

The logarithm of irrational numbers and Beatty sequences

by

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Notation. Throughout, we use $[x]$ to represent the greatest integer not exceeding x , and define $\{x\} := x - [x]$. Unless otherwise specified, the letters a and b represent irrational numbers. Also, A and B represent the *Beatty sequences* $A := ([an])_{n=1}^{\infty}$ and $B := ([bn])_{n=1}^{\infty}$. We will often refer to *integers* when we really mean positive integers, but this will not generate any conceptual ambiguity.

1. Introduction and statement of main result. Sturmian sequences are part of the general study of combinatorial properties of finite and infinite words that play a role in various fields of physics, mathematics, biology and computer science (see [1], [3], [Be] and the references therein). In recent years, Beatty and Sturmian sequences have been studied in various contexts including dynamical systems, prime number distribution, multiplicative functions in general, characters, non-residues and primitive roots, divisor functions, character sums, representation of integers as sums of primes (see [ABS], [BS] [Be], [Br], [F], [GN], [MH], [PS], [St], [Ta] and the references therein).

O’Bryant [Ob] obtains a special representation for a power series whose n th term is nonzero for n in a particular Beatty sequence and zero otherwise. He then uses that representation to generalize the Rayleigh–Beatty Theorem. Other works (see [Ko] and the references therein) also associate Beatty sequences with power series. In another direction, the asymptotic behavior of sums of arithmetic functions involving Beatty sequences and some of their applications have been studied in, e.g., [ABS], [Ab].

2010 *Mathematics Subject Classification*: Primary 11A25, 11B75, 40A05; Secondary 11B85.

Key words and phrases: logarithm, irrational numbers, Beatty sequences, Sturmian sequences, series, Frullani’s integral.

Received 10 May 2011; revised 23 February 2017.

Published online 7 July 2017.

In this paper we study the behavior of an infinite series involving Beatty sequences, establish its convergence and obtain an identity involving the logarithm of their generating irrational slope (Theorem 7.1). The terms of the series are differences of ratios of Beatty sequences, not necessarily complementary (such differences of ratios have interesting properties, one of them being that Sturmian sequences can be defined in terms of them; see Section 9). We find partial sums of arithmetical functions defining Sturmian sequences, and partial sums of products of such arithmetical functions. These are then applied, together with other techniques, to prove our main result.

MAIN THEOREM. *Let $a, b > 1$ be irrational numbers. Let $a_n = [an]$ and $b_n = [bn]$. Then*

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(\frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} \right) + \sum_{n=1}^{\infty} \frac{a\{a^{-1}(n+1)\} - b\{b^{-1}(n+1)\}}{n(n+1)} = \log \frac{a}{b}.$$

Sections 2 through 8 are used to prove this theorem. In Sections 2 through 6 we work with a case particular of interest when the sequences A and B are complementary. There we also introduce a series of lemmas and notation that are used throughout the paper. In Sections 7 and 8 we prove the main result. Specifically, the Main Theorem is proved as a corollary of Theorem 7.1. In the final section we show how Beatty ratios can be used to define inhomogeneous Sturmian sequences, and we make some remarks about Frullani’s integral.

We now concentrate on the limit expression on the left hand side of the Main Theorem. For any fixed integer k , we have

$$(1.1) \quad \frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} = \frac{b_n a_{n+k} - a_n b_{n+k}}{a_n b_n} = \frac{(bn - \{bn\})(a(n+k) - \{a(n+k)\}) - (an - \{an\})(b(n+k) - \{b(n+k)\})}{a_n b_n}.$$

If we expand the numerator and sum it over n , we obtain

$$(1.2) \quad \sum_{n=1}^{\infty} \left(\frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} \right) = \sum_{n=1}^{\infty} \frac{-bn\{a(n+k)\} - an\{bn\} - ak\{bn\} + \{bn\}\{a(n+k)\}}{a_n b_n} + \sum_{n=1}^{\infty} \frac{an\{b(n+k)\} + bn\{an\} + bk\{an\} - \{an\}\{b(n+k)\}}{a_n b_n}.$$

The terms with factor k , upon dividing by k , yield

$$\sum_{n=1}^{\infty} \frac{a\{bn\} - b\{an\}}{a_n b_n}.$$

Also, the terms with the product of two fractional parts, after dividing by k , give $O(1/k)$. The remaining terms, after dividing by k , yield

$$(1.3) \quad \frac{1}{k} \sum_{n=1}^{\infty} \frac{an\{b(n+k)\} - an\{bn\}}{a_n b_n} + \frac{1}{k} \sum_{n=1}^{\infty} \frac{-bn\{a(n+k)\} + bn\{an\}}{a_n b_n}.$$

Consider now the first sum in (1.3) and note that the denominator can be written as $abn^2(1 + O(1/n))$. Hence, for any fixed $M = M(k)$ to be specified later, we can write

$$(1.4) \quad \begin{aligned} \sum_{n=1}^M \frac{an\{b(n+k)\}}{ka_n b_n} - \sum_{n=1}^{M+k} \frac{an\{bn\}}{ka_n b_n} &= \sum_{n=1}^M \frac{an\{b(n+k)\}}{abkn^2} - \sum_{n=1}^{M+k} \frac{an\{bn\}}{abkn^2} + O\left(\frac{1}{k}\right) \\ &= \sum_{n=1}^M \frac{\{b(n+k)\}}{bkn} - \sum_{n=1}^{M+k} \frac{\{bn\}}{bkn} + O\left(\frac{1}{k}\right) \\ &= \sum_{n=1}^M \left(\frac{\{b(n+k)\}}{kbn} - \frac{\{b(n+k)\}}{bk(n+k)} \right) + O\left(\frac{\log k}{k}\right) \\ &= \sum_{n=1}^M \frac{\{b(n+k)\}}{kb} \left(\frac{1}{n} - \frac{1}{n+k} \right) + O\left(\frac{\log k}{k}\right) \\ &= \frac{1}{kb} \sum_{n=1}^M \left(\frac{1}{n} - \frac{1}{n+k} \right) + O\left(\frac{\log k}{k}\right) \\ &= O\left(\frac{\log(M+k)}{k}\right) + O\left(\frac{\log k}{k}\right). \end{aligned}$$

The last equality holds because the only surviving terms inside the telescoping sum in the previous step are $1/n$ for $n = 1, \dots, k$ and $-1/n$ for $n = M + 1, \dots, M + k$. Taking $M = k^2$, we see that the left hand side of (1.4) yields

$$\sum_{n=1}^M \frac{an\{b(n+k)\}}{ka_n b_n} - \sum_{n=1}^{M+k} \frac{an\{bn\}}{ka_n b_n} = O\left(\frac{\log k}{k}\right),$$

which approaches zero as $k \rightarrow \infty$. Hence, as $k \rightarrow \infty$, (1.3) tends to zero.

Thus, if we divide (1.2) by k and let $k \rightarrow \infty$, we obtain

$$(1.5) \quad \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(\frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} \right) = \sum_{n=1}^{\infty} \frac{a\{bn\} - b\{an\}}{a_n b_n}.$$

Putting together (1.5) and the Main Theorem, we get the following corollary.

COROLLARY 1.1. *If $a, b > 1$ are irrational numbers, then*

$$\sum_{n=1}^{\infty} \left(\frac{a\{bn\} - b\{an\}}{a_n b_n} + \frac{a\{a^{-1}(n+1)\} - b\{b^{-1}(n+1)\}}{n(n+1)} \right) = \log \frac{a}{b}.$$

Now, it is reasonable to expect that, in absolute value, the right hand side of (1.5) is approximately $\left| \frac{\pi^2}{6b} - \frac{\pi^2}{6a} \right|$. Thus, since both a and b are greater than 1, it is reasonable to conjecture the following:

CONJECTURE 1. *If $a, b > 1$ are irrational numbers, then*

$$\left| \log \frac{a}{b} - \sum_{n=1}^{\infty} \frac{a\{a^{-1}(n+1)\} - b\{b^{-1}(n+1)\}}{n(n+1)} \right| < \frac{\pi^2}{6}.$$

However, numerical data suggest that the quantity in absolute value is less than 1:

CONJECTURE 2. *If $a, b > 1$ are irrational numbers, then*

$$\left| \log \frac{a}{b} - \sum_{n=1}^{\infty} \frac{a\{a^{-1}(n+1)\} - b\{b^{-1}(n+1)\}}{n(n+1)} \right| < 1.$$

Before proving the Main Theorem, we briefly examine some specific series similar to the one in that theorem, to gain some insight into the heuristics behind the convergence of this type of series. Let $a = (1 + \sqrt{5})/2$, $b = a^2$, $a_n = [an]$ and $b_n = [bn]$, as in Theorem 1. What can we expect from the series $\sum_{n=1}^{\infty} (\{an\}/a_n - \{bn\}/b_n)$? Since $\{xn\}$ is uniformly distributed for x irrational, for large m the m th partial sum of the series behaves like $(\frac{1}{a} - \frac{1}{b})\frac{1}{2} \log m$, and so the series is divergent. How about

$$\sum_{n=1}^{\infty} \left(\frac{a_n}{a_{n+1}} - \frac{b_n}{b_{n+2}} \right) = \frac{1}{2} - \frac{2}{7} + \frac{2}{3} - \frac{5}{10} + \frac{3}{4} - \frac{7}{13} + \dots ?$$

Computing some of its partial sums gives: $s_{10} = 1.22014275$, $s_{50} = 2.49049396$, $s_{100} = 3.149858764$, $s_{500} = 4.731312527$, $s_{1000} = 5.420950626$, $s_{10000} = 7.720369134$, $s_{50000} = 9.329523382$. We thus expect that this series is divergent. However, the series

$$\sum_{n=1}^{\infty} \left(\frac{a_n}{a_{n+1}} - \frac{b_n}{b_{n+1}} \right) = \left(\frac{1}{2} - \frac{2}{5} \right) + \left(\frac{2}{3} - \frac{5}{7} \right) + \left(\frac{3}{4} - \frac{7}{10} \right) + \dots,$$

which involves only a minor change compared to the previous one, behaves “better”. Some of its partial sums are $s_{10} = -0.06921181263$, $s_{50} =$

-0.08157653315 , $s_{100} = -0.08271773217$, $s_{500} = -0.08318152710$, $s_{1000} = -0.08329909024$, $s_{10000} = -0.08340515936$, $s_{50000} = -0.08342404240$. We start to suspect that we have a convergent series. Similarly, if we invert the quotients and consider instead the type of series given in the Main Theorem,

$$S := \sum_{n=1}^{\infty} \left(\frac{a_{n+1}}{a_n} - \frac{b_{n+1}}{b_n} \right) = \left(3 - \frac{5}{2} \right) + \left(\frac{4}{3} - \frac{7}{5} \right) + \left(\frac{3}{2} - \frac{10}{7} \right) + \dots,$$

we see that its partial sums are $s_{100} = 0.5463290032, \dots, s_{10000} = 0.5475731159, \dots$ suggesting convergence, as expected. These types of series happen to have many interesting properties, convergence being one of them. We rewrite the last series in a more suggestive way:

$$S = \frac{1}{2} - \frac{1}{15} + \frac{1}{14} + \frac{1}{30} - \frac{3}{104} + \frac{1}{45} - \frac{2}{99} + \frac{1}{60} + \frac{2}{161} - \frac{3}{208} + \frac{5}{476} + \frac{5}{589} - \frac{4}{357} + \frac{1}{132} - \frac{1}{104} + \dots$$

Notice that this is not an alternating series. The occurrence of positive and negative terms is not periodic. Moreover, the absolute values of the terms are not monotone. Similarly, the positive terms are not monotone; e.g., $\frac{1}{60} > \frac{2}{161}$ but $\frac{2}{161} < \frac{1}{132}$ (nor are the negative terms monotone, in general). However, as can be realized when we rewrite S , the reason this type of series converges is that there is enough cancellation. In fact, as will be evident in the proof, for the m th partial sum, we get $\log m$ cancellation. This is precisely the order of magnitude of the heuristic estimate given above.

A special case here occurs when the irrationals a and b are such that $1 \leq q < a < q + 1$ and $1 < b = r + \{a\}$, for $q, r \in \mathbb{Z}$. In Theorem 2.1 we treat separately the case $a = \phi$ and $b = \phi^2$, where ϕ is the golden ratio, because in addition to being special in the sense we have just described, it also has the special property that it involves two complementary Beatty sequences. The proof we present differs from the one for the general case. The proof for the particular case works, with some minor modification, for any $a, b > 1$ such that $1/a + 1/b = 1$, and a and b have the same fractional parts. Such examples can be created by setting $b = a + t$ for some fixed integer t , and solving the equation $1/a + 1/b = 1$ for a .

2. A particular case

THEOREM 2.1. *Let $a = (1 + \sqrt{5})/2$ and $b = a^2 = a + 1$. Write $a_n := [an]$ and $b_n := [bn]$. Then the series*

$$\sum_{n=1}^{\infty} \left(\frac{a_{n+1}}{a_n} - \frac{b_{n+1}}{b_n} \right)$$

converges.

REMARK 2.2. Note that in Theorem 2.1, $\{a\} = \{b\} = (\sqrt{5} - 1)/2$, and $1 < a < 2$ and $2 < b < 3$. Also $1/a + 1/b = 1$. Thus $A = (a_n)_{n=1}^\infty$ and $B = (b_n)_{n=1}^\infty$ are complementary Beatty sequences.

The proof of this theorem can be divided into two parts. First, we write the series as a sum over all integers, instead of only those of the form $[an]$ and $[bn]$. Second, we use summation by parts to complete the proof. In the first part of the proof we will make use of [1, Ch. 9, equations (2.1)–(2.3)].

DEFINITION 2.3. For a real number α such that $0 < \alpha < 1$, we define the *characteristic function* of α as

$$(2.1) \quad f_\alpha(n) = [\alpha(n + 1)] - [\alpha n].$$

Clearly $f_\alpha(n) = 1$ or $f_\alpha(n) = 0$. Since the sum telescopes, we have

$$(2.2) \quad \sum_{n=1}^m f_\alpha(n) = [\alpha(m + 1)].$$

REMARK 2.4. Notice that (2.2) gives the number of integers $n \leq m$ for which $f_\alpha(n) = 1$.

DEFINITION 2.5. For $1 < \beta \in \mathbb{R} \setminus \mathbb{Q}$, define

$$(2.3) \quad g'_\beta(n) = \begin{cases} 1 & \text{if } n = [k\beta] \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known [1, Lemma 9.1.3] that for all integers n ,

$$(2.4) \quad g'_\beta(n) = f_{1/\beta}(n).$$

2.1. First part of the proof of Theorem 2.1. We divide the argument into a sequence of lemmas. We start by rewriting a_{n+1} and b_{n+1} .

LEMMA 2.6.

$$a_{n+1} = \begin{cases} a_n + 2 & \text{if } n \in A, \\ a_n + 1 & \text{otherwise.} \end{cases}$$

Proof. Note that

$$\begin{aligned} a_{n+1} - a_n &= [a(n + 1)] - [an] = [(1 + \{a\})(n + 1)] - [(1 + \{a\})n] \\ &= [n + 1 + \{a\}(n + 1)] - [n + \{a\}n] \\ &= n + 1 - n + [\{a\}(n + 1)] - [\{a\}n] \\ &= 1 + f_{\{a\}}(n). \end{aligned}$$

From (2.3) and the fact that $1/\{a\} = a$ we find that $a_{n+1} = a_n + 1 + f'_a(n)$, and the result follows. ■

Similarly, but using $2 < b < 3$ and $\{b\} = \{a\}$, we obtain

LEMMA 2.7.

$$b_{n+1} = \begin{cases} b_n + 3 & \text{if } n \in A, \\ b_n + 2 & \text{otherwise.} \end{cases}$$

Using these two lemmas, we immediately have

$$\frac{a_{n+1}}{a_n} = \begin{cases} 1 + 2/a_n & \text{if } n \in A, \\ 1 + 1/a_n & \text{otherwise,} \end{cases} \quad \frac{b_{n+1}}{b_n} = \begin{cases} 1 + 3/b_n & \text{if } n \in A, \\ 1 + 2/b_n & \text{otherwise.} \end{cases}$$

The sets A and B form a partition of the integers. Hence, after rewriting the m th partial sum of the series in Theorem 2.1 as

$$H := \sum_{n=1}^m \left(\frac{a_{n+1}}{a_n} - \frac{b_{n+1}}{b_n} \right) = \sum_{\substack{n \leq m \\ n \in A}} \left(\frac{2}{a_n} - \frac{3}{b_n} \right) + \sum_{\substack{n \leq m \\ n \in B}} \left(\frac{1}{a_n} - \frac{2}{b_n} \right),$$

we obtain

LEMMA 2.8. *Let a and b be as in Theorem 2.1 and define*

$$H'_1 := \sum_{\substack{n \leq m \\ a_n = [ka] \\ k \in A}} \frac{2}{a_n} + \sum_{\substack{n \leq m \\ a_n = [ka] \\ k \in B}} \frac{1}{a_n},$$

$$H'_2 := \sum_{\substack{n \leq m \\ b_n = [kb] \\ k \in A}} \frac{3}{b_n} + \sum_{\substack{n \leq m \\ b_n = [kb] \\ k \in B}} \frac{2}{b_n}.$$

Then $H = H'_1 - H'_2$.

Next, set

$$(2.5) \quad x = x(b, m) = [mb],$$

$$(2.6) \quad y = y(a, m) = [ma].$$

In the rest of this paper, unless otherwise specified, when we write x or y we refer to $x(b, m)$ and $y(a, m)$ respectively. Note that

$$(2.7) \quad [(x + 1)b^{-1}] = [(mb + 1 - \{mb\})b^{-1}] = m + [(1 - \{mb\})b^{-1}] = m.$$

The last equality holds because inside the bracket we have the product of two factors that are each less than one. Similarly

$$(2.8) \quad [(y + 1)a^{-1}] = m.$$

Note that the sums in H'_1 and in H'_2 only run over numbers that are in A and in B , respectively. As can be clearly seen, the a_n 's in the first sum of H'_1 come from multiplying by an element of A , while the a_n 's in the second sum of H'_1 come from multiplying by an element of B . A similar arrangement can be observed in H'_2 . For instance, the integer 10 belongs to B , and it comes from multiplying by an element of A , since $10 = [4b]$ and $4 \in A$. This suggests that in order to write these two partial sums over all the integers, we need to introduce indicator functions that tell us when a particular integer belongs to the sequence A or B , and when that integer

comes from multiplying by an element of A or B . This is our immediate goal.

To do so, we define the arithmetic functions given below in terms of the characteristic functions of $1/a$ and $1/b$. The reasons for extending these two characteristic functions in the form given below are as follows: As can be clearly seen, the sums in H'_1 run over all elements of A in the interval $[a_1, a_m]$, and H'_2 runs over numbers in B in the interval $[b_1, b_m]$. We would like to combine these two sums into a single sum with upper limit given by x , which is the largest limit of summation in H'_1 and H'_2 .

DEFINITION 2.9. Let a and b be the slopes of two Beatty sequences A and B , respectively, and for α irrational, $0 < \alpha < 1$, let f_α be given by Definition 2.3. Also, let $x = x(b, m)$ and $y = y(a, m)$ be as in (2.5) and (2.6), respectively. Define f and g as follows:

$$(2.9) \quad f(n) = \begin{cases} f_{1/a}(n) & \text{if } n \leq y, \\ 0 & \text{otherwise.} \end{cases} \quad g(n) = \begin{cases} g_{1/b}(n) & \text{if } n \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

In this part of the proof we will also make use of the following lemma from [1, proof of Lemma 9.1.3].

LEMMA 2.10. *Let $\alpha > 1$ be any irrational number. Then $n = [\alpha k]$ if and only if $k = [n/\alpha] + 1$ and $[(n + 1)/\alpha] = k$.*

In the following lemma we rewrite the m th partial sum associated to the series in Theorem 2.1 over all positive integers up to $[mb]$. (Note that the statement of the lemma is not symmetric in x and y .)

LEMMA 2.11. *Let a be the golden ratio ($a = \frac{1+\sqrt{5}}{2}$) and b be the golden ratio squared ($b = a^2$). Let $x = [mb]$, and let $f(n)$ and $g(n)$ be given by (2.9). Then $H = H_1 - H_2$ where*

$$H_1 = \sum_{n=1}^x \left(\frac{2f(n)f([n/a] + 1)}{n} + \frac{f(n)g([n/a] + 1)}{n} \right),$$

$$H_2 = \sum_{n=1}^x \left(\frac{3g(n)f([n/a] + 1)}{n} + \frac{2g(n)g([n/a] + 2)}{n} \right).$$

Proof. We will only provide the proof for H_1 because the proof for H_2 is similar. We can assume that $n \leq y$, since $f(n) = 0$ whenever $n > y$, by (2.8) and Definition 2.9. For the first term in H_1 it follows that $n = a_r$, for $r \in A$ and $n \leq y$ if and only if $f(n)f(r) = 1$. From Lemma 2.10, we find that $r = [n/a] + 1$. Thus, the first summand in H_1 corresponds to the first sum in H'_1 . Similarly $n = a_r$, for $r \in B$ and $n \leq y$ if and only if $f(n)g(r) = 1$ and $r = [n/b] + 1$ by Lemma 2.10. Hence the second summand of H_1 corresponds to the second sum in H'_1 . ■

Now that we have written each of the two partial sums in H over all the integers up to x , we are going to use partial summation to estimate each of these four sums. This will complete the proof of Theorem 2.1. But first we need to prove some auxiliary propositions, dealing with general partial sums. We then apply them to the desired cases.

3. Counting special elements of Beatty sequences. Applying summation by parts is one of the important ingredients in the proof of the Main Theorem of this paper and its particular case. The terms of the sums we work with are usually quotients involving characteristic functions (say f and g) of irrational numbers. They are of the form $f(n)g(\lfloor n/c \rfloor + 1)/n$ or $f(n)/n$, or a combination of both. So, we start by obtaining an expression for such types of summatory functions. Then in Theorem 5.5 we find an asymptotic formula for the m th partial sum whose terms are these types of quotients.

The first partial sum we find tells us how many elements there are in a Beatty sequence up to some given number t , and is presented in the following lemma. The first part of the lemma was proved earlier, but we include it here for completeness.

LEMMA 3.1. *Let $0 < c < 1$ be any irrational number, and let $f(n) = f_c(n)$. Then*

$$(3.1) \quad \sum_{n \leq t} f(n) = [c(\lfloor t \rfloor + 1)].$$

If $t = x = \lfloor m/c \rfloor$, then

$$\sum_{n \leq x} f(n) = m.$$

Proof. By (2.2), the left hand side of (3.1) gives $[c(\lfloor x \rfloor + 1)]$. And this gives m by (2.7). ■

We also need to count the elements of a given Beatty sequence that come from multiplying by elements of another Beatty sequence. The lemma below involves sums encoding this type of selection of numbers. For $0 < c, d < 1$, it counts the elements up to t that are in the Beatty sequence with slope $1/c$ and come from multiplying by elements in the Beatty sequence with slope $1/d$.

LEMMA 3.2. *Let $0 < c, d < 1$ be any irrational numbers, and let $f(n) = f_c(n)$ and $g(n) = g_d(n)$ be as in Definition 2.9. Then*

$$\sum_{n \leq t} f(n)g(\lfloor cn \rfloor + 1) = [d(\lfloor c(\lfloor t \rfloor + 1) \rfloor + 1)].$$

REMARK 3.3. As an illustration of this lemma, consider the following example. In the first 13 elements of the Beatty sequence generated by ϕ , where ϕ is the golden ratio, i.e.,

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21,$$

there are five numbers that come from multiplying by an element of the complementary sequence: 3, 8, 11, 16, 21. They are given by $[2\phi]$, $[5\phi]$, $[7\phi]$, $[10\phi]$ and $[13\phi]$. In other words, these numbers come from multiplying by elements in the Beatty complement, namely by 2, 5, 7, 10, 13, respectively. This quantity is encoded in the sum

$$\sum_{n \leq 21} f(n)g\left(\left[\frac{1}{\phi}n\right] + 1\right) = \left[\frac{1}{\phi^2}\left(\left[\frac{1}{\phi}([21] + 1)\right] + 1\right)\right] = 5,$$

and five is precisely the number one would expect according to the explanation given above.

Proof of Lemma 3.2. Let $C = ((1/c)n)_{n=1}^\infty$ and $D = ((1/d)n)_{n=1}^\infty$. Then, by (2.4), f and g are the indicator functions of the Beatty sequences D and C . Note that $f(n)g([nc] + 1) = 1$ if and only if $n \in C$ and $([nc] + 1) \in D$, by the definition of f and g . In other words, n is of the form c_r with $r \in D$. By Lemma 2.10, we see that $r = [nc] + 1$. Now by Remark 2.4, there are $s := [c([t] + 1)]$ integers up to t for which $f(n) = 1$. From these integers, by the same remark, only $[d(s + 1)]$ are of the form $[(1/d)k]$. Hence, the lemma holds. ■

REMARK 3.4. Lemma 3.1 tells us that

$$[c([t] + 1)] = \sum_{n \leq t} f(n) = \sum_{n \leq [t]} f(n),$$

and thus we can apply Lemma 3.1 to Lemma 3.2 to see that

$$\sum_{n \leq t} f(n)g([cn] + 1) = [d([c([t] + 1)] + 1)] = \sum_{n \leq [c([t]+1)]} g(n).$$

Hence

$$\begin{aligned} (3.2) \quad \sum_{n \leq [c([t]+1)]} g(n+k) &= \sum_{k < n \leq [c([t]+1)]+k} g(n) \\ &= \sum_{n \leq [c([t]+1)]+k} g(n) - \sum_{n \leq k} g(n) \\ &= [d([c([t] + 1)] + k + 1)] - [d(k + 1)]. \end{aligned}$$

We are now ready to generalize the previous lemma. Given two Beatty sequences A and B , with characteristic sequences f and g respectively, and given a nonnegative integer k , we can count the integers n that are less than a real number t and that also satisfy the following two conditions: first, n is

in A , and second, if $n = [a_r]$, then $r + k$ is an element of B . Note also that this generalizes the previous lemma, because if $k = 0$, then the integer n is an element of A that is found by multiplying by an element of B . We find this type of partial sum next.

LEMMA 3.5. *Let $0 < c, d < 1$ be any irrational numbers, and let $f(n) = f_c(n)$ and $g(n) = g_d(n)$. Then, for any nonnegative integer k ,*

$$(3.3) \quad \sum_{n \leq t} f(n)g([c(n)] + k + 1) = [d([c([t] + 1)] + k + 1)] - [d(k + 1)].$$

Proof. In light of (2.4), $f(n)g([c(n)] + 1)$ equals 1 if both $n = [j/c]$ and j is in the Beatty sequence with slope $1/d$ (note that $j = [cn] + 1$ by Lemma 2.10). Analogously, $f(n)g([c(n)] + k + 1)$ is 1 if $n = [j/c]$ and $j + k$ is in the Beatty sequence with slope $1/d$. Since there are $[c([t] + 1)]$ integers up to t in the Beatty sequence with slope $1/c$, we have

$$\sum_{n \leq t} f(n)g([c(n)] + k + 1) = \sum_{j \leq [c([t] + 1)]} g(j + k).$$

By (3.2), the last expression gives precisely the right hand side of (3.3). ■

4. Some useful integrals and sums. For real numbers $c, d > 1$, let

$$(4.1) \quad F_1(t) = [c^{-1}([t] + 1)],$$

$$(4.2) \quad F_2(t) = [d^{-1}([c^{-1}([t] + 1)] + 1)].$$

Now rewrite $F_1(t)$ and $F_2(t)$ using $[x] = x - \{x\}$ repeatedly to obtain

$$\begin{aligned} F_1(t) &= \frac{1}{c}(t + 1 - \{t\}) - \{c^{-1}([t] + 1)\}, \\ F_2(t) &= (cd)^{-1}(t + 1 - \{t\}) - d^{-1}\{c^{-1}([t] + 1)\} \\ &\quad + d^{-1} - \{d^{-1}([c^{-1}([t] + 1)] + 1)\}. \end{aligned}$$

LEMMA 4.1. *For any irrational $c > 1$ and $x = x(c, m) = [cm]$ we have*

$$\int_1^x \frac{[c^{-1}([t] + 1)]}{t^2} dt = \frac{1}{c}(\log m + \log c + \gamma) - \sum_{n=1}^{\infty} \frac{\{c^{-1}(n + 1)\}}{n(n + 1)} + O\left(\frac{1}{m}\right),$$

where γ is Euler's constant.

Proof. Integrating F_1 we see that

$$\begin{aligned} \int_1^x \frac{F_1(t)}{t^2} dt &= \int_1^x \frac{\frac{1}{c}(t + 1 - \{t\}) - \{c^{-1}([t] + 1)\}}{t^2} dt \\ &= \frac{1}{c} \log x \Big|_1^x + \int_1^x \frac{\frac{1}{c}(1 - \{t\}) - \{c^{-1}([t] + 1)\}}{t^2} dt. \end{aligned}$$

If we complete the tails of the integral we have

$$\int_1^x \frac{F_1(t)}{t^2} dt = \frac{1}{c} \log t \Big|_1^x + \frac{1}{c} \int_1^\infty \frac{1 - \{t\}}{t^2} dt - \int_1^\infty \frac{\{c^{-1}([t] + 1)\}}{t^2} dt \\ - \int_x^\infty \frac{\frac{1}{c}(1 - \{t\}) - \{c^{-1}([t] + 1)\}}{t^2} dt.$$

The first integral on the right hand side equals $\frac{1}{c}\gamma$ (see [2, p. 56]). The third is $O(1/x)$, and given that $x = [cm]$, this is $O(1/m)$. Also, for the same reason, $\log x = \log(cm - \{cm\}) = \log(cm) + O(1/m) = \log c + \log m + O(1/m)$. Hence

$$\int_1^x \frac{F_1(t)}{t^2} dt = \frac{1}{c}(\log m + \log c + \gamma) + \int_1^\infty \frac{\{c^{-1}([t] + 1)\}}{t^2} dt + O\left(\frac{1}{m}\right).$$

We can write the last integral as an infinite sum:

$$\int_1^\infty \frac{\{c^{-1}([t] + 1)\}}{t^2} dt = \sum_{n=1}^\infty \int_0^1 \frac{\{c^{-1}(n+1)\}}{(n+t)^2} dt \\ = \sum_{n=1}^\infty \{c^{-1}(n+1)\} \int_0^1 \frac{1}{(n+t)^2} dt = \sum_{n=1}^\infty \frac{\{c^{-1}(n+1)\}}{n(n+1)}.$$

This yields the assertion. ■

LEMMA 4.2. *Let $r'_n = [(1/r)n]$ for r a real number. For any irrational $c, d > 1$, and $x = x(c, m) = [cm]$, we have*

$$\int_1^x \frac{[d^{-1}([c^{-1}([t] + 1)] + k + 1)]}{t^2} dt \\ = (cd)^{-1}(\log m + \log c + \gamma) + d^{-1}(k + 1) \\ - \sum_{n=1}^\infty \frac{d^{-1}\{c^{-1}(n+1)\} + \{d^{-1}(c'_{n+1} + k + 1)\}}{n(n+1)} + O\left(\frac{1}{m}\right).$$

Proof. Proceed in a similar manner to how we dealt with Lemma 4.1. ■

5. Applying summation by parts

LEMMA 5.1. *Let $c > 1$ be any irrational number, $f(n)$ be given in terms of $f_{\{c\}}(n)$ as in Definition 2.9, and $c_n = [cn]$. If $C = (c_n)_{n=1}^\infty$ and $x = [mc]$, then*

$$\sum_{n=1}^x \frac{f(n)}{n} = \frac{1}{c} + \frac{1}{c}(\log m + \log c + \gamma) - \sum_{n=1}^\infty \frac{\{c^{-1}(n+1)\}}{n(n+1)} + O\left(\frac{1}{m}\right).$$

REMARK 5.2. We can rewrite Lemma 5.1 as

$$\sum_{n=1}^x \frac{f(n)}{n} = c^{-1} \log m + \alpha_c + O\left(\frac{1}{m}\right),$$

where α_c is a constant.

Proof of Lemma 5.1. We will apply summation by parts. Let $F(t) = \sum_{n \leq t} f(n)$. Then by Lemma 3.1 and by (4.1) we see that

$$F(t) = \sum_{n \leq t} f(n) = [c([t] + 1)] = F_1(t).$$

Thus

$$(5.1) \quad \sum_{n=1}^x \frac{f(n)}{n} = \frac{F_1(x)}{x} + \int_1^x \frac{F_1(t)}{t^2} dt.$$

From the definition of x , we see that

$$\frac{F_1(x)}{x} = \frac{1}{c} + O\left(\frac{1}{x}\right) = \frac{1}{c} + O\left(\frac{1}{m}\right),$$

The result follows by applying Lemma 4.1 to the integral in (5.1). ■

LEMMA 5.3. Let $c > 1$ be any irrational number, and let $d^{-1} = \{c\}$ and $x = [mc]$. Define f and g in terms of $f_{1/c}$ and $g_{1/d}$ respectively, as in Definition 2.9. Then

$$(5.2) \quad \begin{aligned} & \sum_{n=1}^x \frac{f(n)g([n/c] + k + 1)}{n} \\ &= \frac{\{c\}}{c} + (cd)^{-1}(\log m + \log c + \gamma) + \{d^{-1}(1 + k)\} \\ & \quad - \sum_{n=1}^{\infty} \frac{d^{-1}\{c^{-1}(n + 1)\} + \{d^{-1}(c'_{n+1} + k + 1)\}}{n(n + 1)} + O\left(\frac{1}{m}\right). \end{aligned}$$

REMARK 5.4. Notice that we can rewrite Lemma 5.3 as

$$\sum_{n=1}^x \frac{f(n)g([n/c] + 1)}{n} = (cd)^{-1} \log m + \alpha_{c,d} + O\left(\frac{1}{m}\right),$$

where $\alpha_{c,d}$ is a constant.

Proof of Lemma 5.3. Let $F_3(t) = \sum_{n \leq t} f(n)g([n/c] + k + 1)$. We apply Lemma 3.5 with c and d replaced by c^{-1} and d^{-1} respectively to see that

$$F_3(t) = [d^{-1}([c^{-1}([t] + 1)] + k + 1)] - [d^{-1}(k + 1)] = F_2(t) - [d^{-1}(k + 1)].$$

The last equality follows from (4.2) where $F_2(t)$ is defined. Upon applying

summation by parts we obtain

$$\begin{aligned}
 (5.3) \quad & \sum_{n=1}^x \frac{f(n)g([n/c] + k + 1)}{n} \\
 &= \frac{F_3(x)}{x} + \int_1^x \frac{F_3(t)}{t^2} dt \\
 &= \frac{F_3(x)}{x} + \int_1^x \frac{[d^{-1}([c^{-1}([t] + 1)] + k + 1)]}{t^2} dt - \int_1^x \frac{[d^{-1}(k + 1)]}{t^2} dt \\
 &= \frac{F_3(x)}{x} + \int_1^x \frac{F_2(t)}{t^2} dt - [d^{-1}(k + 1)] + O\left(\frac{1}{x}\right).
 \end{aligned}$$

Since $x = [mc]$, it follows that

$$\frac{F_3(x)}{x} = \frac{d^{-1}}{c} + O\left(\frac{1}{x}\right) = \frac{\{c\}}{c} + O\left(\frac{1}{m}\right).$$

Now the integral involving F_2 in (5.3) was evaluated in Lemma 4.2. Also we combine the term $d^{-1}(k+1)$ from Lemma 4.2 with the expression $-[d^{-1}(k+1)]$ in (5.3) to get $\{d^{-1}(k + 1)\}$. The result follows. ■

THEOREM 5.5. *Let $c > 1$ be any irrational number and set $d = \{c\}^{-1}$ and $q = c - \{c\}$. Let $c_n = [cn]$, $C = (c_n)_{n=1}^\infty$, $d_n = [dn]$ and $D = (d_n)_{n=1}^\infty$. Also let $c'_n = [c^{-1}n]$ and $x = x(c, m) = [mc]$. Define $f(n)$ and $g(n)$ in terms of $f_{1/c}(n)$ and $g_{1/d}(n)$ respectively, as in Definition 2.9. Then*

$$\begin{aligned}
 (5.4) \quad & \sum_{n=1}^x \frac{qf(n) + f(n)g([n/c] + k + 1)}{n} \\
 &= 1 + \log m + \log c + \gamma + \{\{c\}(k + 1)\} \\
 &\quad - \sum_{n=1}^\infty \frac{c\{c^{-1}(n + 1)\} + \{\{c\}(c'_{n+1} + k + 1)\}}{n(n + 1)} + O\left(\frac{1}{m}\right),
 \end{aligned}$$

where γ is Euler's constant.

REMARK 5.6. Theorem 5.5 takes a partial sum and transforms it into a main term, $\log m$, and an infinite sum. The partial sum involves only quotients of elements that are in a given sequence, namely C . The infinite sum involves fractional parts of elements of two sequences, C^{-1} and D^{-1} (where $R^{-1} = ([r^{-1}n])_{n=1}^\infty$). Since the infinite sum involves only fractional parts in the numerator, and n^2 in the denominator, it is convergent.

Proof of Theorem 5.5. If we put together Lemmas 5.1 and 5.3, the partial sum in Theorem 5.5 becomes

$$\begin{aligned}
 (5.5) \quad & \sum_{n=1}^x \frac{qf(n) + f(n)g([n/c] + k + 1)}{n} \\
 &= \frac{q + \{c\}}{c} + \left(\frac{q}{c} + (cd)^{-1} \right) (\log m + \log c + \gamma) + \{\{c\}(k + 1)\} \\
 &\quad - \sum_{n=1}^{\infty} \frac{q\{c^{-1}(n + 1)\} + d^{-1}\{c^{-1}(n + 1)\} + \{d^{-1}(c'_{n+1} + k + 1)\}}{n(n + 1)} + O\left(\frac{1}{m}\right).
 \end{aligned}$$

Notice that

$$\frac{q}{c} + (cd)^{-1} = \frac{c - \{c\}}{c} + \frac{\{c\}}{c} = 1 \quad \text{and} \quad \frac{q + \{c\}}{c} = \frac{c}{c} = 1.$$

Also, since $q + d^{-1} = c$, we see that

$$q\{c^{-1}(n + 1)\} + d^{-1}\{c^{-1}(n + 1)\} = c\{c^{-1}(n + 1)\},$$

and thus (5.5) becomes (5.4). ■

6. Completion of proof of Theorem 2.1. It suffices to prove that the partial sum $H = H_1 - H_2$ (from Lemma 2.11) is convergent as $x \rightarrow \infty$. To do so, we apply the remark following Lemma 5.3 to H_1 and H_2 . Since each sum is further divided into two partial sums, we will apply it to each of these partial sums. For the first partial sum of H_1 , the parameters of the lemma are $c = a$, $d = a^{-1}$ and $x = [am]$. Also f and g of Lemma 5.3 are the same in the first partial sum of H_1 , i.e., they are both defined in terms of $f_{1/a}$. So the first term of H_1 gives

$$\begin{aligned}
 (6.1) \quad & \sum_{n=1}^x \frac{2f(n)f([n/a] + 1)}{n} = (cd)^{-1} \log m + \alpha_{c,d} + O\left(\frac{1}{m}\right) \\
 &= 2(ad)^{-1} \log m + \alpha_a + O\left(\frac{1}{m}\right) \\
 &= \frac{2}{a^2} \log m + \alpha_a + O\left(\frac{1}{m}\right).
 \end{aligned}$$

For the second summand, $d = b^{-1}$ and thus g is defined in terms of $g_{1/b}$. The other parameters remain the same. Hence

$$\begin{aligned}
 (6.2) \quad & \sum_{n=1}^x \frac{f(n)g([n/a] + 1)}{n} = (cd)^{-1} \log m + \alpha_{c,d} + O\left(\frac{1}{m}\right) \\
 &= (ad)^{-1} \log m + \alpha_{a,b} + O\left(\frac{1}{m}\right) \\
 &= \frac{1}{ab} \log m + \alpha_{a,b} + O\left(\frac{1}{m}\right).
 \end{aligned}$$

Similarly, in H_2 we have $c = b$ and $d = a^{-1}$ for the first summand. Thus, the roles of f and g in Lemma 5.3 are switched in this first summand. Also, for the second summand of H_2 , we take $c = b$ and $d = b^{-1}$. As a result, in this second summand f and g are the same, i.e., they are both defined in terms of $g_{1/b}$. In the end, the two summands of H_2 give

$$\begin{aligned}
 (6.3) \quad H_2 &= \sum_{n=1}^x \left(\frac{3g(n)f([n/b] + 1)}{n} + \frac{2g(n)g([n/b] + 2)}{n} \right) \\
 &= (3(bd)^{-1} + 2(bd)^{-1}) \log m + \alpha_a + O\left(\frac{1}{m}\right) \\
 &= \left(\frac{3}{ab} + \frac{2}{b^2}\right) \log m + \alpha_{a,b} + O\left(\frac{1}{m}\right).
 \end{aligned}$$

When we put together (6.1)–(6.3), H becomes

$$(6.4) \quad H = H_1 - H_2 = \left(\frac{2}{a^2} + \frac{1}{ab} - \frac{3}{ab} - \frac{2}{b^2}\right) \log m + \alpha_{a,b} + O\left(\frac{1}{m}\right).$$

Using $1/a + 1/b = 1$ and $a^2 = b$, we see that

$$\frac{3}{ab} + \frac{2}{b^2} - \frac{2}{a^2} - \frac{1}{ab} = \frac{2}{b} \left(\frac{1}{a} + \frac{1}{b}\right) + \frac{1}{ab} - \frac{2}{a^2} - \frac{1}{ab} = \frac{2}{b} - \frac{2}{a^2} = 0.$$

Thus if we let m tend to ∞ in (6.4), we find that $H = \alpha_{a,b}$. ■

7. Generalization. We now prove a more general version of Theorem 2.1, one involving any irrational numbers $a, b > 1$, and a_{n+k} for any integer k . This version will give an identity involving the type of series that occurs in Theorem 2.1. Here Theorem 7.1 is presented, and then the Main Theorem (in which the identity is found) is derived as a corollary.

THEOREM 7.1. *Let $a, b > 1$ be irrational numbers. Let $a_n = [an]$ and $b_n = [bn]$. Also, let $a'_n = [a^{-1}n]$ and $b'_n = [b^{-1}n]$. Then, for any integer k ,*

$$\begin{aligned}
 \sum_{n=1}^{\infty} \left(\frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} \right) &= k(\log a - \log b) + \sum_{j=1}^k (\{j\{a\}\} - \{j\{b\}\}) \\
 &\quad - \sum_{n=1}^{\infty} \left(\frac{ak\{a^{-1}(n+1)\} - bk\{b^{-1}(n+1)\}}{n(n+1)} \right) \\
 &\quad - \sum_{n=1}^{\infty} \left(\frac{\sum_{j=1}^k \{\{a\}(a'_{n+1} + j)\} - \sum_{j=1}^k \{\{b\}(b'_{n+1} + j)\}}{n(n+1)} \right).
 \end{aligned}$$

By Weyl’s theorem [Mo, p. 1] for any irrational α , and any fixed integer q ,

the sequences $\{\alpha n\}$ and $\{\alpha(q + n)\}$ are uniformly distributed, and thus

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k \{j\alpha\} = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k \{\{a\}(a'_{n+1} + j)\} = \frac{1}{2}.$$

Upon dividing by k and letting $k \rightarrow \infty$, we deduce the following corollary (which is the Main Theorem).

COROLLARY 7.2. *Let $a, b > 1$ be irrational numbers. Let $a_n = [an]$ and $b_n = [bn]$. Then*

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(\frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} \right) + \sum_{n=1}^{\infty} \frac{a\{a^{-1}(n+1)\} - b\{b^{-1}(n+1)\}}{n(n+1)} = \log \frac{a}{b}.$$

We proceed by rewriting the series in Theorem 7.1 as a sum over all integers. When doing this, we will need to change the strategy used in Theorem 2.1, since in general, we do not have the property that A and B partition the integers (here $A = (a_n)_{n=1}^{\infty}$ and $B = (b_n)_{n=1}^{\infty}$). When applying partial summation, we will use Theorem 5.5 instead of the remarks following Lemmas 5.1 and 5.3.

We begin with the left hand side of Theorem 7.1,

$$(7.1) \quad \sum_{n=1}^{\infty} \left(\frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} \right).$$

Again we divide the proof into several lemmas. Since $a = b$ gives zero trivially, throughout the proof we assume that $a < b$.

LEMMA 7.3. *Let $a > 1$ be any irrational number, and let q be the unique positive integer such that $1 \leq q < a < q + 1$. Set $c^{-1} = \{a\}$ and $f(n) = f_{1/c}(n)$. Then*

$$(7.2) \quad a_{n+1} = a_n + q + f(n).$$

Proof. Note that

$$\begin{aligned} a_{n+1} - a_n &= [a(n+1)] - [an] = [(q + \{a\})(n+1)] - [(q + \{a\})n] \\ &= [qn + q + \{a\}(n+1)] - [qn + \{a\}n] \\ &= qn + q - qn + [\{a\}(n+1)] - [\{a\}n] = q + f_{\{a\}}(n), \end{aligned}$$

hence $a_{n+1} = a_n + q + f(n)$. ■

Similarly, we can prove the following result:

LEMMA 7.4. *Let $b > 1$ be any irrational number, and let r be the unique positive integer such that $1 \leq r < b < r + 1$. Set $d^{-1} = \{b\}$ and $g(n) = g_{1/d}(n)$. Then*

$$(7.3) \quad b_{n+1} = b_n + r + g(n),$$

where $g_{1/d}(n)$ is the characteristic function of $1/d$.

REMARK 7.5. Lemmas 7.3 and 7.4 are written in a slightly different way than the corresponding lemmas in the proof of Theorem 2.1. This will allow us to write the sum over all positive integers without requiring that the sequences A and B partition the integers. In Theorem 2.1 we only needed two characteristic functions, since in that case $a = \{a\}^{-1} = \{b\}^{-1}$. In the present theorem we will need four characteristic functions. The changes in the rest of the proof of Theorem 2.2 are consequences of these variations.

LEMMA 7.6. *Let $1 \leq q < a < q + 1$ be any irrational number. Set $c^{-1} = \{a\}$ and $f(n) = f_{1/c}(n)$. Then for any integer k we have*

$$a_{n+k} = a_n + kq + \sum_{j \leq k} f(n + j).$$

Proof. From (7.1) we have

$$a_{n+k} - a_n = \sum_{j \leq k} (a_{n+j} - a_{n+j-1}) = \sum_{j=0}^{k-1} (q + f(n + j)) = kq + \sum_{j=0}^{k-1} f(n + j). \blacksquare$$

We can now write

$$(7.4) \quad \frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} = \frac{qk + \sum_{j=0}^{k-1} f(n + j)}{a_n} - \frac{rk + \sum_{j=0}^{k-1} g(n + j)}{b_n}.$$

Set $y = y(a, m) = [am]$ and $x = x(b, m) = [bm]$. As we did in Theorem 2.1, we define the following two additional arithmetic functions in terms of the characteristic functions $h_{1/a}$ and $w_{1/b}$, respectively.

DEFINITION 7.7. Define

$$h(n) = \begin{cases} h_{1/c}(n) & \text{if } n \leq y, \\ 0, & \text{otherwise,} \end{cases} \quad w(n) = \begin{cases} w_{1/d}(n) & \text{if } n \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

Now we use (7.4) and Definition 7.7 to write

$$\begin{aligned} \sum_{n \leq m} \left(\frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} \right) &= \sum_{n \leq x} \frac{qk + \sum_{j=0}^{k-1} f(n + j)}{a_n} - \sum_{n \leq y} \frac{rk + \sum_{j=0}^{k-1} g(n + j)}{b_n} \\ &= \sum_{n \leq x} \frac{qkh(a_n) + \sum_{j=0}^{k-1} h(a_n)f(n + j)}{a_n} \\ &\quad - \sum_{n \leq y} \frac{rkw(b_n) + \sum_{j=0}^{k-1} w(b_n)g(n + j)}{b_n}. \end{aligned}$$

Using (2.4) we immediately obtain the following lemma.

LEMMA 7.8. *For any two Beatty sequences a_n with slope a and b_n with slope b , let f and g be the characteristic sequences of $\{a\}$ and $\{b\}$, respectively. Also let h and w be given as in Definition 2.9 in terms of $h_{1/a}$*

and $w_{1/b}$, respectively. Then

$$\sum_{n \leq m} \left(\frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} \right) = \sum_{n \leq x} \frac{qkh(n) + \sum_{j=0}^{k-1} h(n)f([n/a] + j + 1)}{n} - \sum_{n \leq y} \frac{rkw(n) + \sum_{j=0}^{k-1} w(n)g([n/a] + j + 1)}{n},$$

where $q = [a]$ and $r = [b]$.

THEOREM 7.9. *Let $c > 1$ be any irrational number and set $d = \{c\}^{-1}$ and $q = c - \{c\}$. Let $c_n = [cn]$, $C = (c_n)_{n=1}^\infty$, $d = [dn]$ and $D = (d_n)_{n=1}^\infty$. Also let $c'_n = [c^{-1}n]$ and for any integer m , let $x = x(m, c) = [mc]$. Let $f(n)$ and $g(n)$ be given as in Definition 2.9 in terms of $f_{1/c}(n)$ and $g_{1/d}(n)$. Then*

$$\begin{aligned} \sum_{n=1}^x \frac{kqf(n) + \sum_{j=1}^k f(n)g([n/c] + j + 1)}{n} \\ = k(1 + \log m + \log c + \gamma) + \sum_{j=1}^k \{\{c\}j\} \\ - k \sum_{n=1}^\infty \frac{c\{c^{-1}(n+1)\} + \{\{c\}(c'_{n+1} + k + 1)\}}{n(n+1)} + O\left(\frac{k}{m}\right). \end{aligned}$$

Proof. Note that

$$\begin{aligned} \sum_{n=1}^x \frac{kqf(n) + \sum_{j=1}^k f(n)g([n/c] + j + 1)}{n} \\ = \sum_{n=1}^x \frac{\sum_{j=1}^k (qf(n) + f(n)g([n/c] + j + 1))}{n} \\ = \sum_{j=1}^k \sum_{n=1}^x \frac{qf(n) + f(n)g([n/c] + j + 1)}{n}. \end{aligned}$$

Now apply Theorem 5.5 to the inner sum to get

$$\begin{aligned} \sum_{j=1}^k \sum_{n=1}^x \frac{qf(n) + f(n)g([n/c] + j + 1)}{n} \\ = \sum_{j=1}^k \left(O\left(\frac{1}{m}\right) + 1 + \log m + \log c + \gamma + \{\{c\}(k+1)\} \right. \\ \left. - \sum_{n=1}^\infty \frac{c\{c^{-1}(n+1)\} + \{\{c\}(c'_{n+1} + k + 1)\}}{n(n+1)} \right), \end{aligned}$$

and then sum from $j = 1$ to k to complete the proof. ■

8. Proof of Theorem 7.1. From Lemma 7.8 we know that

$$\sum_{n \leq m} \left(\frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} \right) = \sum_{n \leq x} \frac{qkh(n) + \sum_{j=0}^{k-1} h(n)f([n/a] + j + 1)}{n} \\ - \sum_{n \leq y} \frac{rkw(n) + \sum_{j=0}^{k-1} w(n)g([n/b] + j + 1)}{n}.$$

By Theorem 7.9 we see that

$$(8.1) \quad \sum_{n=1}^x \frac{kqh(n) + \sum_{j=1}^k h(n)f([n/a] + j + 1)}{n} \\ = k(1 + \log m + \log a + \gamma) + \sum_{j=1}^k \{\{a\}j\} - \sum_{n=1}^{\infty} \frac{ak\{a^{-1}(n+1)\}}{n(n+1)} \\ - \sum_{n=1}^{\infty} \frac{\sum_{j=1}^k \{\{a\}(a'_{n+1} + j)\}}{n(n+1)} + O\left(\frac{k}{m}\right).$$

Also

$$(8.2) \quad \sum_{n=1}^x \frac{kqw(n) + \sum_{j=1}^k w(n)g([n/b] + j + 1)}{n} \\ = k(1 + \log m + \log b + \gamma) + \sum_{j=1}^k \{\{b\}j\} - \sum_{n=1}^{\infty} \frac{kb\{b^{-1}(n+1)\}}{n(n+1)} \\ - \sum_{n=1}^{\infty} \frac{\sum_{j=1}^k \{\{b\}(b'_{n+1} + j)\}}{n(n+1)} + O\left(\frac{k}{m}\right).$$

We subtract (8.2) from (8.1) to obtain

$$\sum_{n=1}^{\infty} \left(\frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} \right) = k(\log a - \log b) + \sum_{j=1}^k (\{j\{a\}\} - \{j\{b\}\}) \\ - \sum_{n=1}^{\infty} \left(\frac{ak\{a^{-1}(n+1)\} - bk\{b^{-1}(n+1)\}}{n(n+1)} \right) \\ - \sum_{n=1}^{\infty} \left(\frac{\sum_{j=1}^k \{\{a\}(a'_{n+1} + j)\} - \sum_{j=1}^k \{\{b\}(b'_{n+1} + j)\}}{n(n+1)} \right).$$

This completes the proof.

REMARK 8.1. In the last steps of this proof we see the $\log m$ cancellation mentioned in the introductory section.

9. Definition of Sturmian sequences from Beatty ratios and final remarks. The identity of Theorem 7.1 can be written as

$$\sum_{n=1}^{\infty} \left(\frac{a_{n+k}}{a_n} - \frac{b_{n+k}}{b_n} \right) + P(a, b) = k(\log a - \log b),$$

where the constant $P(a, b)$ represents the obvious missing expressions. This identity brings to mind the theorem of Frullani [Fr] which says that, under certain conditions on f ,

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx = A(\log a - \log b).$$

We conclude this paper by providing a new way to define Sturmian sequences in terms of differences of quotients of the form $a_{n+1}/a_n - b_{n+1}/b_n$. Sturmian sequences can also be defined in terms of differences of reciprocals of these quotients and other forms of difference quotients. This is in the spirit of [1], [Ki] and [3] in which the authors present several equivalent ways in which Beatty and Sturmian sequences can be generated.

THEOREM 9.1. *Let a and b be irrational numbers such that $1 < a < b$, and $\theta = \{a\} = \{b\}$. Define*

$$h(n) = \frac{a_{n+1}}{a_n} - \frac{b_{n+1}}{b_n}, \quad f(n) = \begin{cases} 1, & h(n) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and set $\mathbf{f} := f(1)f(2) \dots$. Also, let $g := g_\theta$ be the characteristic (Sturmian) sequence with slope θ defined by (2.3), and set $\mathbf{g} := g(1)g(2) \dots$. Then

$$\mathbf{f} = \mathbf{g}.$$

REMARK 9.2. By (2.4) we see that $h(n) > 0$ if and only if n is in the Beatty sequence B with slope $1/\theta$. Thus, given any Beatty sequence, we can define an infinite word \mathbf{g} with a 1 or a 0 in each position, and such that \mathbf{g} contains a 1 in the n th position if and only if $n \in B$.

Proof of Theorem 9.1. Suppose $b = r + \theta$ and $a = q + \theta$. Write $b_n = [(r + \theta)n] = rn + [\theta n] =: rn + \theta_n$, and similarly write $a_n = qn + \theta_n$. One can use (7.2) and (7.3) to write $h(n)$ as

$$\begin{aligned} h(n) &= \frac{a_{n+1}}{a_n} - \frac{b_{n+1}}{b_n} = \frac{a_n + q + g(n)}{a_n} - \frac{b_n + r + g(n)}{b_n} \\ &= \frac{\theta_n(q - r) + ng(n)(r - q)}{a_n b_n} = \frac{(ng(n) - \theta_n)(r - q)}{a_n b_n}. \end{aligned}$$

Since $\theta_n < n$ (because $\theta < 1$) and $r > q$, it follows that $g(n) = 1$ if and only if $h(n) > 0$. Thus, the theorem holds. ■

Acknowledgements. The author thanks Prof. Kenneth B. Stolarsky for a number of helpful discussions.

This research was partially supported by the Dominican Republic Ministerio de Educación Superior Ciencia y Tecnología (MESCYT), through the Fondo Nacional de Innovación y Desarrollo Científico y Tecnológico (FONDOCYT) (Grant Number 2015-1D2-186).

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