

Primes in arithmetic progressions to spaced moduli. III

by

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1. Introduction. Let

$$E(x, q) = \max_{(a, q)=1} \left| \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n) - \frac{x}{\phi(q)} \right|,$$

where Λ is the von Mangoldt function. Let

$$S_f = \{f(k) : k \in \mathbb{N}\},$$

where f is a polynomial of degree $d \geq 2$ with integer coefficients and positive leading coefficient. In analogy with the Bombieri–Vinogradov theorem, we would like to show that

$$(1.1) \quad \sum_{\substack{q \in S_f \\ Q < q \leq 2Q}} E(x, q) \ll_{A, \varepsilon} x Q^{1/d-1} (\log x)^{-A}$$

for $\varepsilon > 0$, $A > 0$ and $Q \leq x^{1/2-\varepsilon}$. In the general case, (1.1) is known only for $Q \leq x^{9/20-\varepsilon}$, and in the special case $f(X) = X^2$, for $Q \leq x^{43/90-\varepsilon}$ [2].

Here we refine the approach in [2] for $f(X) = X^2$.

THEOREM 1. *Let $f(X) = X^2$. Then (1.1) holds for $Q \leq x^{1/2-\varepsilon}$.*

To prove Theorem 1, we sharpen the auxiliary results on pp. 147–150 of [2]. With a little modification, we are then able to complete the proof of Theorem 1 by arguing as in [2]. The key new result is Lemma 2 below, which strengthens Lemma 11 of [2]. Thanks are due to James Maynard for suggesting in conversation the line of argument used to prove Lemma 2.

NOTATION. We write

$$\|\theta\| = \min_{n \in \mathbb{Z}} |\theta - n|$$

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and, for complex numbers c_1, \dots, c_N ,

$$\|c\|_2 = \left(\sum_{n=1}^N |c_n|^2 \right)^{1/2}.$$

The k th Riesz mean is defined by

$$(1.2) \quad A_k(x, q, a, d) = \frac{1}{k!} \sum_{\substack{\ell \leq x \\ \ell \equiv a \pmod{q} \\ \ell \equiv 0 \pmod{d}}} \left(\log \frac{x}{\ell} \right)^k \quad (k = 0, 1, \dots)$$

and we write

$$(1.3) \quad r_k(x, q, a, d) = A_k(x, q, a, d) - \frac{x}{qd}.$$

It is convenient to write $a^{(q)}$ for an arbitrary integer with $(a^{(q)}, q) = 1$.

We suppose, as we may, that x is large and ε is sufficiently small, and write $\delta = \varepsilon^2$. Except in Lemma 5, implied constants depend at most on ε or, when A appears in the result, on ε and A .

The conductor of a primitive Dirichlet character χ is denoted by $C(\chi)$.

2. The large sieve for square moduli

LEMMA 1. Let $\Delta > 0$ and $Q \geq 1$. For β real, let $\mathcal{N}(\beta)$ denote the number of relatively prime pairs a, q with $1 \leq a \leq q^2$, $q \leq Q$, and

$$\left\| \frac{a}{q^2} - \beta \right\| \leq \Delta.$$

Then

$$\mathcal{N}(\beta) \ll (Q\Delta^{-1})^\varepsilon (Q^3\Delta + Q^{1/2}).$$

Proof. This is due to Baier and Zhao [1, Section 11]. ■

LEMMA 2. Let $Q \geq 1$. Let a_1, \dots, a_N be complex numbers,

$$T(\alpha) = \sum_{n=1}^N a_n e(n\alpha).$$

Let $g \in \mathbb{N}$. Then

$$(2.1) \quad \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a, gq^2)=1}}^{gq^2} \left| T\left(\frac{a}{gq^2}\right) \right|^2 \ll (QN)^\varepsilon \left(1 + \frac{g}{N}\right) (gQ^3 + Q^{1/2}N) \|a\|_2^2.$$

Proof. We first show that, for real α and $\Delta > 0$, the number $\mathcal{M}(\alpha)$ of solutions of

$$(2.2) \quad \left\| \frac{a}{gq^2} - \alpha \right\| \leq \Delta, \quad 0 \leq a \leq gq^2 - 1, (a, gq^2) = 1, 1 \leq q \leq Q,$$

satisfies

$$(2.3) \quad \mathcal{M}(\alpha) \ll (1 + g\Delta)(Q^3(g\Delta) + Q^{1/2})(Q\Delta^{-1})^\varepsilon.$$

To see this, write $a = b + q^2n$, $0 \leq n < g$, $0 \leq b < q^2$. Then (2.2) implies

$$\left\| \frac{b}{q^2} - g\alpha \right\| = \left\| \frac{b + q^2n}{q^2} - g\alpha \right\| \leq g\Delta.$$

The number of possible pairs b, q is

$$\ll (Q^3(g\Delta) + Q^{1/2})(Q\Delta^{-1})^\varepsilon$$

by Lemma 1. Once b is fixed, (2.2) implies

$$\left\| \frac{b}{gq^2} + \frac{n}{g} - \alpha \right\| \leq \Delta.$$

There are at most $2g\Delta + 1$ possible n , and the bound (2.3) follows.

By [4, Theorem 2.1], the left-hand side of (2.1) is bounded by

$$(2.4) \quad \ll (N + \Delta^{-1}) \left(\max_{\alpha \in \mathbb{R}} \mathcal{M}(\alpha) \right) \|a\|_2^2$$

for any $\Delta > 0$. We take $\Delta = N^{-1}$ and apply (2.3) to obtain the lemma. ■

LEMMA 3. Let $Q = x^\theta$ and $0 < \lambda \leq \theta$, $x \geq M \gg x^{\theta-\lambda}$. Let c_1, \dots, c_M be complex numbers. Let

$$T(\lambda) = \sum_{Q < q^2 \leq 2Q} \sum_{\substack{\chi \pmod{q^2} \\ x^\lambda < C(\chi) \leq 2x^\lambda}} \left| \sum_{m=1}^M c_m \chi(m) \right|^2.$$

Then

$$T(\lambda) \ll x^\varepsilon (Q^{1/2}x^\lambda + Q^{3/4}Mx^{-\lambda/2}) \|c\|_2^2.$$

Proof. For a character $\chi \pmod{q^2}$ counted in $T(\lambda)$, induced by a primitive character $\chi' \pmod{C(\chi)}$, we have

$$(2.5) \quad C(\chi) = gk^2 \in (x^\lambda, 2x^\lambda]$$

with g squarefree, $k \in \mathbb{N}$; and

$$\chi(m) = \begin{cases} \chi'(m) & \text{if } (m, q) = 1, \\ 0 & \text{if } (m, q) > 1. \end{cases}$$

Since $C(\chi) \mid q^2$, we have

$$vgk^2 = q^2 \in (x^\theta, 2x^\theta]$$

for a natural number v . Obviously $v = gt^2$, $t \in \mathbb{N}$, and

$$(2.6) \quad x^{\theta/2} < q = gtk \leq (2x^\theta)^{1/2}.$$

It follows that

$$(2.7) \quad T(\lambda) \leq \sum_{\substack{g \geq 1, t \geq 1 \\ x^{\theta-\lambda}/2 < gt^2 \leq 2x^{\theta-\lambda}}} \sum_{\substack{x^{\theta/2}/gt < k \leq 2x^{\theta/2}/gt}} \sum^* \left| \sum_{\substack{m \leq M \\ (m,t)=1}} c_m \chi'(m) \right|^2.$$

Here \sum^* denotes a sum restricted to primitive characters. By a standard inequality [3, Chapter 27, (10)],

$$\sum^* \left| \sum_{\substack{m \leq M \\ (m,t)=1}} c_m \chi'(m) \right|^2 \leq \frac{\phi(gk^2)}{gk^2} \sum_{\substack{a=1 \\ (a,gk^2)=1}}^{gk^2} \left| \sum_{\substack{m=1 \\ (m,t)=1}}^M c_m e\left(\frac{am}{gk^2}\right) \right|^2.$$

Using Lemma 2 for fixed g and t , we see that the sum over k on the right-hand side of (2.7) is

$$\ll (QM)^{\varepsilon/3} \left(g \left(\frac{Q^{1/2}}{gt} \right)^3 + \frac{Q^{1/4}}{(gt)^{1/2}} M \right) \|c\|_2^2.$$

(Note that $g \ll x^{\theta-\lambda} \ll M$ here.) For some $G \geq 1, T \geq 1$ with $GT^2 \asymp x^{\theta-\lambda}$, we have

$$\begin{aligned} T(\lambda) &\ll (\log x)^2 (QM)^{\varepsilon/3} \sum_{G \leq g < 2G} \sum_{T \leq t < 2T} \left\{ g \left(\frac{Q^{1/2}}{gt} \right)^3 + \frac{Q^{1/4}}{(gt)^{1/2}} M \right\} \|c\|_2^2 \\ &\ll x^\varepsilon (Q^{3/2} G^{-1} T^{-2} + Q^{1/4} M G^{1/2} T^{1/2}) \|c\|_2^2 \\ &\ll x^\varepsilon (Q^{1/2} x^\lambda + Q^{3/4} M x^{-\lambda/2}) \|c\|_2^2. \end{aligned}$$

This completes the proof of Lemma 3. ■

LEMMA 4. *Let $1 \leq x^\lambda \leq Q \ll x^{1/2-\varepsilon}$, $x^\lambda \geq Q^{1/2} x^{\varepsilon/6}$. Let H and K satisfy*

$$Qx^{-\lambda} \ll K \ll H \ll x^{3/5}, \quad HK \ll x.$$

Let a_n ($K < n \leq 2K$) and b_m ($H < m \leq 2H$) be complex numbers with $a_n \ll x^\delta$ and $b_m \ll x^\delta$. Let

$$\begin{aligned} K(s, \chi) &= \sum_{K < n \leq 2K} a_n \chi(n) n^{-s}, \quad H(s, \chi) = \sum_{H < m \leq 2H} b_m \chi(m) m^{-s}, \\ S &= \sum_{Q < q^2 \leq 2Q} \sum_{\substack{\chi \pmod{q^2} \\ x^\lambda < C(\chi) \leq 2x^\lambda}} \left| H\left(\frac{1}{2} + it, \chi\right) K\left(\frac{1}{2} + it, \chi\right) \right|. \end{aligned}$$

Then

$$S \ll x^{1/2-\varepsilon/20} Q^{1/2}.$$

Proof. We apply the Cauchy–Schwarz inequality to S , followed by applications of Lemma 3 to each of the two sums over q, χ . The conditions

$$H \gg x^{\theta-\lambda}, \quad K \gg x^{\theta-\lambda}$$

are fulfilled since $H \gg K \gg Qx^{-\lambda}$. As $\sum_m |b_m m^{-1/2-it}|^2 \ll x^{2\delta}$ and similarly for $\sum_n |a_n n^{-1/2-it}|^2$, we have

$$\begin{aligned} S &\ll x^{3\delta} (Q^{1/4} x^{\lambda/2} + Q^{3/8} K^{1/2} x^{-\lambda/4}) (Q^{1/4} x^{\lambda/2} + Q^{3/8} H^{1/2} x^{-\lambda/4}) \\ &\ll x^{3\delta} (Q^{1/2} x^\lambda + Q^{3/4} x^{1/2-\lambda/2} + Q^{5/8} x^{\lambda/4} H^{1/2}) \\ &\ll x^{3\delta} (Q^{3/2} + Q^{3/4} x^{1/2-\lambda/2} + Q^{7/8} x^{3/10}). \end{aligned}$$

Each of these three terms is $\ll x^{1/2-\varepsilon/20} Q^{1/2}$:

$$\begin{aligned} Q^{3/2} x^{3\delta} (x^{1/2-\varepsilon/20} Q^{1/2})^{-1} &\ll Q x^{\varepsilon/20+3\delta-1/2} \ll 1, \\ Q^{3/4} x^{1/2-\lambda/2+3\delta} (x^{1/2-\varepsilon/4} Q^{1/2})^{-1} &\ll Q^{1/4} x^{-\lambda/2+3\delta+\varepsilon/20} \ll 1, \\ Q^{7/8} x^{3/10+3\delta} (Q^{1/2} x^{1/2-\varepsilon/20})^{-1} &\ll Q^{3/8} x^{-1/5+\varepsilon/20+3\delta} \ll 1. \end{aligned}$$

This completes the proof of Lemma 4. ■

3. Proof of Theorem 1. It is convenient to write $S(Q) = \{q^2 : Q < q^2 \leq 2Q\}$.

LEMMA 5. *Let $0 < \gamma < 1$. There is a subset $F(Q)$ of $S(Q)$ with*

$$\#F(Q) \ll Q^{1/2-\beta}$$

such that for $q^2 \in S(Q) \setminus F(Q)$, χ a nonprincipal character (mod q^2) and $\operatorname{Re} s = 1/2$, we have

$$\sum_{n \leq N} \chi(n) n^{-s} \ll |s| N^{1/2-\beta} \quad (N \geq q^\gamma).$$

Here $\beta = \beta(\gamma) > 0$. The implied constants depend on γ .

Proof. This is a special case of [2, Lemma 6]. ■

We shall refer to $F(Q)$ in the remaining lemmas. The following lemma is a variant of [2, Proposition 1].

LEMMA 6. *Let M_1, \dots, M_{15} be numbers with $M_1 \geq \dots \geq M_{15} \geq 1$, and suppose that $\{1, \dots, 15\}$ has a partition into subsets A, B such that*

$$\prod_{i \in A} M_i \ll x^{1/2-3\varepsilon/4}, \quad \prod_{i \in B} M_i \ll x^{1/2-3\varepsilon/4}.$$

Let $a_i(m)$ ($M_i/2 < m \leq M_i$, $1 \leq i \leq 15$) be complex sequences with

$$|a_i(m)| \leq \log m \quad (1 \leq i \leq 15, M_i/2 < m \leq M_i).$$

Suppose that whenever $M_i > x^{1/8}$, then $a_i(m)$ is 1 ($M_i/2 < m \leq M_i$) or

$\log m$ ($M_i/2 < m \leq M_i$). Let

$$M_i(s, \chi) = \sum_{M_i/2 < m \leq M_i} a_i(m) \chi(m) m^{-s},$$

$$L = x/(M_1 \dots M_{15}), \quad B_1(s, \chi) = \sum_{Lx^{-\varepsilon} < n \leq L} \chi(n) n^{-s}.$$

Then for $\operatorname{Re} s = 1/2$ and $Q \ll x^{1/2-\varepsilon}$,

$$S := \sum_{q \in S(Q) \setminus F(Q)} \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} |B_1(s, \chi) M_1(s, \chi) \dots M_{15}(s, \chi)| \ll |s|^3 Q^{1/2} x^{1/2-3\delta}.$$

Proof. It suffices to show for $0 \leq \lambda \leq \theta$ that

$$(3.1) \quad S(\lambda) \ll |s|^3 Q^{1/2} x^{1/2-4\delta},$$

where $S(\lambda)$ is the subsum of S defined by the additional condition

$$x^\lambda < C(\lambda) \leq 2x^\lambda.$$

Arguing exactly as in the proof of [2, Lemma 10], we deduce (3.1) unless (writing as usual $Q = x^\theta$) we have

$$(3.2) \quad \lambda > (5\theta + \varepsilon)/6,$$

We now suppose that (3.2) holds. We decompose $B_1(s, \chi)$ into $O(\log x)$ subsums $M_{16}(x, \chi)$ defined by a condition

$$M_{16}/2 < n \leq M_{16},$$

where $Lx^{-\varepsilon} \leq M_{16} < L$. It suffices to prove the analogue of (3.1) with $B(s, \chi)$ replaced by $M_i(s, \chi)$, and 6δ in place of 4δ .

Rearranging M_1, \dots, M_{16} as $N_1 \geq \dots \geq N_{16}$, write $N_i(s, \chi)$ for the corresponding Dirichlet polynomials and

$$N_i = x^{\beta_i}.$$

Then $\beta_1 \geq \dots \geq \beta_{16} \geq 0$ and $1 - \varepsilon \leq \beta_1 + \dots + \beta_{16} \leq 1$.

We can use the argument in the proof of [2, Lemma 15] to complete the present proof whenever $\beta_1 + \beta_2 > 3/5$. Suppose now that

$$\beta_1 + \beta_2 < 3/5.$$

As shown in the proof of [2, Lemma 15], there is a subset W of $\{1, \dots, 16\}$ such that

$$x^{1/2} \ll H := \prod_{j \in W} 2M_j \ll x^{3/5}.$$

Let $K := \prod_{j \leq 16, j \notin W} 2M_j$. We see that

$$x^{2/5-\varepsilon} \ll K \ll H, \quad HK \ll x.$$

Let

$$H(s, \chi) = \prod_{j \in W} M_j(s, \chi), \quad K(s, \chi) = \prod_{\substack{1 \leq j \leq 16 \\ j \notin W}} M_j(s, \chi).$$

We note that $K \gg x^{\theta-\lambda}$, since $\theta - \lambda < 1/12$. Hence we may apply Lemma 3 to obtain the desired bound in the form

$$\sum_{q \in S(Q)} \sum_{\chi \pmod{q}} |H(s, \chi)K(s, \chi)| \ll x^{1/2-6\delta} Q^{1/2}. \blacksquare$$

Our final lemma is a variant of [2, Lemma 18].

LEMMA 7. *Let $a_i(m)$ ($1 \leq i \leq 15$) be nonnegative sequences satisfying the hypotheses of Lemma 6. Let*

$$u_d = \sum_{\substack{d=m_1 \dots m_{15} \\ M_i/2 < m_i \leq M_i \ \forall_i}} a_1(m_1) \dots a_{15}(m_{15})$$

for $D_1 < d \leq D$, with $D = M_1 \dots M_{15}$, $D_1 = 2^{-15}D$. Let $Q \ll x^{1/2-\varepsilon}$. Then for every $A > 0$,

$$\sum_{q \in S(Q) \setminus F(Q)} \left| \sum_{D_1 < d \leq D} u_d r_0(x, q, a^{(q)}, d) \right| \ll \frac{x}{Q^{1/2}(\log x)^A}.$$

Proof. Just as in the proof of [2, Lemma 18], it suffices to show that

$$\sum_{q \in S(Q) \setminus F(Q)} \left| \sum_{D_1 < d \leq D} u_d r_4(x, q, a^{(q)}, d) \right| \ll x^{1-\delta} Q^{-1/2}.$$

The condition from small ℓ in (1.2), (1.3) to r_4 is negligible:

$$\begin{aligned} \sum_{D_1 < d \leq D} |u_d| |r_4(x^{1-\varepsilon}, q, a^{(q)}, d)| &\ll x^{\varepsilon/3} \left\{ \sum_{\substack{md \leq x^{1-\varepsilon} \\ md \equiv a \pmod{q}}} 1 + \sum_{d \leq x^{1-2\varepsilon/3}} \frac{x^{1-\varepsilon}}{qd} \right\} \\ &\ll x^{1-\delta} Q^{-1}, \end{aligned}$$

thus it suffices to show that

$$(3.3) \quad \sum_{q \in S(Q) \setminus F(Q)} \sum_{D_1 < d \leq D} u_d (r_4(x, q, a^{(q)}, d) - r_4(x^{1-\varepsilon}, q, a^{(q)}, d)) \ll x^{1-\delta} Q^{-1/2}.$$

We now follow the argument in the proof of [2, Lemma 18] to show that (3.3) follows from

$$(3.4) \quad \int_{\operatorname{Re} s = 1/2} \sum_{q \in S(Q) \setminus F(Q)} \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} \left| \sum_{D_1 < d \leq D} u_d \chi(d) d^{-s} \right| |B_1(s, \chi)| \frac{|ds|}{|s|^5} \ll x^{1/2-\delta} Q^{-1/2}.$$

Here $B_1(s, \chi)$ is the Dirichlet polynomial in Lemma 6. At this point we see that (3.4) follows from Lemma 6. ■

Proof of Theorem 1. Just as in [2], we reduce this to showing that

$$(3.5) \quad \sum_{q \in S(Q) \setminus F(Q)} \left| \sum_{m, n \leq Qx^{\varepsilon/4}} \Lambda(m) \mu(n) r_0(x, q, a^{(q)}, mn) \right| \ll xQ^{-1/2} (\log x)^{-A}$$

for every $A > 0$. We use Heath-Brown's decomposition of $\Lambda(m)$, and a slight variant of this decomposition for $\mu(n)$, to show that (3.5) follows from Lemma 7; full details are given on p. 158 of [2]. ■

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