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SYMPLECTIC NON-SQUEEZING FOR MASS SUBCRITICAL FOURTH-ORDER SCHRÖDINGER EQUATIONS

BY

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Abstract. Applying strategies of R. Killip et al. (2016), we establish symplectic non-squeezing for the mass subcritical fourth-order Schrödinger equations $iu_t - \Delta^2 u = \pm|u|^p u$ with $3/2 < p < 8$ in dimension one.

1. Introduction. We study symplectic non-squeezing for the fourth-order Schrödinger equations of the form

$$\begin{cases} (i\partial_t - \Delta^2)u = \pm|u|^p u, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x) \in L^2_x(\mathbb{R}), \end{cases} \tag{1.1}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$. *Symplectic non-squeezing* means that the flow associated to (1.1) does not carry any ball of radius R into any cylinder whose cross-section has radius $r < R$ (see also Remark 1.3 below).

The class of solutions to (1.1) is invariant under the scaling

$$u(t,x) \mapsto \lambda^{4/p} u(\lambda^4 t, \lambda x), \qquad \lambda > 0. \tag{1.2}$$

This scaling defines a notion of *criticality* for (1.1). In particular, one can check that the only homogeneous L^2_x-based Sobolev space that is invariant under (1.2) is $\dot{H}^{s_c}_x(\mathbb{R})$, where the *critical regularity index* s_c is given by $s_c := 1/2 - 4/p$. If we take $u_0 \in L^2_x(\mathbb{R})$, then for $s_c = 0$ we call problem (1.1) *mass critical*. For $s_c < 0$ it is *mass subcritical*, while for $s_c > 0$ it is *mass supercritical*.

If a solution u of (1.1) has sufficient decay at infinity and smoothness, it conserves mass:

$$M(u) = \int_{\mathbb{R}} |u(t,x)|^2 \, dx = M(u_0). \tag{1.3}$$

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In this article we focus on the initial-value problem (1.1) in the mass subcritical regime, that is, when $s_c < 0$ (i.e. $p < 8$). We prove the symplectic non-squeezing property for solutions to (1.1) with $3/2 < p < 8$.

To begin, we need the definition of a solution.

DEFINITION 1.1. A function $u : I \times \mathbb{R} \to \mathbb{C}$ on a non-empty time interval $I \ni 0$ is a *solution* to problem (1.1) if it belongs to $C_t(K, L^2_x(\mathbb{R})) \cap L^{10}_{t,x}(K \times \mathbb{R})$ for any compact interval $K \subset I$ and obeys the Duhamel formula

$$u(t) = e^{-it\Delta^2} u_0 - i \int_0^t e^{-i(t-s)\Delta^2} (\pm |u|^p u(s))\, ds \tag{1.4}$$

for each $t \in I$. We call I the *lifespan* of u. We say that u is a *maximal-lifespan solution* if it cannot be extended to any strictly larger interval. We call u *global* if $I = \mathbb{R}$.

Fourth-order Schrödinger equations have been intensively studied in recent papers [1, 5, 14, 16, 17, 18]. Ben-Artzi, Koch and Saut [1] give sharp dispersive estimates for the biharmonic Schrödinger operator $e^{-it\Delta^2}$ which lead to the Strichartz estimates for the fourth-order Schrödinger equation (see also [15, 17, 18]). Using the Strichartz estimates and a standard fixed point argument, we easily obtain the global well-posedness for problem (1.1) in both the defocusing and focusing cases, that is, with + and − signs of the non-linearity, respectively.

Now, we state our main result.

THEOREM 1.2. *Let $3/2 < p < 8$. Fix $z_* \in L^2(\mathbb{R})$, $l \in L^2(\mathbb{R})$ with $\|l\|_{L^2(\mathbb{R})} = 1$, $\alpha \in \mathbb{C}$, $0 < r < R < \infty$, and $T > 0$. Then there exists $u_0 \in B(z_*, R)$ such that the solution u to the initial-value problem* (1.1) *with initial data $u(0) = u_0$ satisfies*

$$|\langle l, u(T)\rangle - \alpha| > r. \tag{1.5}$$

REMARK 1.3. (i) Theorem 1.2 implies that the flow associated to (1.1) does not carry any ball of radius R into any cylinder whose cross-section has radius $r < R$ (see Figure 1).

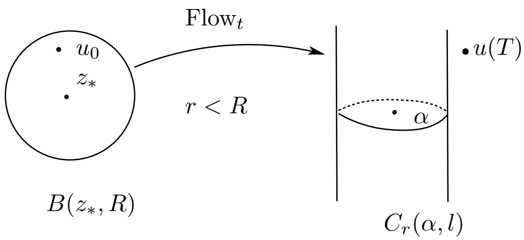

Fig. 1

(ii) The restriction $p > 3/2$ stems from the persistence of positive regularity (see Lemma 3.1 below).

(iii) By the same argument as for Theorem 1.2, one can also obtain the same result for the initial-value problem (1.1) with $1 < p < 8/d$ in dimensions $2 \le d \le 7$. Here the restriction on the dimension also comes from the persistence of positive regularity.

The study of non-squeezing for non-linear Hamiltonian PDE was initiated by Kuksin [12], whose approach was to develop a variant of Gromov's theorem in Hilbert space and then verify the hypotheses of his theorem for several PDE examples, including the non-linear Klein–Gordon equations with weak non-linearities. Symplectic non-squeezing was later proved for certain subcritical non-linear Klein–Gordon equations by Bourgain [3]. Also, Bourgain [2] extended these results to the cubic NLS in dimension one, where the full equation is approximated by a finite-dimensional flow to which Gromov's finite-dimensional non-squeezing result is applied. Symplectic non-squeezing was also proven for other models [4, 7, 13, 20]. All previous non-squeezing results for non-linear PDE were proved for problems posed on tori. Recently, using the finite-dimensional approximation strategy, Killip, Visan and Zhang [10, 11] established the symplectic non-squeezing for the mass critical and mass subcritical non-linear Schrödinger equation on $\mathbb{R}^d$, which is the first symplectic non-squeezing result for a Hamiltonian PDE in infinite volume. Applying the strategies from [10, 11], we establish symplectic non-squeezing for the mass subcritical fourth-order Schrödinger equations, i.e. Theorem 1.2.

1.1. Outline of the proof of Theorem 1.2. Fix $z_* \in L^2(\mathbb{R})$, $l \in L^2(\mathbb{R})$ with $\|l\|_2 = 1$, $\alpha \in \mathbb{C}$, $0 < r < R < \infty$, and $T > 0$. Write $M := \|z_*\|_2 + R$. Let $N_n, L_n \to \infty$, and $\delta \in (0, (R-r)/8)$. By density there exist $\tilde{z}_*$, $\tilde{l} \in C_c^\infty(\mathbb{R})$ such that

$$\|z_* - \tilde{z}_*\|_{L^2} \le \delta \quad \text{and} \quad \|l - \tilde{l}\|_{L^2} \le \delta M^{-1} \quad \text{with } \|\tilde{l}\|_2 = 1. \tag{1.6}$$

Since $L_n \to \infty$, the supports of $\tilde{z}_*$ and $\tilde{l}$ are contained in the interval $[-L_n/2, L_n/2]$ for n sufficiently large. Hence, we can view $\tilde{z}_*$ and $\tilde{l}$ as functions on the torus $\mathbb{T}_n = \mathbb{R}/L_n\mathbb{Z}$ (see Subsection 2.1).

We now consider the initial-value problem

$$\begin{cases} (i\partial_t - \Delta^2)u_n = P_{\le N_n}^{L_n} F(P_{\le N_n}^{L_n} u_n), & (t,x) \in \mathbb{R} \times \mathbb{T}_n, \\ u_n(0) \in \mathcal{H}_n = \{f \in L^2(\mathbb{T}_n) : P_{>2N_n}^{L_n} f = 0\}, \end{cases} \tag{1.7}$$

where $P_{\le N_n}^{L_n}$ denotes the Fourier multiplier $P_{\le N_n}$ on $\mathbb{T}_n$. By Lemma 5.2 below and mass conservation, we know that problem (1.7) has global solutions. We observe that (1.7) is a finite-dimensional Hamiltonian system with respect to the standard Hilbert-space symplectic structure on $\mathcal{H}_n$; the Hamiltonian is

$$H(u) = \int_{\mathbb{T}_n} \left(\frac{1}{2}|\partial_x u|^2 \pm \frac{1}{p+2}|u|^{p+2} \right) dx. \tag{1.8}$$

Hence, using Gromov's symplectic non-squeezing theorem (see Theorem 2.3 below), we deduce that there exist initial data

$$u_{0,n} \in B_{\mathcal{H}_n}(P^{L_n}_{\le N_n} \tilde{z}_*, R - 4\delta) \tag{1.9}$$

such that the solution to problem (1.7) with $u_n(0) = u_{0,n}$ obeys

$$|\langle \tilde{l}, u_n(T)\rangle_{L^2(\mathbb{T}_n)} - \alpha| > r + 4\delta. \tag{1.10}$$

On the other hand, we will utilize the stability result to show in Section 5 that for n sufficiently large, solutions u_n to (1.7) can be well approximated by solutions to the corresponding system posed on $\mathbb{R}$ on the fixed time interval $[-T, T]$:

$$\begin{cases} (i\partial_t - \Delta^2)\tilde{u}_n = P_{\le N_n} F(P_{\le N_n}\tilde{u}_n), & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ \tilde{u}_n(0,x) = \chi^0_n(x) u_{0,n}(x + \mathbb{T}_n), \end{cases} \tag{1.11}$$

where $P_{\le N_n}$ is the usual Littlewood–Paley projection operator, and χ^0_n is a smooth cutoff function defined in Section 5 below. Moreover, since the sequence $\{\tilde{u}_n(0)\}$ is uniformly bounded in $L^2_x(\mathbb{R})$, there exists a subsequence of n such that

$$\chi^0_n u_{0,n} \rightharpoonup u_{0,\infty} \quad \text{weakly in } L^2(\mathbb{R}).$$

By Lemma 3.1 below, we know that there is a global solution $u_\infty(t,x)$ to (1.1) with initial data $u_\infty(0) = u_{0,\infty}$. Using the local smoothing estimate and precompactness in $L^p(\mathbb{R}^d)$, we see that, up to a subsequence,

$$\tilde{u}_n(T) \rightharpoonup u_\infty(T) \quad \text{weakly in } L^2(\mathbb{R}).$$

Collecting the above estimates, we can show that the solution $u_\infty(t,x)$ satisfies $u_\infty(0) \in B(z_*, R)$ and

$$|\langle l, u_\infty(T)\rangle - \alpha| > r. \tag{1.12}$$

Thus, we deduce Theorem 1.2 by choosing $u(t,x) = u_\infty(t,x)$.

The paper is organized as follows. In Section 2, as a preliminary section, we give some notation, recall the Strichartz estimate and prove the local smoothing estimate. In Section 3, we establish well-posedness for some fourth-order Schrödinger equations. In Section 4, we prove approximation in the weak topology. In Section 5, we give a finite-dimensional approximation. Finally, we prove Theorem 1.2 in Section 6.

2. Preliminaries

2.1. Some notation. For non-negative quantities X and Y, we will write $X \lesssim Y$ to denote the estimate $X \le CY$ for some $C > 0$. If $X \lesssim Y \lesssim X$, we will write $X \sim Y$. Dependence of implicit constants on the power p will be suppressed; dependence on additional parameters will be indicated by subscripts. For example, $X \lesssim_u Y$ indicates $X \le CY$ for some $C = C(u)$. We will use $X_\pm$ to denote $X \pm \varepsilon$ for any $\varepsilon > 0$.

For a space-time slab $I \times \mathbb{R}$, we write $L_t^q L_x^r(I \times \mathbb{R})$ for the Banach space of functions $u : I \times \mathbb{R} \to \mathbb{C}$ equipped with the norm

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R})} := \Big(\int_I \|u(t)\|_{L_x^r(\mathbb{R})}\, dt\Big)^{1/q},$$

with the usual adjustments when q or r is infinity. When $q = r$, we abbreviate $L_t^q L_x^q = L_{t,x}^q$. We will also often abbreviate $\|f\|_{L_x^r(\mathbb{R})}$ to $\|f\|_{L_x^r}$. For $1 \le r \le \infty$, we use r' to denote the dual exponent to r, i.e. the solution to $1/r + 1/r' = 1$.

We define the Fourier transform on $\mathbb{R}$ by

$$\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\cdot\xi} f(x)\, dx.$$

We can then define the fractional differentiation operators $|\nabla|^s$ and $\langle\nabla\rangle^s$ for $s \in \mathbb{R}$ via

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat{f}(\xi), \qquad \widehat{\langle\nabla\rangle^s f}(\xi) := \langle\xi\rangle^s \hat{f}(\xi),$$

where $\langle\xi\rangle := 1 + |\xi|$. This enables us to define the homogeneous and inhomogeneous Sobolev norms

$$\|f\|_{\dot{H}_x^s(\mathbb{R})} := \big\||\xi|^s \hat{f}\big\|_{L_x^2(\mathbb{R})}, \qquad \|f\|_{H_x^s(\mathbb{R})} := \|\langle\xi\rangle^s \hat{f}\|_{L_x^2(\mathbb{R})}.$$

For $L \in \mathbb{R}^+$, the distance on the torus $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$ is defined by

$$\operatorname{dist}(x, y) = \operatorname{dist}(x - y, \mathbb{R}/L\mathbb{Z}). \tag{2.1}$$

For a function $f \in L^2(\mathbb{R})$ with $\operatorname{supp} f \subset I$, $|I| \le L$, we define

$$(p_* f)(x + \mathbb{T}_L) := \sum_{k\in\mathbb{Z}} f(x + kL), \tag{2.2}$$

in particular

$$(p_* f)(x) = f(x), \quad x \in I.$$

Consequently, we can view the function on $\mathbb{R}$ that is supported in an interval of length L naturally as a function on the torus $\mathbb{T}_L$. Hence, we can denote $p_* f(x)$ by $f(x)$ for brevity.

Conversely, for $g \in L^2(\mathbb{T}_L)$ and a smooth cutoff function χ on $\mathbb{R}$ that is supported in an interval of length L, we define

$$p^*(\chi g)(x) := \chi(x) g(x + \mathbb{T}_L).$$

Consequently, we can view χg as a function on $\mathbb{R}$.

2.2. Basic analysis. We will make frequent use of the Littlewood–Paley projection operators. Specifically, we let φ be a radial bump function supported on the ball $|\xi| \le 1.5$ and equal to 1 on the ball $|\xi| \le 1.4$. For $N \in 2^{\mathbb{Z}}$, we define the *Littlewood–Paley projection operators* by

$$\begin{aligned}\widehat{P_{\le N} f}(\xi) &:= \widehat{f_{\le N}}(\xi) := \varphi(\xi/N)\widehat{f}(\xi),\\ \widehat{P_{> N} f}(\xi) &:= \widehat{f_{> N}}(\xi) := (1-\varphi(\xi/N))\widehat{f}(\xi),\\ \widehat{P_{N} f}(\xi) &:= \widehat{f_{N}}(\xi) := (\varphi(\xi/N)-\varphi(2\xi/N))\widehat{f}(\xi).\end{aligned}$$

The Littlewood–Paley operators commute with the derivative operators, the free propagator, and the conjugation operation. These operators are self-adjoint and bounded on every L^p_x and $\dot{H}^s_x$ space for $1 \le p \le \infty$ and $s \ge 0$. They also obey the following standard Bernstein estimates: For $1 \le r \le \infty$ and $s \ge 0$,

$$\begin{aligned}\big\||\nabla|^{\pm s} P_N f\big\|_{L^r_x(\mathbb{R})} &\sim N^{\pm s}\|P_N f\|_{L^r_x(\mathbb{R})},\\ \big\||\nabla|^{s} P_{\le N} f\big\|_{L^r_x(\mathbb{R})} &\lesssim N^{s}\|P_{\le N} f\|_{L^r_x(\mathbb{R})}.\end{aligned}$$

Fix $L \in \mathbb{R}^+$. Then $P^L_{\le N}$ denotes the Fourier multiplier $P_{\le N}$ on the torus $\mathbb{T}_L$, that is,

$$P^L_{\le N} f(x) = \int_0^L K^L(x,y) f(y)\, dy,$$

with

$$K^L(x,y) = \frac{1}{L}\sum_{n\in\mathbb{Z}} e^{2\pi i (x-y)\cdot n/L}\varphi\Big(\frac{n}{NL}\Big). \tag{2.3}$$

Lemma 2.1 (Midpoint rule error). *Given $\xi_0 \in \mathbb{R}^d$ and $L > 0$, let Q denote the cube $Q = \xi_0 + \big[-\frac{1}{2L}, \frac{1}{2L}\big)^d$ of side-length L^{-1} centered at ξ_0. Then*

$$\Big|\int_Q h(\xi)\, d\xi - \frac{1}{L^d} h(\xi_0)\Big| \lesssim \frac{1}{L^{d+2}}\|\partial^2 h\|_{L^\infty(Q)}. \tag{2.4}$$

We refer to [10] for the proof of this lemma.

Next, we recall a criterion for a set $\mathcal{F} \subset L^p(\mathbb{R}^d)$ to be precompact. This will serve as an important technical tool when we prove well-posedness in the weak topology in Section 4.

Proposition 2.2 (precompactness in $L^p(\mathbb{R}^d)$, cf. [9, 19]). *Fix $1 \le p < \infty$. Then a family $\mathcal{F} \subset L^p(\mathbb{R}^d)$ is precompact in this topology if and only if it obeys the following three conditions:*

(i) Uniform boundedness: *There exists $A > 0$ such that $\|f\|_{L^p(\mathbb{R}^d)} \le A$ for all $f \in \mathcal{F}$.*

(ii) Equicontinuity: *For any* $\epsilon > 0$, *there exists* $\delta > 0$ *with the property that* $\int_{\mathbb{R}^d} |f(x+y) - f(x)|^p \, dx < \epsilon$ *for all* $f \in \mathcal{F}$ *and* $|y| < \delta$.

(iii) Tightness: *For any* $\epsilon > 0$, *there exists* $R > 0$ *such that* $\int_{|x|>R} |f(x)|^p \, dx < \epsilon$ *for all* $f \in \mathcal{F}$.

Finally, we recall the non-squeezing theorem of Gromov (see [6, Corollary 0.3.A]), which implies the non-squeezing property for the solutions to the non-linear fourth-order Schrödinger equations on the torus.

THEOREM 2.3 (Gromov's non-squeezing theorem [6]). *Fix* $0 < r < R < \infty$ *and* $\alpha \in \mathbb{C}$. *Let* $B(z_*, R)$ *denote the ball of radius* R *centered at* $z_* \in \mathbb{C}^n$, *let* $\ell \in \mathbb{C}^n$ *have unit length, and suppose* $\phi : B(z_*, R) \to \mathbb{C}^n$ *is a smooth symplectomorphism* (*with respect to the standard structure*). *Then there exists* $z \in B(z_*, R)$ *such that*

$$|\langle \ell, \phi(z) \rangle - \alpha| > r.$$

Equivalently, ϕ *does not map* $B(z_*, R)$ *wholly into the cylinder* $\{\zeta \in \mathbb{C}^n : |\langle \ell, \zeta \rangle - \alpha| < r\}$.

2.3. Strichartz estimate and local smoothing estimate. The *biharmonic Schrödinger semigroup* is defined for any tempered distribution g by

$$e^{-it\Delta^2} g = \mathcal{F}^{-1} e^{-it|\xi|^4} \mathcal{F} g.$$

Now we recall the dispersive estimate for the biharmonic Schrödinger operator.

LEMMA 2.4 (Dispersive estimate, cf. [1]). *Let* $2 \le q \le \infty$. *Then we have the dispersive estimate*

$$\|e^{-it\Delta^2} f\|_{L^q_x(\mathbb{R})} \le C|t|^{-\frac{1}{4}(1-2/q)} \|f\|_{L^{q'}_x(\mathbb{R})} \tag{2.5}$$

for all $t \neq 0$ *and* $1/q + 1/q' = 1$.

The Strichartz estimates involve the following definition.

DEFINITION 2.5 (Strichartz spaces). We define the *Strichartz norm* of a space-time function via

$$\|u\|_{S(I\times\mathbb{R})} := \|u\|_{C_t L^2_x(I\times\mathbb{R})} + \|u\|_{L^8_t L^\infty_x(I\times\mathbb{R})},$$

and the dual norm via

$$\|G\|_{N(I\times\mathbb{R})} := \inf_{G=G_1+G_2} \|G_1\|_{L^1_t L^2_x(I\times\mathbb{R})} + \|G_2\|_{L^{8/7}_t L^1_x(I\times\mathbb{R})}.$$

We define Strichartz spaces on the torus analogously.

According to the above dispersive estimate, the abstract duality and an interpolation argument (see [8]), we have the following Strichartz estimates.

Proposition 2.6 (Strichartz estimates for fourth-order Schrödinger equation [15, 17]). *Suppose that $u(t,x)$ is a solution on $[0,T]$ to the initial-value problem*

$$\begin{cases} (i\partial_t - \Delta^2)u(t,x) = h, & (t,x) \in [0,T] \times \mathbb{R}, \\ u(0) = u_0(x) \end{cases} \tag{2.6}$$

for some data u_0 and $T > 0$. Then we have the Strichartz estimates

$$\|u\|_{S(I\times\mathbb{R}))} \lesssim \|u_0\|_{L^2(\mathbb{R})} + \|h\|_{N(I\times\mathbb{R}))}, \tag{2.7}$$

and for $s \geq 0$,

$$\big\||\nabla|^s u\big\|_{L_t^\infty L_x^2(I\times\mathbb{R}))} \lesssim \|u_0\|_{\dot{H}^s(\mathbb{R})} + \big\||\nabla|^{s-1/2}h\big\|_{L_t^{4/3}L_x^1(I\times\mathbb{R})}. \tag{2.8}$$

Lemma 2.7 (Local smoothing estimate). *Suppose $u: \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is a solution to (2.6). Then, for every $T, R > 0$,*

$$\|u\|_{L^2_{t,x}([-T,T]\times[-R,R])} \lesssim R^{1/2}\big\{\big\||\nabla|^{-3/2}u(0)\big\|_{L_x^2} + \big\||\nabla|^{-3/2}h\big\|_{L_t^1L_x^2([0,T]\times\mathbb{R})}\big\}. \tag{2.9}$$

Proof. By the Duhamel formula (1.4), we are reduced to showing

$$\int_{\mathbb{R}}\int_{|x|\leq R} |e^{-it\Delta^2}u_0|^2\,dx\,dt \lesssim R\int_{\mathbb{R}} \big||\nabla|^{-3/2}u_0\big|^2\,dx. \tag{2.10}$$

Using Fourier transform and changing variables, we get

$$\begin{aligned} e^{-it\Delta^2}&u_0(x) \\ &= \int_{\mathbb{R}} e^{ix\xi}e^{-it|\xi|^4}\hat{u}_0(\xi)\,d\xi \\ &= \int_0^\infty e^{ix\xi}e^{-it|\xi|^4}\hat{u}_0(\xi)\,d\xi - \int_0^\infty e^{-ix\xi}e^{-it|\xi|^4}\hat{u}_0(-\xi)\,d\xi \\ &= \frac{1}{4}\int_0^\infty e^{-it\rho}[e^{ix\rho^{1/4}}\hat{u}_0(\rho^{1/4}) - e^{-ix\rho^{1/4}}\hat{u}_0(-\rho^{1/4})]\rho^{-3/4}\,d\rho \\ &= \frac{1}{4}\mathcal{F}_{\rho\mapsto t}\big[\chi_{[0,\infty)}(\rho)\big(e^{ix\rho^{1/4}}\hat{u}_0(\rho^{1/4}) - e^{-ix\rho^{1/4}}\hat{u}_0(-\rho^{1/4})\big)\rho^{-3/4}\big](t), \end{aligned}$$

which implies

$$\begin{aligned} \int_{\mathbb{R}} |e^{-it\Delta^2}u_0|^2\,dt &= c\int_0^\infty \big(e^{ix\rho^{1/4}}\hat{u}_0(\rho^{1/4}) - e^{-ix\rho^{1/4}}\hat{u}_0(-\rho^{1/4})\big)^2\rho^{-3/2}\,d\rho \\ &\lesssim \int_0^\infty \big(|\hat{u}_0(\rho^{1/4})|^2 + |\hat{u}_0(-\rho^{1/4})|^2\big)\rho^{-3/2}\,d\rho \\ &\lesssim \int_{\mathbb{R}} \big||\xi|^{-3/2}\hat{u}_0(\xi)\big|^2\,d\xi \lesssim \int_{\mathbb{R}} \big||\nabla|^{-3/2}u_0\big|^2\,dx, \end{aligned}$$

and so (2.10) follows. ■

As a consequence of the local smoothing estimate, we obtain an equicontinuity condition.

LEMMA 2.8. *Fix $T > 0$, and suppose $u : [-2T, 2T] \times \mathbb{R} \to \mathbb{C}$ satisfies*

$$\|u\|_{\tilde{S}} := \|u\|_{L_t^\infty L_x^2([-2T,2T]\times\mathbb{R})} + \|(i\partial_t - \Delta^2)u\|_{L_t^{8/p} L_x^2([-2T,2T]\times\mathbb{R})} < \infty. \tag{2.11}$$

Then

$$\|u(t+\tau, x+y) - u(t,x)\|_{L^2_{t,x}([-T,T]\times[-R,R])} \lesssim_{R,T} \left\{|\tau|^{3/11} + |\tau|^{1-p/8} + |y|^{3/5}\right\}\|u\|_{\tilde{S}}$$

uniformly for $|\tau| \le T$ and $y \in \mathbb{R}$.

Proof. By the triangle inequality, it suffices to prove

$$\|u(t, x+y) - u(t,x)\|_{L^2_{t,x}([-T,T]\times[-R,R])} \lesssim_{R,T} |y|^{3/5}\|u\|_{\tilde{S}} \tag{2.12}$$

and

$$\|u(t+\tau, x) - u(t,x)\|_{L^2_{t,x}([-T,T]\times[-R,R])} \lesssim_{R,T} \left(|\tau|^{3/11} + |\tau|^{1-p/8}\right)\|u\|_{\tilde{S}}. \tag{2.13}$$

First, we use the local smoothing estimate (2.9) and Bernstein's and Hölder's inequalities to estimate the high-frequency portion:

$$\begin{aligned} \|u_{>N}(t+\tau, x+y) - u_{>N}(t,x)\|_{L^2_{t,x}([-T,T]\times[-R,R])} &\lesssim R^{1/2}\big(\||\nabla|^{-3/2}u_{>N}\|_{L_t^\infty L_x^2([-2T,2T]\times\mathbb{R})} \\ &\quad + \big\||\nabla|^{-3/2}(i\partial_t - \Delta^2)u_{>N}\big\|_{L_t^1 L_x^2([-2T,2T]\times\mathbb{R})}\big) \\ &\lesssim R^{1/2}N^{-3/2}(1 + T^{1-p/8})\|u\|_{\tilde{S}}. \end{aligned} \tag{2.14}$$

Next we turn to the low-frequency contribution. Using the Bernstein and Hölder inequalities, we have

$$\begin{aligned} \|u_{\le N}(t, x+y) - u_{\le N}(t,x)\|_{L^2_{t,x}([-2T,2T]\times\mathbb{R})} &\lesssim NT^{1/2}|y|\, \|u(t)\|_{L_t^\infty L_x^2([-2T,2T]\times\mathbb{R})} \lesssim NT^{1/2}|y|\, \|u\|_{\tilde{S}}. \end{aligned}$$

Setting $N = |y|^{-2/5}$, and using (2.14), we obtain

$$\|u(t, x+y) - u(t,x)\|_{L^2_{t,x}([-T,T]\times[-R,R])} \lesssim (R^{1/2} + RT^{1/2}T^{1-p/8} + T^{1/2})|y|^{3/5}\|u\|_{\tilde{S}},$$

and so (2.12) follows.

Noting

$$\big\|P_{\le N}[e^{-i\tau\Delta^2} - 1]e^{-it\Delta^2}u(0)\big\|_{L_x^2(\mathbb{R})} \lesssim N^4|\tau|\, \|u(0)\|_{L_x^2(\mathbb{R})}$$

and the Duhamel formula

$$u(t) = e^{-it\Delta^2}u(0) - i\int_0^t e^{-i(t-s)\Delta^2}(i\partial_s - \Delta^2)u(s)\,ds$$

and using the Strichartz estimate, we get

$$\begin{aligned}
&\big\|u_{\le N}(t+\tau) - u_{\le N}(t)\big\|_{L^2_x(\mathbb{R})} \\
&\lesssim \big\|P_{\le N}[e^{-i\tau\Delta^2} - 1]e^{-it\Delta^2}u(0)\big\|_{L^2_x} + \Big\|\int_t^{t+\tau} e^{-i(t-s)\Delta^2}(i\partial_s - \Delta^2)u(s)\,ds\Big\|_{L^2_x} \\
&\lesssim N^4|\tau|\,\|u(0)\|_{L^2_x(\mathbb{R})} + \|(i\partial_t - \Delta^2)u\|_{L^1_t L^2_x([t,t+\tau]\times\mathbb{R})} \\
&\lesssim \{N^4|\tau| + |\tau|^{1-p/8}\}\|u\|_{\tilde{S}}.
\end{aligned}$$

Combining this with (2.14) and choosing $N = |\tau|^{-2/11}$ yields

$$\begin{aligned}
&\big\|u(t+\tau) - u(t)\big\|_{L^2_{t,x}([-T,T]\times[-R,R])} \\
&\qquad\lesssim (R^{1/2} + R^{1/2}T^{1-p/8} + T^{1/2})(|\tau|^{3/11} + |\tau|^{1-p/8})\|u\|_{\tilde{S}}.
\end{aligned}$$

This implies (2.13). ■

3. Well-posedness for fourth-order Schrödinger equations. In this section, we consider the well-posedness of the equation

$$i\partial_t u - \Delta^2 u = \mathcal{P}F(\mathcal{P}u), \tag{3.1}$$

where $F(u) = \pm|u|^p u$, and $\mathcal{P}$ is either the identity or the projection $P_{\le N}$ with $N \in 2^{\mathbb{Z}}$.

A simple computation shows that the solutions to (3.1) are unique and conserve both mass and energy:

$$\int_{\mathbb{R}} |u(t,x)|^2\,dx \quad \text{and} \quad E(u(t)) := \int_{\mathbb{R}} \frac{1}{2}|\Delta u(t,x)|^2 \pm \frac{1}{p+2}|\mathcal{P}u(t,x)|^{p+2}\,dx,$$

respectively. In fact, (3.1) is the Hamiltonian evolution associated to $E(u)$ via the standard symplectic structure:

$$\omega : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathbb{R} \quad \text{with} \quad \omega(u,v) = \operatorname{Im}\int_{\mathbb{R}} u(x)\bar{v}(x)\,dx.$$

LEMMA 3.1 (Well-posedness). *Let $1 < p < 8$, $u_0 \in L^2(\mathbb{R})$ with $\|u_0\|_2 \le M$. Assume that the operator $\mathcal{P}$ is L^r bounded for all $1 \le r \le \infty$. Then there exists a unique global solution $u : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ to (3.1) with initial data $u(0) = u_0$. Moreover, for any $T > 0$,*

$$\|u\|_{S([0,T]\times\mathbb{R})} \lesssim_T M. \tag{3.2}$$

If $u_0 \in H^k(\mathbb{R})$ *with* $k \in \{1,2,3\}$, *then*

$$\|\partial_x^k u\|_{S([0,T]\times\mathbb{R})} \lesssim_{M,T} \|\partial_x^k u_0\|_{L^2(\mathbb{R})} \tag{3.3}$$

when either $2 < p < \infty$, *or* $3/2 < p < 2$ *with* $k \in \{1,2\}$; *and for* $3/2 < p < 2$ *with* $k = 3$, *we have*

$$\|\langle\nabla\rangle^3 u\|_{L_t^\infty L_x^2([0,T]\times\mathbb{R})} \lesssim_{M,T} \|u_0\|_{H^3(\mathbb{R})}. \tag{3.4}$$

Proof. We apply a fixed point argument. First we define the map

$$\Phi(u(t)) = e^{-it\Delta^2}u_0 - i\int_0^t e^{-i(t-s)\Delta^2}\mathcal{P}F(\mathcal{P}u)\,ds \tag{3.5}$$

on the complete metric space

$$B := \{u \in C_t(I;L_x^2) : \|u\|_{S(I\times\mathbb{R})} \le 2C\|u_0\|_{L_x^2}\}$$

with the metric $d(u,v) = \|u-v\|_{L_t^8 L_x^\infty(I\times\mathbb{R})}$.

It suffices to prove that Φ is a contraction map on B. If $u \in B$, then by the Strichartz estimate, we have

$$\begin{aligned}\|\Phi(u)\|_{S(I\times\mathbb{R})} &\le C\|u_0\|_{L_x^2} + C\|\mathcal{P}F(\mathcal{P}u)\|_{L_t^1 L_x^2(I\times\mathbb{R})}\\ &\le C\|u_0\|_{L_x^2} + C|I|^{1-p/8}\|u\|^{p+1}_{L_t^{8(p+1)/p}L_x^{2(p+1)}(I\times\mathbb{R})}\\ &\le C\|u_0\|_{L_x^2} + C|I|^{1-p/8}\big(2C\|u_0\|_{L_x^2}\big)^{p+1} \le 2C\|u_0\|_{L_x^2}\end{aligned}$$

provided that

$$(2C)^{p+1}\|u_0\|^p_{L_x^2}|I|^{1-p/8} < 1.$$

On the other hand, by the same argument as before, for $u,v \in B$ we have

$$\begin{aligned}&d\big(\Phi(u),\Phi(v)\big)\\ &\quad\le C\|\mathcal{P}F(\mathcal{P}u) - \mathcal{P}F(\mathcal{P}v)\|_{L_t^1 L_x^2(I\times\mathbb{R})}\\ &\quad\le C|I|^{1-p/8}\|u-v\|_{L_t^8 L_x^\infty(I\times\mathbb{R})}\big(\|u\|_{L_t^{8p/p-1}L_x^{2p}(I\times\mathbb{R})} + \|v\|_{L_t^{8p/p-1}L_x^{2p}(I\times\mathbb{R})}\big)^p\\ &\quad\le C|I|^{1-p/8}(2C\|u_0\|_{L_x^2})^p d(u,v),\end{aligned}$$

which yileds

$$d\big(\Phi(u),\Phi(v)\big) \le \tfrac{1}{2}d(u,v),$$

on choosing $|I|$ so small that

$$C|I|^{1-p/8}(2C\|u_0\|_{L_x^2})^p \le 1/2.$$

A standard fixed point argument gives us a unique solution u of (3.1) on $I\times\mathbb{R}$ which satisfies the bound

$$\|u\|_{S(I\times\mathbb{R})} \le 2C\|u_0\|_{L_x^2} \lesssim M.$$

Since the time interval I only depends on the $L^2_x(\mathbb{R})$-norm of the initial data and on the Strichartz constants, we obtain global well-posedness by using mass conservation.

Next, we turn to the proof of (3.3). By (3.2) and interpolation, we have

$$\|u\|_{L_t^{8(p+1)/p}L_x^{2(p+1)}([0,T]\times\mathbb{R})} \lesssim M.$$

Let η be a small constant to be chosen later depending only on the Strichartz constants. Divide $[0,T]$ into $J = O\big(1+(M/\eta)^{8(p+1)/p}\big)$ subintervals $I_j = [t_{j-1}, t_j]$ such that

$$\|u\|_{L_t^{8(p+1)/p}L_x^{2(p+1)}(I_j\times\mathbb{R})} \le \eta.$$

On each subinterval we apply the Strichartz estimate and interpolation to obtain, for $p \ge 2$ or $3/2 < p < 2$ with $k \in \{1,2\}$,

$$\begin{aligned}\|\partial_x^k u\|_{S(I_j\times\mathbb{R})} &\lesssim \|u(t_{j-1})\|_{\dot H^k} + \|\partial_x^k \mathcal{P}F(\mathcal{P}u)\|_{L_t^1L_x^2(I_j\times\mathbb{R})}\\ &\lesssim \|u(t_{j-1})\|_{\dot H^k} + T^{1-p/8}\|u\|^p_{L_t^{8(p+1)/p}L_x^{2(p+1)}(I_j\times\mathbb{R})}\|\partial_x^k u\|_{L_t^{8(p+1)/p}L_x^{2(p+1)}(I_j\times\mathbb{R})}\\ &\lesssim \|u(t_{j-1})\|_{\dot H^k} + \eta^p T^{1-p/8}\|\partial_x^k u\|_{S(I_j\times\mathbb{R})},\end{aligned}$$

while for $3/2 < p < 2$ and $k = 3$, we have

$$\begin{aligned}\|\langle\nabla\rangle^3 u\|_{L_t^\infty L_x^2(I_j\times\mathbb{R})} &\lesssim \|u(t_{j-1})\|_{H^3} + \|\langle\nabla\rangle^{5/2}\mathcal{P}F(\mathcal{P}u)\|_{L_t^{4/3}L_x^1(I_j\times\mathbb{R})}\\ &\lesssim \|u(t_{j-1})\|_{H^3}\\ &\quad + T^{(7-p)/8}\|u\|^p_{L_t^{8p/(p-1)}L_x^{2p}(I_j\times\mathbb{R})}\|\langle\nabla\rangle^{5/2}u\|_{L_t^\infty L_x^2(I_j\times\mathbb{R})}\\ &\lesssim \|u(t_{j-1})\|_{H^3} + \eta T^{1-p/8}\|\langle\nabla\rangle^3 u\|_{L_t^\infty L_x^2(I_j\times\mathbb{R})}.\end{aligned}$$

Choosing η small enough we get, for $p \ge 2$ or $3/2 < p < 2$ with $k \in \{1,2\}$,

$$\|\partial_x^k u\|_{S(I_j\times\mathbb{R})} \lesssim \|u(t_{j-1})\|_{\dot H^k},$$

and for $3/2 < p < 2$ and $k = 3$,

$$\|\langle\nabla\rangle^3 u\|_{L_t^\infty L_x^2(I_j\times\mathbb{R})} \lesssim \|u(t_{j-1})\|_{H^3}.$$

Iterating this process J times we derive (3.3) and (3.4). ∎

4. Well-posedness in the weak topology. In this section, we will prove the following approximation in the weak topology.

THEOREM 4.1 (Approximation in the weak topology). *Let $N_n \to \infty$ and $u_n : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ be a sequence of solutions to the system*

$$\begin{cases} i\partial_t u_n - \Delta^2 u_n = P_{\le N_n}F(P_{\le N_n}u_n),\\ u_n(0,x) = u_n(0).\end{cases} \tag{4.1}$$

Assume that

$$u_n(0) \rightharpoonup u_{\infty,0} \quad \text{weakly in } L^2(\mathbb{R}) \tag{4.2}$$

and define u_∞ *to be the solution to* (1.1) *with* $u_\infty(0) = u_{\infty,0}$. *Then there exists a subsequence of* n *such that*

$$u_n(t) \rightharpoonup u_\infty(t) \quad \text{weakly in } L^2(\mathbb{R}) \tag{4.3}$$

for all $t \in \mathbb{R}$.

To do so, we need the following proposition.

PROPOSITION 4.2. *Let* $1 \le p < 8$ *and* $u_n : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ *be a sequence of solutions to* (4.1). *Assume that*

$$M := \sup_n \|u_n(0)\|_{L_x^2(\mathbb{R})} < \infty. \tag{4.4}$$

Then there exist a function $v : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ *and a subsequence of* n *such that*

$$\lim_{n\to\infty} \|u_n - v\|_{L_{t,x}^{p+2}([-T,T]\times[-R,R])} = 0 \tag{4.5}$$

for all $R, T > 0$.

Proof. By a diagonal argument, we only need to consider a single fixed pair $R, T > 0$. In what follows, the implicit constants will be permitted to depend on R, T, and M.

By (4.4) and Lemma 3.1, we get

$$\sup_n \|u_n\|_{S([-4T,4T]\times\mathbb{R})} \lesssim 1. \tag{4.6}$$

Consequently, if we define the smooth cutoff function $\chi : \mathbb{R}^2 \to [0,1]$ as

$$\chi(t,x) = \begin{cases} 1 & \text{if } |t| \le T \text{ and } |x| \le R, \\ 0 & \text{if } |t| > 2T \text{ or } |x| > 2R, \end{cases}$$

then Hölder's inequality yields

$$\|\chi u_n\|_{L_{t,x}^{p+2}(\mathbb{R}\times\mathbb{R})} \lesssim T^{\frac{8-p}{8(p+2)}} \|u_n\|_{L_t^{8(p+2)/p} L_x^{p+2}([-2T,2T]\times\mathbb{R})} \lesssim 1.$$

Thus, the sequence $\{\chi u_n\}$ is uniformly bounded in $L_{t,x}^{p+2}(\mathbb{R}\times\mathbb{R})$. Moreover, by Lemma 2.8 and (4.6),

$$\begin{aligned}
&\|\chi u_n(t+\tau, x+y) - \chi u_n(t,x)\|_{L_{t,x}^2(\mathbb{R}\times\mathbb{R})} \\
&\qquad \lesssim \|u_n(t+\tau, x+y) - u_n(t,x)\|_{L_{t,x}^2([-2T,2T]\times[-2R,2R])} \\
&\qquad \lesssim \{|\tau|^{3/11} + |\tau|^{1-p/8} + |y|^{3/5}\} \|u_n\|_{\tilde{S}} \\
&\qquad \lesssim \{|\tau|^{3/11} + |\tau|^{1-p/8} + |y|^{3/5}\} \\
&\qquad\qquad \times \left(\|u_n\|_{L_t^\infty L_x^2([-2T,2T]\times\mathbb{R})} + \||u_n|^p u_n\|_{L_t^{8/p} L_x^2([-2T,2T]\times\mathbb{R})} \right) \\
&\qquad \lesssim |\tau|^{3/11} + |\tau|^{1-p/8} + |y|^{3/5}.
\end{aligned}$$

Using interpolation and (4.6), we get

$$\begin{aligned}
&\|\chi u_n(t+\tau,x+y)-\chi u_n(t,x)\|_{L^{p+2}_{t,x}(\mathbb{R}\times\mathbb{R})}\\
&\quad\lesssim \|\chi u_n(t+\tau,x+y)-\chi u_n(t,x)\|^{\theta}_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R})}\|\chi u_n(t+\tau,x+y)-\chi u_n(t,x)\|^{1-\theta}_{L^2_{t,x}}\\
&\quad\lesssim \|u_n\|^{\theta}_{L^{10}_{t,x}([-3T,3T]\times\mathbb{R})}\big(|\tau|^{3/11}+|\tau|^{1-p/8}+|y|^{3/5}\big)^{1-\theta}\\
&\quad\lesssim \big(|\tau|^{3/11}+|\tau|^{1-p/8}+|y|^{3/5}\big)^{1-\theta},
\end{aligned}$$

with $\theta=\frac{5p}{4(p+2)}\in[0,1]$. Hence, the sequence $\{\chi u_n\}$ is also equicontinuous. Since χu_n is compactly supported, it is also tight. Thus, from Proposition 2.2, $\{\chi u_n\}$ is precompact in $L^{p+2}_{t,x}(\mathbb{R}\times\mathbb{R})$, and so there is a subsequence such that (4.5) holds. ∎

Proof of Theorem 4.1. We may restrict attention to a fixed time interval $[-T,T]$. Throughout the proof, all space-time norms will be taken over $[-T,T]\times\mathbb{R}$ unless explicitly stated otherwise. In what follows, the implicit constants will be permitted to depend on T and on the uniform bound on $\|u_n(0)\|_{L^2_x}$.

From the weak convergence of the initial data (4.2), we know that $\{u_n(0)\}$ is bounded in $L^2(\mathbb{R})$. Hence, by Lemma 3.1 and interpolation,

$$\sup_n(\|u_n\|_{L^\infty_t L^2_x}+\|u_n\|_{L_t^{8(p+1)/p}L_x^{2(p+1)}}+\|u_n\|_{L^{p+2}_{t,x}})\lesssim 1. \tag{4.7}$$

On the other hand, using Proposition 4.2, we may assume that (up to a subsequence) (4.5) holds for some v, all $R>0$, and our chosen T. This together with (4.7) and a duality argument yields

$$\|v\|_{L^\infty_t L^2_x}+\|v\|_{L^{p+2}_{t,x}}\lesssim 1. \tag{4.8}$$

Moreover, as $L^2(\mathbb{R})$ has a countable dense collection of C_c^∞ functions, (4.5) guarantees that

$$u_n(t)\rightharpoonup v(t)\quad \text{weakly in } L^2(\mathbb{R}) \text{ for all } t\in\Omega, \tag{4.9}$$

where $\Omega\subseteq[-T,T]$ is of full measure.

Now, we wish to pass to weak limits on both sides of the Duhamel formula

$$u_n(t)=e^{-it\Delta^2}u_n(0)-i\int_0^t e^{-i(t-s)\Delta^2}P_{\le N_n}F\big(P_{\le N_n}u_n(s)\big)\,ds.$$

It is easy to see that $e^{-it\Delta^2}u_n(0)\rightharpoonup e^{-it\Delta^2}u_{\infty,0}$ weakly in $L^2(\mathbb{R})$. Indeed, for any $\varphi\in L^2(\mathbb{R})$, by (4.2) we have

$$\langle\varphi,e^{-it\Delta^2}u_n(0)-e^{-it\Delta^2}u_{\infty,0}\rangle=\langle e^{it\Delta^2}\varphi,u_n(0)-u_{\infty,0}\rangle\to 0$$

as $n\to\infty$.

Next, we claim that

$$\operatorname*{w-lim}_{n\to\infty}\int_0^t e^{-i(t-s)\Delta^2}P_{\leq N_n}F\big(P_{\leq N_n}u_n(s)\big)\,ds=\int_0^t e^{-i(t-s)\Delta^2}F(v(s))\,ds, \tag{4.10}$$

where the weak limit is in the $L^2(\mathbb{R})$ topology. Before we prove this claim, let us use it to complete the proof of Theorem 4.1. Collecting the above estimates yields

$$\operatorname*{w-lim}_{n\to\infty}u_n(t)=e^{-it\Delta^2}u_\infty(0)-i\int_0^t e^{-i(t-s)\Delta^2}F(v(s))\,ds. \tag{4.11}$$

The right-hand side of (4.11) is continuous in t, with values in $L^2(\mathbb{R})$, and the left-hand side agrees with $v(t)$ for almost every t by (4.9). Consequently, after altering v on a space-time set of measure zero, we obtain a $v\in C([-T,T];L^2(\mathbb{R}))$ that still obeys (4.8) but now also satisfies

$$v(t)=e^{-it\Delta^2}u_\infty(0)-i\int_0^t e^{-i(t-s)\Delta^2}F(v(s))\,ds\quad\text{and}\quad\operatorname*{w-lim}_{n\to\infty}u_n(t)=v(t) \tag{4.12}$$

for all $t\in[-T,T]$. By the definition of the strong solution and Lemma 3.1, we deduce that $v=u_\infty$ on $[-T,T]$ and (4.3) holds for $t\in[-T,T]$.

Finally, it remains to show the claim (4.10). Fix $\psi\in L^2(\mathbb{R})$ and decompose

$$\Big\langle\psi,\int_0^t e^{-i(t-s)\Delta^2}\big[F(v(s))-P_{\leq N_n}F\big(P_{\leq N_n}u_n(s)\big)\big]\,ds\Big\rangle$$

$$=\Big\langle\psi,\int_0^t e^{-i(t-s)\Delta^2}\big[F\big(P_{\leq N_n}u_n(s)\big)-P_{\leq N_n}F\big(P_{\leq N_n}u_n(s)\big)\big]\,ds\Big\rangle \tag{4.13}$$

$$-\Big\langle\psi,\int_0^t e^{-i(t-s)\Delta^2}\chi_R^c\big[F\big(P_{\leq N_n}u_n(s)\big)-F(v(s))\big]\,ds\Big\rangle \tag{4.14}$$

$$-\Big\langle\psi,\int_0^t e^{-i(t-s)\Delta^2}\chi_R\big[F\big(P_{\leq N_n}u_n(s)\big)-F(v(s))\big]\,ds\Big\rangle, \tag{4.15}$$

where χ_R denotes the indicator function of $[-R,R]$ and χ_R^c denotes the indicator of the complementary set.

By Hölder's inequality, (4.7), and the Dominated Convergence Theorem,

$$\begin{aligned}|(4.13)|&\leq\int_0^t|\langle e^{i(t-s)\Delta^2}P_{>N_n}\psi,F(P_{\leq N_n}u_n(s))\rangle|\,ds\\&\lesssim\sqrt{T}\,\|P_{>N_n}\psi\|_{L_x^2}\|u_n\|^{p+1}_{L_t^{8(p+1)/p}L_x^{2(p+1)}}\to0\quad\text{as }n\to\infty.\end{aligned}$$

Similarly,

$$\sup_n |(4.14)| \lesssim T^{\frac{8-p}{8(p+2)}} \|\chi_R^c e^{-it\Delta^2}\psi\|_{L_t^{8(p+2)/p} L_x^{p+2}} \Big[\|v\|_{L_{t,x}^{p+2}}^{p+1} + \sup_n \|u_n\|_{L_{t,x}^{p+2}}^{p+1}\Big] \to 0 \quad \text{as } R \to \infty.$$

An easy application of Schur's test shows that

$$\|\chi_R P_{\le N_n} \chi_{2R}^c f\|_{L^p(\mathbb{R})} \lesssim_\beta (N_n R)^{-\beta} \|f\|_{L^p(\mathbb{R})}$$

for any $1 \le p \le \infty$ and any $\beta > 0$. Hence,

$$\|\chi_R P_{\le N_n}(u_n - v)\|_{L_{t,x}^{p+2}} \lesssim \big\|\chi_{2R}(u_n - v)\big\|_{L_{t,x}^{p+2}} + (N_n R)^{-\beta}\{\|u_n\|_{L_{t,x}^{p+2}} + \|v\|_{L_{t,x}^{p+2}}\}.$$

Using this estimate together with (4.7), (4.8), (4.5), and the fact that $N_n \to \infty$, we deduce that

$$|(4.15)| \lesssim T^{\frac{8-p}{8(p+2)}} \|\psi\|_{L_x^2} \|\chi_R P_{\le N_n}(u_n - v)\big\|_{L_{t,x}^{p+2}} \big(\|v\|_{L_{t,x}^{p+2}} + \|u_n\|_{L_{t,x}^{p+2}}\big)^p \to 0 \quad \text{as } n \to \infty.$$

Thus (4.10) follows, completing the proof of Theorem 4.1. ∎

5. Finite-dimensional approximation. In this section, we will utilize the stability result to show that for n sufficiently large, solutions to the finite-dimensional Hamiltonian systems

$$\begin{cases} (i\partial_t - \Delta^2)u_n = P_{\le N_n}^{L_n} F(P_{\le N_n}^{L_n} u_n), & (t,x) \in \mathbb{R} \times \mathbb{T}_n, \\ u_n(0) = u_{0,n} \end{cases} \tag{5.1}$$

can be well approximated by solutions to the corresponding systems posed on $\mathbb{R}$ on the fixed time interval $[-T,T]$. To this end, we first establish the Strichartz estimates and stability theory on the torus.

5.1. Strichartz estimates and stability on the torus

PROPOSITION 5.1 (Local-in-time dispersive and Strichartz estimates). *Given $T > 0$ and $1 \le N \in 2^{\mathbb{Z}}$, there exists $L_0 = L_0(T,N) \ge 1$ sufficiently large such that for $L \ge L_0$,*

$$\|e^{-it\Delta^2} P_{\le N}^L f\|_{L_x^\infty(\mathbb{T}_L)} \lesssim |t|^{-1/4} \|f\|_{L_x^1(\mathbb{T}_L)} \quad \text{uniformly for } t \in [-T,T] \setminus \{0\}, \tag{5.2}$$

and

$$\|e^{-it\Delta^2} P_{\le N}^L f\|_{L_t^q L_x^r([-T,T]\times\mathbb{T}_L)} \lesssim_{p,q} \|f\|_{L_x^2(\mathbb{T}_L)}. \tag{5.3}$$

Here, $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$, (q,r) is a biharmonic admissible pair, in the sense that

$$4/q = 1/2 - 1/r \quad \text{with } 8 < q \le \infty,$$

and $P_{\le N}^L$ denotes the Fourier multiplier $P_{\le N}$ on $\mathbb{T}_L$.

Proof. The proof of [10, Proposition 7.2] can be adopted verbatim, but we give a sketch for completeness. It suffices to prove (5.2), since (5.3) follows from the usual TT^* argument.

Observing that

$$e^{-it\Delta^2}P^L_{\le N}f(x) = \frac{1}{L}\sum_{n\in\mathbb{Z}}\int_{\mathbb{T}_L} f(y)e^{i(2\pi(x-y)\cdot n/L-16\pi^4 t|n/L|^4)}\varphi\left(\frac{n}{NL}\right)dy,$$

we derive that the convolution kernel associated to the operator $e^{-it\Delta^2}P^L_{\le N}$ is

$$k(x) := \frac{1}{L}\sum_{n\in\mathbb{Z}} e^{i\Phi(n)}\varphi\left(\frac{n}{NL}\right) \quad \text{with } \Phi(n) := 2\pi x\cdot\frac{n}{L} - 16\pi^4 t\left|\frac{n}{L}\right|^4.$$

We are reduced to proving

$$|k(x)| \lesssim |t|^{-1/4} \tag{5.4}$$

uniformly for $0<|t|\le T$ and $x\in[-L/2,L/2]$. To do so, we consider two cases: $|x|\le R$ and $|x|>R$ with $R=C(1+N^3T)$.

Case 1: $|x|\le R$. We write

$$\begin{aligned} k(x) &= \sum_{n\in\mathbb{Z}}\left(\frac{1}{L}e^{2\pi i x\cdot n/L-16\pi^4 it(n/L)^4}\varphi\left(\frac{n}{NL}\right) - \int_{Q_n} e^{2\pi i x\cdot\xi-16\pi^4 it|\xi|^4}\varphi\left(\frac{\xi}{N}\right)d\xi\right) && (5.5)\\ &\quad + \int_{\mathbb{R}} e^{2\pi i x\cdot\xi-16\pi^4 it|\xi|^4}\varphi\left(\frac{\xi}{N}\right)d\xi, && (5.6)\end{aligned}$$

where $Q_n := [n/L-1/(2L), n/L+1/(2L)]$. By Lemma 2.4, we easily get

$$|(5.6)| \lesssim |t|^{-1/4}.$$

Applying Lemma 2.1, we have

$$|(5.5)| \lesssim L^{-3}\sum_{|n|\lesssim NL}(R^2+N^6T^2+N^2T+N^{-2}) \lesssim L^{-3}R^2NL \lesssim T^{-1/4},$$

provided we choose $L\gg\sqrt{N}\,T^{1/8}(1+N^3T)$. Hence, we obtain (5.4) in this case.

Case 2: $|x|>R$ *and* $x\in[-L/2,L/2]$. Using the identity

$$e^{i\Phi(n)} = \frac{e^{i\Phi(n+1)}-e^{i\Phi(n)}}{e^{i[\Phi(n+1)-\Phi(n)]}-1} =: \frac{e^{i\Phi(n+1)}-e^{i\Phi(n)}}{\Psi(n)}$$

and applying a change of variables, we can write

$$k(x) = \frac{1}{L}\sum_{n\in\mathbb{Z}} \frac{e^{i\Phi(n+1)} - e^{i\Phi(n)}}{\Psi(n)}\varphi\Big(\frac{n}{NL}\Big)$$

$$= \frac{1}{L}\sum_{n\in\mathbb{Z}} e^{i\Phi(n)}\bigg[\frac{\varphi(\frac{n-1}{NL})}{\Psi(n-1)} - \frac{\varphi(\frac{n}{NL})}{\Psi(n)}\bigg]$$

$$= \frac{1}{L}\sum_{n\in\mathbb{Z}} e^{i\Phi(n)}\frac{1}{\Psi(n-1)}\bigg[\varphi\Big(\frac{n-1}{NL}\Big) - \varphi\Big(\frac{n}{NL}\Big)\bigg] \tag{5.7}$$

$$+ \frac{1}{L}\sum_{n\in\mathbb{Z}} e^{i\Phi(n)}\varphi\Big(\frac{n}{NL}\Big)\bigg[\frac{1}{\Psi(n-1)} - \frac{1}{\Psi(n)}\bigg]. \tag{5.8}$$

From

$$\Phi(n+1) - \Phi(n) = 2\pi x/L - 16\pi^4 t((n+1)^4 - n^4)/L$$

and the definition of R, we have

$$|\Phi(n+1) - \Phi(n)| \geq \sqrt{2}\,\pi R/L - O(N^3T/L) \geq R/L$$

provided $L \gg 1$. This also implies

$$|\Psi(n-1)|^{-1} + |\Psi(n)|^{-1} = |e^{i[\Phi(n)-\Phi(n-1)]} - 1|^{-1} + |e^{i[\Phi(n+1)-\Phi(n)]} - 1|^{-1} \lesssim L/R. \tag{5.9}$$

From the Fundamental Theorem of Calculus we have

$$\bigg|\varphi\Big(\frac{n-1}{NL}\Big) - \varphi\Big(\frac{n}{NL}\Big)\bigg| \lesssim \frac{1}{NL},$$

$$\bigg|\frac{1}{\Psi(n-1)} - \frac{1}{\Psi(n)}\bigg| \leq \frac{|\Phi(n+1) - 2\Phi(n) + \Phi(n-1)|}{|\Psi(n)\Psi(n-1)|} \lesssim \frac{N^3T}{L^2R^2}.$$

Inserting these estimates into (5.7) and (5.8), we obtain

$$|k(x)| \lesssim \frac{1}{L}NL\bigg[\frac{L}{R}\,\frac{1}{NL} + \frac{N^3T}{L^2R^2}\bigg] \lesssim \frac{1}{R} + \frac{N^4T}{L^2R^2} \lesssim T^{-1/4}.$$

This completes the proof of Proposition 5.1. ∎

As a consequence of the Strichartz estimates, we can obtain a stability theory for the following frequency-localized NLS on the torus $\mathbb{T}_L$:

$$\begin{cases} (i\partial_t - \Delta^2)u = P^L_{\leq N}F(P^L_{\leq N}u), & (t,x) \in \mathbb{R}\times\mathbb{T}_L, \\ u(0) = P^L_{\leq N}u_0. \end{cases} \tag{5.10}$$

Lemma 5.2 (Stability). *Fix $T > 0$ and $1 \leq N \in 2^{\mathbb{Z}}$. Let L_0 be as in Proposition* 5.1. *Given $L \geq L_0$, let $\tilde{u}$ be an approximate solution to* (5.10) *on* $[-T,T]$ *in the sense that*

$$\begin{cases} (i\partial_t - \Delta^2)\tilde{u} = P^L_{\le N} F(P^L_{\le N}\tilde{u}) + e, \\ \tilde{u}(0) = P^L_{\le N}\tilde{u}_0 \end{cases}$$

for some function e *and* $\tilde{u}_0 \in L^2(\mathbb{T}_L)$*. Assume that* $\|\tilde{u}\|_{S([-T,T]\times\mathbb{T}_L)} \le M$ *and*

$$\|u_0 - \tilde{u}_0\|_{L^2(\mathbb{T}_L)} \le \epsilon \quad \text{and} \quad \|e\|_{N([-T,T]\times\mathbb{T}_L)} \le \epsilon. \tag{5.11}$$

Then if $\epsilon \le \epsilon_0(M,T)$*, there exists a unique solution* u *to* (5.10) *such that*

$$\|u - \tilde{u}\|_{S([-T,T]\times\mathbb{T}_L)} \le C(M,T)\epsilon.$$

Proof. Since $\|\tilde{u}\|_{S([-T,T]\times\mathbb{T}_L)} \le M$, we may subdivide $[-T,T]$ into $C(M,\varepsilon_0)$ time intervals $I_j = [t_j, t_{j+1}]$ such that

$$\|\tilde{u}\|_{L_t^{8(p+1)/p} L_x^{2(p+1)}(I_j\times\mathbb{T}_L)} \le \epsilon_0 \ll 1, \quad 1 \le j \le C(M,\varepsilon_0).$$

Let $\gamma(t) = u(t) - \tilde{u}(t)$, and

$$\gamma_j(t) = e^{-i(t-t_j)\Delta^2}\gamma(t_j), \quad 1 \le j \le C(M,\varepsilon_0) - 1.$$

Then γ satisfies the differential equation

$$\begin{cases} (i\partial_t - \Delta^2)\gamma = P^L_{\le N} F\big(P^L_{\le N}(\tilde{u}+\gamma)\big) - P^L_{\le N} F(P^L_{\le N}\tilde{u}) - e, \\ \gamma(t_j) = \gamma_j(t_j), \end{cases}$$

which implies that

$$\gamma(t) = \gamma_j(t) - i\int_{t_j}^{t} e^{-i(t-s)\Delta^2}\big(P^L_{\le N}F(P^L_{\le N}(\tilde{u}+\gamma)) - P^L_{\le N}F(P^L_{\le N}\tilde{u}) - e\big)\,ds,$$

$$\gamma_{j+1}(t) = \gamma_j(t) - i\int_{t_j}^{t_{j+1}} e^{-i(t-s)\Delta^2}\big(P^L_{\le N}F(P^L_{\le N}(\tilde{u}+\gamma)) - P^L_{\le N}F(P^L_{\le N}\tilde{u}) - e\big)\,ds.$$

By Proposition 5.1, we have

$$\begin{aligned} &\|\gamma - \gamma_j\|_{S(I_j\times\mathbb{T}_L)} + \|\gamma_{j+1} - \gamma_j\|_{S([-T,T]\times\mathbb{T}_L)} \\ &\lesssim \|P^L_{\le N}F\big(P^L_{\le N}(\tilde{u}+\gamma)\big) - P^L_{\le N}F(P^L_{\le N}\tilde{u})\|_{N(I_j\times\mathbb{T}_L)} + \|e\|_{N([-T,T]\times\mathbb{T}_L)} \\ &\lesssim \epsilon + T^{1-8/p}\|\gamma\|^{p+1}_{L_t^{8(p+1)/p}L_x^{2(p+1)}(I_j\times\mathbb{T}_L)} \\ &\quad + T^{1-8/p}\|\gamma\|_{L_t^{8(p+1)/p}L_x^{2(p+1)}(I_j\times\mathbb{T}_L)}\|\tilde{u}\|^p_{L_t^{8(p+1)/p}L_x^{2(p+1)}(I_j\times\mathbb{T}_L)} \\ &\lesssim \epsilon + T^{1-8/p}\|\gamma\|^{p+1}_{S(I_j\times\mathbb{T}_L)} + T^{1-8/p}\epsilon_0^p\|\gamma\|_{S(I_j\times\mathbb{T}_L)}. \end{aligned} \tag{5.12}$$

Therefore, assuming that

$$\|\gamma\|_{S(I_j\times\mathbb{T}_L)} \le 2C\epsilon, \quad \forall j = 1,\dots,C(M,\varepsilon_0) - 1, \tag{5.13}$$

by (5.12), we have

$$\|\gamma\|_{S(I_j\times\mathbb{T}_L)} + \|\gamma_{j+1}\|_{S((t_{j+1},t_N)\times\mathbb{T}_L)} \le C\|\gamma_j\|_{S((t_j,t_N)\times\mathbb{T}_L)} + C\epsilon \tag{5.14}$$

for some absolute constant $C > 0$. By (5.11) and iteration on j, we obtain

$$\|\gamma\|_{S(I\times\mathbb{T}_L)} \le C\epsilon. \tag{5.15}$$

Hence (5.13) is justified by continuity in t and induction on j. ∎

5.2. Approximation by finite-dimensional PDE. Fix $M, T > 0$ and $\eta_n \to 0$. Let $N_n \to \infty$ be given and let $L_n = L_n(M, T, N_n, \eta_n)$ be large constants to be chosen later; in particular, we will have $L_n \to \infty$. Let $\mathbb{T}_n := \mathbb{R}/L_n\mathbb{Z}$ and let

$$\begin{aligned} &u_{0,n} \in \mathcal{H}_n := \big\{f \in L^2(\mathbb{T}_n) : P^{L_n}_{>2N_n} f = 0\big\},\\ &\|\partial_x^k u_{0,n}\|_{L^2(\mathbb{T}_n)} \le M N_n^k \end{aligned} \tag{5.16}$$

for $k \in \{0, 1, 2, 3\}$. We consider the finite-dimensional Hamiltonian systems

$$\begin{cases} (i\partial_t - \Delta^2)u_n = P^{L_n}_{\le N_n} F(P^{L_n}_{\le N_n} u_n), & (t, x) \in \mathbb{R}\times\mathbb{T}_n,\\ u_n(0) = u_{0,n}. \end{cases} \tag{5.17}$$

By Lemma 5.2 and mass conservation, we know that (5.17) has a global solution.

Taking L_n such that

$$L_n \gg \frac{M^2}{\eta_n}\cdot\frac{1}{\eta_n}N_n^3 T, \tag{5.18}$$

we can subdivide the interval $[L_n/4, L_n/2]$ into at least $16M^2/\eta_n$ subintervals of length $(20/\eta_n)N_n^3 T$. By the pigeonhole principle and (5.16), there exists a subinterval, which we denote by

$$I_n := [c_n - (10/\eta_n)N_n^3 T, c_n + (10/\eta_n)N_n^3 T],$$

such that

$$\|u_{0,n}\chi_{I_n}\|_{L^2_x} \le \tfrac14\sqrt{\eta_n}. \tag{5.19}$$

For $0 \le j \le 4$, we choose a smooth cutoff function $\chi_n^j : \mathbb{R} \to [0, 1]$ with

$$\chi_n^j(x) = \begin{cases} 1, & x \in \big[c_n - L_n + \frac{10-2j}{\eta_n}N_n^3 T, c_n - \frac{10-2j}{\eta_n}N_n^3 T\big],\\ 0, & x \in \big(-\infty, c_n - L_n + \frac{10-2j-1}{\eta_n}N_n^3 T\big) \cup \big(c_n - \frac{10-2j-1}{\eta_n}N_n^3 T, \infty\big). \end{cases}$$

Thus,

$$\operatorname{supp}\chi_n^j(x) \subset \bigg[c_n - L_n + \frac{10-2j-1}{\eta_n}N_n^3 T, c_n - \frac{10-2j-1}{\eta_n}N_n^3 T\bigg].$$

For $0 \le i < j \le 4$, we easily deduce from the definition that $\chi_n^i \equiv 1$ on $\operatorname{supp} \chi_n^j$, and so

$$\chi_n^j \chi_n^i = \chi_n^j. \tag{5.20}$$

Similarly,

$$\|\partial_x^k \chi_n^j\|_{L^\infty} \lesssim \big((N_n^3 T)^{-k} \eta_n^k\big) \quad \text{for each } k \ge 0. \tag{5.21}$$

Noting that the periodic property and (5.19) give

$$\begin{aligned}
\|u_{0,n} - \chi_n^j u_{0,n}\|_{L^2(\mathbb{T}_n)} &= \|u_{0,n} - \chi_n^j u_{0,n}\|_{L^2([c_n - L_n + \frac{10-2j-1}{\eta_n} N_n^3 T, c_n - \frac{10-2j-1}{\eta_n} N_n^3 T])} \\
&\lesssim \|u_{0,n}\|_{L^2([c_n - L_n + \frac{10-2j-1}{\eta_n} N_n^3 T, c_n - L_n + \frac{10-2j}{\eta_n} N_n^3 T])} \\
&\quad + \|u_{0,n}\|_{L^2([c_n - \frac{10-2j}{\eta_n} N_n^3 T, c_n - \frac{10-2j-1}{\eta_n} N_n^3 T])} \\
&\lesssim \|u_{0,n}\|_{L^2(I_n)} \lesssim \sqrt{\eta_n},
\end{aligned}$$

we obtain

$$\|(1 - \chi_n^j) u_{0,n}\|_{L^2(\mathbb{T}_n)} \lesssim \sqrt{\eta_n} \quad \text{for all } 0 \le j \le 4. \tag{5.22}$$

To handle the frequency truncations appearing in (5.17) and in (5.27), we recall the control interactions between these cutoffs and Littlewood–Paley operators.

Lemma 5.3 (Littlewood–Paley estimates, cf. [10, 11]). *For L_n sufficiently large and all $0 \le j \le 4$,*

$$\|\chi_n^j (P_{\le N_n} - P_{\le N_n}^{L_n}) \chi_n^j\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \lesssim N_n^{-1}, \tag{5.23}$$

$$\|[\chi_n^j, P_{\le N_n}^{L_n}]\|_{L^2(\mathbb{T}_n) \to L^2(\mathbb{T}_n)} + \|[\chi_n^j, P_{\le N_n}]\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \lesssim N_n^{-4}, \tag{5.24}$$

$$\|[P_{\le N_n}, (1 - \chi_n^j)^2]\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \lesssim N_n^{-4}, \tag{5.25}$$

as $n \to \infty$. Moreover, if $i > j$ then

(5.26)

$$\|\chi_n^j P_{\le N_n}^{L_n} (1 - \chi_n^i)\|_{L^2(\mathbb{T}_n) \to L^2(\mathbb{T}_n)} + \|\chi_n^j P_{\le N_n} (1 - \chi_n^i)\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \lesssim N_n^{-4}.$$

Now let $\tilde{u}_n$ denote the solution to

$$\begin{cases} (i\partial_t - \Delta^2) \tilde{u}_n = P_{\le N_n} F(P_{\le N_n} \tilde{u}_n), & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ \tilde{u}_n(0,x) = \chi_n^0(x) u_{0,n}(x + L_n \mathbb{Z}), \end{cases} \tag{5.27}$$

where $u_{0,n} \in L^2(\mathbb{T}_n)$ is as in (5.16). From Lemma 3.1, we know that these solutions are global and satisfy

$$\|\partial_x^k \tilde{u}_n\|_{S([-T,T] \times \mathbb{R})} \lesssim_T M N_n^k \tag{5.28}$$

uniformly in n and $k \in \{0, 1, 2, 3\}$.

Lemma 5.4 (Mass localization for $\tilde{u}_n$). *Let $\tilde{u}_n$ be the solution to* (5.27) *as above. Then for every* $0 \le j \le 4$,

$$\|(1-\chi_n^j)\tilde{u}_n\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})} = o(1) \quad \text{as } n \to \infty.$$

Proof. We need only present the proof for $j = 1$. To this end, we define

$$\mathcal{M}(t) := \int_{\mathbb{R}} |1-\chi_n^1(x)|^2 |\tilde{u}_n(t,x)|^2 \, dx.$$

By (5.20), we get $\mathcal{M}(0) = 0$. Moreover,

$$\frac{d}{dt}\mathcal{M}(t) = 2\,\mathrm{Re}\int_{\mathbb{R}^2} |1-\chi_n^1|^2 \overline{\tilde{u}_n}\partial_t \tilde{u}_n \, dx$$

$$= 2\,\mathrm{Im}\int_{\mathbb{R}} |1-\chi_n^1|^2 \overline{\tilde{u}_n}\Delta^2 \tilde{u}_n \, dx \tag{5.29}$$

$$+ 2\,\mathrm{Im}\int_{\mathbb{R}} |1-\chi_n^1|^2 \overline{\tilde{u}_n} P_{\le N_n} F(P_{\le N_n}\tilde{u}_n) \, dx. \tag{5.30}$$

Integrating by parts, we see that

$$(5.29) = 4\,\mathrm{Im}\int_{\mathbb{R}} (1-\chi_n^1)\partial_x\chi_n^1 \overline{\tilde{u}_n}\partial_x^3\tilde{u}_n \, dx - 4\,\mathrm{Im}\int_{\mathbb{R}} (1-\chi_n^1)\partial_x\chi_n^1 \partial_x\overline{\tilde{u}_n}\partial_x^2\tilde{u}_n \, dx.$$

This together with(5.28) yields

$$\begin{aligned}|(5.29)| &\lesssim \|(1-\chi_n^1)\overline{\tilde{u}_n}\|_{L_x^2}\|\partial_x\chi_n^1\|_{L^\infty}\|\partial_x^3\tilde{u}_n\|_{L_x^2} + \|\partial_x\chi_n^1\|_{L^\infty}\|\partial_x\tilde{u}_n\|_{L_x^2}\|\partial_x^2\tilde{u}_n\|_{L_x^2}\\ &\le C(M)(\eta_n/T)\mathcal{M}^{1/2}(t) + C(M)\eta_n.\end{aligned}$$

Next we estimate (5.30). We write

$$\begin{aligned}(5.30) &= 2\,\mathrm{Im}\int_{\mathbb{R}} F(P_{\le N_n}\tilde{u}_n)P_{\le N_n}(|1-\chi_n^1|^2\overline{\tilde{u}_n})\, dx\\ &= 2\,\mathrm{Im}\int_{\mathbb{R}} F(P_{\le N_n}\tilde{u}_n)[P_{\le N_n},(1-\chi_n^1)^2]\overline{\tilde{u}_n}\, dx\\ &\quad + 2\,\mathrm{Im}\int_{\mathbb{R}} F(P_{\le N_n}\tilde{u}_n)(1-\chi_n^1)^2 P_{\le N_n}\overline{\tilde{u}_n}\, dx\\ &= 2\,\mathrm{Im}\int_{\mathbb{R}} F(P_{\le N_n}\tilde{u}_n)[P_{\le N_n},(1-\chi_n^1)^2]\overline{\tilde{u}_n}\, dx.\end{aligned}$$

Using the Hölder inequality and (5.25), we estimate

$$\begin{aligned}|(5.30)| &\lesssim \|[P_{\le N_n},(1-\chi_n^1)^2]\tilde{u}_n\|_2\|F(P_{\le N_n}\tilde{u}_n)\|_2\\ &\le \|[P_{\le N_n},(1-\chi_n^1)^2]\|_{L^2\to L^2}\|\tilde{u}_n\|_2\|P_{\le N_n}\tilde{u}_n\|_{L_x^{2(p+1)}}^{p+1}\\ &\lesssim N_n^{-4}\|\tilde{u}_n\|_2\|P_{\le N_n}u_n\|_{\dot{H}^{p/(2(p+1))}}^{p+1} \le C(M)N_n^{-4}N_n^{p/2}\\ &\le C(M,T)N_n^{-(8-p)/2}.\end{aligned}$$

Putting the above estimates together we get

$$\frac{d}{dt}\mathcal{M}(t) \le C(M,T)\eta_n \mathcal{M}^{1/2}(t) + C(M,T)\eta_n + C(M,T)N_n^{-(8-p)/2}.$$

This combined with $\mathcal{M}(0) = 0$ implies

$$\mathcal{M}(t) \lesssim_{M,T} \eta_n + N_n^{-(8-p)/2}, \qquad \forall t \in [-T,T].$$

Thus, we complete the proof of the lemma. ∎

With these preliminaries in hand, we now turn to the main goal of this section.

THEOREM 5.5 (Approximation). *Fix $M > 0$ and $T > 0$. Let $N_n \to \infty$ and let L_n be sufficiently large depending on M, T, N_n. Assume $u_{0,n} \in \mathcal{H}_n$ with $\|u_{0,n}\|_{L^2(\mathbb{T}_n)} \le M$. Let u_n and $\tilde{u}_n$ be the solutions to* (5.17) *and* (5.27), *respectively. Then*

$$\lim_{n\to\infty} \|P^{L_n}_{\le 2N_n}(\chi_n^2 \tilde{u}_n) - u_n\|_{S([-T,T]\times\mathbb{T}_n)} = 0. \tag{5.31}$$

Proof. Denote

$$z_n := P^{L_n}_{\le 2N_n}(\chi_n^2 \tilde{u}_n).$$

We will deduce (5.31) by the stability result (Lemma 5.2). We are reduced to verifying:

$$\sup_n \|z_n\|_{S([-T,T]\times\mathbb{T}_n)} \lesssim M, \tag{5.32}$$

$$\lim_{n\to\infty} \|z_n(0) - u_n(0)\|_{L^2(\mathbb{T}_n)} = 0, \tag{5.33}$$

$$\lim_{n\to\infty} \|(i\partial_t - \Delta^2)z_n - P^{L_n}_{\le N_n}F(P^{L_n}_{\le N_n}z_n)\|_{N([-T,T]\times\mathbb{T}_n)} = 0. \tag{5.34}$$

Applying Lemma 3.1, we easily obtain

$$\|z_n\|_{S([-T,T]\times\mathbb{T}_n)} \lesssim \|\tilde{u}_n\|_{S([-T,T]\times\mathbb{R})} \lesssim C(M,T),$$

and so (5.32) follows.

From $u_{0,n} \in \mathcal{H}_n$ and (5.22), we have

$$\begin{aligned}\|z_n(0) - u_n(0)\|_{L^2(\mathbb{T}_n)} &= \|P^{L_n}_{\le 2N_n}(\chi_n^2 u_{0,n} - u_{0,n})\|_{L^2(\mathbb{T}_n)} \\ &\lesssim \|(1-\chi_n^2)u_{0,n}\|_{L^2(\mathbb{T}_n)} \lesssim \sqrt{\eta_n}.\end{aligned}$$

This implies (5.33).

It remains to verify (5.34). A simple computation shows

$$(i\partial_t - \Delta^2)z_n - P^{L_n}_{\le N_n}F(P^{L_n}_{\le 2N_n}z_n)$$

$$= P^{L_n}_{\le 2N_n}[\chi_n^2\partial_x^4\tilde{u}_n - \partial_x^4(\chi_n^2\tilde{u}_n)] \tag{5.35}$$

$$+ P^{L_n}_{\le 2N_n}[\chi_n^2 P_{\le N_n}F(P_{\le N_n}\tilde{u}_n) - P^{L_n}_{\le N_n}F(P^{L_n}_{\le N_n}(\chi_n^2\tilde{u}_n))]. \tag{5.36}$$

In view of the boundedness of $P^{L_n}_{\le 2N_n}$, it suffices to show that the terms in square brackets converge to zero in $N([-T,T]\times\mathbb{T}_n)$ as $n\to\infty$.

Noting that

$$\partial_x^4(\chi_n^2\tilde u_n)-\chi_n^2\partial_x^4\tilde u_n=(\partial_x^4\chi_n^2)\tilde u_n+4(\partial_x^3\chi_n^2)\partial_x\tilde u_n+6(\partial_x^2\chi_n^2)\partial_x^2\tilde u_n+4(\partial_x\chi_n^2)\partial_x^3\tilde u_n$$

and combining this with (5.22) and (5.28) yields

$$\begin{aligned}
&\|(\partial_x^4\chi_n^2)\tilde u_n\|_{L_t^1L_x^2([-T,T]\times\mathbb{T}_n)}\lesssim T\|\partial_x^4\chi_n^2\|_{L_x^\infty(\mathbb{R})}\|\tilde u_n\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})}\lesssim_T N_n^{-12},\\
&\|(\partial_x^3\chi_n^2)\partial_x\tilde u_n\|_{L_t^1L_x^2([-T,T]\times\mathbb{T}_n)}\\
&\qquad\lesssim T\|\partial_x^3\chi_n^2\|_{L_x^\infty(\mathbb{R})}\|\partial_x\tilde u_n\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})}\lesssim_{T,M} N_n^{-8},\\
&\|(\partial_x^2\chi_n^2)\partial_x^2\tilde u_n\|_{L_t^1L_x^2([-T,T]\times\mathbb{T}_n)}\\
&\qquad\lesssim T\|\partial_x^2\chi_n^2\|_{L_x^\infty(\mathbb{R})}\|\partial_x^2\tilde u_n\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})}\lesssim_{T,M} N_n^{-4},\\
&\|(\partial_x\chi_n^2)\partial_x^3\tilde u_n\|_{L_t^1L_x^2([-T,T]\times\mathbb{T}_n)}\\
&\qquad\lesssim T\|\partial_x\chi_n^2\|_{L_x^\infty(\mathbb{R})}\|\partial_x^3\tilde u_n\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})}\lesssim_{T,M} \sqrt{\eta_n}.
\end{aligned}$$

It remains to estimate (5.36). We decompose

$$\chi_n^2P_{\le N_n}F(P_{\le N_n}\tilde u_n)-P_{\le N_n}^{L_n}F(P_{\le N_n}^{L_n}(\chi_n^2\tilde u_n))$$

$$=\chi_n^2P_{\le N_n}\big[F(P_{\le N_n}\tilde u_n)-F(P_{\le N_n}(\chi_n^2\tilde u_n))\big] \tag{5.37}$$

$$+\chi_n^2P_{\le N_n}(1-\chi_n^3)F(P_{\le N_n}(\chi_n^2\tilde u_n))] \tag{5.38}$$

$$+\chi_n^2P_{\le N_n}\chi_n^3\big[F(P_{\le N_n}(\chi_n^2\tilde u_n))-F(P_{\le N_n}^{L_n}(\chi_n^2\tilde u_n))\big] \tag{5.39}$$

$$+\chi_n^2\big(P_{\le N_n}-P_{\le N_n}^{L_n}\big)\chi_n^3F(P_{\le N_n}^{L_n}(\chi_n^2\tilde u_n)) \tag{5.40}$$

$$+[\chi_n^2,P_{\le N_n}^{L_n}]\chi_n^3F(P_{\le N_n}^{L_n}(\chi_n^2\tilde u_n)) \tag{5.41}$$

$$+P_{\le N_n}^{L_n}(\chi_n^2-1)F(P_{\le N_n}^{L_n}(\chi_n^2\tilde u_n)). \tag{5.42}$$

First, we estimate (5.37). By Hölder's inequality and (5.22), we obtain

$$\begin{aligned}
&\|(5.37)\|_{N([-T,T]\times\mathbb{T}_n)}\\
&\qquad\lesssim\|F(P_{\le N_n}\tilde u_n)-F(P_{\le N_n}(\chi_n^2\tilde u_n))\|_{L_t^1L_x^2([-T,T]\times\mathbb{R})}\\
&\qquad\lesssim T^{1-p/8}\|(1-\chi_n^2)\tilde u_n\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})}\|\tilde u_n\|^p_{L_t^8L_x^\infty([-T,T]\times\mathbb{R})}\\
&\qquad\lesssim\sqrt{\eta_n}.
\end{aligned}$$

We next turn to (5.38). By Lemma 5.3, we get

$$\begin{aligned}
&\|(5.38)\|_{L_t^1L_x^2([-T,T]\times\mathbb{T}_n)}\\
&\lesssim\|\chi_n^2P_{\le N_n}(1-\chi_n^3)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}T^{1-p/8}\|\tilde u_n\|^{p+1}_{L_t^{8(p+1)/p}L_x^{2(p+1)}([-T,T]\times\mathbb{R})}\lesssim N_n^{-1}.
\end{aligned}$$

We now consider (5.39). Using (5.23) and (5.28), we estimate

$$
\begin{aligned}
&\|(5.39)\|_{L_t^1 L_x^2([-T,T]\times\mathbb{T}_n)}\\
&\lesssim T^{1-p/8}\|\chi_n^3(P_{\le N_n}-P_{\le N_n}^{L_n})\chi_n^2\tilde{u}_n\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{T}_n)}\|\chi_n^2\tilde{u}_n\|^p_{L_t^8 L_x^\infty([-T,T]\times\mathbb{R})}\\
&\lesssim T^{1-p/8}\|\chi_n^3(P_{\le N_n}-P_{\le N_n}^{L_n})\\
&\quad\times\chi_n^3\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\|\chi_n^2\tilde{u}_n\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})}\|\tilde{u}_n\|^p_{L_t^8 L_x^\infty([-T,T]\times\mathbb{R})}\\
&\lesssim N_n^{-1}.
\end{aligned}
$$

Next we turn to (5.40). Using (5.22), (5.23), and (5.28), we get

$$
\begin{aligned}
&\|(5.40)\|_{L_t^1 L_x^2([-T,T]\times\mathbb{T}_n)}\\
&\lesssim \|\chi_n^3(P_{\le N_n}-P_{\le N_n}^{L_n})\chi_n^3\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\|\chi_n^4 F(P_{\le N_n}^{L_n}(\chi_n^2\tilde{u}_n))\|_{L_t^1 L_x^2([-T,T]\times\mathbb{R})}\\
&\lesssim N_n^{-1}T^{1-p/8}\|\tilde{u}_n\|^{p+1}_{L_t^{8(p+1)/p}L_x^{2(p+1)}([-T,T]\times\mathbb{R})}\lesssim N_n^{-1}.
\end{aligned}
$$

To estimate (5.41), we apply (5.24) and (5.28) as follows:

$$
\begin{aligned}
&\|(5.41)\|_{L_t^1 L_x^2([-T,T]\times\mathbb{T}_n)}\\
&\qquad\lesssim \|[\chi_n^2, P_{\le N_n}^{L_n}]\|_{L^2(\mathbb{T}_n)\to L^2(\mathbb{T}_n)}\|\chi_n^3 F(P_{\le N_n}^{L_n}(\chi_n^2\tilde{u}_n))\|_{L_t^1 L_x^2([-T,T]\times\mathbb{T}_n)}\\
&\qquad\lesssim N_n^{-1}T^{1-p/8}\|\tilde{u}_n\|^{p+1}_{L_t^{8(p+1)/p}L_x^{2(p+1)}([-T,T]\times\mathbb{R})}\lesssim N_n^{-1}.
\end{aligned}
$$

Finally, we estimate (5.42). Write $\tilde{u}_n = \chi_n^1\tilde{u}_n + (1-\chi_n^1)\tilde{u}_n$ and make use of (5.28), (5.26), and (5.22) to get

$$
\begin{aligned}
&\|(5.42)\|_{N([-T,T]\times\mathbb{T}_n)}\\
&\qquad\lesssim \|(\chi_n^2-1)F(P_{\le N_n}^{L_n}(\chi_n^2\tilde{u}_n))\|_{L_t^1 L_x^2([-T,T]\times\mathbb{R})}\\
&\qquad\lesssim T^{1-p/8}\|(1-\chi_n^2)P_{\le N_n}^{L_n}\chi_n^1\tilde{u}_n\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})}\|\tilde{u}_n\|^p_{L_t^8 L_x^\infty([-T,T]\times\mathbb{R})}\\
&\qquad\quad + T^{1-p/8}\|(1-\chi_n^1)\tilde{u}_n\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})}\|\tilde{u}_n\|^p_{L_t^8 L_x^\infty([-T,T]\times\mathbb{R})}\\
&\qquad\lesssim N_n^{-1}+\sqrt{\eta_n}.
\end{aligned}
$$

Thus, we complete the proof of Theorem 5.5. ∎

6. Proof of Theorem 1.2. Fix $z_* \in L^2(\mathbb{R})$, $l \in L^2(\mathbb{R})$ with $\|l\|_2 = 1$, $\alpha \in \mathbb{C}$, $0 < r < R < \infty$, and $T > 0$. Write $M := \|z_*\|_{L^2} + R$. Let $N_n \to \infty$, $\eta_n \to 0$ and choose L_n as in Section 5. Let $\delta \in (0, (R-r)/8)$.

It follows from the density that there exist functions $\tilde{z}_*, \tilde{l} \in C_c^\infty(\mathbb{R})$ such that

$$
\|z_* - \tilde{z}_*\|_{L^2} \le \delta \quad\text{and}\quad \|l - \tilde{l}\|_{L^2} \le \delta M^{-1} \quad\text{with } \|\tilde{l}\|_2 = 1. \tag{6.1}
$$

Since $L_n \to \infty$, the supports of $\tilde{z}_*$ and $\tilde{l}$ are contained inside the interval $[-L_n/2, L_n/2]$ for n sufficiently large. Hence, we can view $\tilde{z}_*$ and $\tilde{l}$ as functions on the torus $\mathbb{T}_n = \mathbb{R}/L_n\mathbb{Z}$. Using Bernstein's inequality, we have

$$\|\tilde{z}_* - P^{L_n}_{\le N_n}\tilde{z}_*\|_{L^2(\mathbb{T}_n)} \lesssim N_n^{-1}\|\tilde{z}_*\|_{H^1(\mathbb{T}_n)} \to 0 \quad \text{as } n \to \infty, \tag{6.2}$$

and

$$\lim_{n\to\infty} \|P^{L_n}_{>2N_n}\tilde{l}\|_{L^2(\mathbb{T}_n)} = 0. \tag{6.3}$$

Now, we consider a finite-dimensional Hamiltonian system

$$\begin{cases} (i\partial_t - \Delta^2)u_n = P^{L_n}_{\le N_n}F(P^{L_n}_{\le N_n}u_n), \quad (t,x) \in \mathbb{R}\times\mathbb{T}_n, \\ u_n(0) \in \mathcal{H}_n = \{f \in L^2(\mathbb{T}_n) : P^{L_n}_{>2N_n}f = 0\}. \end{cases} \tag{6.4}$$

Using Theorem 2.3, we deduce that there exist initial data

$$u_{0,n} \in B_{\mathcal{H}_n}(P^{L_n}_{\le N_n}\tilde{z}_*, R - 4\delta) \tag{6.5}$$

such that the solutions to (6.4) with $u_n(0) = u_{0,n}$ obey

$$|\langle \tilde{l}, u_n(T)\rangle_{L^2(\mathbb{T}_n)} - \alpha| > r + 4\delta. \tag{6.6}$$

Next, by Lemma 3.1, we derive that there is a unique global solution of

$$\begin{cases} (i\partial_t + \Delta)\tilde{u}_n = P_{\le N_n}F(P_{\le N_n}\tilde{u}_n), \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \\ \tilde{u}_n(0,x) = \chi^0_n(x)u_{0,n}(x + \mathbb{T}_n), \end{cases}$$

denoted by $\tilde{u}_n : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$, and write $z_n := P^{L_n}_{\le 2N_n}(\chi^2_n\tilde{u}_n)$. Then, by Theorem 5.5, we obtain

$$\lim_{n\to\infty} \|z_n - u_n\|_{L^\infty_t L^2_x([-T,T]\times\mathbb{T}_n)} = 0. \tag{6.7}$$

Thanks to the triangle inequality, (5.22), (6.5), (6.2), and (6.1), we deduce that

$$\begin{aligned} \|\chi^0_n u_{0,n} - z_*\|_{L^2(\mathbb{R})} &\le \|(\chi^0_n - 1)u_{0,n}\|_{L^2(\mathbb{T}_n)} + \|u_{0,n} - P^{L_n}_{\le N_n}\tilde{z}_*\|_{L^2(\mathbb{T}_n)} \\ &\quad + \|P^{L_n}_{\le N_n}\tilde{z}_* - \tilde{z}_*\|_{L^2(\mathbb{T}_n)} + \|\tilde{z}_* - z_*\|_{L^2(\mathbb{R})} \\ &\le \sqrt{\eta_n} + R - 4\delta + \sqrt{\eta_n} + \delta \le R - \delta, \end{aligned}$$

provided we take n sufficiently large. Thus, up to a subsequence, we may assume that

$$\chi^0_n u_{0,n} \rightharpoonup u_{0,\infty} \quad \text{weakly in } L^2(\mathbb{R}).$$

Moreover, using Fatou's lemma we get

$$u_{0,\infty} \in B(z_*, R). \tag{6.8}$$

By Lemma 3.1 again, we know that there is a global solution $u_\infty(t,x)$ to (1.1) with initial data $u_\infty(0) = u_{0,\infty}$. Up to a subsequence, by Theorem 4.1 we have

$$\tilde{u}_n(T) \rightharpoonup u_\infty(T) \quad \text{weakly in } L^2(\mathbb{R}).$$

This together with Lemma 5.4 implies that

$$\chi_n^2 \tilde{u}_n(T) \rightharpoonup u_\infty(T) \quad \text{weakly in } L^2(\mathbb{R}).$$

Combining this with (6.3), the definition of z_n, (6.7), and (6.6), we get

$$\begin{aligned} \big|\langle \tilde{l}, u_\infty(T)\rangle_{L^2(\mathbb{R})} - \alpha\big| &= \lim_{n\to\infty} \big|\langle \tilde{l}, \chi_n^2 \tilde{u}_n(T)\rangle_{L^2(\mathbb{T}_n)} - \alpha\big| \\ &= \lim_{n\to\infty} \big|\langle P^{L_n}_{\le 2N_n} \tilde{l}, \chi_n^2 \tilde{u}_n(T)\rangle_{L^2(\mathbb{T}_n)} - \alpha\big| \\ &= \lim_{n\to\infty} \big|\langle \tilde{l}, z_n(T)\rangle_{L^2(\mathbb{T}_n)} - \alpha\big| \\ &= \lim_{n\to\infty} \big|\langle \tilde{l}, u_n(T)\rangle_{L^2(\mathbb{T}_n)} - \alpha\big| \ge r + 4\delta. \end{aligned} \tag{6.9}$$

On the other hand, by mass conservation, we have

$$\|u_\infty(T)\|_{L^2(\mathbb{R})} = \|u_{0,\infty}\|_{L^2(\mathbb{R})} < R + \|z_*\|_{L^2(\mathbb{R})} = M.$$

Applying (6.9) and (6.1), we derive that

$$\begin{aligned} \big|\langle l, u_\infty(T)\rangle_{L^2(\mathbb{R})} - \alpha\big| &\ge \big|\langle \tilde{l}, u_\infty(T)\rangle_{L^2(\mathbb{R})} - \alpha\big| - \|l - \tilde{l}\|_{L^2(\mathbb{R})} \|u_\infty(T)\|_{L^2(\mathbb{R})} \\ &\ge r + 4\delta - \delta \ge r. \end{aligned}$$

Combining this with (6.8), we know that $u_\infty(t,x)$ satisfies Theorem 1.2.

Therefore, by choosing $u(t,x) = u_\infty(t,x)$, we complete the proof of the theorem.

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