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Capitulation in abelian extensions of number fields

by

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Introduction. Let K be a number field and L a finite abelian extension of K. In this paper, we investigate the "capitulation kernel," the kernel of the map j from the ideal class group Cl_K of K to the ideal class group Cl_L of L induced by the lifting of ideals from K to L. Elements of the kernel are said to *capitulate* in L. This problem has a long history, most notably when L is an unramified abelian extension of K. For example, when L/K is unramified and cyclic, Hilbert's Theorem 94 says that the order of the capitulation kernel is divisible by the degree $[L : K]$. This result was extended to all unramified abelian extensions by Suzuki [26]. (See also [12].) The well-known Principal Ideal Theorem states that if L is the Hilbert class field of K, then every ideal of K becomes principal in L.

More recently González-Avilés [8] and Schoof and Washington [25] have studied the capitulation kernel for both unramified and ramified extensions. In [8], the cokernel of j is studied as well. In the present paper we will study the kernel and cokernel of j ($\operatorname{Ker} j$ and $\operatorname{Coker} j$) for arbitrary abelian extensions L over K. In particular, we describe them in terms of Galois cohomology groups. We also look at the subgroup J of $\operatorname{Ker} j$ consisting of those ideal classes of K that are in the image of the norm map from Cl_L to Cl_K and that capitulate in L. We then study the quotient group of $\operatorname{Ker} j$ by J and show that when $J = (1)$, $\operatorname{Ker} j$ is isomorphic to the Galois group of M over K where M is the intersection of L and the Hilbert class field of K. We give some criteria under which $\operatorname{Ker} j$ is as large as possible, namely, when it equals the set of elements of Cl_K which are killed by $[L : K]$. In particular, we prove that this happens when $[L : K]$ is relatively prime to the order of the relative class group of L over K. We discuss "number knots" as described by Jehne [14] and use them to prove a version of Hilbert's Theorem 90 for

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ideal classes, generalizing results of Furtwängler and Bembom [2]. We then restrict our attention to quadratic extensions L over K, computing $\operatorname{Ker} j$ when the 2-part of Cl_L is cyclic or elementary, and we give examples when K is an imaginary quadratic field.

0. Preliminaries. If G is a finite group and A is a G-module, then:

(1) $A^G = \{a \in A \mid \sigma a = a \text{ for all } \sigma \in G\}$ and $I_G A$ is the subgroup of A generated by elements of the form $\frac{\sigma a}{a}$, where $a \in A$, $\sigma \in G$.
(2) $H^n(G, A)$, $n \in \mathbb{Z}$, is the nth *Tate cohomology group* of G with respect to A. When G is understood, we write $H^n(A)$ for $H^n(G, A)$.
(3) $N_G : A \to A^G$ is the *G-norm*, defined by $N_G(a) = \prod_{\sigma \in G} \sigma a$. $N_G A$, the range of N_G, is contained in A^G, so N_G is well defined.
(4) The most commonly used cohomology groups in this paper are $H^0(G, A) = A^G/N_G A$ and $H^{-1}(G, A) = \operatorname{Ker} N_G / I_G A$.
(5) If A is a finite abelian group and n a positive integer, let $A_n = \{a \in A \mid a^n = 1\}$. For a prime p, $A(p)$ is the *p-primary part* of A. We denote the order of A by $|A|$.
(6) If $f : A \to B$ is a homomorphism of G-modules, then $\operatorname{Im} f$ denotes the image of f, $\operatorname{Ker} f$ its kernel, and $f^G : A^G \to B^G$ is the restriction of f to A^G.

Let F be a field.

(7) F^* is the *multiplicative group of nonzero elements* of F, E_F is the group of *units* of the ring of integers of F, and H_F the *Hilbert class field* of F, that is, H_F is the maximal unramified abelian extension of F. For prime p, $H_F^{(p)}$ is the *p-Hilbert class field* of F, the maximal unramified abelian extension of F of p-power order.
(8) D_F, P_F, and Cl_F are, respectively, the groups of *fractional ideals*, *principal ideals*, and *ideal classes* of F; h_F is the *class number* of F.
(9) J_F, U_F, and C_F are, respectively, the groups of *ideles*, *idele units*, and *idele classes* of F; $\Gamma_F = U_F/E_F$ is the group of *idele class units*.
(10) If E is a Galois extension of F, then $G(E/F)$ denotes the *Galois group* of E over F.

Throughout this paper, K is a number field and L a finite *abelian* extension of K with Galois group $G = G(L/K)$. Unless otherwise noted, we let $n = [L : K]$. The groups listed in definitions (7) through (9) above, with $F = L$, are all G-modules.

(11) $N_{L/K} : Cl_L \to Cl_K$ is the map induced by the norm map on ideals. The range of $N_{L/K}$ will be denoted by B:

$$B = N_{L/K} Cl_L. \tag{0.1}$$

The kernel of $N_{L/K}$, the *relative class group* of L over K, will be denoted by

$$Cl_{L/K} = \operatorname{Ker} N_{L/K}. \tag{0.2}$$

(12) $j : Cl_K \to Cl_L^G$ is the map induced by the lifting of ideals from K to L, and $\operatorname{Ker} j$ is its kernel. Since $\operatorname{Im} j \subset Cl_L^G$, $\operatorname{Coker} j$ is defined to be $Cl_L^G/\operatorname{Im} j$.
(13) We let $M = H_K \cap L$, the *maximal unramified abelian extension* of K in L.
(14) We let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}^+$ the set of positive integers. For $n \in \mathbb{Z}^+$, $\mathbb{Z}/n\mathbb{Z}$ is the ring of integers modulo n.

Remark 0.1. (1) The following diagrams are commutative:

$$\mathbf{D1}\quad \begin{array}{ccc} Cl_L & \xrightarrow{N_{L/K}} & Cl_K \\ \downarrow{\scriptstyle N_G} & & \downarrow{\scriptstyle j} \\ Cl_L^G & \xrightarrow{\text{id}} & Cl_L^G \end{array} \qquad \text{and} \qquad \mathbf{D2}\quad \begin{array}{ccc} Cl_K & \xrightarrow{j} & Cl_L^G \\ \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle N_{L/K}|Cl_L^G} \\ Cl_K & \xrightarrow{\text{id}} & Cl_K \end{array}$$

where id is the identity map, $N_{L/K}|Cl_L^G$ is the restriction of $N_{L/K}$ to Cl_L^G, and $n : Cl_K \to Cl_K$ is raising to the nth power.

(2) Diagram **D2** shows that $\operatorname{Ker} j \subset (Cl_K)_n$. In fact, $\operatorname{Ker} j \subset (Cl_K)_d$ where $d = \gcd(n, h_K)$. In Section 3, we will give conditions for $\operatorname{Ker} j = (Cl_K)_d$.

(3) If G is cyclic, then $H^i(G, A) \cong H^{i+2}(G, A)$ for any G-module A and $i \in \mathbb{Z}$.

We assume the basic results of class field theory (see [10] or [21]). In particular, Cl_K is isomorphic, via the Artin map, to $G(H_K/K)$. Also note that $G(M/K) \cong G(L/K)/G(L/M)$ is annihilated by n.

The following result is an important consequence of class field theory and is fundamental to the main results of this paper.

Proposition 0.2.

(1) *There exists a canonical isomorphism of finite abelian n-torsion groups $Cl_K/B \cong G(M/K)$.*
(2) $Cl_{L/K} = \operatorname{Ker} N_{L/K} \cong G(H_L/LH_K)$.
(3) $B = N_{L/K} Cl_L \cong G(H_K/M)$.

Proof. The above isomorphisms follow from the fact that the norm map $N_{L/K} : Cl_L \to Cl_K$ corresponds via class field theory to the natural projection $G(H_L/L) \to G(LH_K/L)$. ■

Corollary 0.3. *If $M = H_K \cap L = K$, then $N_{L/K} : Cl_L \to Cl_K$ is surjective.*

Suppose that L is a finite abelian extension of K such that $n = [L : K]$ is a power of a prime p. If p divides $|Cl_K|$, we can restrict the extension of ideal classes map j to the p-primary part $Cl_K(p)$ of Cl_K. We then have a map $j(p) : Cl_K(p) \to Cl_L^G(p)$. Since $\operatorname{Ker} j \subset (Cl_K)_n$, it follows that $\operatorname{Ker} j(p) = \operatorname{Ker} j$.

The following lemma will prove useful in this and subsequent sections.

LEMMA 0.4.

(1) *Let* $(1) \to A \to B \to C \to (1)$ *be an exact sequence of abelian groups and* n *a positive integer. Then we have an exact sequence*

$$(1) \to A_n \to B_n \to C_n \to A/A^n \to B/B^n \to C/C^n \to (1). \tag{0.3}$$

(2) *Let* $f : A \to B$ *be a homomorphism of torsion abelian groups and* p *a prime. Let* $f(p) : A(p) \to B(p)$ *be the map induced by* f. *Then* $\operatorname{Ker} f(p) = (\operatorname{Ker} f)(p)$ *and* $\operatorname{Im} f(p) = (\operatorname{Im} f)(p)$.

(3) *Let* $A \xrightarrow{f} B \xrightarrow{g} C$ *be an exact sequence of torsion abelian groups. Then the induced sequence* $A(p) \xrightarrow{f(p)} B(p) \xrightarrow{g(p)} C(p)$ *is exact.*

Proof. Apply the torsion functor $\operatorname{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ to the given short exact sequence. Now it is known that, for any abelian group Q,

$$\operatorname{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, Q) = (1) \quad \text{for } i \geq 2,$$
$$\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, Q) = Q_n,$$
$$\operatorname{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, Q) = Q/Q^n \cong \mathbb{Z}/n\mathbb{Z} \otimes Q$$

(see Weibel [27, Calculation 3.1.1]). Sequence (0.3) now follows.

By (1), for each positive integer n, the following sequences are exact:

$$(1) \to (\operatorname{Ker} f)_{p^n} \to A_{p^n} \to (\operatorname{Im} f)_{p^n} \to \mathbb{Z}/p^n\mathbb{Z} \otimes (\operatorname{Ker} f),$$
$$(1) \to (\operatorname{Im} f)_{p^n} \to B_{p^n} \to (\operatorname{Coker} f)_{p^n} \to \mathbb{Z}/p^n\mathbb{Z} \otimes (\operatorname{Im} f).$$

Taking direct limits over n, we get the exact sequences

$$(1) \to (\operatorname{Ker} f)(p) \to A(p) \to (\operatorname{Im} f)(p) \to \mathbb{Q}_p/\mathbb{Z}_p \otimes (\operatorname{Ker} f),$$
$$(1) \to (\operatorname{Im} f)(p) \to B(p) \to (\operatorname{Coker} f)(p) \to \mathbb{Q}_p/\mathbb{Z}_p \otimes (\operatorname{Im} f).$$

Now $\operatorname{Ker} f$ and $\operatorname{Im} f$ are torsion and $\mathbb{Q}_p/\mathbb{Z}_p$ is a divisible abelian group. Thus $\mathbb{Q}_p/\mathbb{Z}_p \otimes \operatorname{Ker} f$ and $\mathbb{Q}_p/\mathbb{Z}_p \otimes \operatorname{Im} f$ are both trivial. (2) and (3) follow. ∎

We can now state a p-version of Proposition 0.2. Let $N_{L/K}^{(p)} : Cl_L(p) \to Cl_K(p)$ be the restriction of the norm map to $Cl_L(p)$.

PROPOSITION 0.5. *Suppose that* L *is a finite abelian extension of* K *such that* $n = [L : K]$ *is a power of a prime* p. *Then:*

(1) *There exists a canonical isomorphism of finite abelian* n*-torsion groups* $Cl_K(p)/B(p) \cong G(M/K)$.

(2) $Cl_{L/K}(p) = \operatorname{Ker} N^{(p)}_{L/K} \cong G(H^{(p)}_L/LH^{(p)}_K)$.
(3) $B(p) = N^{(p)}_{L/K} Cl_L(p) \cong G(H^{(p)}_K/M)$.

DEFINITION 0.6. We let e_f be the product of all the ramification indices $e_\wp$ where $\wp$ runs over all finite primes of L, e_∞ the product of the ramification indices $e_\wp$ over all infinite primes of L, and $e = e_f e_\infty$. We will also let e_0 be the least common multiple of all the ramification indices, finite and infinite. We will refer to e as the *ramification product* and e_0 as the *ramification lcm.*

For the field L, we have the following exact sequences of G-modules:

(S1) $(1) \to E_L \to L^* \to P_L \to (1)$, **(S2)** $(1) \to U_L \to J_L \to D_L \to (1)$,
(S3) $(1) \to \varGamma_L \to C_L \to Cl_L \to (1)$, **(S4)** $(1) \to E_L \to U_L \to \varGamma_L \to (1)$,
(S5) $(1) \to L^* \to J_L \to C_L \to (1)$, **(S6)** $(1) \to P_L \to D_L \to Cl_L \to (1)$.

The *nontrivial* maps above will be referred to as the *canonical maps* and will be labeled as they are used. There is a corresponding collection of exact sequences over the field K. If A_L and B_L are two of these G-modules with a canonical map $A_L \to B_L$, then we have induced homomorphisms $H^q(G, A_L) \to H^q(G, B_L)$ for each q in $\mathbb{Z}$. These maps will also be referred to as *canonical maps.*

For each of these G-modules A_L, there is a norm map $A_L \to A_K$ which will be denoted by $N_{L/K}$ when it is clear from the context which norm map is meant. When two or more norm maps are being used, we will introduce appropriate subscripts to distinguish them.

We now state some results on the cohomology of some of the modules in the above sequences **(S1)** to **(S6)** that will be used throughout this paper. They are well known and based on both global and local class field theory (see [10] or [21]).

If P is a prime of K and $\wp$ is a prime of L lying above P, let $G_P = G(L_\wp/K_P)$ be the corresponding *decomposition subgroup* of G, where K_P and $L_\wp$ represent the usual completions. Let $T_\wp$ be the *inertia group* of $\wp$ over P, $U_\wp$ the group of local units in $L_\wp$ and $e_\wp$ the *ramification index* of $\wp$ over P.

PROPOSITION 0.7. *Let L be a finite abelian extension of K.*

(1) $H^1(L^*) = (1)$ *(Hilbert's Theorem* 90*) and* $H^{-1}(D_L) = H^1(D_L) = (1)$.
(2) *For any* $q \in \mathbb{Z}$, $H^q(C_L) \cong H^{q-2}(\mathbb{Z})$ *and in particular* $H^0(C_L) \cong H^{-2}(\mathbb{Z}) \cong G$. *Morever,* $C_L^G = C_K$, $H^1(C_L) = (1)$, *and* $H^2(C_L)$ *is cyclic of order* $[L : K]$.
(3) *For any* $q \in \mathbb{Z}$, $H^q(J_L) \cong \prod_P H^q(G_P, L^*_\wp) \cong \prod_P H^{q-2}(G_P, \mathbb{Z})$ *where the product is over all primes P of K. In particular, we have* $H^0(J_L) \cong \prod_P H^{-2}(G_P, \mathbb{Z}) \cong \prod_P G_P$ *and* $H^1(J_L) = (1)$. *In addition,* $J_L^G = J_K$.

(4) $H^1(U_L) \cong \prod_P H^1(G_P, U_\wp) \cong \prod_P \mathbb{Z}/e_\wp\mathbb{Z}$ *where the product is over all finite primes* P *of* K.
(5) $U_L^G = U_K$ *and* $H^0(U_L) \cong \prod_P T_\wp$ *where the product is over all primes* P *of* K, *finite and infinite.*
(6) *If* L *is an unramified extension of* K, *then* $H^q(U_L) = (1)$ *for all* $q \in \mathbb{Z}$.

REMARK 0.8. (1) It is well known that $G(L/M)$ is the subgroup of G generated by the inertia groups $T_\wp$. When $G(L/M)$ is cyclic, $[L : M] = e_0$.

(2) Because each ramification index $e_\wp$ divides $n = [L : K]$, the ramification lcm e_0 divides n as well.

Since M is an unramified abelian extension of K, by Suzuki's Theorem, at least $[M : K]$ ideal classes of K become trivial in M and hence in L. Combining this fact with the remarks in the preceding paragraph, we have:

PROPOSITION 0.9 (Suzuki). $|\mathrm{Ker}\, j|$ *is divisible by* $[M : K]$. *Moreover, if* L *is a cyclic extension of* M, *then* $|\mathrm{Ker}\, j|$ *is divisible by* n/e_0 *where* e_0 *is the ramification lcm.*

1. Cohomological characterizations of $\mathrm{Ker}\, j$ **and** $\mathrm{Coker}\, j$**.** In this section, we characterize $\mathrm{Ker}\, j$ and $\mathrm{Coker}\, j$ in terms of cohomology groups. In particular, we give necessary and sufficient conditions for $\mathrm{Coker}\, j$ to be isomorphic to $H^1(P_L)$. (Recall that P_L is the group of principal ideals of L.) As always, we assume L is a finite abelian extension of K. The exact sequences **(S1)** to **(S6)** cited here are from Section 0.

By sequence **(S3)**, we have a commutative diagram with exact rows

D3
$$\begin{array}{ccccccccccc}
(1) & \longrightarrow & \Gamma_K & \xrightarrow{\gamma_K} & C_K & \longrightarrow & Cl_K & \longrightarrow & (1) & & \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle j} & & & & \\
(1) & \longrightarrow & \Gamma_L^G & \xrightarrow{\gamma_L^G} & C_K & \xrightarrow{\pi_L^G} & Cl_L^G & \longrightarrow & H^1(\Gamma_L) & \longrightarrow & (1)
\end{array}$$

where the left vertical map α is clearly injective, the middle vertical map is the identity and j is the right vertical map. So Γ_K can be regarded as a subgroup of Γ_L^G. The map $Cl_L^G \to H^1(\Gamma_L)$ is surjective since $H^1(C_L) = (1)$ by Proposition 0.7(2). By the Snake Lemma, we have the exact sequence

$$(1) \to \mathrm{Ker}\, j \to \mathrm{Coker}\, \alpha \to \mathrm{Coker}\, \mathrm{id} \to \mathrm{Coker}\, j \to H^1(\Gamma_L) \to (1).$$

We have proved the following.

PROPOSITION 1.1. (1) $\mathrm{Coker}\, j \cong H^1(\Gamma_L)$. (2) $\mathrm{Ker}\, j \cong \Gamma_L^G/\Gamma_K$.

The following result is equivalent to [8, Theorem 2.4]. The fact that $\mathrm{Ker}\, j \cong \mathrm{Ker}[H^1(E_L) \to H^1(U_L)]$ was proved in Schoof–Washington [25].

THEOREM 1.2. *There exists an exact sequence*

$$(1) \to \operatorname{Ker} j \to H^1(E_L) \xrightarrow{\theta} H^1(U_L) \to \operatorname{Coker} j \to H^1(P_L) \to (1)$$

where θ is induced by the canonical embedding $E_L \to U_L$. Thus $\operatorname{Coker} j \cong H^1(P_L)$ *if and only if θ is surjective, and* $\operatorname{Ker} j \cong H^1(E_L)$ *if and only if θ is trivial.*

Proof. The exact sequence **(S4)**: $(1) \to E_L \to U_L \to \Gamma_L \to (1)$ and the fact that $U_L^G = U_K$ (Proposition 0.7(5)) induce the exact sequence $(1) \to \Gamma_K \to \Gamma_L^G \to H^1(E_L) \xrightarrow{\theta} H^1(U_L) \to H^1(\Gamma_L) \xrightarrow{g} H^2(E_L)$. By Proposition 1.1, $\operatorname{Ker} j \cong \Gamma_L^G/\Gamma_K \cong \operatorname{Im}[\Gamma_L^G \to H^1(E_L)] = \operatorname{Ker}\theta$. So we have the exact sequence

$$(1) \to \operatorname{Ker} j \to H^1(E_L) \xrightarrow{\theta} H^1(U_L) \to H^1(\Gamma_L) \xrightarrow{g} H^2(E_L).$$

If $f : H^1(P_L) \to H^2(E_L)$ is the canonical map coming from sequence **(S1)**, it is easy to see that $\operatorname{Im} f = \operatorname{Im} g$ by looking at the commutative square

$$\begin{array}{ccc} H^0(Cl_L) & \longrightarrow & H^1(P_L) \\ \big\downarrow & & \big\downarrow f \\ H^1(\Gamma_L) & \xrightarrow{g} & H^2(E_L) \end{array}$$

The left vertical map comes from **(S3)** and the top horizontal map from **(S6)**. These last two maps are surjective because $H^1(C_L) = H^1(D_L) = (1)$ by Proposition 0.7. Moreover, f is injective by Hilbert's Theorem 90. Therefore, $\operatorname{Im} g \cong H^1(P_L)$. Recall also that $H^1(\Gamma_L) \cong \operatorname{Coker} j$ by Proposition 1.1. The desired sequence now follows. ∎

COROLLARY 1.3. *If L is an unramified abelian extension of K, then* $\operatorname{Ker} j \cong H^1(E_L)$ *and* $\operatorname{Coker} j \cong H^1(P_L)$. *If, in addition, L/K is cyclic, then* $\operatorname{Coker} j \cong E_K/N_{L/K}E_L$ *and* $|\operatorname{Ker} j| = [L : K]\,|\operatorname{Coker} j|$.

Proof. When L/K is unramified, we have $H^i(U_L) = (1)$ for all $i \in \mathbb{Z}$ (Proposition 0.7(6)), so the first statement follows from Theorem 1.2. If L/K is cyclic as well, then, using **(S4)**, $H^1(\Gamma_L) \cong H^2(E_L) \cong H^0(E_L)$, so $\operatorname{Coker} j \cong H^0(E_L)$. Finally, $|\operatorname{Ker} j| = [L : K]\,|H^0(E_L)|$ by the Herbrand quotient for E_L. ∎

REMARK 1.4. (1) The fact that $\operatorname{Ker} j \cong H^1(E_L)$ for unramified abelian extensions is a well-known result of Iwasawa.

(2) Since $\operatorname{Ker} j$ is isomorphic to a subgroup of $H^1(E_L)$, we have another proof that $\operatorname{Ker} j \subset (Cl_K)_n$ with $n = [L : K]$.

(3) It follows from (2) that if F is a field such that $K \subset F \subset L$, and $x \in Cl_K$ capitulates in L, then $x^{[L:F]}$ capitulates in F.

Corollary 1.3 establishes that when L/K is unramified, $\operatorname{Coker} j \cong H^1(P_L)$. We now show that this can also happen for some ramified extensions, generalizing a result in Cornell–Rosen [6].

Let $P_1, \ldots, P_t$ be the primes of K that ramify in L. For each $i = 1, \ldots, t$ we have $P_i O_L = \wp_{i_1}^{e_i} \cdots \wp_{i_g}^{e_i}$ where O_L is the ring of integers of L and $\wp_{i_1}, \wp_{i_2}, \ldots, \wp_{i_g}$ are the distinct primes of L that lie above P_i. Let $I_i = \wp_{i_1} \cdots \wp_{i_g}$. Then $P_i O_L = I_i^{e_i} = \eta(P_i)$ where $\eta : D_K \to D_L^G$ is the obvious inclusion. Then any element I in D_L^G can be written as $I = I_1^{m_1} \cdots I_t^{m_t} I_0$ where $0 \le m_i < e_i$ for each i and $I_0 \in D_K$, identifying D_K with its image $\eta(D_K)$ in D_L^G. Let $\rho : D_L^G \to Cl_L^G$ be the map that sends an element of D_L^G to its corresponding ideal class, that is, $\rho(I) = IP_L$. We have:

Lemma 1.5. $\operatorname{Im} \rho = \operatorname{Im} j$ *if and only if for each* $i = 1, \ldots, t$ *the ideal class* $I_i P_L$ *is in* $\operatorname{Im} j$. *In particular, if each* I_i *is principal in* O_L, *then* $\operatorname{Im} \rho = \operatorname{Im} j$.

Proposition 1.6. *Suppose that* K *has* t *ramified primes* $P_1, \ldots, P_t$ *and for each* $i = 1, \ldots, t$, *let* I_i *be the product of all primes of* L *that lie above* P_i. *Then* $\operatorname{Coker} j \cong H^1(P_L)$ *if and only if for each* i, *the ideal class* $I_i P_L$ *is in* $\operatorname{Im} j$.

Proof. Using sequences **(S1)–(S6)** from Section 0, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
(1) & \longrightarrow & E_K & \longrightarrow & K^* & \longrightarrow & P_L^G & \longrightarrow & H^1(E_L) & \longrightarrow & (1) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \theta & & \\
(1) & \longrightarrow & U_K & \longrightarrow & J_K & \longrightarrow & D_L^G & \longrightarrow & H^1(U_L) & \longrightarrow & (1) \\
 & & \downarrow & & \downarrow & & \downarrow \rho & & \downarrow & & \\
(1) & \longrightarrow & \Gamma_L^G & \longrightarrow & C_K & \xrightarrow{\pi_L^G} & Cl_L^G & \longrightarrow & H^1(\Gamma_L) & \longrightarrow & (1)
\end{array}$$

Since the canonical map $\pi_K : C_K \to Cl_K$ is surjective and $\pi_L^G = j \circ \pi_K$, we have $\operatorname{Im} \pi_L^G = \operatorname{Im} j$. Because $J_K \to C_K$ is surjective, $\operatorname{Im} \pi_L^G \subset \operatorname{Im} \rho$. By diagram chasing, we see that $\operatorname{Im} \rho \subset \operatorname{Im} \pi_L^G$ if and only if $\theta : H^1(E_L) \to H^1(U_L)$ is surjective. By Theorem 1.2 and Lemma 1.5, the proposition follows. ∎

Combining Proposition 1.6 with Corollary 1.3, we have:

Theorem 1.7. $\operatorname{Coker} j \cong H^1(P_L)$ *if and only if one of the following is true:*

(1) *L is an unramified extension of K.*
(2) *K has t ramified primes $P_1, \ldots, P_t$ and for each $i = 1, \ldots, t$, if I_i is the product of all primes of L that lie above P_i, then the ideal class $I_i P_L$ is in* $\operatorname{Im} j$.

LEMMA 1.8. *Suppose that L is a cyclic extension of K. Then $H^1(P_L) \cong E_K \cap N_{L/K}L^*/N_{L/K}E_L$ and $|H^0(E_L)| = |H^1(P_L)|[E_K : E_K \cap N_{L/K}L^*]$.*

Proof. Using the sequence **(S1)**: $(1) \to E_L \to L^* \to P_L \to (1)$, Hilbert's Theorem 90, and the fact that L/K is cyclic, we have an exact sequence $(1) \to H^{-1}(P_L) \to H^0(E_L) \xrightarrow{\alpha} H^0(L^*)$. Now α is defined by $\alpha(xN_{L/K}E_L) = xN_{L/K}L^*$ for $x \in E_K$. Thus

$$H^1(P_L) \cong H^{-1}(P_L) \cong \operatorname{Ker}\alpha = E_K \cap N_{L/K}L^*/N_{L/K}E_L.$$

The lemma follows. ∎

PROPOSITION 1.9. *Suppose that L is a cyclic extension of K. Let e, e_f, and e_∞ be the ramification products as defined in Definition 0.6. Then*

$$|\operatorname{Ker} j| = \frac{[L:K]\,|H^0(E_L)|\,|\operatorname{Coker} j|}{e|H^1(P_L)|}$$

and $|H^0(E_L)|/|H^1(P_L)|$ is an integer. In particular, if one of the two conditions of Theorem 1.7 holds, then:

(1) $|\operatorname{Ker} j| = [L:K]\,|H^0(E_L)|/e$.
(2) $\operatorname{Im} j = Cl_L^G$ *if and only if every unit of K that is a global norm from L is the norm of a unit of L.*

Proof. The Herbrand quotient of E_L is well known to be $e_\infty/[L:K]$, and $|H^1(U_L)| = e_f$ by Proposition 0.7(4). The formula for $|\operatorname{Ker} j|$ now follows from Theorem 1.2. The proof of Lemma 1.8 shows that $H^1(P_L)$ is isomorphic to a subgroup of $H^0(E_L)$. Statement (1) is clear, and since $\operatorname{Im} j = Cl_L^G$ is equivalent to $\operatorname{Coker} j = H^1(P_L) = (1)$ by Theorem 1.7, Lemma 1.8 yields (2). ∎

COROLLARY 1.10. *Suppose that L is a cyclic extension of K and $E_K = N_{L/K}E_L$. Then $|\operatorname{Ker} j|$ divides $[L:K]/e_\infty$. Moreover, $|\operatorname{Coker} j|$ divides e_f. If, in addition, L/K is unramified, then $|\operatorname{Ker} j| = [L:K]$ and $\operatorname{Im} j = Cl_L^G$.*

Proof. By Theorem 1.2, $|\operatorname{Ker} j|$ divides $|H^1(E_L)|$, and $|H^1(E_L)| = [L:K]/e_\infty$ since $H^0(E_L) = (1)$. That $|\operatorname{Coker} j|$ divides e_f follows from Proposition 1.9. ∎

COROLLARY 1.11. *Let K be an imaginary quadratic field whose 2-class number is greater than 1. Let $L = K(\sqrt{d})$ where d is a square-free integer, $d \geq 2$. Assume that $K \cap \mathbb{Q}(\sqrt{d}) = \mathbb{Q}$. Assume also that the norm of the fundamental unit ε of $\mathbb{Q}(\sqrt{d})$ is -1. Then $|\operatorname{Ker} j| = 1$ or 2. Thus if L/K is unramified, then $|\operatorname{Ker} j| = 2$.*

Proof. We have $N_{L/K}\varepsilon = N_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}\varepsilon = -1$, and since $E_K = \{-1, 1\}$, we have $E_K = N_{L/K}E_L$. Now $e_\infty = 1$. So by Corollary 1.10, $|\operatorname{Ker} j|$ divides 2. ∎

2. A quotient of $\operatorname{Ker} j$**.** In this section, we look at the quotient group $\operatorname{Ker} j/J$ where J is the subgroup of $\operatorname{Ker} j$ consisting of those ideal classes in $B = N_{L/K} Cl_L$ that capitulate in L; that is,

$$J = \operatorname{Ker} j \cap B. \tag{2.1}$$

We will show that there exists an unramified abelian extension R of K such that $\operatorname{Ker} j/J \cong G(M/R)$, $M = H_K \cap L$, and $|J|$ is divisible by $[R : K]$.

Remark 2.1. By Proposition 0.2, B is identified with $G(H_K/M)$, so $J = \operatorname{Ker} j \cap G(H_K/M)$. Thus when $M = H_K \cap L = K$, we have $\operatorname{Ker} j = J$.

If we restrict the norm map $N_{L/K} : Cl_L \to Cl_K$ to the subgroup $\operatorname{Ker} N_G$, diagram **D1** from Section 0 shows that we have a well-defined surjective homomorphism $N'_{L/K} : \operatorname{Ker} N_G \to J$ with kernel $Cl_{L/K} = \operatorname{Ker} N_{L/K}$. So we have:

Proposition 2.2. $J \cong \operatorname{Ker} N_G/Cl_{L/K}$.

Theorem 2.3.

(1) *There is a canonical exact sequence of finite abelian n-torsion groups*

$$(1) \to J \to \operatorname{Ker} j \to G(M/K) \xrightarrow{\bar{j}} H^0(Cl_L) \xrightarrow{\varphi} \operatorname{Coker} j \to (1). \tag{2.2}$$

(2) *There exists an unramified abelian extension R of K such that* $\operatorname{Ker} j/J \cong G(M/R)$ *and* $G(R/K) \cong \operatorname{Im} j/N_G Cl_L$.

(3) $|J|$ *is divisible by* $[R : K]$. *Consequently, if* $J = (1)$, *then* $\operatorname{Ker} j \cong G(M/K)$.

Proof. The map $j : Cl_K \to Cl_L^G$ induces $\bar{j} : Cl_K/N_{L/K} Cl_L \to Cl_L^G/N_G Cl_L$. By diagram **D1** in Section 0, $\bar{j}$ is well defined and clearly $\operatorname{Ker} \bar{j} = \operatorname{Ker} j/J$. Also, $\operatorname{Im} \bar{j} = \operatorname{Im} j/N_G Cl_L$, the kernel of the natural map $\varphi : Cl_L^G/N_G Cl_L \to Cl_L^G/\operatorname{Im} j$. Sequence (2.2) now follows by identifying Cl_K/B with $G(M/K)$. Statement (2) follows easily from (1), and (3) from Suzuki's Theorem. ∎

Although $|\operatorname{Ker} j|$ is divisible by $[M : K]$, Suzuki's Theorem does not say that $G(M/K)$ can be embedded in $\operatorname{Ker} j$. The next corollary gives a sufficient condition for $G(M/K)$ to be isomorphic to a subgroup of $\operatorname{Ker} j$. We supplement that with an example where no subgroup of $\operatorname{Ker} j$ is isomorphic to $G(M/K)$ (see Example 2.6 below). We also give an example where $\operatorname{Ker} j \cong G(M/K)$ even though $R \neq K$ (Example 2.7).

For further discussion of the embedding of $G(M/K)$ in $\operatorname{Ker} j$, see Section 3.2 of Bembom's thesis [2], as well as Examples 1 and 2 there. The next corollary gives a sufficient condition for the existence of such an embedding. We first prove a lemma.

Lemma 2.4. *Let A and B be finite abelian groups and C a subgroup of B. Suppose that A is isomorphic to B/C. Then A is isomorphic to a subgroup of B.*

Proof. We use the well-known fact that a finite abelian group B is isomorphic (noncanonically) to its character group $\chi(B) = \mathrm{Hom}(B, \mathbb{C}^*)$, $\mathbb{C}^*$ being the group of nonzero complex numbers. Also, if C is a subgroup of B, then $\chi(B/C) \cong C^\wedge = \{\alpha \in \chi(B) \mid \alpha(x) = 1\ \forall x \in C\}$. Thus $A \cong B/C \cong \chi(B/C) \cong C^\wedge$ and $C^\wedge$ is isomorphic to a subgroup of B since $B \cong \chi(B)$. ∎

Corollary 2.5. *There is a canonical exact sequence*

$$(1) \to G(R/K) \to H^0(Cl_L) \to \operatorname{Coker} j \to (1).$$

Consequently, $\operatorname{Coker} j = H^0(Cl_L)$ *if and only if* $R = K$. *Moreover, if* $R = K$, *then* $G(M/K)$ *is isomorphic* (*noncanonically*) *to a subgroup of* $\operatorname{Ker} j$.

Proof. The exact sequence follows easily from Theorem 2.3. If $R = K$, then by Theorem 2.3(2) and Lemma 2.4, $G(M/K)$ is isomorphic to a subgroup of $\operatorname{Ker} j$. ∎

Example 2.6. This example can be found in Scholz and Taussky [24] as well as Bembom [2]. Let $K = \mathbb{Q}(\sqrt{-3299})$. Then $Cl_K \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Let x and y be generators of Cl_K of orders 9 and 3 respectively. Let T be the subgroup of Cl_K generated by x^3 and y, and E the fixed field of T. Let L be an unramified abelian extension of degree 9 over K. (There are four such extensions, exactly three of them cyclic.) By the structure of Cl_K, we have $E \subset L$. Let $B = G(H_K/L)$ and $j : Cl_K \to Cl_L$ the extension of ideal classes map. In [24] it is shown that x^3 does not capitulate in E. Then by Remark 1.4(3), $\operatorname{Ker} j$ contains no element of order 9, and hence $\operatorname{Ker} j = T \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Thus for those three cyclic extensions L, no subgroup of $\operatorname{Ker} j$ is isomorphic to $G(L/K)$.

Note that for each of these four extensions, $J = \operatorname{Ker} j \cap B = B$. Hence by Theorem 2.3, $R = E$.

Example 2.7. Let $K = \mathbb{Q}(\sqrt{-5703})$. (See [2, Sec. 3.2] for a discussion of this and similar examples.) As in the previous example, $Cl_K \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, but the capitulation results are quite different. Keeping the notation from Example 2.6, for the three cyclic extensions L of degree 9 over K, we have $\operatorname{Ker} j \cong \mathbb{Z}/9\mathbb{Z}$, $J = (1)$, and $R = K$. For the fourth, $\operatorname{Ker} j \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $J = B$, and $R = E$. So $\operatorname{Ker} j \cong G(L/K)$ in all four cases, and in one of them $R \neq K$.

We have seen that if $J = (1)$, then $R = K$. We give a partial converse in Corollary 2.9 below, which is another corollary of Theorem 2.3. We use the following lemma. For a proof of this lemma, see Wong [28]. (See also [7].)

Lemma 2.8. *Let G be a finite nilpotent group and A a finite G-module. If $H^m(G,A) = (1)$ for some integer m, then $H^n(G,A) = (1)$ for all n.*

Corollary 2.9. *Suppose that* $\operatorname{Im} j = Cl_L^G$ *and $R = K$. Then $J = (1)$. In particular, if $H_K \cap L = M = K$ and* $\operatorname{Im} j = Cl_L^G$, *then* $\operatorname{Ker} j = (1)$.

Proof. From Corollary 2.5, we have $H^0(Cl_L) = (1)$, and hence $H^{-1}(Cl_L) = (1)$ by Lemma 2.8. Thus $\operatorname{Ker} N_G = I_G Cl_L = \operatorname{Ker} N_{L/K}$, and $J = (1)$ by Proposition 2.2. ∎

One way to characterize the field R is as the fixed field of the subgroup $(\operatorname{Ker} j)(N_{L/K} Cl_L)$ of Cl_K. In particular, we have:

Proposition 2.10. $(\operatorname{Ker} j)(N_{L/K} Cl_L) \cong G(H_K/R)$.

Proof. Let $A = \operatorname{Ker} j$. As before, we let $B = N_{L/K} Cl_L$. Then $AB/B \cong \operatorname{Ker} j/J \cong G(H_K/R)/G(H_K/M)$ by Theorem 2.3(2). As $|B| = |G(H_K/M)|$, it follows that $|AB| = |G(H_K/R)|$. Now $AB = G(H_K/F)$ for some field F such that $K \subset F \subset H_K$. Since $B \subset AB$, we have $F \subset M$. Let $\pi : Cl_K \to G(M/K)$ be the canonical map and π_0 its restriction to AB. If we identify AB with $G(H_K/F)$, π_0 maps an automorphism of $G(H_K/F)$ to its restriction to M. Since $\operatorname{Ker} \pi_0 = B$ and $\operatorname{Im} \pi_0 = \pi(AB) \cong G(M/R)$, we have the exact sequence $(1) \to G(H_K/M) \to G(H_K/F) \to G(M/R) \to (1)$, giving $F = R$. ∎

Corollary 2.11.

(1) *If $J = (1)$, then $Cl_K \cong$* $\operatorname{Ker} j \times N_{L/K} Cl_L$.
(2) *If $J = B = N_{L/K} Cl_L$, then* $\operatorname{Ker} j = G(H_K/R)$.

Another consequence of $J = (1)$ is the following:

Proposition 2.12. *If $J = (1)$, then every ideal of K that capitulates in L in fact capitulates in M.*

Proof. Let $j_{M/K} : Cl_K \to Cl_M$ be the map induced by lifting of ideals. Let $J_{M/K} = \operatorname{Ker} j_{M/K} \cap N_{M/K} Cl_M$. Clearly, $\operatorname{Ker} j_{M/K} \subset \operatorname{Ker} j$. By Corollary 0.3, $N_{L/M} : Cl_L \to Cl_M$ is surjective. Hence $N_{L/K} Cl_L = N_{M/K} Cl_M$ and $J_{M/K} \subset J$. So if $J = (1)$, then applying Theorem 2.3 to M/K, we obtain $\operatorname{Ker} j_{M/K} = \operatorname{Ker} j$. ∎

3. The semisimple case. As noted in Remark 0.1(2), $\operatorname{Ker} j \subset (Cl_K)_n$ where $n = [L : K]$. We let

$$d = \gcd(n, h_K). \tag{3.1}$$

Clearly, $(Cl_K)_d = (Cl_K)_n$. As defined in Section 0, we let $Cl_{L/K} = \operatorname{Ker} N_{L/K}$, the *relative class group*, and $B = N_{L/K} Cl_L$, the image of $N_{L/K}$.

We let $h_{L/K}$ be the order of $Cl_{L/K}$; $h_{L/K}$ is the *relative class number*. So $|Cl_{L/K}| = h_{L/K} = [H_L : LH_K]$ and $|B| = [LH_K : L] = h_L/h_{L/K}$.

In this section, we give some necessary and sufficient conditions for the equality $\operatorname{Ker} j = (Cl_K)_d$. We will see, for example, that $\operatorname{Ker} j = (Cl_K)_d$ if n is relatively prime to either $h_L/h_{L/K}$ (Proposition 3.7) or $h_{L/K}$ (Corollary 3.10).

Remark 3.1. (1) Since $\operatorname{Ker} j \subset (Cl_K)_d$, we have $J = \operatorname{Ker} j \cap B \subset B_d$.
(2) By diagram **D2** in Remark 0.1, $(Cl_K)^d = (Cl_K)^n \subset B$.

We consider the following maps: $\alpha : (Cl_K)_d \to Cl_K/B$ is defined by $\alpha(x) = xB$, $x \in (Cl_K)_d$; $\beta : Cl_K/B \to B/B^d$ is defined by $\beta(xB) = N_{L/K}j(x)B^d = x^nB^d$, $x \in Cl_K$; $\delta : B/B^d \to Cl_K/(Cl_K)^d$ is defined by $\delta(bB^d) = b(Cl_K)^d$, $b \in B$; and $\varepsilon : Cl_K/(Cl_K)^d \to Cl_K/B$ is defined by $\varepsilon(x(Cl_K)^d) = xB$, $x \in Cl_K$. With the help of Remark 3.1(2) above, it is easy to see that these maps are well defined.

Lemma 3.2. *The following sequence is exact:*

$$(1) \to B_d \to (Cl_K)_d \xrightarrow{\alpha} Cl_K/B \xrightarrow{\beta} B/B^d \xrightarrow{\delta} Cl_K/(Cl_K)^d \xrightarrow{\varepsilon} Cl_K/B \to (1). \tag{3.2}$$

Proof. Apply Lemma 0.4 to the sequence $(1) \to B \to Cl_K \to Cl_K/B \to (1)$ and use the fact that $(Cl_K)^d \subset B$ from Remark 3.1(2). ∎

Let $N_1 : Cl_L^G/N_G Cl_L \to B/B^d$ be defined by $N_1(zN_G Cl_L) = N_{L/K}(z)B^d$ and $N_2 : Cl_L^G/\operatorname{Im} j \to Cl_K/(Cl_K)^d$ by $N_2(y \operatorname{Im} j) = N_{L/K}(y)(Cl_K)^d$. By Diagram **D2** and Remark 3.1(2), N_1 and N_2 are well defined.

Proposition 3.3. *The following diagram* **D4** *is commutative with exact rows:*

$$\begin{array}{ccccccccccccccc}
(1) & \to & J & \to & \operatorname{Ker} j & \xrightarrow{\alpha_0} & G(M/K) & \xrightarrow{\bar{j}} & H^0(Cl_L) & \xrightarrow{\varphi} & \operatorname{Coker} j & \to & (1) \\
 & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle N_1} & & \downarrow{\scriptstyle N_2} & & \downarrow \\
(1) & \to & B_d & \to & (Cl_K)_d & \xrightarrow{\alpha} & G(M/K) & \xrightarrow{\beta} & B/B^d & \xrightarrow{\delta} & Cl_K/(Cl_K)^d & \xrightarrow{\varepsilon} & G(M/K) & \to & (1)
\end{array}$$

Proof. The top sequence comes from Theorem 2.3(1) and the maps $\bar{j}$ and φ are defined in the proof of Theorem 2.3. The bottom sequence is (3.2) with $G(M/K)$ identified with Cl_K/B, and i_1 and i_2 are inclusion maps. The equalities $N_1 \circ \bar{j} = \beta$, $N_2 \circ \varphi = \delta \circ N_1$, and $\varepsilon \circ N_2 = (1)$ follow easily from the definitions of the various maps. ∎

The right side of diagram **D4** of Proposition 3.3 induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & G(M/K) & \xrightarrow{\bar{j}} & H^0(Cl_L) & \xrightarrow{\varphi} & \operatorname{Coker} j & \longrightarrow & (1) \\ & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle N_1} & & \downarrow{\scriptstyle N_2} & & \downarrow \\ (1) & \longrightarrow & \operatorname{Im}\beta & \xrightarrow{i} & B/B^d & \xrightarrow{\delta} & Cl_K/(Cl_K)^d & \xrightarrow{\varepsilon} & G(M/K) \longrightarrow (1) \end{array}$$

where $\beta' : G(M/K) \to \operatorname{Im}\beta$ is the map β with restricted codomain. Note that β' is surjective and $\operatorname{Ker}\beta' = \operatorname{Ker}\beta$.

We apply a version of the Snake Lemma due to Lemmermeyer [17, Proposition 1.3.7] to arrive at the following exact sequence:

$$(1) \to \operatorname{Ker}\bar{j} \to \operatorname{Ker}\beta' \to \operatorname{Ker}N_1 \xrightarrow{\varphi'} \operatorname{Ker}N_2 \to \operatorname{Coker}\beta' \\ \to \operatorname{Coker}N_1 \to \operatorname{Coker}N_2 \to \operatorname{Coker}\delta \to (1).$$

Now $\operatorname{Ker}\bar{j} = \operatorname{Ker} j/J$, $\operatorname{Ker}\beta = (Cl_K)_d/B_d$, $\operatorname{Coker}\beta' = (1)$, and $\operatorname{Coker}\delta = \operatorname{Im}\varepsilon = G(M/K)$. Note also that φ' is the restriction of φ to $\operatorname{Ker}N_1$.

So we have the exact sequence

$$(3.3) \qquad (1) \to \operatorname{Ker} j/J \xrightarrow{i_0} (Cl_K)_d/B_d \xrightarrow{\psi} \operatorname{Ker}N_1 \xrightarrow{\varphi'} \operatorname{Ker}N_2 \to (1).$$

Let $I = \operatorname{Im}\alpha$ and $I_0 = \operatorname{Im}\alpha_0$ where α and α_0 come from diagram **D4**. Then the left side of **D4** induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} (1) & \longrightarrow & J & \longrightarrow & \operatorname{Ker} j & \xrightarrow{\alpha_0} & I_0 & \longrightarrow & (1) \\ & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} & & \\ (1) & \longrightarrow & B_d & \longrightarrow & (Cl_K)_d & \xrightarrow{\alpha} & I & \longrightarrow & (1) \end{array}$$

Applying the Snake Lemma to this diagram and using the fact that the vertical maps are injective, we have an exact sequence

$$(1) \to B_d/J \to (Cl_K)_d/\operatorname{Ker} j \to I/I_0 \to (1).$$

Thus

$$\operatorname{Coker}[B_d/J \to (Cl_K)_d/\operatorname{Ker} j] = I/I_0 = \operatorname{Coker}[\operatorname{Ker} j/J \to (Cl_K)_d/B_d].$$

Therefore (3.3) is equivalent to the exact sequence

$$(3.4) \qquad (1) \to B_d/J \xrightarrow{i^*} (Cl_K)_d/\operatorname{Ker} j \xrightarrow{\psi^*} \operatorname{Ker}N_1 \xrightarrow{\varphi'} \operatorname{Ker}N_2 \to (1)$$

where i^* is induced by the inclusion and ψ^* is defined by $\psi^*(x\operatorname{Ker} j) = j(x)N_G Cl_L$.

LEMMA 3.4. *ψ^* is the trivial map if and only if $(Cl_K)_d \subset G(H_K/R)$.*

Proof. By (3.4) and the fact that $\operatorname{Ker} j \cap B_d = J$, we see that ψ^* is the trivial map $\Leftrightarrow (Cl_K)_d = (\operatorname{Ker} j)B_d \Leftrightarrow (Cl_K)_d \subset (\operatorname{Ker} j)B$. But $(\operatorname{Ker} j)B = G(H_K/R)$ by Proposition 2.10. ■

Combining Lemma 3.4 and sequence (3.4), we obtain:

THEOREM 3.5. $\operatorname{Ker} j = (Cl_K)_d$ *if and only if* $J = B_d$ *and* $(Cl_K)_d \subset G(H_K/R)$.

As an immediate consequence of Theorem 3.5, we have:

COROLLARY 3.6.

(1) *If* $J = B_d$ *and* $R = K$, *then* $\operatorname{Ker} j = (Cl_K)_d$.
(2) *If* $\operatorname{Ker} j = (Cl_K)_d = Cl_K$, *then* $J = B_d = B$ *and* $R = K$.

PROPOSITION 3.7. *Suppose that* $\gcd(n, h_L/h_{L/K}) = 1$. *Then* $J = (1)$ *and* $\operatorname{Ker} j = (Cl_K)_d$.

Proof. If $\gcd(n, h_L/h_{L/K}) = 1$, then $J = B_n = (1)$, so $R = K$ by Theorem 2.3(3), and $\operatorname{Ker} j = (Cl_K)_d$ by Corollary 3.6(1). ∎

PROPOSITION 3.8. *Suppose that* Cl_K *is cyclic and* L/K *is unramified abelian. Then* $\operatorname{Ker} j = (Cl_K)_d$ *and* $|J| = \gcd(d, h_L/h_{L/K}) = [R : K]$.

Proof. $|\operatorname{Ker} j| \le |(Cl_K)_d| = d$ since Cl_K is cyclic. Since L/K is unramified, $d = n = [L : K]$, so $|\operatorname{Ker} j| \ge d$ by Suzuki's Theorem. Thus $\operatorname{Ker} j = (Cl_K)_d$, $J = B_d$, and $|J| = \gcd(d, h_L/h_{L/K})$. By Theorem 2.3(2), this implies $|\operatorname{Ker} j| = |J|\,[L : R]$, so $|J| = [R : K]$. ∎

The following theorem generalizes [25, Lemma 4(i)]:

THEOREM 3.9. *Let* t *be a positive divisor of* n *that is relatively prime to* $h_{L/K}$. *Then* $(Cl_K)_t \subset \operatorname{Ker} j$ *and* J *contains a subgroup isomorphic to* $(Cl_L)_t$. *In particular, if* $\gcd(d, h_{L/K}) = 1$, *then* $\operatorname{Ker} j = (Cl_K)_d$ *and* J *contains a subgroup isomorphic to* $(Cl_L)_d$.

Proof. Let $x \in (Cl_K)_t$. Because $N_{L/K} \circ j$ is raising to the nth power, $j(x)$ is killed by both t and $h_{L/K}$, and therefore is trivial. So we have $(Cl_K)_t \subset \operatorname{Ker} j$ and $B_t \subset J$. Thus the norm map $N_{L/K}$ gives an injection $(Cl_L)_t \to B_t$, whose image is in J, so J contains a subgroup isomorphic to $(Cl_L)_t$. If $\gcd(d, h_{L/K}) = 1$, then letting $t = d$, from the above we have $\operatorname{Ker} j = (Cl_K)_d$. ∎

COROLLARY 3.10. *Suppose that* n *and* $h_{L/K}$ *are relatively prime. Then* $\operatorname{Ker} j = (Cl_K)_d$, $J = B_d \cong (Cl_L)_n$, $Cl_L^G \cong B$, *and* $(Cl_L)_n \subset Cl_L^G$.

Proof. By Theorem 3.9, $\operatorname{Ker} j = (Cl_K)_d$ and $J = B_d$. The sequence of G-modules $(1) \to Cl_{L/K} \to Cl_L \to B \to (1)$ is exact and G acts trivially on B. Since $\gcd(n, h_{L/K}) = 1$, we have $H^0(Cl_{L/K}) = Cl_{L/K}^G = (1)$ and $H^1(Cl_{L/K}) = (1)$, so $N_{L/K}^G : Cl_L^G \to B$ is an isomorphism. Since $(Cl_{L/K})_n = (1)$, we have an injection $N'_{L/K} : (Cl_L)_n \to B_n$ induced by $N_{L/K}$. It follows that $|B_n| = |(Cl_L^G)_n| \le |(Cl_L)_n| \le |B_n|$. Thus $N'_{L/K}$ is an isomorphism. Finally, $|(Cl_L^G)_n| = |(Cl_L)_n|$, so $(Cl_L)_n \subset Cl_L^G$. ∎

Remark 3.11. (1) The conclusions of Corollary 3.10 hold whenever $N_{L/K} : Cl_L \to Cl_K$ is injective.

(2) Some results on the calculation of $Cl_{L/K}$ can be found in Lemmermeyer [18]. (See Proposition 1 and Corollary 2 of that paper. Proposition 16 thereof will be used in Corollary 5.4 below.)

The last result of this section is an easy consequence of the injection $i_0 : \operatorname{Ker} j/J \to (Cl_K)_d/B_d$ from sequence (3.3) above. Replacing d by p, we have:

Proposition 3.12. *Let L be an extension of K of prime degree p. Assume that the p-rank of $Cl_K(p)$ equals the p-rank of $B(p)$. Then $R = L \cap H_K$ and* $\operatorname{Ker} j = J$.

Proof. Let $d = \gcd(p, h_K)$. Recall that by Theorem 2.3(2), there exists a field R such that $K \subset R \subset M$ and $\operatorname{Ker} j/J \cong G(M/R)$ where $M = L \cap H_K$. If $d = 1$, then $M = R = K$, and thus $\operatorname{Ker} j = J = (1)$. Suppose $d = p$. From sequence (3.3), we have an injection $i_0 : \operatorname{Ker} j/J \to (Cl_K)_p/B_p$. But if $Cl_K(p)$ and $B(p)$ have the same p-rank, then $(Cl_K)_p = B_p$, and thus $\operatorname{Ker} j = J$. By Theorem 2.3(2), $R = M$. ∎

Remark 3.13. The converse of Proposition 3.12 is false. Let K be a field such that $Cl_K(2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and L an unramified quadratic extension of K. Suppose that $G(H_L^{(2)}/K)$ is the generalized quaternion group Q_k of order 2^k, $k \geq 4$, where $H_L^{(2)}$ is the 2-Hilbert class field of L. (See Section 5 for the definition of Q_k.) Kisilevsky [15] proves that $\operatorname{Ker} j = J \cong \mathbb{Z}/2\mathbb{Z}$, so that $R = M$ and 2-rank $Cl_K \neq$ 2-rank B.

4. Knot groups. As in the previous sections, L is an abelian extension of K of degree n. We introduce two important subfields of H_L. The *abelian genus field* L_g of L over K is the maximal unramified abelian extension of L that is of the form EL where E is an abelian extension of K. The *central class field* L_c of L over K is the maximal unramified extension of L such that L_c is Galois over K and $G(L_c/L)$ is contained in the center of $G(L_c/K)$. Clearly, $L_g \subset L_c \subset H_L$. We let $U = G(H_L/K)$ and $A_L = G(H_L/L)$. (Recall that $A_L \cong Cl_L$.)

The action of G on A_L is by conjugation and is compatible, by class field theory, with the action of G on Cl_L: if $a \in A_L$, $\sigma \in G$, then $\sigma a = u^{-1}au$ where σ is the canonical image of $u \in U$. Since G is abelian, we see that $G(H_L/L_g) \cong U'$, the commutator subgroup of U, and $G(H_L/L_c) \cong [U, A_L]$, the subgroup of U' generated by the commutators of the form $vxv^{-1}x^{-1}$, $v \in U$, $x \in A_L$. Thus $G(H_L/L_c) \cong I_G A_L$ and $G(L_c/LH_K) \cong Cl_{L/K}/I_G Cl_L$.

Jehne [14] introduces various "knot groups"; in particular,

$$\nu = \nu_{L/K} = K^* \cap N_{L/K}J_L/N_{L/K}L^*, \tag{4.1}$$

$$\gamma = \gamma_{L/K} = \Gamma_K \cap N_{L/K}C_L/N_{L/K}\Gamma_L. \tag{4.2}$$

REMARK 4.1. (1) By the definition of ν, $\nu = (1)$ if and only if the *Hasse Norm Principle* (HNP) holds for the extension L over K, i.e. any element of K^* that is a norm everywhere locally is a global norm from L.

(2) We see from their definitions that $\nu \subset H^0(L^*)$ and $\gamma \subset H^0(\Gamma_L)$. It follows that ν and γ are both n-torsion groups with $n = [L : K]$.

(3) Let $\delta = \delta_{L/K} = P_K \cap N_{L/K}D_L/N_{L/K}P_L$. Then there is a natural map $\nu \to \delta$ and we let δ^0 be its image.

For the proof of the following proposition, see [14, Theorems 1 and 3 and Proposition 1].

PROPOSITION 4.2.

(1) *There is an exact sequence* $(1) \to \omega^0 \to \nu \to \delta^0 \to (1)$ *where* $\omega^0 = E_K \cap N_{L/K}U_L/E_K \cap N_{L/K}L^*$.
(2) $\gamma \cong G(L_g/LH_K)$ *and* $\delta^0 \cong G(L_c/L_g)$.
(3) *Let e be the ramification product. Then*

$$|\gamma| = [L_g : LH_K] = \frac{e}{[L : M][E_K : E_K \cap N_{L/K}U_L]}. \tag{4.3}$$

COROLLARY 4.3.

(1) *HNP holds for L over K if and only if $L_c = L_g$ and $E_K \cap N_{L/K}U_L = E_K \cap N_{L/K}L^*$. Consequently, if HNP holds for L over K, then* $(1) \to I_G Cl_L \to Cl_{L/K} \to \gamma \to (1)$ *is exact and*

$$|\gamma| = [L_g : LH_K] = \frac{e}{[L : M][E_K : E_K \cap N_{L/K}L^*]}. \tag{4.4}$$

(2) *If L is an unramified abelian extension of K, then $\gamma = (1)$.*

REMARK 4.4. It is known that HNP holds for L over K in each of the following cases:

(1) L is a cyclic extension of K;
(2) there is a prime P_0 of K such that $G_{P_0} = G$;
(3) the least common multiple of the local degrees $[L_\wp : K_P]$ equals $[L : K]$.

(Condition (3), less known, is proved in [14].)

Recall that $H^{-1}(Cl_L) = \operatorname{Ker} N_G/I_G Cl_L$ and $J = \operatorname{Ker} j \cap N_{L/K}Cl_L = \operatorname{Ker} j \cap B$. Define $\lambda : H^{-1}(Cl_L) \to Cl_K$ by $\lambda(xI_G Cl_L) = N_{L/K}(x)$ for $x \in \operatorname{Ker} N_G$. It is easy to see that λ is a well-defined homomorphism, $\operatorname{Im}\lambda = J$, and $\operatorname{Ker}\lambda = \operatorname{Ker} N_{L/K}/I_G Cl_L \cong G(L_c/LH_K)$. So if L is an unramified abelian extension of K, then $\operatorname{Ker}\lambda \cong \delta^0 \cong G(L_c/L_g)$. From now on, we will consider λ as a map from $H^{-1}(Cl_L)$ to J, so that λ is in fact surjective.

From Corollary 4.3(1) and the previous remarks, we have:

PROPOSITION 4.5. *There is a surjective homomorphism* $\lambda : H^{-1}(Cl_L) \to J$ *such that* $\operatorname{Ker}\lambda = Cl_{L/K}/I_G Cl_L \cong G(L_c/LH_K)$. *If HNP holds for* L *over* K *and* $\gamma = (1)$, *then* $Cl_{L/K} = I_G Cl_L$ *and* $J \cong H^{-1}(Cl_L)$.

THEOREM 4.6. *Suppose that* L *is a cyclic extension of* $M = H_K \cap L$. *Let* e_0 *be the ramification lcm. Suppose that* $e_0 = e$ *and HNP holds for* L *over* K. *Then* $\gamma = (1)$, $J \cong H^{-1}(Cl_L)$, $E_K \subset N_{L/K}L^*$, *and* $Cl_{L/K} = I_G Cl_L$. *Moreover, if* $J = (1)$, *then* $\operatorname{Im} j = Cl_L^G$.

Proof. By (4.4) and the fact that $e = e_0 = [L : M]$ (Remark 0.8(1)), we have $\gamma = (1)$ and $E_K \subset N_{L/K}L^*$. By Proposition 4.5, $J \cong H^{-1}(Cl_L)$ and $Cl_{L/K} = I_G Cl_L$. Thus if $J = (1)$, then $H^{-1}(Cl_L) = (1)$, and $\operatorname{Coker} j = H^0(Cl_L) = (1)$ by Corollary 2.5 and Lemma 2.8. ∎

REMARK 4.7. The conclusion that $Cl_{L/K} = I_G Cl_L$ is a version of Hilbert's Theorem 90 for ideal classes. Furtwängler proved it for unramified cyclic extensions. For a discussion of this theorem, see Bembom [2, Section 2.3].

In what follows we give some consequences of the previous results of this section when L is a cyclic extension of K.

PROPOSITION 4.8. *Suppose that* L *is a cyclic extension of* K *and* e *is the ramification product.*

(1) $Cl_L/Cl_L^G \cong I_G Cl_L$ *and* $|Cl_L^G| = |N_{L/K}Cl_L|\,|\gamma| = [L_g : L]$.
(2) *We have*

$$|Cl_L^G| = \frac{h_K e}{[L : K][E_K : E_K \cap N_{L/K}L^*]}$$

where h_K *is the class number of* K.

Proof. The map $\varphi : Cl_L \to I_G Cl_L$ defined by $\varphi(x) = \frac{\sigma x}{x}$, σ a generator of G, proves that $Cl_L/Cl_L^G \cong I_G Cl_L$. Since G is cyclic, we have $L_c = L_g$ and (1) follows. Statement (2) follows from (1) and (4.4). ∎

Statement (2) of Proposition 4.8 is known as the *Ambiguous Class Number Formula.* Other proofs can be found in Cornell–Rosen [6] and Gras [10].

COROLLARY 4.9. *Suppose that* L *is a cyclic extension of* K. *Then* $|\operatorname{Ker} j| = [M : K]\,|\operatorname{Coker} j|/|\gamma|$ *and* $|\gamma|$ *divides* $|\operatorname{Coker} j|$.

Proof. The first statement follows from Proposition 4.8(1). Since $|\operatorname{Ker} j|$ is divisible by $[M : K]$, we see that $|\gamma|$ divides $|\operatorname{Coker} j|$. ∎

Corollary 4.10. *Suppose that L is a cyclic extension of K and $e_0 = e$. Then* $|\mathrm{Ker}\, j| = [L : K]\, |\mathrm{Coker}\, j|/e$ *and* $H^1(P_L) \cong H^0(E_L)$. *Moreover, if* $\mathrm{Im}\, j = Cl_L^G$, *then every unit of K is the norm of a unit of L and this will happen whenever $J = (1)$.*

Proof. Since L/K is cyclic, by Theorem 4.6 we have $\gamma = (1)$, $E_K \subset N_{L/K}L^*$, and if $J = (1)$, then $\mathrm{Im}\, j = Cl_L^G$. By Corollary 4.9 and Remark 0.8(2), $|\mathrm{Ker}\, j| = (n/e)|\mathrm{Coker}\, j|$. (Note that n/e is an integer.) By Lemma 1.8 and the fact that $E_K \subset N_{L/K}L^*$, we see that $H^1(P_L) \cong H^0(E_L)$. If $\mathrm{Im}\, j = Cl_L^G$, then $H^1(P_L) = (1)$, so $E_K = N_{L/K}E_L$. ∎

The following corollary is a slight generalization of [6, Theorem 7].

Corollary 4.11. *Suppose that L is a cyclic extension of K. Suppose that there is exactly one prime P_0, a finite prime of K, that is totally ramified in L. Suppose also that the ideal class of $\wp_0$, the prime of L lying above P_0, is in the image of j. Then $M = K$,* $|\mathrm{Ker}\, j| = [E_K : N_{L/K}E_L]$, *and: j is injective if and only if* $\mathrm{Im}\, j = Cl_L^G$. *In particular, if K is an imaginary quadratic field and $[L : K]$ is odd, then* $\mathrm{Ker}\, j = (1)$ *and* $\mathrm{Im}\, j = Cl_L^G$.

Proof. We have $[L : M] = e_0 = e = [L : K]$. Therefore, $M = K$. By Corollary 4.10 and Theorem 1.7, $|\mathrm{Ker}\, j| = |\mathrm{Coker}\, j| = |H^1(P_L)| = |H^0(E_L)|$. The corollary now follows. ∎

Example 4.12. The following specialization of Corollary 4.11 is well known. Let p be a prime that is congruent to 3 mod 4. Let $K = \mathbb{Q}(\sqrt{-p})$ and $L = \mathbb{Q}(\zeta_p)$ where ζ_p is a primitive pth root of unity. Then $pO_L = (1-\zeta_p)^{p-1}$, so the prime P_K of K lying above p is totally ramified in L and the ideal class of $P_L = (1-\zeta_p)$, being trivial, is in the image of j. Thus $j : Cl_{\mathbb{Q}(\sqrt{-p})} \to Cl_{\mathbb{Q}(\zeta_p)}$ is injective and $\mathrm{Im}\, j = Cl_{\mathbb{Q}(\zeta_p)}^G$, $G = G(L/K)$. (Note that by Theorem 4.6, $\gamma = (1)$ even though L is not unramified over K.)

We now look at some of the ideas of this section when L is a *cyclic p-extension* of K for a prime p. We define $L_g^{(p)}$ to be the maximal unramified abelian p-extension of L of the form LF where F is an abelian p-extension of K. Clearly $L_g^{(p)} \subset H_L^{(p)}$, where $H_L^{(p)}$ is the p-Hilbert class field of L. Then $G(H_L^{(p)}/L) \cong Cl_L(p)$. As in the case of the genus field L_g, when G is cyclic, we have $G(H_L^{(p)}/L_g^{(p)}) \cong I_G(Cl_L(p))$ and $G(L_g^{(p)}/L) \cong Cl_L(p)/I_G(Cl_L(p))$ (see [23, Proposition 1.2]).

Let $U^{(p)} = G(H_L^{(p)}/K)$, $(U^{(p)})'$ its commutator subgroup, and $A^{(p)} = G(H_L^{(p)}/L)$. The action of G on $A^{(p)}$ is by conjugation, so $(U^{(p)})' = I_G(Cl(p))$.

Let $\gamma^{(p)} = G(L_g^{(p)}/LH_K^{(p)})$. As in Section 0, we let $N_{L/K}^{(p)} : Cl_L(p) \to Cl_K(p)$ be the restriction of the norm map to $Cl_L(p)$. Recall from Proposition 0.5(2) that $Cl_{L/K}(p) = \mathrm{Ker}\, N_{L/K}^{(p)} = G(H_L^{(p)}/LH_K^{(p)})$. Hence we have an exact sequence

$$(4.5)\qquad (1) \to I_G(Cl_L(p)) \to \operatorname{Ker} N_{L/K}^{(p)} \to \gamma^{(p)} \to (1).$$

PROPOSITION 4.13. *Let L be a cyclic p-extension of K. Then $\gamma^{(p)} \cong \gamma$. Consequently,*

$$(4.6)\qquad |\gamma^{(p)}| = [L_g^{(p)} : LH_K^{(p)}] = \frac{e}{[L:M][E_K : E_K \cap N_{L/K}L^*]}.$$

Proof. We first prove that $(I_G Cl_L)(p) = I_G(Cl_L(p))$. Applying Lemma 0.4(3) to the exact sequence $(1) \to Cl_L^G \to Cl_L \xrightarrow{\varphi} I_G Cl_L \to (1)$ from Proposition 4.8(1), we have an exact sequence $(1) \to Cl_L^G(p) \to Cl_L(p) \xrightarrow{\varphi^{(p)}} (I_G Cl_L)(p) \to (1)$. Now $\varphi^{(p)}$ is the restriction to $Cl_L(p)$ of the map φ in the proof of Proposition 4.8(1), so $\operatorname{Im}\varphi^{(p)} = I_G(Cl_L(p))$. We conclude that $(I_G Cl_L)(p) = I_G(Cl_L(p))$. By Corollary 4.3(1) and Lemma 0.4(3), the sequence $(1) \to (I_G Cl_L)(p) \to Cl_{L/K}(p) \to \gamma(p) \to (1)$ is exact. As noted above, $Cl_{L/K}(p) = \operatorname{Ker} N_{L/K}^{(p)}$. Also $\gamma(p) = \gamma$ since γ is a p-group by Remark 4.1(2). Therefore, $(1) \to I_G(Cl_L(p)) \to \operatorname{Ker} N_{L/K}^{(p)} \to \gamma \to (1)$ is exact, and combining this sequence with (4.5), we have $\gamma^{(p)} \cong \gamma$. ∎

REMARK 4.14. (1) The proof of Proposition 4.13 shows that when L is a cyclic p-extension of K, we have $|(Cl_L(p))^G| = [L_g^{(p)} : L] = |\gamma^{(p)}|\,|B(p)|$.

(2) The map $\lambda : H^{-1}(Cl_L) \to J$ defined earlier has an analog for p-extensions L/K: $\lambda^{(p)} : H^{-1}(Cl_L(p)) \to J$ defined by $\lambda^{(p)}(xI_G(Cl_L(p))) = N_{L/K}^{(p)}(x)$. Now $J = \operatorname{Ker} j \cap B = \operatorname{Ker} j \cap B(p)$, so $\lambda^{(p)}$ is surjective with $\operatorname{Ker}\lambda^{(p)} = \operatorname{Ker} N_{L/K}^{(p)}/I_G(Cl_L(p))$. Note also that $J \cong \operatorname{Ker} N_G^{(p)}/\operatorname{Ker} N_{L/K}^{(p)}$ where $N_G^{(p)}$ is the restriction of N_G to $Cl_L(p)$.

5. Quadratic extensions. For the remainder of this paper, for a field k, $Cl_k^{(2)}$ will denote the 2-primary component of Cl_k, and $H_k^{(2)}$ the 2-*Hilbert class field* of k, that is, the maximal unramified abelian 2-extension of k. We let $h_k^{(2)} = |Cl_k^{(2)}|$.

In this section, we consider the case where L is a *quadratic* extension of K and $Cl_L^{(2)}$ is either cyclic or elementary. Note that $M = H_K \cap L = H_K^{(2)} \cap L$.

We let $B^{(2)} = N_{L/K}^{(2)} Cl_L^{(2)} = B(2)$ and $Cl_{L/K}^{(2)} = \operatorname{Ker} N_{L/K}^{(2)}$ where $N_{L/K}^{(2)}$ is the restriction of $N_{L/K}$ to $Cl_L^{(2)}$. Adjusting previous notation, we also let $2^m = |B^{(2)}| = [LH_K^{(2)} : L] = [H_K^{(2)} : M]$, $2^s = |I_G Cl_L^{(2)}| = [H_L^{(2)} : L_g^{(2)}]$, and $2^t = |\gamma^{(2)}| = [L_g^{(2)} : LH_K^{(2)}]$ where $L_g^{(2)}$ is the 2-genus field of L over K as defined in Section 4. Thus $2^{s+t} = |Cl_{L/K}^{(2)}| = [H_L^{(2)} : LH_K^{(2)}]$, and 2^{m+s+t}

$= |Cl_L^{(2)}|$. Recall that by Remark 4.14(1), $|(Cl_L^{(2)})^G| = |\gamma^{(2)}|\,|B^{(2)}| = 2^{m+t}$. Note that $m \geq 1$ since we are assuming that $H_K^{(2)} \neq M$. Since $[L:K] = 2$, either L is unramified over K or $M = L \cap H_K^{(2)} = K$. Note also that by Remark 4.1(2), every element of $\gamma^{(2)}$ is killed by 2. By (4.6), when $M = K$ we have

$$|\gamma^{(2)}| = \frac{e}{2[E_K : E_K \cap N_{L/K}L^*]} \tag{5.1}$$

where e is the ramification product.

THEOREM 5.1. *Let L be a quadratic extension of K. Suppose that $Cl_L^{(2)}$ is cyclic and that $m + t \geq 2$. Then $0 \leq s + t \leq 1$. If $s = t = 0$, then* $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$ *when $M = K$, and* $\operatorname{Ker} j = (Cl_K)_2 \cong \mathbb{Z}/2\mathbb{Z}$ *or* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *when $M = L$. If $s = 0$, $t = 1$, then* $\operatorname{Ker} j = (1)$. *If $s = 1$, $t = 0$, then* $\operatorname{Ker} j = J = (1)$ *when $M = K$, and* $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$ *when $M = L$.*

Proof. First note that if $M = K$, then $Cl_K^{(2)}$ is cyclic of order 2^m, and if $M = L$, then $Cl_K^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^m\mathbb{Z}$ or $\mathbb{Z}/2^{m+1}\mathbb{Z}$. Note also that $|\gamma^{(2)}| \leq 2$ since $Cl_L^{(2)}$ is cyclic and $\gamma^{(2)}$ is 2-elementary. If $s = t = 0$, then $[L:K]$ is relatively prime to the relative class group $Cl_{L/K}$, so $\operatorname{Ker} j = (Cl_K)_2$ by Corollary 3.10. Therefore if $M = K$, then $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$, and if $M = L$, then $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ depending on whether $Cl_K^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^m\mathbb{Z}$ or $\mathbb{Z}/2^{m+1}\mathbb{Z}$. Suppose that $s = 0$ and $t = 1$. Then $|\operatorname{Ker} N_{L/K}^{(2)}| = |\gamma^{(2)}| = 2$, so L/K is not unramified by Corollary 4.3(2), and thus $M = K$. Since $s = 0$, G acts trivially on $Cl_L^{(2)}$, giving $\operatorname{Ker} N_G^{(2)} = (Cl_L^{(2)})_2 \cong \mathbb{Z}/2\mathbb{Z}$. Recall that $J \cong \operatorname{Ker} N_G^{(2)} / \operatorname{Ker} N_{L/K}^{(2)}$. (See Remark 4.14(2).) We conclude that $\operatorname{Ker} j = J = (1)$.

Now assume $s \geq 1$ and $m + t \geq 2$. Let $h = m + s + t$. Note that in the unramified case $Cl_K^{(2)}$ cannot be cyclic because the 2-class tower would have length one in that case. So when $M = L$, we have $Cl_K^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^m\mathbb{Z}$. Let x be a generator of $Cl_L^{(2)}$ and σ a generator of G. Then $(Cl_L^{(2)})^G = \langle x^{2^s} \rangle$ and $I_G Cl_L^{(2)} = \langle x^{2^{m+t}} \rangle$. Therefore $\sigma x = x^{2^{m+t}i+1}$ for some positive integer i and $N_G^{(2)}(x) = x\sigma x = x^{2^{m+t}i+2} = x^{2(2^{m+t-1}i+1)}$. Since $N_G^{(2)}(x) \in (Cl_L^{(2)})^G$, it follows that $N_G^{(2)}(x) = x^{2^s k}$ for some positive integer k. Hence $2^{m+t-1}i + 1 \equiv 2^{s-1}k \pmod{2^{h-1}}$. Now $h - 1 \geq 2$ and $m + t - 1 \geq 1$, giving a contradiction if $s \geq 2$. Therefore $s = 1$ and k is odd. Thus $N_G^{(2)}(x) = x^{2k}$, a generator of $(Cl_L^{(2)})^G$. It follows that $|\operatorname{Ker} N_G^{(2)}| = 2 = |I_G Cl_L^{(2)}|$, giving $|\operatorname{Ker} N_{L/K}^{(2)}| = 2$ and $t = 0$. Hence $J = (1)$ and the last statement of the theorem follows from Theorem 2.3(3). ∎

Corollary 5.2. *Suppose that* $Cl_K^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^m\mathbb{Z}$ *where* $m \geq 2$. *Let* L *be an unramified quadratic extension of* K *such that* $Cl_L^{(2)}$ *is cyclic. Then* $s = 0$ *or* 1. *If* $s = 0$, *then* $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. *If* $s = 1$, *then* $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$.

Corollary 5.3. *Suppose that* $Cl_K^{(2)} \cong \mathbb{Z}/2^m\mathbb{Z}$. *Let* L *be a quadratic extension of* K *such that* $L \cap H_K^{(2)} = K$, $Cl_L^{(2)}$ *is cyclic, and* $m+t \geq 2$. *Then* $s+t = 0$ *or* 1. *If* $s+t = 0$, *then* $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$. *If* $s+t = 1$, *then* $\operatorname{Ker} j = (1)$.

The only case for cyclic $Cl_L^{(2)}$ not covered above is $m = 1$, $t = 0$, and $s \geq 1$. The unramified case, that is, when $Cl_K^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, has been done by Kisilevsky [15]. We discuss his results below and then use his methods to do the ramified case. The following corollary shows that the $m = 1$, $t = 0$, $s \geq 1$ case does not always occur.

Corollary 5.4. *Let* L *be a cyclic quartic extension of a field* F *and suppose* K *is the unique intermediate field. Suppose that* $Cl_L^{(2)}$ *is cyclic and* $s + t \neq 0$. *Then* $s + t = 1$ *and if* $s = 1$, *then* $m \geq 2$. *Therefore, if* $Cl_K^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^m\mathbb{Z}$, *then* $J = (1)$ *and* $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$. *If* $Cl_K^{(2)} \cong \mathbb{Z}/2^m\mathbb{Z}$, *then* $\operatorname{Ker} j = (1)$.

Proof. It follows from [18, Proposition 16] that $|Cl_{L/K}^{(2)}| = 1$ or 2. Hence $s + t = 1$. It suffices to show that if $m = 1$, then $s = 0$. Let τ generate $G(L/F)$ and σ generate G. We have $Cl_L = \langle x \rangle$ where x has order 4. Then $\tau(x) = x$ or x^3, giving $\sigma(x) = \tau^2(x) = x$. The rest of the corollary follows from Corollaries 5.2 and 5.3. ∎

Before discussing the $m = 1$, $t = 0$, $s \geq 1$ case in general, we obtain some results when $Cl_L^{(2)}$ is an *elementary* 2-group.

Theorem 5.5. *Suppose that* L *is a quadratic extension of* K *such that* $Cl_L^{(2)}$ *is an elementary* 2*-group. Then* $J \cong (\mathbb{Z}/2\mathbb{Z})^{m-s}$, *so* $s \leq m$. *Thus:*

(1) *If* $L \cap H_K^{(2)} = K$, *then* $Cl_K^{(2)} \cong (\mathbb{Z}/2\mathbb{Z})^m$ *and* $\operatorname{Ker} j = J \cong (\mathbb{Z}/2\mathbb{Z})^{m-s}$.
(2) *If* L/K *is unramified and* $s = m$, *then* $Cl_K^{(2)} \cong (\mathbb{Z}/2\mathbb{Z})^{m+1}$ *and* $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$.
(3) *If* L/K *is unramified,* $s < m$, *and* $Cl_K^{(2)} \cong \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{m-1}$, *then* $\operatorname{Ker} j = J \cong (\mathbb{Z}/2\mathbb{Z})^{m-s}$.
(4) *If* L/K *is unramified and* $s = 0$, *then* $\operatorname{Ker} j = (Cl_K)_2$.

Proof. First note that if $M = K$, then $Cl_K^{(2)} \cong (\mathbb{Z}/2\mathbb{Z})^m$, and if $M = L$, then $Cl_K^{(2)} \cong (\mathbb{Z}/2\mathbb{Z})^{m+1}$ or $\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{m-1}$. Let σ generate G. If $y \in \operatorname{Ker} N_G^{(2)}$, then $y^2 = 1 = N_G^{(2)}(y) = y\sigma y$, giving $\sigma y = y$ and $y \in (Cl_L^{(2)})^G$. On the other hand, because $Cl_L^{(2)}$ is 2-elementary, we have $(Cl_L^{(2)})^G \subset \operatorname{Ker} N_G^{(2)}$. Hence $(Cl_L^{(2)})^G = \operatorname{Ker} N_G^{(2)}$ and by Remark 4.14(2), $|J| = 2^{m-s}$. Statement (1) now follows. If L/K is unramified and $s = m$, then $\operatorname{Ker} j \cong G(L/K) \cong \mathbb{Z}/2\mathbb{Z}$ by Theorem 2.3(3). By Corollary 2.11(1), $Cl_K^{(2)} \cong \operatorname{Ker} j \times B(2) \cong (\mathbb{Z}/2\mathbb{Z})^{m+1}$. Statement (3) follows from Proposition 3.12, and (4) by Corollary 3.10. ∎

Example 5.6. Let $K = \mathbb{Q}(\sqrt{-5})$ and $L = K(\sqrt{41})$. Then K has class number 2, and as 41 splits completely in K, there are two primes of K that ramify in L. Thus the ramification product e is 4. Since $N_{L/K}(32 + 5\sqrt{41}) = -1$, by (5.1), $|\gamma^{(2)}| = 2$. It will follow from Theorem 6.8 below that $Cl_L^{(2)} \cong (\mathbb{Z}/2\mathbb{Z})^3$. Thus $m = t = s = 1$, and $\operatorname{Ker} j = (1)$ by Theorem 5.5(1).

We now consider the case where $m = 1$, $t = 0$, $s \geq 1$ and no prior constraints on the structure of $Cl_L^{(2)}$ are given. Thus $Cl_K^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ when L/K is unramified and $Cl_K^{(2)} \cong \mathbb{Z}/2\mathbb{Z}$ when $L \cap H_K^{(2)} = K$. Kisilevsky [15] has established that in the unramified case $Cl_L^{(2)}$ is either cyclic or the Klein 4-group. We prove that this is true in the ramified case as well.

We define the quaternion, dihedral, and semidihedral groups of order 2^k:

$$Q_k = \langle x, y \mid x^{2^{k-2}} = y^2 = a \neq 1,\ a^2 = 1,\ y^{-1}xy = x^{-1} \rangle,$$
$$D_k = \langle x, y \mid x^{2^{k-1}} = y^2 = 1,\ y^{-1}xy = x^{-1} \rangle,$$
$$S_k = \langle x, y \mid x^{2^{k-1}} = y^2 = 1,\ y^{-1}xy = x^{2^{k-2}-1} \rangle.$$

We let $Q = Q_3$ and $D = D_3$.

Kisilevsky uses the following result of Gorenstein [9]: *Let U be a group of order 2^k, $k \geq 3$, U' its commutator subgroup. Suppose that $U/U' \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then $U \cong Q_k$, D_k, or S_k.*

Let L be a quadratic extension of K for which $Cl_K^{(2)} \cong \mathbb{Z}/2\mathbb{Z}$, $L \cap H_K^{(2)} = K$, and $L_g^{(2)} = LH_K^{(2)}$. Using the notation and remarks following Example 4.12, we let $U^{(2)} = G(H_L^{(2)}/K)$ and $A^{(2)} = G(H_L^{(2)}/L) \cong Cl_L^{(2)}$. Then $(U^{(2)})' \cong I_G Cl_L^{(2)}$. Recall that the action of G on $A^{(2)}$ is by conjugation. (See the remarks preceding Proposition 4.13.) Since

$$U^{(2)}/(U^{(2)})' \cong G(LH_K^{(2)}/K) \cong G(L/K) \times G(H_K^{(2)}/K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

by Gorenstein's Theorem, $U^{(2)}$ is isomorphic to Q_k, D_k, or S_k where $k = s + 2 \geq 3$.

$U^{(2)}$ has three subgroups of index 2: $A_1 = \langle x \rangle$, $A_2 = \langle x^2, y \rangle$, and $A_3 = \langle x^2, xy \rangle$ where x and y are the generators of $U^{(2)}$ as defined above. Let L_1, L_2, and L_3 be the respective fixed fields. So $A^{(2)} = A_i$ and $L = L_i$ for some $i = 1, 2, 3$. In addition, $(U^{(2)})' = \langle x^2 \rangle$ and $A_0 = \langle x^4 \rangle$ is the unique subgroup of index 2 in $(U^{(2)})'$. Let L_0 be the fixed field of A_0. Then it is easy to see that $G(L_0/K)$ is a nonabelian group of order 8. In fact, $G(L_0/K) \cong D = D_3$ except when $U^{(2)} = Q = Q_3$, in which case $x^4 = 1$ and $G(L_0/K) \cong Q$. Note that when $k = 3$, S_k is abelian, so we must have $U^{(2)} = Q_3 = Q$ or $U^{(2)} = D_3 = D$ in that case.

THEOREM 5.7. *Let L be a quadratic extension of K. Suppose that $Cl_K^{(2)} \cong \mathbb{Z}/2\mathbb{Z}$ and $L \cap H_K^{(2)} = K$. Suppose also that $L_g^{(2)} = LH_K^{(2)}$; that is, $m = 1$, $t = 0$, and $s \geq 1$. Then $Cl_L^{(2)}$ is either cyclic or the Klein 4-group. Moreover:*

(1) *Suppose that $s \geq 2$. Then $G(L_0/K) \cong D$, $Cl_L^{(2)} \cong A_1 = \langle x \rangle$ and* $\operatorname{Ker} j = J \cong \mathbb{Z}/2\mathbb{Z}$ *when $U^{(2)} = Q_k$ or D_k and* $\operatorname{Ker} j = J = (1)$ *when $U^{(2)} = S_k$.*
(2) *Suppose that $s = 1$. Then $U^{(2)} = Q$ or D. If $U^{(2)} = Q$, then $Cl_L^{(2)} \cong \mathbb{Z}/4\mathbb{Z}$ and* $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$. *If $U^{(2)} = D$ and $A^{(2)} = A_1$, then* $\operatorname{Ker} j = J \cong \mathbb{Z}/2\mathbb{Z}$, *and if $U^{(2)} = D$ and $A^{(2)} = A_2$ or A_3, then $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and* $\operatorname{Ker} j = J = (1)$.

Proof. First recall that $\operatorname{Ker} j = J$ since $M = K$ and $J \cong H^{-1}(Cl_L)$ by Proposition 4.5. Since G is cyclic and Cl_L is finite, we have $|J| = |H^{-1}(Cl_L)| = |H^0(Cl_L)|$. Also $|(Cl_L^{(2)})^G| = 2$ by Remark 4.14(1). Let σ generate G. As above, we let $A^{(2)} = G(H_L^{(2)}/L)$.

(1) Suppose that $s \geq 2$. That $G(L_0/K) \cong D$ follows from the remarks preceding Theorem 5.7. Suppose that $A^{(2)} = A_2 = \langle x^2, y \rangle$ or $A^{(2)} = A_3 = \langle x^2, xy \rangle$. If $U^{(2)} = Q_k$ or D_k, then $y^{-1}xy = x^{-1}$, so $y^{-1}x^2y = x^{-2}$. If $U^{(2)} = S_k$, then $y^{-1}xy = x^{2^{k-2}-1}$, so $y^{-1}x^2y = x^{2^{k-1}-2} = x^{-2}$ in this case as well. But $A^{(2)}$ is abelian, so $x^2y = yx^2$, giving $x^4 = 1$ and $k = 3$. This is a contradiction, so $Cl_L^{(2)} \cong A_1 = \langle x \rangle$. For $a \in A^{(2)}$, we have $\sigma a = y^{-1}ay$. If $U^{(2)} = Q_k$ or D_k, then $\sigma x = y^{-1}xy = x^{-1}$. Hence $N_G Cl_L^{(2)} = (1)$ and $|J| = |H^0(Cl_L^{(2)})| = |(Cl_L^{(2)})^G| = 2$. We thus have $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$. If $U^{(2)} = S_k$, then $\sigma x = y^{-1}xy = x^{2^{k-2}-1}$, so $N_G(x) = x^{2^{k-2}}$. Thus $|N_G Cl_L^{(2)}| = 2$ and $\operatorname{Ker} j = (1)$.

(2) Suppose that $s = 1$. As noted above, $U^{(2)} = Q$ or D.

(a) Suppose that $U^{(2)} = Q$. If $A^{(2)} = A_1 = \langle x \rangle$, then as in the proof of (1), $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$. It is easy to see that $A_2 = \langle x^2, y \rangle$ is cyclic generated by y and $A_3 = \langle x^2, xy \rangle$ is cyclic generated by xy. If $A^{(2)} = A_2$, then $\sigma y = x^{-1}yx = y^3$ and $N_G Cl_L^{(2)} = (1)$. If $A^{(2)} = A_3$, $\sigma(xy) = y^{-1}xyy = y^{-1}xy^2$ and again $N_G Cl_L^{(2)} = (1)$. So in all cases we can conclude that $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$.

(b) Suppose that $U^{(2)} = D$. If $A^{(2)} = A_1 = \langle x \rangle$, then again $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$. If $A^{(2)} = A_2$ or A_3, then $A^{(2)} = \{e, x^2, y, x^2y\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and by Theorem 5.5(1), $\operatorname{Ker} j = (1)$. ∎

COROLLARY 5.8. *Under the assumptions of Theorem* 5.7, *suppose that* $|Cl_L^{(2)}| = 4$. *Then* $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$ *when* $Cl_L^{(2)}$ *is cyclic, and* $\operatorname{Ker} j = (1)$ *when* $Cl_L^{(2)}$ *is the Klein* 4*-group.*

REMARK 5.9. Theorem 5.7 is a generalization of [3, Proposition 5], which in turn is based on work of Gras [11].

6. Imaginary quadratic fields. In this section, we look at examples illustrating Theorems 5.1, 5.5, and 5.7 when $K = \mathbb{Q}(\sqrt{-p})$ where p is a prime such that $p \equiv 1 \pmod 4$ and $L = K(\sqrt{d})$ where $d = q$ or $d = -q$, q a prime distinct from p. The group $Cl_K^{(2)}$ is cyclic since only two rational primes ramify in K. When $p \equiv 1 \pmod 8$, $h_K^{(2)}$ is divisible by 4, and when $p \equiv 5 \pmod 8$, we have $h_K^{(2)} = 2$. (When $p \equiv 3 \pmod 4$, $\mathbb{Q}(\sqrt{-p})$ has odd class number.)

A key source for this section is the work of McCall, Parry, and Ranalli [19, 20], who determine all imaginary bicyclic biquadratic fields with 2-class group cyclic or the Klein 4-group.

As in Section 5, the numbers m, s, and t are defined by $2^m = |B^{(2)}| = [H_K^{(2)} : M]$, $2^s = |I_G Cl_L^{(2)}| = [H_L^{(2)} : L_g^{(2)}]$, and $2^t = |\gamma^{(2)}| = [L_g^{(2)} : LH_K^{(2)}]$. Also, $h_k^{(2)} = |Cl_k^{(2)}|$ for any field k. Note that for K and L as defined above, $M = L \cap H_K^{(2)} = K$ so that $|Cl_K^{(2)}| = 2^m$. We also let Q_k, D_k, and S_k be the quaternion, dihedral, and semidihedral groups of order 2^k as defined in Section 5.

The three quadratic subfields of L are $K = \mathbb{Q}(\sqrt{-p})$, $F = \mathbb{Q}(\sqrt{d})$, and $T = \mathbb{Q}(\sqrt{-pd})$. We let $\delta_L = [E_L : E_K E_F E_T]$ be the unit index of the biquadratic field L. The Kuroda class number formula (see [16] for a proof) says that $h_L = \frac{1}{2}\delta_L h_K h_F h_T$. It is known [16] that $\delta_L = 1$ or 2, so

$$h_L^{(2)} = \tfrac{1}{2}\delta_L h_K^{(2)} h_F^{(2)} h_T^{(2)}. \tag{6.1}$$

To apply Theorem 5.7, we need $\gamma^{(2)} = (1)$. By (5.1),

$$|\gamma^{(2)}| = \frac{e}{2[E_K : E_K \cap N_{L/K}L^*]}.$$

So we need to compute the ramification product e as well as determine when $-1 \in N_{L/K}L^*$. The following two lemmas will address these questions.

If ℓ is a rational prime, we let $P(\ell)$ be a prime of K above ℓ, and $\wp(\ell)$ a prime of L above $P(\ell)$. We also let $e_{\wp(\ell)}$ be the ramification index of $\wp(\ell)$ over $P(\ell)$. So e is the product over all primes $P(\ell)$ of $e_{\wp(\ell)}$. (Note that the infinite prime is unramified.) If a is an integer prime to ℓ, $\left(\frac{a}{\ell}\right)$ is the *Legendre symbol.*

Lemma 6.1. *Let* $K = \mathbb{Q}(\sqrt{-p})$ *where* p *is a prime such that* $p \equiv 1 \pmod 4$. *Let* $L = K(\sqrt{d})$ *where* d *is a square-free integer,* $d \neq 0, 1, -1$. *Let* u *be the number of odd prime divisors* ℓ *of* d *such that* $\ell \neq p$ *and* $\left(\frac{-p}{\ell}\right) = -1$, *and let* v *be the number of such primes* ℓ *with* $\left(\frac{-p}{\ell}\right) = +1$. *Then* $e = 2^{u+2v+w}$ *where* $w = 0$ *if* d *is odd, and* $w = 1$ *if* d *is even.*

Proof. Let $F = \mathbb{Q}(\sqrt{d})$ and $T = \mathbb{Q}(\sqrt{-pd})$ be the other two quadratic subfields of L. If ℓ is an odd rational prime that does not divide pd, then ℓ is unramified in each intermediate subfield, and therefore in L. So $e_{\wp(\ell)} = 1$. The prime p ramifies in K and T but not in F. So $P(p)$ does not ramify in L and we have $e_{\wp(p)} = 1$. Now suppose that ℓ is an odd prime divisor of d. Then ℓ ramifies in F and T but not in K. Thus $e(\wp(\ell)/\ell) = 2$ and since $e(P(\ell)/\ell) = 1$, we have $e_{\wp(\ell)} = e(\wp(\ell)/P(\ell)) = 2$. Note: ℓ splits in K if and only if $\left(\frac{-p}{\ell}\right) = +1$. Finally, we consider the prime 2. If d is odd, then d or $-pd$ is congruent to 1 mod 4. So 2 ramifies in K but does not ramify in some intermediate subfield. Thus 2 is not totally ramified in L, so $e(\wp(2)/2) = e(P(2)/2) = 2$ and $e_{\wp(2)} = e(\wp(2)/P(2)) = 1$. If d is even, then 2 ramifies in each intermediate field and so 2 is totally ramified in L. Therefore, in this case $e_{\wp(2)} = 2$. The lemma follows. ∎

If $d = a^2 + b^2$, then $N_{L/K}\left(\frac{a+\sqrt{d}}{b}\right) = N_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}\left(\frac{a+\sqrt{d}}{b}\right) = \frac{a^2-d}{b^2} = -1$. In particular, when $d = 2$ or $d = q \neq p$ where q is a prime and $q \equiv 1 \pmod 4$, then -1 is a global norm from L. When d is not the sum of two squares, then -1 is not a norm from $\mathbb{Q}(\sqrt{d})$ but it may still be a norm from L. We have:

Lemma 6.2. *Let* $K = \mathbb{Q}(\sqrt{-p})$ *and* $L = \mathbb{Q}(\sqrt{-p}, \sqrt{d})$ *where* p *is a prime,* $p \equiv 1 \pmod 4$, *and* d *is a square-free integer. If* d *is positive and contains no prime factor* q *such that* $q \equiv 3 \pmod 4$, *then* -1 *is a global norm from* L. *If* $d = \pm q$, q *a prime such that* $q \equiv 3 \pmod 4$, *then* $-1 \in N_{L/K}L^*$ *if and only if* $\left(\frac{p}{q}\right) = +1$.

Proof. The first statement is clear from the preceding remarks. Suppose $d = q$ or $d = -q$ where q is a prime such that $q \equiv 3 \pmod 4$. The only primes of K that ramify in L are the ones above q. Let Q be one of those primes. The local norm residue symbol $\left(\frac{(-1,d)}{Q}\right)$ equals $(-1)^{(NQ-1)/2}$, N the ideal norm. But $NQ = q^2$ if q is inert in K, and $NQ = q$ if q splits in K. So $\left(\frac{(-1,q)}{Q}\right) = 1$ if and only if $\left(\frac{-p}{q}\right) = -1$. ∎

The following lemma gives two criteria for the unit index to be 1. The first is well known. For a proof of the second, see Ranalli [22, Chapter 6, p. 33].

Lemma 6.3. *Let* L *be an imaginary biquadratic field and* F *its real quadratic subfield. If the norm of the fundamental unit of* F *is* -1, *then*

$\delta_L = 1$. *If there is an odd prime* ℓ *that divides the discriminant of* L *but does not divide the discriminant of* F, *then* $\delta_L = 1$.

PROPOSITION 6.4. *Let* $K = \mathbb{Q}(\sqrt{-p})$ *and let* $L = \mathbb{Q}(\sqrt{-p}, \sqrt{q})$ *or* $L = \mathbb{Q}(\sqrt{-p}, \sqrt{-q})$ *where* p *and* q *are primes,* $q \neq p$, *and* $p \equiv 1 \pmod 4$. *If* $q \equiv 1 \pmod 4$ *and* $\left(\frac{p}{q}\right) = 1$, *then* $|\gamma^{(2)}| = 2$. *Otherwise,* $\gamma^{(2)} = (1)$.

Proof. We apply formula (5.1) and Lemmas 6.1 and 6.2. If $q = 2$, or if $q \equiv 1 \pmod 4$ and $\left(\frac{p}{q}\right) = -1$, or if $q \equiv 3 \pmod 4$ and $\left(\frac{p}{q}\right) = 1$, then $e = 2$ and -1 is a global norm from L. Thus $\gamma^{(2)} = (1)$. If $q \equiv 3 \pmod 4$ and $\left(\frac{p}{q}\right) = -1$, then $e = 4$ and -1 is not a global norm from L, so $\gamma^{(2)} = (1)$ in this case as well. If $q \equiv 1 \pmod 4$ and $\left(\frac{p}{q}\right) = 1$, we have $e = 4$ and -1 is a global norm from L, giving $|\gamma^{(2)}| = 2$. ∎

Statement (1) of the following proposition is a direct consequence of [19, Theorem 5 and Tables I–III]. Statement (2) follows from [20, Theorem 15(r)].

PROPOSITION 6.5. *Let* p *and* q *be distinct primes where* $p \equiv 1 \pmod 4$. *Let* $L = \mathbb{Q}(\sqrt{-p}, \sqrt{q})$.

(1) $Cl_L^{(2)}$ *is cyclic if and only if one of the following cases holds:*
   (a) $q \equiv 3, 5,$ *or* $7 \pmod 8$ *and* $\left(\frac{p}{q}\right) = -1$;
   (b) $p \equiv 5 \pmod 8$ *and either* $q = 2$, *or* $q \equiv 3 \pmod 4$ *and* $\left(\frac{p}{q}\right) = 1$.

(2) $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *if and only if* $p \equiv 5 \pmod 8$, $q \equiv 1 \pmod 8$, *and* $\left(\frac{p}{q}\right) = -1$.

The following lemma is helpful for computing the Kuroda class number formula (6.1) in the following two theorems. Proofs of statements (1) and (2) can be found in Conner and Hurrelbrink [4, Corollary 19.6]. Statement (3) is proved in Kisilevsky [15].

LEMMA 6.6. *Let* p, q *be primes where* $p \equiv 1 \pmod 4$. *Let* $T = \mathbb{Q}(\sqrt{-pq})$.

(1) *Suppose that* $q \equiv 3 \pmod 4$. *If* $\left(\frac{p}{q}\right) = -1$, *then* $|Cl_T^{(2)}| = 2$. *If* $\left(\frac{p}{q}\right) = 1$, *then* $Cl_T^{(2)}$ *is cyclic of order divisible by* 4.
(2) *Suppose that* $p \equiv 5 \pmod 8$ *and* $q = 2$. *Then* $|Cl_T^{(2)}| = 2$.
(3) *Suppose that* $q \equiv 1 \pmod 4$ *and* $\left(\frac{p}{q}\right) = -1$. *If* $pq \equiv 1 \pmod 8$, *then* $Cl_T^{(2)}$ *is noncyclic of order* 2^r, $r \geq 3$. *If* $pq \equiv 5 \pmod 8$, *then* $Cl_T^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

THEOREM 6.7. *Suppose that* $K = \mathbb{Q}(\sqrt{-p})$ *where* p *is a prime such that* $p \equiv 1 \pmod 8$. *Let* $L = K(\sqrt{q})$ *where* q *is an odd prime distinct from* p, $q \not\equiv 1 \pmod 8$, *and* $\left(\frac{p}{q}\right) = -1$. *If* $q \equiv 3 \pmod 4$, *then* $\operatorname{Ker} j = (Cl_K^{(2)})_2 \cong \mathbb{Z}/2\mathbb{Z}$. *If* $q \equiv 5 \pmod 8$, *then* $\operatorname{Ker} j = (1)$.

Proof. Let $F = \mathbb{Q}(\sqrt{q})$ and $T = \mathbb{Q}(\sqrt{-pq})$. By Proposition 6.4, $\gamma^{(2)} = (1)$, and by Proposition 6.5(1)(a), $Cl_L^{(2)}$ is cyclic. The unit index δ_L of L is 1 by Lemma 6.3 because the prime p divides the discriminant of L but not that of the real quadratic subfield $\mathbb{Q}(\sqrt{q})$. Now F has odd class number and $h_K^{(2)} = 2^m$, $m \geq 2$. By Lemma 6.6 and the Kuroda class number formula, $h_L^{(2)} = 2^m$ when $q \equiv 3 \pmod 4$, and $h_L^{(2)} = 2^{m+1}$ when $q \equiv 5 \pmod 8$. In the former case, $s = t = 0$, and $\operatorname{Ker} j = (Cl_K^{(2)})_2 \cong \mathbb{Z}/2\mathbb{Z}$ by Corollary 3.10. In the latter case, $s = 1$, and $\operatorname{Ker} j = (1)$ by Corollary 5.3. ∎

THEOREM 6.8. *Suppose that $K = \mathbb{Q}(\sqrt{-p})$ where p is a prime such that $p \equiv 5 \pmod 8$. Let $F = \mathbb{Q}(\sqrt{q})$, $T = \mathbb{Q}(\sqrt{-pq})$, and $L = K(\sqrt{q})$ where q is a prime distinct from p. Let $U^{(2)} = G(H_L^{(2)}/K)$.*

(1) *Suppose that $q \equiv 1 \pmod 8$ and $\left(\frac{p}{q}\right) = -1$. Then $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\operatorname{Ker} j = (1)$.*

(2) *Suppose that $q \equiv 1 \pmod 8$ and $\left(\frac{p}{q}\right) = 1$. Then $Cl_L^{(2)}$ has 2-rank 3. Thus $Cl_L^{(2)}$ is 2-elementary if and only if $h_T^{(2)} = 8$ and in that case $\operatorname{Ker} j = (1)$.*

(3) *Suppose that $q \equiv 5 \pmod 8$ and $\left(\frac{p}{q}\right) = -1$. Then $Cl_L^{(2)} \cong \mathbb{Z}/2^{k-1}\mathbb{Z}$ for some $k \geq 4$, and $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $U^{(2)} \cong D_k$ or Q_k, and $\operatorname{Ker} j = (1)$ if and only if $U^{(2)} \cong S_k$.*

(4) *Suppose that $q = 2$, or $q \equiv 3 \pmod 4$ and $\left(\frac{p}{q}\right) = -1$. Then $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z}$ and $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$.*

(5) *Suppose that $q \equiv 3 \pmod 4$ and $\left(\frac{p}{q}\right) = 1$. Then $Cl_L^{(2)} \cong \mathbb{Z}/2^{k-1}\mathbb{Z}$ for some $k \geq 3$, and $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $U^{(2)} \cong D_k$ or Q_k, and $\operatorname{Ker} j = (1)$ if and only if $U^{(2)} \cong S_k$. If $k = 3$, then $U^{(2)} \cong D_3$ or Q_3 and $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$. In particular, $k = 3$ when $h_T^{(2)} = 4$.*

Proof. $h_F^{(2)} = 1$ and since $p \equiv 5 \pmod 8$, we have $h_K^{(2)} = 2$. By Lemma 6.3 and the Kuroda class number formula, $h_L^{(2)} = h_T^{(2)}$.

(1) By Proposition 6.5(2), $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. By Proposition 6.4, $\gamma^{(2)} = (1)$. Hence $m = s = 1$. By Theorem 5.5, $\operatorname{Ker} j = (1)$.

(2) The fact that the 2-rank of $Cl_L^{(2)}$ is 3 follows from [19, Proposition 4]. By the Kuroda class number formula, $h_L^{(2)} = h_T^{(2)} = 2^r$ for some $r \geq 3$. Suppose $r = 3$. By Proposition 6.4, $|\gamma^{(2)}| = 2$. Thus $m = t = s = 1$ and by Theorem 5.5(1), $\operatorname{Ker} j = (1)$.

(3) By Lemma 6.6(3) and Proposition 6.5(1)(a), $Cl_L^{(2)} \cong \mathbb{Z}/2^{k-1}\mathbb{Z}$ for some $k \geq 4$. By Proposition 6.4, $\gamma^{(2)} = (1)$. So we have $m = 1$, $t = 0$ and $s \geq 1$. By Theorem 5.7, $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$ if $U^{(2)} \cong D_k$ or Q_k, and $\operatorname{Ker} j = (1)$ if $U^{(2)} \cong S_k$.

(4) By Lemma 6.6, $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z}$. Hence $s = t = 0$, so $\operatorname{Ker} j = (Cl_L^{(2)})_2 \cong \mathbb{Z}/2\mathbb{Z}$ by Corollary 3.10.

(5) By Proposition 6.5(1)(b) and Lemma 6.6(1), $Cl_L^{(2)} \cong \mathbb{Z}/2^{k-1}\mathbb{Z}$ for some $k \geq 3$. By Proposition 6.4, $\gamma^{(2)} = (1)$. The result follows from Theorem 5.7. ∎

PROPOSITION 6.9. *Let p and q be primes where $p \equiv 1 \pmod 4$. Let $L = \mathbb{Q}(\sqrt{-p}, \sqrt{-q})$.*

(1) *$Cl_L^{(2)}$ is cyclic if and only if one of the following holds:*
   (a) *$q \equiv 3 \pmod 4$ and $\left(\frac{p}{q}\right) = -1$; if, in addition, $p \equiv 5 \pmod 8$, then $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z}$;*
   (b) *$p \equiv 5 \pmod 8$ and $q = 2$; in this case, $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z}$;*
   (c) *$pq \equiv 5 \pmod 8$ and $\left(\frac{p}{q}\right) = -1$.*

(2) *$Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if and only if either $p \equiv 5 \pmod 8$, $q \equiv 3 \pmod 4$, and $\left(\frac{p}{q}\right) = 1$, or $p \equiv q \equiv 5 \pmod 8$ and $\left(\frac{p}{q}\right) = -1$.*

Proof. Statements (1)(a) and (1)(b) follow from [19, Theorem 5 and Tables I and II]. Statements (1)(c) and (1)(d) follow from [19, Theorem 7]. Statement (2) is a consequence of [20, Theorem 15, parts (w) and (o) respectively]. ∎

The proof of the following lemma can be found in Conner and Hurrelbrink [4, Corollaries 19.7 and 19.8 and Proposition 19.9].

LEMMA 6.10. *Let $F = \mathbb{Q}(\sqrt{pq})$ where p and q are primes and $p \equiv 1 \pmod 4$. Let ε be the fundamental unit of F.*

(1) *If $\left(\frac{p}{q}\right) = -1$, then $h_F^{(2)} = 2$. If, in addition, $q \equiv 1 \pmod 4$, then $N_{F/\mathbb{Q}}\varepsilon = -1$.*
(2) *Suppose that $p \equiv 5 \pmod 8$. If $q \equiv 3 \pmod 4$ or $q = 2$, then $h_F^{(2)} = 2$. If $q = 2$, then $N_{F/\mathbb{Q}}\varepsilon = -1$.*

THEOREM 6.11. *Suppose that $K = \mathbb{Q}(\sqrt{-p})$ where p is a prime such that $p \equiv 5 \pmod 8$. Let $L = K(\sqrt{-q})$ where q is a prime $\neq p$ and $U^{(2)} = G(H_L^{(2)}/K)$. Then:*

(1) *If $q = 2$ or $q \equiv 3 \pmod 4$ and $\left(\frac{p}{q}\right) = -1$, then $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z}$ and $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$.*
(2) *If $q \equiv 3 \pmod 4$ and $\left(\frac{p}{q}\right) = +1$, or if $q \equiv 5 \pmod 8$ and $\left(\frac{p}{q}\right) = -1$, then $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\operatorname{Ker} j = (1)$.*
(3) *If $q \equiv 1 \pmod 8$ and $\left(\frac{p}{q}\right) = -1$, then $Cl_L^{(2)} \cong \mathbb{Z}/2^{k-1}\mathbb{Z}$ for some $k \geq 4$, and $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $U \cong D_k$ or Q_k, and $\operatorname{Ker} j = (1)$ if and only if $U \cong S_k$.*

Proof. Let $F = \mathbb{Q}(\sqrt{pq})$ and $T = \mathbb{Q}(\sqrt{-q})$. Since $p \equiv 5 \pmod 8$, $h_K^{(2)} = 2$. By Lemma 6.10, $h_F^{(2)} = 2$, and by Proposition 6.4, $\gamma^{(2)} = (1)$. Thus $m = 1$ and $t = 0$.

(1) $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z}$ by Proposition 6.9(1)(a)&(b). Thus $s = t = 0$, and $\operatorname{Ker} j = (Cl_K^{(2)})_2 \cong \mathbb{Z}/2\mathbb{Z}$ by Corollary 3.10.

(2) By Proposition 6.9(2), $Cl_L^{(2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence $s = 1$, and $\operatorname{Ker} j = (1)$ by Theorem 5.5(1).

(3) By Lemma 6.10(1) and the Kuroda class number formula, $h_L^{(2)}$ is divisible by 8. We have $m = 1$, $t = 0$, and $s \geq 2$. Thus by Proposition 6.9(1)(c), $Cl_L^{(2)}$ is cyclic of order 2^{k-1} for some $k \geq 4$. Statement (3) now follows from Theorem 5.7(1). ∎

Lemma 6.12. *Suppose that* $L = \mathbb{Q}(\sqrt{-p}, \sqrt{-q})$ *where* $p \equiv 1 \pmod 8$, $q \equiv 3 \pmod 4$, *and* $\left(\frac{p}{q}\right) = -1$. *Then* $\delta_L = 1$.

Proof. Let $\varepsilon = x + y\sqrt{pq}$ be the fundamental unit of $F = \mathbb{Q}(\sqrt{pq})$. Clearly $N_{F/\mathbb{Q}}\varepsilon = 1$ and x and y have opposite parity. Let O_F be the ring of integers of F. In [1, Remarque 15] Azizi shows that 2ε is the square of an element of O_F. It follows easily that x must be even and y odd. If $\delta_L = 2$, then by [13, Proposition 1], the square-free part of $N_{F/\mathbb{Q}}(\varepsilon + 1)$ is p or q, from which it follows that x is odd, a contradiction. Thus $\delta_L = 1$. ∎

Theorem 6.13. *Suppose that* $K = \mathbb{Q}(\sqrt{-p})$ *where* p *is a prime such that* $p \equiv 1 \pmod 8$. *Let* $L = K(\sqrt{-q})$ *where* q *is a prime* $\neq p$ *and* $\left(\frac{p}{q}\right) = -1$. *If* $q \equiv 3 \pmod 4$, *then* $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$. *If* $q \equiv 5 \pmod 8$, *then* $\operatorname{Ker} j = (1)$.

Proof. Let $F = \mathbb{Q}(\sqrt{pq})$ and $T = \mathbb{Q}(\sqrt{-q})$. We have $h_K^{(2)} = 2^m$, $m \geq 2$, since $p \equiv 1 \pmod 8$, and Lemma 6.10 yields $h_F^{(2)} = 2$. By Proposition 6.9, $Cl_L^{(2)}$ is cyclic.

(1) Suppose that $q \equiv 3 \pmod 4$. Then $h_T^{(2)} = 1$. By Lemma 6.12, $\delta_L = 1$ and the Kuroda class number formula gives $h_L^{(2)} = 2^m$, $m \geq 2$. Thus $s = t = 0$, and $\operatorname{Ker} j \cong \mathbb{Z}/2\mathbb{Z}$ by Corollary 3.10.

(2) Suppose that $q \equiv 5 \pmod 8$. In this case, $h_T^{(2)} = 2$. By Lemma 6.10, $\delta_L = 1$. Now by the Kuroda class number formula, $h_L^{(2)} = 2^{m+1}$. Since $\left(\frac{p}{q}\right) = -1$, we have $\gamma^{(2)} = (1)$ by Proposition 6.4. So $t = 0$ and $s = 1$. By Theorem 5.1, $\operatorname{Ker} j = (1)$. ∎

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