  $\text{Tor}_i^R(R/J, H_I^{t-1}(M)) = 0$  for all  $i \geq 0$ . Therefore, Proposition 3.2 yields  $\text{Ext}_R^i(R/J, H_I^{t-1}(M)) = 0$  for

all  $i \geq 0$ . Proceeding in the same way we see that  $\text{Ext}_R^j(R/J, H_I^i(M)) = 0$  for all  $i \geq 2$  and  $j \geq 0$ .

- (ii) Use [AB, Lemma 2.2] and (i).
- (iii) follows from (ii).
- (iv) follows from (ii) using Proposition 3.2. ■

**THEOREM 4.2.** *Let  $R$  be a Noetherian domain and let  $I$  and  $J$  be non-zero proper ideals of  $R$  such that  $\text{cd}(J, R) = 1$  and  $J \not\subseteq \text{Rad}(I)$ . Then*

$$\text{cd}(I \cap J, R) = \max\{i \in \mathbb{Z} : \text{Supp } H_I^i(R) \not\subseteq V(J)\}.$$

*Proof.* Since by hypothesis we have  $J \not\subseteq \text{Rad}(I)$ , it follows that

$$J \not\subseteq \bigcap_{\mathfrak{p} \in \text{mAss}_R R/I} \mathfrak{p},$$

which implies that  $J \not\subseteq \mathfrak{q}$  for some  $\mathfrak{q} \in \text{mAss}_R R/I$ . Let  $k := \text{height } \mathfrak{q}$ . Then by Grothendieck's Non-Vanishing Theorem we have

$$0 \neq H_{\mathfrak{q}R_{\mathfrak{q}}}^k(R_{\mathfrak{q}}) = H_{IR_{\mathfrak{q}}}^k(R_{\mathfrak{q}}) \simeq (H_I^k(R))_{\mathfrak{q}},$$

which gives  $\mathfrak{q} \in \text{Supp } H_I^k(R)$ . Hence  $\text{Supp } H_I^k(R) \not\subseteq V(J)$ , and so

$$k \in \{i \in \mathbb{Z} : \text{Supp } H_I^i(R) \not\subseteq V(J)\} =: S(R).$$

In particular,  $S(R) \neq \emptyset$ .

Now, set  $\ell := \max S(R)$  and  $t := \text{cd}(I \cap J, R)$ . Then, as by [BS, Corollary 3.3.3] we have  $\ell \leq \text{ara}(I)$  and  $t \leq \text{ara}(I \cap J)$ , it follows that  $0 \leq \ell < \infty$  and  $0 \leq t < \infty$ . By the Mayer–Vietoris exact sequence for each  $i > t$  we have the exact sequence

$$H_{I+J}^i(R) \rightarrow H_I^i(R) \oplus H_J^i(R) \rightarrow H_{I \cap J}^i(R),$$

which gives the exact sequence

$$H_{I+J}^i(R) \rightarrow H_I^i(R) \oplus H_J^i(R) \rightarrow 0,$$

and hence

$$\text{Supp } H_I^i(R) \subseteq \text{Supp } H_{I+J}^i(R) \subseteq V(I + J) \subseteq V(J).$$

So, it is clear that  $\ell \leq t$ .

On the other hand, since by hypothesis  $R$  is a domain and  $I \cap J \neq 0$ , it follows that  $\ell, t \geq 1$ . Hence, if  $t = 1$ , then it is clear that  $\ell = 1 = t$ . Now, assume that  $t \geq 2$  and  $\ell < t$ . Then  $\text{Supp } H_I^t(R) \subseteq V(J)$ . Also, the Mayer–Vietoris exact sequence gives the exact sequence

$$H_I^t(R) \oplus H_J^t(R) \rightarrow H_{I \cap J}^t(R) \rightarrow H_{I+J}^{t+1}(R),$$

which implies that

$$\begin{aligned} \text{Supp } H_{I \cap J}^t(R) &\subseteq \text{Supp } H_J^t(R) \cup \text{Supp } H_I^t(R) \cup \text{Supp } H_{I+J}^{t+1}(R) \\ &\subseteq V(J). \end{aligned}$$

So, the non-zero  $R$ -module  $H_{I \cap J}^t(R)$  is  $J$ -torsion, and hence

$$\text{Hom}_R(R/J, H_{I \cap J}^t(R)) \neq 0.$$

But, by Theorem 4.1,

$$\text{Hom}_R(R/J, H_{I \cap J}^t(R)) = 0,$$

a contradiction. So,  $\ell = t$  whenever  $t \geq 2$ . ■

**PROPOSITION 4.3.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and let  $I$  and  $J$  be proper ideals of  $R$  such that  $\text{cd}(J, R) = 1$ . Let  $k \geq 2$  be an integer such that  $\text{Supp } H_I^k(R) \not\subseteq V(J)$ . Then the  $R$ -module  $H_{I \cap J}^k(R)$  is not  $I \cap J$ -cofinite. In particular,  $H_{I \cap J}^k(R) \neq 0$ .*

*Proof.* By the Mayer–Vietoris exact sequence we have the exact sequence

$$H_{I+J}^k(R) \rightarrow H_I^k(R) \oplus H_J^k(R) \rightarrow H_{I \cap J}^k(R).$$

Since  $\text{Supp } H_{I+J}^k(R) \subseteq V(I+J) \subseteq V(J)$ , we have  $\text{Supp } H_{I \cap J}^k(R) \not\subseteq V(J)$ . (Note that  $\text{Supp } H_I^k(R) \not\subseteq V(J)$  by hypothesis.) In particular,  $H_{I \cap J}^k(R) \neq 0$ .

To prove the main assertion, assume the opposite and

$$\dim \text{Supp } H_{I \cap J}^k(R) = d.$$

Then, in view of [M, Theorem 2.9], we have  $H_{\mathfrak{m}}^d(H_{I \cap J}^k(R)) \neq 0$ . On the other hand, Theorem 4.1 yields  $\text{Ext}_R^j(R/\mathfrak{m}, H_{I \cap J}^k(R)) = 0$  for all  $j \geq 0$ . Hence it follows from [AM, Theorem 2.9] that  $H_{\mathfrak{m}}^j(H_{I \cap J}^k(R)) = 0$  for all  $j \geq 0$ . Therefore,  $H_{\mathfrak{m}}^d(H_{I \cap J}^k(R)) = 0$ , a contradiction. ■

**REMARK.** There are examples of Noetherian local rings  $(R, \mathfrak{m})$  with proper ideals  $I$  for which  $\text{cd}(I, R) = 1$  and  $\text{ara}(I) \geq 2$ . For instance, the following example is given by Hellus and Stückrad [HS2].

**EXAMPLE 4.4.** Let  $k$  be a field and  $S = k[[x, y, z, w]]$ , where  $x, y, z, w$  are independent indeterminates over  $k$ . Let

$$f = xw - yz, \quad g = y^3 - x^2z, \quad h = z^3 - w^2y.$$

Let

$$R = S/fS \quad \text{and} \quad I = (f, g, h)S/fS.$$

Then  $R$  is a Noetherian local ring of dimension 3 with maximal ideal  $\mathfrak{m} = (x, y, z, w)S/fS$ . Also,  $\text{cd}(I, R) = 1$  and  $\text{ara}(I) = 2$  for the ideal  $I$  of  $R$ . (See [HS2, Remark 2.1(ii)].)

**Acknowledgements.** The authors are deeply grateful to the referee for a very careful reading of the manuscript and many valuable suggestions. Also, we would like to thank Professors Kamran Divaani-Aazar and Reza Naghipour for their careful reading of the first draft and many helpful suggestions.

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