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NEW LINEAR AND MULTILINEAR VERSIONS OF PUHL'S QUOTIENT FORMULA

BY

DUMITRU POPA (Constanţa)

Abstract. We prove new linear and multilinear versions of Puhl's quotient formula.

1. Introduction and background. The theory of absolutely summing operators was originated by seminal ideas of Grothendieck, in the 50's. In 1968, Lindenstrauss and Pełczyński [5] published a classical paper in Studia Mathematica, revisiting Grothendieck's results, and since then the absolutely summing operators has been an important field of research in functional analysis. The extension of the linear theory to the multilinear setting is not easy and there are several different "natural" approaches. This line of research was initiated by Pietsch in the 80's and now it is an active research topic.

In a 1977 paper, J. Puhl [9, Satz 8] proved the *quotient formula* $\Pi_1 \circ \Pi_q^{-1} = \Pi_{q^*}$, $1 < q < \infty$, in other words: a bounded linear operator $T : X \to Y$ is q-summing if and only if for all Banach spaces Z and all q^*-summing operators $V : Z \to X$, $T \circ V$ is 1-summing. Moreover,

$$\pi_q(T) = \sup_{\pi_{q^*}(V) \le 1} \pi_1(T \circ V).$$

The main purpose of this paper is to prove new linear and multilinear versions of Puhl's quotient formula (see Theorem 2 and Corollaries 2–4).

Let us fix some notation and terminology. For X a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, X^* is the dual of X, and a typical member of X^* will be denoted by x^*. For $1 \le p < \infty$ and $x_1, \ldots, x_m \in X$, we write $w_p((x_i)_{1 \le i \le m})$ for $\sup_{\|x^*\| \le 1} (\sum_{i=1}^m |x^*(x_i)|^p)^{1/p}$. A bounded linear operator $T : X \to Y$ is *p-summing* if there exists a constant $C \ge 0$ such that for all $x_1, \ldots, x_m \in X$,

$$\Big(\sum_{i=1}^m \|T(x_i)\|^p\Big)^{1/p} \le C w_p((x_i)_{1 \le i \le m});$$

the *p-summing norm* of T is $\pi_p(T) = \inf\{C \mid C \text{ as above}\}$ [2, 3, 7, 10].

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Also for $0 < p < \infty$, X a Banach space and $x_1, \ldots, x_m \in X$, we write $l_p((x_i)_{1 \le i \le m})$ to denote $(\sum_{i=1}^m \|x_i\|^p)^{1/p}$.

Let n be a natural number, $1 \le p_1, \ldots, p_n < \infty$ and $0 < t < \infty$ be such that $1/t \le 1/p_1 + \cdots + 1/p_n$. A bounded n-linear operator $U : X_1 \times \cdots \times X_n \to Y$ is $(t; p_1, \ldots, p_n)$-*summing* if there exists $C \ge 0$ such that for all $(x_i^j)_{1 \le i \le m} \subset X_j$ $(1 \le j \le n)$,

$$\Big(\sum_{i=1}^m \|U(x_i^1, \ldots, x_i^n)\|^t\Big)^{1/t} \le C \prod_{j=1}^n w_{p_j}\big((x_i^j)_{1 \le i \le m}\big).$$

We write $\pi_{t;p_1,\ldots,p_n}(U) = \inf\{C \ge 0 \mid C \text{ as above}\}$. Note that $\pi_{t;p_1,\ldots,p_n}$ is a norm if $t \ge 1$ (and a t-norm if $t < 1$). If $1 \le p_1, \ldots, p_n < \infty$ and $0 < t < \infty$ with $1/t = 1/p_1 + \cdots + 1/p_n$, $(t; p_1, \ldots, p_n)$-summing operators are called $(p_1, \ldots, p_n)$-*dominated*, and we write $\varDelta_{p_1,\ldots,p_n}$ instead of $\pi_{t;p_1,\ldots,p_n}$. If $1 \le p_1 = \cdots = p_n = p < \infty$ and $0 < t < \infty$ with $1/t \le n/p$, $(t; p, \ldots, p)$-summing operators are called $(t;p)$-*summing* and we write $\pi_{t;p}$ instead of $\pi_{t;p,\ldots,p}$. For a comparative study of various classes of multilinear summing operators we recomend [6, 8].

If $1 < p < \infty$ we write p^* to denote the conjugate exponent of p, that is, $1/p + 1/p^* = 1$. Also if $x = (x_n)_{n \in \mathbb{N}}$ and $y = (y_n)_{n \in \mathbb{N}}$ are scalar sequences we write xy for the sequence $(x_n y_n)_{n \in \mathbb{N}}$.

All undefined notation and terminology used in this paper is standard (see e.g. [2, 3, 7, 10]).

2. Preliminary results. The next result is well-known. For the sake of completeness we include its proof.

PROPOSITION 1.

(i) *Let $n \in \mathbb{N}$, $0 < p_1, \ldots, p_n < \infty$ and $0 < p < \infty$ be such that $1/p > 1/p_1 + \cdots + 1/p_n$ and define $0 < s < \infty$ by $1/p = 1/p_1 + \cdots + 1/p_n + 1/s$. Let a be a sequence of scalars and $M \ge 0$ be such that*

$$\|a x_1 \cdots x_n\|_p \le M \|x_1\|_{p_1} \cdots \|x_n\|_{p_n}$$

for all $x_1 \in l_{p_1}, \ldots, x_n \in l_{p_n}$. Then $a \in l_s$ and $\|a\|_s \le M$.

(ii) *Let $n \in \mathbb{N}$ and $0 < t < \infty$ be such that $1/t \le n$. If $T : c_0 \times \overset{n}{\cdots} \times c_0 \to Y$ is $(t; 1, \ldots, 1)$-summing, then*

$$\Big(\sum_{k=1}^\infty \|T(e_k, \ldots, e_k)\|^t\Big)^{1/t} \le \pi_{t;1,\ldots,1}(T).$$

Proof. (i) Define $r_1 = p_1/p, \ldots, r_n = p_n/p$ and $1/r = 1/r_1 + \cdots + 1/r_n$. Then from the condition $1/p > 1/p_1 + \cdots + 1/p_n$ we get $1 < r < \infty$. Define also $b = (|a_k|^p)_{k \in \mathbb{N}}$. Let $x_1 = (x_k^1)_{k \in \mathbb{N}} \in l_{r_1}, \ldots, x_n = (x_k^n)_{k \in \mathbb{N}} \in l_{r_n}$. Then

$(|x_k^1|^{1/p})_{k\in\mathbb{N}} \in l_{p_1}, \ldots, (|x_k^n|^{1/p})_{k\in\mathbb{N}} \in l_{p_n}$, and by hypothesis

$$\Big(\sum_{k=1}^{\infty} |a_k|^p |x_k^1| \cdots |x_k^n|\Big)^{1/p} \le M\Big(\sum_{k=1}^{\infty} |x_k^1|^{r_1}\Big)^{1/p_1} \cdots \Big(\sum_{k=1}^{\infty} |x_k^n|^{r_n}\Big)^{1/p_n}$$
$$= M(\|x_1\|_{r_1} \cdots \|x_n\|_{r_n})^{1/p},$$

that is,

$$\sum_{k=1}^{\infty} |a_k|^p |x_k^1| \cdots |x_k^n| \le M^p \|x_1\|_{r_1} \cdots \|x_n\|_{r_n}.$$

Let $x \in l_r$. Since $1/r = 1/r_1 + \cdots + 1/r_n$, there exist $x_1 = (x_k^1)_{k\in\mathbb{N}} \in l_{r_1}, \ldots, x_n = (x_k^n)_{k\in\mathbb{N}} \in l_{r_n}$ such that $x = x_1 \cdots x_n$ and $\|x\|_r = \|x_1\|_{r_1} \cdots \|x_n\|_{r_n}$. Then by what we have proved above,

$$\sum_{k=1}^{\infty} |a_k|^p |x_k| \le M^p \|x_1\|_{r_1} \cdots \|x_n\|_{r_n} = M^p \|x\|_r.$$

As is well-known, this implies $\sum_{k=1}^{\infty} |a_k|^{pr^*} < \infty$ and $(\sum_{k=1}^{\infty} |a_k|^{pr^*})^{1/r^*} \le M^p$, i.e. $(\sum_{k=1}^{\infty} |a_k|^{pr^*})^{1/(pr^*)} \le M$. Since

$$\frac{1}{pr^*} = \frac{1}{p} - \frac{1}{pr} = \frac{1}{p} - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right) = \frac{1}{s},$$

we get the statement.

(ii) follows from $w_1((e_k)_{k\in\mathbb{N}}; c_0) = 1$ and the definition of $(t; 1, \ldots, 1)$-summing operators. ∎

We need the following variations on the Gelfand Lemma (see [4] or [1, p. 36]). For the sake of completeness we give the proofs.

LEMMA 1. *Let X be a Banach space, $0 < \omega \le 1$ and $p : X \to [0, \infty)$ be subadditive, i.e. $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$, and positive ω-homogeneous, i.e. $p(\lambda x) = \lambda^\omega p(x)$ for all $\lambda \ge 0$ and $x \in X$. If p is lower semicontinuous, then* $\sup_{\|x\|\le 1} p(x) < \infty$.

Proof. For each $\lambda \ge 0$ set $A_\lambda = \{x \in X \mid p(x) \le \lambda\}$. Let $x \in X$. Since $p(x) \in [0, \infty)$, there exists $n \in \mathbb{N}$ such that $n \ge p(x)$, i.e. $x \in A_n$, and thus $X = \bigcup_{n\in\mathbb{N}} A_n$. By the Baire category theorem (X is a Banach space), there exists $n_0 \in \mathbb{N}$ such that $\operatorname{int}(\overline{A_{n_0}}) \ne \emptyset$. Since all the sets A_n are closed (p is lower semicontinuous), we get $\operatorname{int}(A_{n_0}) \ne \emptyset$, i.e. there exists $\delta > 0$ such that $\{x \in X \mid \|x - x_0\| \le \delta\} \subseteq A_{n_0}$. Now take $x \in X$ with $\|x\| \le 1$. From $\|(x_0 + \delta x) - x_0\| \le \delta$ we deduce $x_0 + \delta x \in A_{n_0}$, i.e. $p(x_0 + \delta x) \le n_0$. Then since p is subadditive and positive ω-homogeneous, we have $\delta^\omega p(x) = p(\delta x) \le p(x_0 + \delta x) + p(-x_0) \le n_0 + p(-x_0)$, i.e. $p(x) \le (n_0 + p(-x_0))/\delta^\omega = M$. ∎

LEMMA 2. *Let $X_1, \ldots, X_n$ be Banach spaces, $0 < \omega \le 1$ and $p : X_1 \times \cdots \times X_n \to [0, \infty)$ be such that: p is subadditive in each variable, that is, for* *each* $1 \le i \le n$ *and each* $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$ *and* $y_i \in X_i$,

$$\begin{aligned} & p(x_1, \ldots, x_{i-1}, x_i + y_i, x_{i+1}, \ldots, x_n) \\ & \quad \le p(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) + p(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n); \end{aligned}$$

p *is positive* ω*-homogeneous in each variable, that is, for all* $\lambda \ge 0$ *and* $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$,

$$p(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_n) = \lambda^{\omega} p(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_n);$$

and p *is lower semicontinuous. Then*

$$\sup_{\|x_1\| \le 1, \ldots, \|x_n\| \le 1} p(x_1, \ldots, x_n) < \infty.$$

Proof. The proof is by induction on n. The case $n = 1$ was shown in Lemma 1. Suppose that the assertion is true for $n - 1$, where $n \ge 2$. Let $(x_1, \ldots, x_{n-1}) \in X_1 \times \cdots \times X_{n-1}$. Then $p(x_1, \ldots, x_{n-1}, \cdot) : X_n \to [0, \infty)$ satisfies the hypotheses in Lemma 1. Hence $\sup_{\|x_n\| \le 1} p(x_1, \ldots, x_{n-1}, x_n) < \infty$. This means that $q : X_1 \times \cdots \times X_n \to [0, \infty)$ defined by $q(x_1, \ldots, x_{n-1}) = \sup_{\|x_n\| \le 1} p(x_1, \ldots, x_{n-1}, x_n)$ is well defined. Since p is subadditive, positive ω-homogeneous in each variable and lower semicontinuous, we deduce that q is subadditive, positive ω-homogeneous in each variable and lower semicontinuous. Then, since the assertion is true for $n - 1$, we get

$$\sup_{\|x_1\| \le 1, \ldots, \|x_{n-1}\| \le 1} q(x_1, \ldots, x_{n-1}) < \infty,$$

as desired. ■

COROLLARY 1. *Let* $X_1, \ldots, X_n$ *be Banach spaces,* $0 < \omega \le 1$ *and* $p : X_1 \times \cdots \times X_n \to [0, \infty)$ *be such that:* p *is positive homogeneous in each variable, i.e. for all* $1 \le i \le n$, $\lambda \ge 0$ *and* $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$,

$$p(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_n) = \lambda p(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_n);$$

p *is* ω*-subadditive in each variable, i.e. for all* $1 \le i \le n$, $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$ *and* $y_i \in X_i$,

$$\begin{aligned} & [p(x_1, \ldots, x_{i-1}, x_i + y_i, x_{i+1}, \ldots, x_n)]^{\omega} \\ & \quad \le [p(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)]^{\omega} + [p(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)]^{\omega}; \end{aligned}$$

and p *is lower semicontinuous. Then*

$$\sup_{\|x_1\| \le 1, \ldots, \|x_n\| \le 1} p(x_1, \ldots, x_n) < \infty.$$

Proof. Define $s : X_1 \times \cdots \times X_n \to [0, \infty)$ by setting $s(x_1, \ldots, x_n) = [p(x_1, \ldots, x_n)]^{\omega}$. Then s satisfies the hypotheses in Lemma 2 and the conclusion follows. ■

3. The results. Pietsch's famous composition theorem asserts that $\Pi_q \circ \Pi_r \subset \Pi_p$ for $1 \leq p < q, r < \infty$ such that $1/p = 1/q + 1/r$ [2, 3, 7, 10]. Our first result is a multilinear version of Pietsch's composition theorem.

THEOREM 1. *Let $n \in \mathbb{N}$, $1 \leq s_j < p_j, r_j < \infty$ be such that $1/s_j = 1/p_j + 1/r_j$ for all $1 \leq j \leq n$ and $0 < p < \infty$ such that $1/p \leq 1/p_1 + \cdots + 1/p_n$. If all $X_j \xrightarrow{V_j} Y_j$ are r_j-summing $(1 \leq j \leq n)$ and $T : X_1 \times \cdots \times X_n \to Y$ is $(p; p_1, \ldots, p_n)$-summing, then $T \circ (V_1, \ldots, V_n)$ is $(s; s_1, \ldots, s_n)$-summing and*

$$\pi_{s;p_1,\ldots,p_n}(T \circ (V_1, \ldots, V_n)) \leq \pi_{q;q_1,\ldots,q_n}(T)\pi_{r_1}(V_1) \cdots \pi_{r_n}(V_n)$$

where $1/s = 1/r_1 + \cdots + 1/r_n + 1/p$.

Proof. Let $(x_i^j)_{1\leq i\leq m} \subset X_j$ $(1 \leq j \leq n)$. Since V_j $(1 \leq j \leq n)$ is r_j-summing, by the splitting property [3, Lemma 2.23, p. 53], [7, Theorem 20.1.10, p. 281], [10, Lemma 9.14, p. 55], there exist $\alpha_j = (\alpha_i^j)_{1\leq i\leq m} \subset \mathbb{K}$ and $(y_i^j)_{1\leq i\leq m} \subset Y_j$ such that

$$V_j(x_i^j) = \alpha_i^j y_i^j \quad \text{for each } 1 \leq i \leq m, \tag{3.1}$$

$$\|\alpha_j\|_{r_j} w_{p_j}((y_i^j)_{1\leq i\leq m}) \leq \pi_{r_j}(V_j) w_{s_j}((x_i^j)_{1\leq i\leq m}). \tag{3.2}$$

Then from $1/s = 1/r_1 + \cdots + 1/r_n + 1/p$, Hölder's inequality and (3.1) we deduce

$$\begin{aligned} l_s\big(T \circ (V_1, \ldots, V_n)(x_i^1, \ldots, x_i^n) \bigm| 1 \leq i \leq m\big) \\ &= l_s\big(\alpha_i^1 \cdots \alpha_i^n T(y_i^1, \ldots, y_i^n) \mid 1 \leq i \leq m\big) \\ &\leq \|\alpha_1\|_{r_1} \cdots \|\alpha_n\|_{r_n} l_p(T(y_i^1, \ldots, y_i^n) \mid 1 \leq i \leq m) \end{aligned} \tag{3.3}$$

Since T is $(p; p_1, \ldots, p_n)$-summing, we have

$$l_p(T(y_i^1, \ldots, y_i^n) \mid 1 \leq i \leq m) \leq \pi_{p;p_1,\ldots,p_n}(T) \prod_{j=1}^{n} w_{p_j}((y_i^j)_{1\leq i\leq m}). \tag{3.4}$$

From (3.2)–(3.4) we get

$$\begin{aligned} &l_s\big(T \circ (V_1, \ldots, V_n)(x_i^1, \ldots, x_i^n) \bigm| 1 \leq i \leq m\big) \\ &\quad \leq \pi_{p;p_1,\ldots,p_n}(T)\|\alpha_1\|_{r_1} w_{p_1}((y_i^1)_{1\leq i\leq m}) \cdots \|\alpha_n\|_{r_n} w_{p_n}((y_i^n)_{1\leq i\leq m}) \\ &\quad \leq \pi_{p;p_1,\ldots,p_n}(T)\pi_{r_1}(V_1) \cdots \pi_{r_n}(V_n) \prod_{j=1}^{n} w_{s_j}((x_i^j)_{1\leq i\leq m}). \end{aligned}$$

The definition of $(s; s_1, \ldots, s_n)$-summing operators ends the proof. ∎

The next result is a multilinear version of J. Puhl's theorem [9, Satz 8].

THEOREM 2. *Let $n \in \mathbb{N}$, $1 < p_1, \ldots, p_n < \infty$ and $0 < t < \infty$ be such that $1/t \leq 1/p_1 + \cdots + 1/p_n$ and let $0 < v_n < \infty$ be defined by $1/v_n =$*

$1/t + 1/p_1^* + \cdots + 1/p_n^*$. Let $X_1 \times \cdots \times X_n \xrightarrow{V} Y$ be a bounded n-linear operator. The following assertions are equivalent:

- (i) *V is $(t; p_1, \ldots, p_n)$-summing.*
- (ii) *For any Banach spaces Z_j and p_j^*-summing operators $Z_j \xrightarrow{U_j} X_j$ $(1 \leq j \leq n)$, $V \circ (U_1, \ldots, U_n)$ is $(v_n; 1, \ldots, 1)$-summing.*
- (iii) *For any p_j^*-summing operators $c_0 \xrightarrow{S_j} X_j$ $(1 \leq j \leq n)$, $V \circ (S_1, \ldots, S_n)$ is $(v_n; 1, \ldots, 1)$-summing. Moreover,*

$$\sup_{\pi_{p_1^*}(U_1) \leq 1, \ldots, \pi_{p_n^*}(U_n) \leq 1} \pi_{v_n;1,\ldots,1}(U \circ (V_1, \ldots, V_n))$$
$$= \sup_{\pi_{p_1^*}(S_1) \leq 1, \ldots, \pi_{p_n^*}(S_n) \leq 1} \pi_{v_n;1,\ldots,1}(U \circ (S_1, \ldots, S_n)) = \pi_{t;p_1,\ldots,p_n}(U),$$

where the first supremum is taken over all Banach spaces Z_j and all p_j^*-summing operators $Z_j \xrightarrow{U_j} X_j$ $(1 \leq j \leq n)$.

Proof. (i)⇒(ii) follows from Theorem 1 together with the inequality

$$\sup_{\pi_{p_1^*}(U_1) \leq 1, \ldots, \pi_{p_n^*}(U_n) \leq 1} \pi_{v_n;1,\ldots,1}(U \circ (V_1, \ldots, V_n)) \leq \pi_{t;p_1,\ldots,p_n}(U).$$

(ii)⇒(iii) is obvious and moreover

$$\sup_{\pi_{p_1^*}(S_1) \leq 1, \ldots, \pi_{p_n^*}(S_n) \leq 1} \pi_{v_n;1,\ldots,1}(U \circ (S_1, \ldots, S_n))$$
$$\leq \sup_{\pi_{p_1^*}(U_1) \leq 1, \ldots, \pi_{p_n^*}(U_n) \leq 1} \pi_{v_n;1,\ldots,1}(U \circ (V_1, \ldots, V_n)).$$

(iii)⇒(i). From (iii) and Corollary 1 it follows that

$$M_V := \sup_{\pi_{p_1^*}(S_1) \leq 1, \ldots, \pi_{p_n^*}(S_n) \leq 1} \pi_{v_n;1,\ldots,1}(V \circ (S_1, \ldots, S_n)) < \infty. \tag{3.5}$$

Now for $1 \leq j \leq n$ let $A_j : l_{p_j^*} \to X_j$ be a bounded linear operator. Let also $a_j = (a_i^j)_{i \in \mathbb{N}} \in l_{p_j^*}$ and note that the multiplication operator $M_{a_j} : c_0 \to l_{p_j^*}$ defined by $M_{a_j}(\xi) = a_j \xi$ is p_j^*-summing, with $\pi_{p_j^*}(M_{a_j}) = \|a_j\|_{p_j^*}$. Thus $S_j = A_j \circ M_{a_j} : c_0 \to X_j$ is p_j^*-summing with $\pi_{p_j^*}(S_j) \leq \|a_j\|_{p_j^*} \|A_j\|$. Hence from (iii), $V \circ (S_1, \ldots, S_n) : c_0 \times \overset{n}{\cdots} \times c_0 \to Y$ is $(v_n; 1, \ldots, 1)$-summing and

$$\begin{aligned} \pi_{v_n;1,\ldots,1}(V \circ (S_1, \ldots, S_n)) &\leq M_V \pi_{p_1^*}(S_1) \cdots \pi_{p_n^*}(S_n) \\ &= M_V \|a_1\|_{p_1^*} \cdots \|a_n\|_{p_n^*} \|A_1\| \cdots \|A_n\|. \end{aligned}$$

But, as observed in Proposition 1(ii),

$$\Big(\sum_{k=1}^{\infty} \|(V \circ (S_1, \ldots, S_n))(e_k, \ldots, e_k)\|^{v_n}\Big)^{1/v_n} \leq \pi_{v_n;1,\ldots,1}(V \circ (S_1, \ldots, S_n)).$$

Thus

$$\Big(\sum_{k=1}^{\infty}(|a_k^1|\cdots|a_k^n|\,\|V(S_1(e_k),\dots,S_n(e_k))\|)^{v_n}\Big)^{1/v_n} \le M_V\|a_1\|_{p_1^*}\cdots\|a_n\|_{p_n^*}\|A_1\|\cdots\|A_n\|.$$

Since $1/v_n = 1/t + 1/p_1^* + \dots + 1/p_n^*$, from Proposition 1(i) it follows that

$$\Big(\sum_{k=1}^{\infty}\|V(S_1(e_k),\dots,S_n(e_k))\|^t\Big)^{1/t} \le M_V\|A_1\|\cdots\|A_n\|.$$

Then (see [3, Proposition 2.7, p. 39]) V is $(t;p_1,\dots,p_n)$-summing and

$$\pi_{t;p_1,\dots,p_n}(V) \le M_V = \sup_{\pi_{p_1^*}(S_1)\le 1,\ \dots,\pi_{p_n^*}(S_n)\le 1} \pi_{v_n;1,\dots,1}(V\circ(S_1,\dots,S_n)). \ ■$$

Corollary 2. *Let $n\in\mathbb{N}$, let $X_1\times\dots\times X_n \xrightarrow{V} Y$ be a bounded n-linear operator and $1 < p_1,\dots,p_n < \infty$. The following assertions are equivalent:*

- (i) *V is $(p_1,\dots,p_n)$-dominated.*
- (ii) *For any Banach spaces Z_j and any p_j^*-summing operators $Z_j \xrightarrow{U_j} X_j$ $(1\le j\le n)$, $V\circ(U_1,\dots,U_n)$ is 1-dominated.*
- (iii) *For any p_j^*-summing operators $c_0 \xrightarrow{S_j} X_j$ $(1\le j\le n)$, $V\circ(S_1,\dots,S_n)$ is 1-dominated. Moreover,*

$$\sup_{\pi_{p_1^*}(U_1)\le 1,\ \dots,\pi_{p_n^*}(U_n)\le 1} \varDelta_1(V\circ(U_1,\dots,U_n)) = \sup_{\pi_{p_1^*}(S_1)\le 1,\ \dots,\pi_{p_n^*}(S_n)\le 1} \varDelta_1(U\circ(S_1,\dots,S_n)) = \varDelta_{p_1,\dots,p_n}(V)$$

where the first supremum is taken over all Banach spaces Z_j and all p_j^*-summing operators $Z_j \xrightarrow{U_j} X_j$ $(1\le j\le n)$.

Proof. Note that 'V is $(p_1,\dots,p_n)$-dominated' means that V is $(t;p_1,\dots,p_n)$-summing, where $1/t = 1/p_1+\dots+1/p_n$. The statement follows from Theorem 2 since in this case $1/v_n = 1/t + 1/p_1^* + \dots + 1/p_n^* = n$, $v_n = 1/n$ and $\varPi_{1/n;1,\dots,1} = \varDelta_1$. ■

Corollary 3. *Let $n\in\mathbb{N}$, $1 < p < \infty$ and $0 < t < \infty$ be such that $1/t \le n/p$, and let $0 < v_n < \infty$ be defined by $1/v_n = 1/t + n/p^*$. Let $X_1\times\dots\times X_n \xrightarrow{V} Y$ be a bounded n-linear operator. The following assertions are equivalent:*

- (i) *V is $(t;p)$-summing.*
- (ii) *For any Banach spaces Z_j and any p_j^*-summing operators $Z_j \xrightarrow{U_j} X_j$ $(1\le j\le n)$, $V\circ(U_1,\dots,U_n)$ is $(v_n;1)$-summing.*

(iii) *For any* p_j^**-summing operators* $c_0 \overset{S_j}{\to} X_j$ $(1 \le j \le n)$, $V \circ (S_1, \dots, S_n)$ *is* $(v_n; 1)$*-summing. Moreover,*

$$\sup_{\pi_{p_1^*}(U_1)\le 1,\dots,\pi_{p_n^*}(U_n)\le 1} \pi_{v_n;1}(U \circ (V_1, \dots, V_n)) = \sup_{\pi_{p_1^*}(S_1)\le 1,\dots,\pi_{p_n^*}(S_n)\le 1} \pi_{v_n;1}(U \circ (S_1, \dots, S_n)) = \pi_{t;p}(U),$$

where the first supremum is taken over all Banach spaces Z_j *and all* p_j^**-summing operators* $Z_j \overset{U_j}{\to} X_j$ $(1 \le j \le n)$.

In the case $n = 1$ from Corollary 3 we get the following result not covered by Puhl's theorem.

COROLLARY 4. *Let* $1 < p \le q < \infty$ *and let* $1 \le v < \infty$ *be defined by* $1/v = 1/q + 1/p^*$. *Let* $X \overset{V}{\to} Y$ *be a bounded linear operator. The following assertions are equivalent:*

(i) V *is* (q, p)*-summing.*

(ii) *For each Banach spaces* Z *and each* p^**-summing operator* $Z \overset{U}{\to} X$, $V \circ U$ *is* $(v, 1)$*-summing.*

(iii) *For each* p^**-summing operator* $c_0 \overset{S}{\to} X$, $V \circ S$ *is* $(v, 1)$*-summing. Moreover,*

$$\sup_{\pi_{p^*}(U)\le 1} \pi_{v,1}(U \circ V) = \sup_{\pi_{p^*}(S)\le 1} \pi_{v,1}(U \circ S) = \pi_{q,p}(U),$$

where the first supremum is taken over all Banach spaces Z *and all* p^**-summing operators* $Z \overset{U}{\to} X$.

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Dumitru Popa
Department of Mathematics
Ovidius University of Constanţa
Bd. Mamaia 124
900527 Constanţa, Romania
E-mail: dpopa@univ-ovidius.ro