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On algebraic integers in short intervals and near smooth curves

by

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1. Introduction. Many problems in the theory of Diophantine approximation are related to the distribution of algebraic numbers and algebraic integers [15, 25]. In this paper we wish to investigate the distribution of algebraic integers on the real line and the distribution of the points with algebraic conjugate integer coordinates in the Euclidean plane.

Let $P(t) = a_n t^n + \cdots + a_1 t + a_0$ be a polynomial with integer coefficients of degree n. By the *height* of P we mean $H(P) = \max_{0\le j\le n} |a_j|$. Suppose P is irreducible with coprime coefficients. Its roots α are *algebraic numbers* of degree n and height $H(\alpha) = H(P)$. When $a_n = 1$, these roots are called *algebraic integers* of degree n and height $H(\alpha) = H(P)$.

Let $\# S$ denote the cardinality of a finite set S and $\mu_k D$ the Lebesgue measure of a measurable set $D \subset \mathbb{R}^k$, $k \in \mathbb{N}$. We define the following class of polynomials:

$$\mathcal{P}_n(Q) = \{P \in \mathbb{Z}[t] : \deg P \le n,\ H(P) \le Q\}.$$

We emphasize that we restrict our attention to the case when $Q > Q_0$ is a sufficiently large integer. Furthermore, we will denote by $c_j > 0$, $j \in \mathbb{N}$, positive real numbers independent of $H(P)$ and Q.

The first part of this paper is devoted to the study of the one-dimensional case, namely algebraic integers. Over the last 20 years, new results providing a deeper insight into the distribution of algebraic numbers have been obtained. In particular, lower and upper bounds for the distances between algebraic conjugate numbers and the roots of different integer polynomials were obtained in [3, 10, 14, 20].

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Let us consider an interval $I \subset [-1/2; 1/2]$ of length $|I| = c_1 Q^{-1}$. It is of interest to know whether I contains algebraic numbers α with $\deg \alpha \le n$ and $H(\alpha) \le Q$. In case of a positive answer we are also interested in finding a lower bound for the number of such α's. These problems were solved for $n = 3$ by V. Bernik, N. Budarina and H. O'Donnell [9], and a general result for all integer n was proved by V. Bernik and F. Götze [6]. The result of [6] states that for any integer $Q \ge 1$ there exists an interval I of length $\frac{1}{2}Q^{-1}$ which contains no algebraic number α of height $\le Q$. On the other hand, for $Q > Q_0(n)$ sufficiently large any interval I with $|I| \ge c_1 Q^{-1}$ contains at least $c_2 Q^{n+1}|I|$ real algebraic numbers of degree $\le n$ and height $\le Q$. Furthermore, these algebraic numbers form a regular system [13].

Our purpose is to obtain a similar result in the case of algebraic integers.

THEOREM 1.1. *For any integer $Q \ge 1$ there exists an interval I of length $\frac{1}{2}Q^{-1}$ which contains no algebraic integers of height $\le Q$ and degree ≥ 2.*

It is easily seen that Theorem 1.1 follows from the results in [6], since the algebraic integers form a subset of the set of algebraic numbers.

THEOREM 1.2. *For any integer $n \ge 2$ there exist constants c_3 and c_4 such that any interval I of length $c_3 Q^{-1}$ contains at least $c_4 Q^4 |I|$ real algebraic integers of degree n and height $\le Q$ with $Q > Q_0(n)$.*

REMARK 1. It should be mentioned that the condition $I \subset [-1/2; 1/2]$ is not essential to the proof and can be dropped (see [2, 12] for more details).

REMARK 2. Another way of stating Theorem 1.2 is to say that the set of real algebraic integers of degree n forms a regular system.

DEFINITION. Let Γ be a countable set of real numbers and $N : \Gamma \to \mathbb{R}^+$ be a positive-valued function. The pair (Γ, N) is called a *regular system* if there exists a constant $c_5 = c_5(\Gamma, N) > 0$ such that for every interval $I \subset \mathbb{R}$ the following property is satisfied: for a sufficiently large number $T_0 = T_0(\Gamma, N, I) > 0$ and every integer $T > T_0$ there exist $\gamma_1, \dots, \gamma_t \in \Gamma \cap I$ satisfying

- $N(\gamma_i) \le T$, $1 \le i \le t$,
- $|\gamma_i - \gamma_j| > T^{-1}$, $1 \le i < j \le t$,
- $t > c_5 T|I|$.

A simple example of a regular system is the set of non-zero rational numbers p/q together with the function $N(p/q) := q^2$. Similarly, the set of real algebraic numbers α of degree n forms a regular system with respect to the function $N(\alpha) = (H(\alpha)/(1 + |\alpha|)^n)^{n+1}$, and the set of real algebraic integers α of degree n forms a regular system with respect to the function $N(\alpha) = (H(\alpha)/(1 + |\alpha|)^{n-1})^n$ (see [1, 2, 12]). The interest of Theorem 1.2

is that in contrast to the result of [12] it allows one to clarify the relation between the parameter T_0 and the length of the interval I.

The results mentioned above have many interesting applications. For example, regular systems of algebraic numbers are used to obtain lower bounds for the Hausdorff dimension of various algebraic number sets [1, 18] and to prove Khinchin-type theorems in the case of divergence [2, 5, 11].

In the second part of our paper we study a two-dimensional analogue of Theorem 1.2. An interesting result related to the distribution of points with algebraic conjugate coordinates in the Euclidean plane was obtained in [8, 7]. Let us consider a rectangle $E = I_1 \times I_2$, where I_1, I_2 are intervals of lengths $|I_1| = Q^{-s_1}$, $|I_2| = Q^{-s_2}$ with $0 < s_1 + s_2 < 1$. Furthermore, from now on we make the assumption that

$$E \cap \{(x, y) \in \mathbb{R}^2 : |x - y| \le \varepsilon\} = \emptyset, \tag{1.1}$$

where $\varepsilon > 0$ is a sufficiently small constant. Since the distance between algebraic conjugate numbers is bounded below [14, 20], this condition is not particularly restrictive, but it will simplify our argument. We call a point (α, β) an *algebraic point* if α and β are algebraic conjugate numbers, and an *algebraic integer point* if α and β are algebraic conjugate integers. In [7] it is shown that for $Q > Q_0(n, \varepsilon, s_1, s_2)$ any rectangle E of size $\mu_2 E = Q^{-s_1-s_2}$ with $0 < s_1 + s_2 < 1$ contains at least $c_6 Q^{n+1} \mu_2 E$ algebraic points (α, β) with $\deg \alpha = \deg \beta \le n$, $n \ge 2$ and $H(\alpha) = H(\beta) \le Q$.

We prove that a similar estimate holds in the case of algebraic integer points.

THEOREM 1.3. *Let $E = I_1 \times I_2$ be a rectangle with sides $|I_1| = Q^{-s_1}$ and $|I_2| = Q^{-s_2}$ satisfying* (1.1). *Then for any integer $n \ge 4$ and $0 < s_1 + s_2 < 1$ there exists a constant c_7 such that E contains at least $c_7 Q^n \mu_2 E$ algebraic integer points (α, β) with $\deg \alpha = \deg \beta = n$ and $H(\alpha) = H(\beta) \le Q$ with $Q > Q_0(n, \varepsilon, s_1, s_2)$.*

REMARK 3. It should be noted that the position of the rectangle E is assumed to be fixed, namely its midpoint (d_1, d_2) is independent of Q. Therefore, the values c_7 and Q_0 may depend on d_1 and d_2.

This theorem deals with a simple figure like a rectangle, but it allows one to obtain analogous estimates for more complicated shapes. In particular, a number of interesting problems arise when the distribution of algebraic points in neighborhoods of smooth curves is investigated [22]. Let us mention several recent results in this area. Upper and lower bounds of the same order for the number of rational points near smooth curves have been obtained in [4] and [26]. The 2014 paper [8] presents a lower estimate for the number of algebraic points of arbitrary degree in neighborhoods of smooth curves.

Our main theorem is a restatement of the results of [8] in terms of algebraic integers.

Theorem 1.4. *Let $y = f(x)$ be a continuous differentiable function on an interval $J = [a; b]$ such that $\sup_{x\in J} |f'(x)| < \infty$. Denote*

$$L_J(Q, \lambda) = \{(x, y) \in \mathbb{R}^2 : x_1 \in J,\ |y - f(x)| < c_8 Q^{-\lambda}\}$$

for $0 < \lambda < 1/2$. Then for any integer $n \geq 4$ there exists a constant c_9 such that for $Q > Q_0(n, J, f, \lambda)$ the set $L_J(Q, \lambda)$ contains at least $c_9 Q^{n-\lambda}$ algebraic integer points (α, β) with $\deg \alpha = \deg \beta = n$ and $H(\alpha) = H(\beta) \leq Q$.

Proof. We give only the main ideas of the proof. For more details we refer the reader to [7].

Consider the graph of $y = f(x)$ and the strip $L_J(Q, \lambda)$ for a fixed $0 < \lambda < 1/2$. Divide the strip $L_J(Q, \lambda)$ into segments

$$T_i = \{(x, y) \in \mathbb{R}^2 : x \in J_i,\ |y - f(x)| \leq Q^{-\lambda}\},$$

where $J_i = [x_{i-1}, x_i]$, $x_i = x_{i-1} + c_{10} Q^{-\lambda}$, $x_0 = a$ and $1 \leq i \leq m$. It is easy to check that $m > c_{11} Q^{\lambda}$ for $Q > Q_0$. Let

$$\bar{f}_i = \tfrac{1}{2}\Big(\max_{x\in J_i} f(x) + \min_{x\in J_i} f(x)\Big).$$

Consider the rectangles

$$E_i = \{(x, y) \in \mathbb{R}^2 : x \in J_i,\ |y - \bar{f}_i| \leq c_{12} Q^{-\lambda}\},$$

where c_{12} is so chosen that $E_i \subset T_i$.

From Theorem 1.3 it follows that every rectangle E_i, $i = 1, \ldots, m$, contains at least $c_{13} Q^{n-2\lambda}$ algebraic integer points of degree n and height at most Q. Since $m > c_{11} Q^{\lambda}$, there must be at least $c_9 Q^{n-\lambda}$ algebraic integer points $(\alpha, \beta) \in L_J(Q, \lambda)$. ∎

2. Auxiliary statements. In this section we collect some lemmas which will be used to prove Theorems 1.2 and 1.3. The first paper discussing approximation by algebraic integers was written by H. Davenport and W. M. Schmidt [17]. Recently, their approach has been further developed by Y. Bugeaud [12]. In our paper we are going to apply some of his ideas. The main geometric ingredient is Minkowski's theorems from the geometry of numbers.

Lemma 2.1 (Minkowski's 2nd theorem on successive minima). *Let K be a bounded centrally symmetric convex body in $\mathbb{R}^n$ with successive minima $\tau_1, \ldots, \tau_n$. Then*

$$2^n/n! \leq \tau_1 \ldots \tau_n V(K) \leq 2^n.$$

The best general references here are [16, p. 203], [21, p. 59].

Lemma 2.2 (Bertrand postulate). *For any integer $n \geq 2$ there exists a prime p such that $n < p < 2n$.*

This was proved by P. Chebyshev in 1850 (see for instance [23, Theorem 2.4]).

LEMMA 2.3 (Eisenstein criterion). *Let $P(t) = a_n t^n + \dots + a_1 t + a_0$ be a polynomial with integer coefficients. If there exists a prime number p such that*

$$\begin{cases} a_n \not\equiv 0 \bmod p, \\ a_i \equiv 0 \bmod p, \quad i = 0, \dots, n-1, \\ a_0 \not\equiv 0 \bmod p^2, \end{cases} \tag{2.1}$$

then P is irreducible over the rational numbers.

For a proof see [19].

LEMMA 2.4. *Consider a point $x \in \mathbb{R}$ and a polynomial P with zeros $\alpha_1, \dots, \alpha_n$, where $|x - \alpha_1| = \min_i |x - \alpha_i|$. Then*

$$|x - \alpha_1| \le n|P(x)| \cdot |P'(x)|^{-1}.$$

Proof. We have

$$|P'(x)|\,|P(x)|^{-1} \le \sum_{i=1}^{n} |x - \alpha_i|^{-1} \le n|x - \alpha_1|^{-1}. \blacksquare$$

LEMMA 2.5 (see [6]). *Let $I \subset \mathbb{R}$ be an interval of length $c_{14} Q^{-1}$, where $c_{14} > c_0$. Denote by $\mathcal{L}_n^1 = \mathcal{L}_n^1(Q, \delta_0, I)$ the set of points $x \in I$ such that there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying*

$$\begin{cases} |P(x)| < Q^{-n}, \\ |P'(x)| < \delta_0 Q. \end{cases}$$

Then $\mu_1 \mathcal{L}_n^1 < \frac{1}{4}|I|$ for $\delta_0 = \delta_0(n) > 0$ sufficiently small and $Q > Q_0$.

REMARK 4. It suffices to take $\delta_0(n) = 2^{-n-8} n^{-2}$ (see [6] for more details).

This lemma is a base for the proof of Theorem 1.2.

LEMMA 2.6 (see [7]). *Let $E = I_1 \times I_2$ be a rectangle with midpoint (d_1, d_2) and $|I_i| = Q^{-s_i}$ with $0 < s_1 + s_2 < 1$. Given positive v_1, v_2 satisfying $v_1 + v_2 = n - 1$, let $\mathcal{L}_n^2 = \mathcal{L}_n^2(Q, \delta_0, E, v_1, v_2)$ be the set of points $(x, y) \in E$ such that there exists $P \in \mathcal{P}_n(Q)$ satisfying*

$$\begin{cases} |P(x)| < h_1 Q^{-v_1}, \quad |P(y)| < h_2 Q^{-v_2}, \\ \min\{|P'(x)|, |P'(y)|\} < \delta_0 Q, \end{cases} \tag{2.2}$$

where $h_i = ((|d_i| + 1)^{n+1} - 1)|d_i|^{-1}$, $i = 1, 2$. Then $\mu_2 \mathcal{L}_n^2 < \frac{1}{4}\mu_2 E$ for $\delta_0 = \delta_0(n, d_1, d_2) > 0$ sufficiently small and $Q > Q_0$.

Remark 5. An easy computation shows that for every $(x, y) \in E$ and all $P \in \mathcal{P}_n(Q)$ we have

$$|P(x)| < h_1 Q, \qquad |P(y)| < h_2 Q.$$

Hence the values v_1 and v_2 lie between -1 and n.

Remark 6. It is easily seen (for example from Lemma 2.4) that for a fixed polynomial P the set of points $(x, y) \in \mathbb{R}^2$ satisfying (2.2) is contained in a rectangle $\sigma_P = J_1 \times J_2$ with $\mu_2 \sigma_P \le \frac{1}{4}\mu_2 E$ (see [7]). If $I_1 \subset J_1$ or $I_2 \subset J_2$, we consider the rectangle $I_1 \times J_2$ or $J_1 \times I_2$ instead of σ_P to estimate the measure of $\mathcal{L}_n^2$.

3. Proof of Theorem 1.2. Let $\mathcal{L}_{n-1}^1 = \mathcal{L}_{n-1}^1(Q, \delta_0, I)$ be the set of $x \in I$ such that there exists $P \in \mathcal{P}_{n-1}(Q)$ satisfying

$$\begin{cases} |P(x)| < Q^{-n+1}, \\ |P'(x)| < \delta_0 Q. \end{cases} \tag{3.1}$$

From Lemma 2.5 it follows that

$$\mu \mathcal{L}_{n-1}^1 \le \tfrac{1}{4}|I|$$

for $Q > Q_0$ and $\delta_0 < 2^{-n-7}(n-1)^{-2}$.

Consider the set $B^1 = I \setminus \mathcal{L}_{n-1}^1$. Since for any $x \in I$ there exists $P \in \mathcal{P}_{n-1}(Q)$ satisfying $|P(x)| < Q^{-n+1}$, we conclude that for any $x_0 \in B^1$ and $P \in \mathcal{P}_{n-1}(Q)$, the system

$$\begin{cases} |P(x_0)| < Q^{-n+1}, \\ |P'(x_0)| \ge \delta_0 Q \end{cases}$$

is satisfied, and $\mu_1 B^1 \ge \frac{3}{4}|I|$.

Pick $x_0 \in B^1$ and examine the successive minima $\tau_1, \ldots, \tau_n$ of the compact convex set K defined by the inequalities

$$\begin{cases} |a_{n-1}x_0^{n-1} + \cdots + a_1 x_0 + a_0| \le Q^{-n+1}, \\ |(n-1)a_{n-1}x_0^{n-2} + \cdots + 2a_2 x_0 + a_1| \le Q, \\ |a_{n-1}|, \ldots, |a_2| \le Q. \end{cases} \tag{3.2}$$

Suppose $\tau_1 \le \delta_0$. Then for δ_0 sufficiently small there exists a non-zero $P_0 \in \mathcal{P}_{n-1}(Q)$ satisfying

$$\begin{cases} |P_0(x_0)| \le \delta_0 Q^{-n+1} < Q^{-n+1}, \\ |P_0'(x_0)| \le \delta_0 Q, \\ H(P_0) \le Q. \end{cases}$$

This contradicts $x_0 \in B^1 = I \setminus \mathcal{L}_{n-1}^1$, implying that $\tau_{n-1} \ge \cdots \ge \tau_1 > \delta_0$. Since the volume $V(K)$ of the set K is equal to 2^n, we deduce from Lemma 2.1 that $\tau_1 \ldots \tau_n \le 1$, and hence $\tau_n \le \delta_0^{-n+1}$. Therefore we can choose n linearly independent polynomials with integer coefficients $P_i(t) = a_{i,n-1}t^{n-1} + \cdots + a_{i,1}t + a_{i,0}$, $1 \le i \le n$, satisfying

$$\begin{cases} |P_i(x_0)| \le \delta_0^{-n+1} Q^{-n+1}, \\ |P_i'(x_0)| \le \delta_0^{-n+1} Q, \\ |a_{i,j}| \le \delta_0^{-n+1} Q, \quad 2 \le j \le n-1. \end{cases} \tag{3.3}$$

Applying well-known estimates from the geometry of numbers (see [16, p. 219]) we obtain

$$\Delta = |\det((a_{i,j-1})_{i,j=1}^n)| \le n!.$$

Moreover, from Lemma 2.2 it follows that there exists a prime p which does not divide Δ and satisfies

$$n! < p < 2n!. \tag{3.4}$$

Our next goal is to construct an irreducible monic polynomial of degree n using the polynomials P_i. Consider the following system of linear equations in n variables $\theta_1, \ldots, \theta_n$:

$$\begin{cases} x_0^n + p \sum_{i=1}^n \theta_i P_i(x_0) = p(n+1)\delta_0^{-n+1} Q^{-n+1}, \\ n x_0^{n-1} + p \sum_{i=1}^n \theta_i P_i'(x_0) = pQ + p \sum_{i=1}^n |P_i'(x_0)|, \\ \sum_{i=1}^n \theta_i a_{i,j} = 0, \quad 2 \le j \le n-1. \end{cases} \tag{3.5}$$

In order to find the determinant $\hat{\Delta}$ of this system, it is convenient to transform it as follows. Multiply the kth equation, where $k = 3, \ldots, n$, by $p \cdot x_0^{k-1}$ and subtract it from the first equation. Similarly, multiply the kth equation, where $k = 3, \ldots, n$, by $p \cdot (k-1)x_0^{k-2}$ and subtract it from the second equation. After making these transformations the determinant $\hat{\Delta}$ may be written as

$$\hat{\Delta} = p^2 \cdot \begin{vmatrix} a_{1,1}x_0 + a_{1,0} & \cdots & a_{n,1}x_0 + a_{n,0} \\ a_{1,1} & \cdots & a_{n,1} \\ \vdots & \ddots & \vdots \\ a_{1,n-1} & \cdots & a_{n,n-1} \end{vmatrix}$$

Since the P_i are linearly independent, we conclude that $\hat{\Delta} = p^2 \Delta \ne 0$. Hence, there exists a unique solution $(\theta_1, \ldots, \theta_n)$ of the system (3.5).

For integers $k_1, \dots, k_n$ consider the following polynomial of degree n with integer coefficients:

$$P(t) = t^n + p\sum_{i=1}^{n} k_i P_i(t) = t^n + p(a_{n-1}t^{n-1} + \dots + a_1 t + a_0),$$

where $a_j = \sum_{i=1}^{n} k_i a_{i,j}$, $0 \le j \le n-1$, and k_i, $1 \le i \le n$, satisfy

$$|\theta_i - k_i| \le 1. \tag{3.6}$$

We next show that there exist k_i such that P is irreducible. From (3.6) we have two possible values for every k_i, which will be denoted by k_i^1 and $k_i^2 = k_i^1+1$. Therefore, by Lemma 2.3, it suffices to show that we can choose k_i such that all a_j satisfy (2.1). It is easily seen that the first and second conditions of (2.1) hold for any k_i. It remains to show that $a_0 = k_1 a_{1,0} + \dots + k_n a_{n,0}$ is not divisible by p. Since p does not divide Δ, there exists $1 \le i \le n$ such that $a_{i,0}$ is not divisible by p and hence either $a_0^1 = k_1 a_{1,0} + \dots + a_{i,0}k_i^1 + \dots + a_{n,0}k_n$ or $a_0^2 = k_1 a_{1,0} + \dots + a_{i,0}k_i^2 + \dots + a_{n,0}k_n$ is not divisible by p. Therefore, choosing k_i in this manner yields an irreducible polynomial P.

We now proceed to estimate $|P(x_0)|$, $|P'(x_0)|$ and $H(P)$. Combining (3.3) and (3.6) with the system (3.5) we obtain the following inequalities.

From the first equation of the system it follows that

$$p\delta_0^{-n+1}Q^{-n+1} \le |P(x_0)| \le p(2n+1)\delta_0^{-n+1}Q^{-n+1}. \tag{3.7}$$

Similarly, from the second equation we have

$$pQ \le |P'(x_0)| \le (p + 2pn\delta_0^{-n+1})Q, \tag{3.8}$$

and the remaining equations give

$$|a_j| \le n\delta_0^{-n+1}Q, \qquad 2 \le j \le n-1. \tag{3.9}$$

Finally, applying (3.7)–(3.9) and the inequality $|x_0| \le 1/2$ yields

$$\begin{aligned} |a_1| &\le |P'(x_0)| + n|x_0|^{n-1} + \sum_{j=2}^{n-1} j|x_0|^{j-1}|a_j| \\ &\le (p + 2pn\delta_0^{-n+1})Q + \left(n\delta_0^{-n+1}\sum_{k=1}^{n-1}\frac{k+1}{2^k}\right)Q \\ &\le (p + (2pn+3n)\delta_0^{-n+1})Q, \end{aligned} \tag{3.10}$$

$$\begin{aligned} |a_0| &\le |P(x_0)| + |x_0|^n + |a_1 x_0| + \sum_{j=2}^{n} |x_0|^j|a_j| \\ &\le \tfrac{1}{2}pQ + \left(\tfrac{1}{2}p + (pn + \tfrac{3}{2}n)\delta_0^{-n+1}\right)Q + \tfrac{1}{2}n\delta_0^{-n+1}Q \\ &\le (p + (pn+4n)\delta_0^{-n+1})Q. \end{aligned} \tag{3.11}$$

From (3.9)–(3.11) and (3.4) we conclude that

$$H(P) \le 2n!(2n\delta_0^{-n+1} + 1)Q =: Q_1. \tag{3.12}$$

Consider the roots $\alpha_1, \dots, \alpha_n$ of P, where $|x_0 - \alpha_1| = \min_i |x_0 - \alpha_i|$. In view of Lemma 2.4,

$$|x_0 - \alpha_1| \le n|P(x_0)|\,|P'(x_0)|^{-1}. \tag{3.13}$$

Substituting (3.7) and (3.8) into (3.13) we obtain

$$|x_0 - \alpha_1| \le n(2n+1)\delta_0^{-n+1}Q^{-n} = c_{15}Q^{-n}. \tag{3.14}$$

If α_1 is a complex root of P, then its complex conjugate is also a root of P. Hence, by (3.12), (3.14) and well-known estimates for the roots, namely $|\alpha_i| \le H(P) + 1$, $1 \le i \le n$ (see [24, Theorem 1.1.2]), we deduce that

$$|P(x_0)| = \prod_{i=1}^{n} |x_0 - \alpha_i| \le c_{15}^2 Q^{-2n}\bigl(2 + 2n!(2n\delta_0^{-n+1} + 1)Q\bigr)^{n-2}.$$

This inequality contradicts (3.7) for $Q > Q_0$. Thus, all roots of P are real.

Finally, take a maximal system $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ of real algebraic integers such that $|\gamma_i - \gamma_j| > c_{15}Q^{-n}$, $1 \le i \ne j \le m$. Let us show that for any $x_0 \in B^1$ there exists $\gamma \in \Gamma$ such that $|x_0 - \gamma| \le 2c_{15}Q^{-n}$. By the above arguments and (3.14), for any $x_0 \in B^1$ there exists a real algebraic integer $\alpha_1 \in I$ such that $|x_0 - \alpha_1| \le c_{15}Q^{-n}$. If $\alpha_1 \in \Gamma$, then we can take $\gamma = \alpha_1$; otherwise, there exists $\gamma_i \in \Gamma$ such that $|\alpha_1 - \gamma_i| \le c_{15}Q^{-n}$, and hence

$$|x_0 - \gamma_i| \le |x_0 - \alpha_1| + |\alpha_1 - \gamma_i| \le 2c_{15}Q^{-n}.$$

In this case, we can take $\gamma = \gamma_i$. Therefore, $B^1 \subset \bigcup_{i=1}^m \{x \in I : |x - \gamma_i| \le 2c_{15}Q^{-n}\}$ and

$$4mc_{15}Q^{-n} \ge \mu_1\Bigl(\bigcup_{i=1}^{m}\{x \in I : |x - \gamma_i| \le 2c_{15}Q^{-n}\}\Bigr) \ge \mu_1 B^1 \ge \tfrac{3}{4}|I|.$$

This implies that the number of algebraic integers $\alpha \in I$ with $\deg \alpha = n$ and $H(\alpha) \le Q_1$ is no smaller than

$$m > \tfrac{3}{16}c_{15}^{-1}Q^n|I| = \tfrac{3}{16}c_{15}^{-1}\bigl(2n!(2n\delta_0^{-n+1} + 1)\bigr)^{-1}Q_1^n|I| = c_4Q_1^n|I|$$

for $Q_1 > Q_0$, and the proof is complete.

From the proof of Theorem 1.2 it follows that the set of algebraic integers of degree n forms a regular system with respect to the function $N(\alpha) = (H(\alpha)/(1 + |\alpha|)^{n-1})^n$ and $T_0 = c_{16}|I|^{-n}$, where the constant c_{16} is independent of $|I|$.

4. Proof of Theorem 1.3. The proof of Theorem 1.3 uses the same method as the proof of Theorem 1.2, but it contains some non-trivial elements which require special attention.

The proof applies Lemma 2.6, which is a two-dimensional analogue of Lemma 2.5. Given positive v_1 and v_2 satisfying $v_1 + v_2 = n - 2$, consider the system of inequalities

$$\begin{cases} |P(x)| < \hat{h}_1 Q^{-v_1}, \quad |P(y)| < \hat{h}_2 Q^{-v_2}, \\ \min\{|P'(x)|, |P'(y)|\} < \delta_0 Q, \end{cases} \tag{4.1}$$

where $\hat{h}_i = \max\{((|d_i| + 1)^n - 1)|d_i|^{-1}, \frac{1}{4}|d_1 - d_2|^{-2}\}$, $i = 1, 2$. Lemma 2.6 implies that the measure of the set $\mathcal{L}^2_{n-1} = \mathcal{L}^2_{n-1}(Q, \delta_0, E, v_1, v_2)$ of points $(x, y) \in E$ such that there exists $P \in \mathcal{P}_{n-1}(Q)$ satisfying (4.1) can be estimated as

$$\mu_2 \mathcal{L}^2_{n-1} \leq \tfrac{1}{4} \mu_2 E$$

for $Q > Q_0$ and δ_0 sufficiently small.

It is easy to check, using for example Minkowski's theorem on linear forms [16, p. 73], that for any $(x, y) \in E$ there exists $P \in \mathcal{P}_{n-1}(Q)$ satisfying $|P(x)| < \hat{h}_1 Q^{-v_1}$ and $|P(y)| < \hat{h}_2 Q^{-v_2}$. From this it follows that for any $(x, y) \in B^2 = E \setminus \mathcal{L}^2_{n-1}$ we may choose $P \in \mathcal{P}_{n-1}(Q)$ such that

$$\begin{cases} |P(x)| < \hat{h}_1 Q^{-v_1}, & |P(y)| < \hat{h}_2 Q^{-v_2}, \\ |P'(x)| \geq \delta_0 Q, & |P'(y)| \geq \delta_0 Q, \end{cases}$$

and $\mu_2 B^2 \geq \frac{3}{4} \mu_2 E$.

As in the proof of Theorem 1.2, choose $(x_0, y_0) \in B^2$ and examine the successive minima $\tau_1, \ldots, \tau_n$ of the compact convex set K defined by

$$\begin{cases} |a_{n-1} x_0^{n-1} + \cdots + a_1 x_0 + a_0| \leq \hat{h}_1 Q^{-v_1}, \\ |a_{n-1} y_0^{n-1} + \cdots + a_1 y_0 + a_0| \leq \hat{h}_2 Q^{-v_2}, \\ |(n-1) a_{n-1} x_0^{n-2} + \cdots + 2 a_2 x_0 + a_1| \leq Q, \\ |(n-1) a_{n-1} y_0^{n-2} + \cdots + 2 a_2 y_0 + a_1| \leq Q, \\ |a_i| \leq Q, \quad 4 \leq i \leq n-1. \end{cases}$$

Assume $\tau_1 \leq \delta_0$. Then for δ_0 sufficiently small there exists $P_0 \in \mathcal{P}_{n-1}(Q)$ satisfying

$$\begin{cases} |P_0(x_0)| < \delta_0 \hat{h}_1 Q^{-v_1} < \hat{h}_1 Q^{-v_1}, \quad |P_0(y_0)| < \delta_0 \hat{h}_2 Q^{-v_2} < \hat{h}_2 Q^{-v_2}, \\ |P_0'(x_0)| < \delta_0 Q, \quad |P_0'(y_0)| < \delta_0 Q, \\ H(P_0) < Q. \end{cases}$$

contrary to $(x_0, y_0) \in B^2$. Thus, $\tau_1 > \delta_0$. This fact and the estimate $V(K) > 2^n$ allow us to use Lemma 2.1, namely the inequality $\tau_1 \ldots \tau_n \leq 1$, to conclude that $\tau_n \leq \delta_0^{-n+1}$. Hence, there exist n linearly independent polynomials $P_i(t) = a_{i,n-1} t^{n-1} + \cdots + a_{i,1} t + a_{i,0}$, $1 \leq i \leq n$, with integer coefficients satisfying

$$(4.2)\qquad \begin{cases} |P_i(x_0)| \le \delta_0^{-n+1}\hat{h}_1 Q^{-v_1}, \quad |P_i(y_0)| \le \delta_0^{-n+1}\hat{h}_2 Q^{-v_2}, \\ |P_i'(x_0)| \le \delta_0^{-n+1} Q, \quad |P_i'(y_0)| \le \delta_0^{-n+1} Q, \\ |a_{i,j}| \le \delta_0^{-n+1} Q, \quad 4 \le j \le n-1. \end{cases}$$

Analysis similar to that in the proof of Theorem 1.2 shows that there exists a prime p which does not divide $\Delta = |\det((a_{i,j-1})_{i,j=1}^n)|$ and satisfies

$$(4.3)\qquad n! < p < 2n!.$$

Next, consider a system of linear equations in n variables $\theta_1, \ldots, \theta_n$,

$$(4.4)\qquad \begin{cases} x_0^n + p\sum_{i=1}^n \theta_i P_i(x_0) = p(n+1)\delta_0^{-n+1}\hat{h}_1 Q^{-v_1}, \\ y_0^n + p\sum_{i=1}^n \theta_i P_i(y_0) = p(n+1)\delta_0^{-n+1}\hat{h}_2 Q^{-v_2}, \\ nx_0^{n-1} + p\sum_{i=1}^n \theta_i P_i'(x_0) = pQ + p\sum_{i=1}^n |P_i'(x_0)|, \\ ny_0^{n-1} + p\sum_{i=1}^n \theta_i P_i'(y_0) = pQ + p\sum_{i=1}^n |P_i'(y_0)|, \\ \sum_{i=1}^n \theta_i a_{i,j} = 0, \quad 4 \le j \le n-1. \end{cases}$$

Our goal is to show that the determinant $\hat{\Delta}$ of this system does not vanish. Let us transform (4.4) as follows. Multiply the kth equation, where $k = 5, 6, \ldots, n$, by px_0^{k-1} (respectively py_0^{k-1}) and subtract the result from the first (respectively second) equation of (4.4). Similarly, multiply the kth equation, where $k = 5, 6, \ldots, n$, by $p(k-1)x_0^{k-2}$ (respectively $p(k-1)y_0^{k-2}$) and subtract the result from the third (respectively fourth) equation. After these transformations the determinant of (4.4) may be written as

$$p^4 \begin{vmatrix} a_{1,3}x_0^3 + a_{1,2}x_0^2 + a_{1,1}x_0 + a_{1,0} & \cdots & a_{n,3}x_0^3 + a_{n,2}x_0^2 + a_{n,1}x_0 + a_{n,0} \\ a_{1,3}y_0^3 + a_{1,2}y_0^2 + a_{1,1}y_0 + a_{1,0} & \cdots & a_{n,3}y_0^3 + a_{n,2}y_0^2 + a_{n,1}y_0 + a_{n,0} \\ 3a_{1,3}x_0^2 + 2a_{1,2}x_0 + a_{1,1} & \cdots & 3a_{n,3}x_0^2 + 2a_{n,2}x_0 + a_{n,1} \\ 3a_{1,3}y_0^2 + 2a_{1,2}y_0 + a_{1,1} & \cdots & 3a_{n,3}y_0^2 + 2a_{n,2}y_0 + a_{n,1} \\ a_{1,4} & \cdots & a_{n,4} \\ \vdots & \ddots & \vdots \\ a_{1,n-1} & \cdots & a_{n,n-1} \end{vmatrix}.$$

We proceed to show that $\hat{\Delta}$ is equal to Δ up to a multiple depending only on x_0, y_0 and p. Multiply the third (respectively fourth) row by $\frac{1}{3}x_0$ (respectively $\frac{1}{3}y_0$) and subtract the result from the first (respectively second) row. Then subtracting the first (respectively third) row from the second (respectively fourth) row gives

$$\hat{\Delta} = \frac{p^4(y_0 - x_0)^2}{9} \begin{vmatrix} a_{1,2}x_0^2 + 2a_{1,1}x_0 + 3a_{1,0} & \dots & a_{n,2}x_0^2 + 2a_{n,1}x_0 + 3a_{n,0} \\ a_{1,2}(y_0 + x_0) + 2a_{1,1} & \dots & a_{n,2}(y_0 + x_0) + 2a_{n,1} \\ 3a_{1,3}x_0^2 + 2a_{1,2}x_0 + a_{1,1} & \dots & 3a_{n,3}x_0^2 + 2a_{n,2}x_0 + a_{n,1} \\ 3a_{1,3}(y_0 + x_0) + 2a_{1,2} & \dots & 3a_{n,3}(y_0 + x_0) + 2a_{n,2} \\ a_{1,4} & \dots & a_{n,4} \\ \vdots & \ddots & \vdots \\ a_{1,n-1} & \dots & a_{n,n-1} \end{vmatrix}.$$

Now subtract the second row multiplied by x_0 from the first row and the fourth row multiplied by $1/2$ from the third row. Then subtract the third row multiplied by $(y_0 + x_0)/x_0^2$ from the fourth row, and finally subtract the fourth row multiplied by x_0y_0, $y_0 + x_0$ and $\frac{3}{2}x_0 - \frac{1}{2}y_0$ from the first, second and third row respectively. Finally, we obtain

$$\hat{\Delta} = p^4(y_0 - x_0)^4 \cdot \begin{vmatrix} a_{1,0} & \dots & a_{n,0} \\ \vdots & \ddots & \vdots \\ a_{1,n-1} & \dots & a_{n,n-1} \end{vmatrix} = p^4(y_0 - x_0)^4 \Delta > 0,$$

because the polynomials P_i, $1 \le i \le n$, are linearly independent and $|y_0 - x_0| > \varepsilon > 0$. Hence, the system (4.4) has a unique solution $(\theta_1, \dots, \theta_n)$. Moreover, there exist integers $k_1, \dots, k_n$ satisfying

$$|\theta_i - t_i| \le 1, \quad i = 1, \dots, n, \tag{4.5}$$

such that the polynomial with integer coefficients

$$P(t) = t^n + p\sum_{i=1}^{n} k_i P_i(t) = t^n + p(a_{n-1}t^{n-1} + \dots + a_1 t + a_0),$$

where $a_j = \sum_{i=1}^{n} k_i a_{i,j}$, $0 \le j \le n-1$, is irreducible. This follows by the same arguments as in the previouse section.

Let us estimate the values $|P(x_0)|$, $|P(y_0)|$, $|P'(x_0)|$ and $|P'(y_0)|$. From (4.2), (4.5) and the first four equations of (4.4) we see that

$$p\delta_0^{-n+1}\hat{h}_1 Q^{-v_1} \le |P(x_0)| \le p(2n+1)\delta_0^{-n+1}\hat{h}_1 Q^{-v_1}, \tag{4.6}$$

$$p\delta_0^{-n+1}\hat{h}_2 Q^{-v_2} \le |P(y_0)| \le p(2n+1)\delta_0^{-n+1}\hat{h}_2 Q^{-v_2}, \tag{4.7}$$

$$pQ \le |P'(x_0)| \le (p + 2pn\delta_0^{-n+1})Q, \tag{4.8}$$

$$pQ \le |P'(y_0)| \le (p + 2pn\delta_0^{-n+1})Q. \tag{4.9}$$

Finally, we need to estimate the height $H(P)$. By the fourth to nth equations of (4.4) and inequalities (4.2), (4.5), we have

$$|a_j| \le n\delta_0^{-n+1}Q, \quad 4 \le j \le n-1. \tag{4.10}$$

It remains to estimate $|a_j|$, $0 \le j \le 3$. By (4.6)–(4.10) and the inequalities $|x_0| \le |d_1| + 1/2$, $|y_0| \le |d_2| + 1/2$, for $Q > Q_0$ we have

$$\begin{aligned}|a_3x_0^3 + a_2x_0^2 + a_1x_0 + a_0| &\le |P(x_0)| + \sum_{j=4}^{n-1} |x_0|^j|a_j| + |x_0|^n \\ &< 3pn\delta_0^{-n+1}\hat{h}_1Q^{-v_1} + \Big(n\delta_0^{-n+1}\sum_{j=4}^{n}(|d_1| + 1/2)^j\Big)Q < 4pn\delta_0^{-n+1}\hat{h}_1Q,\end{aligned}$$

and similarly

$$|a_3y_0^3 + a_2y_0^2 + a_1y_0 + a_0| < 4pn\delta_0^{-n+1}\hat{h}_2Q.$$

Then

$$\begin{aligned}|3a_3x_0^2 + 2a_2x_0 + a_1| &\le |P'(x_0)| + \sum_{j=4}^{n-1} j|x_0|^{j-1}|a_j| + n|x_0|^{n-1} \\ &< (p + 2pn\delta_0^{-n+1})Q + \Big(n\delta_0^{-n+1}\sum_{j=4}^{n} j(|d_1| + 1/2)^{j-1}\Big)Q \\ &< (p + 2pn\delta_0^{-n+1} + n^2\hat{h}_1\delta_0^{-n+1})Q,\end{aligned}$$

and similarly

$$|3a_3y_0^2 + 2a_2y_0 + a_1| \le (p + 2pn\delta_0^{-n+1} + n^2\hat{h}_2\delta_0^{-n+1})Q.$$

We emphasize that for simplicity we do not care about the accuracy of the constants. Consider the following system of linear equations in a_0, a_1, a_2 and a_3:

$$\begin{cases} a_3x_0^3 + a_2x_0^2 + a_1x_0 + a_0 = l_1, \\ a_3y_0^3 + a_2y_0^2 + a_1y_0 + a_0 = l_2, \\ 3a_3x_0^2 + 2a_2x_0 + a_1 = l_3, \\ 3a_3y_0^2 + 2a_2y_0 + a_1 = l_4. \end{cases} \tag{4.11}$$

According to the above computations the determinant of the system (4.11) does not vanish. Thus, the system has a unique solution, which may be found by using Cramer's rule. Combining this with the estimates above one can easily verify that

$$|a_j| < c_{17}n\delta_0^{-n+1}Q, \quad 0 \le j \le 3,$$

where $c_{17} = 2^8p\varepsilon^{-3}(\hat{h}_1 + \hat{h}_2)(\max\{|d_1|, |d_2|\})^3$. Applying (4.3) and (4.10) now yields

$$H(P) < c_{18}n\delta_0^{-n+1}Q =: Q_1,$$

where $c_{18} = \max\{1, c_{17}\}$.

Consider the roots $\alpha_1, \dots, \alpha_n$ of P, where $|x_0 - \alpha_1| = \min_i |x_0 - \alpha_i|$, and let $\beta_1, \dots, \beta_n$ be a permutation of these roots such that $|y_0 - \beta_1| = \min_i |y_0 - \beta_i|$. By Lemma 2.4 and estimates (4.6)–(4.8), we have

$$\begin{cases} |x_0 - \alpha_1| < n(2n+1)\delta_0^{-n+1}\hat{h}_1 Q^{-v_1-1} = c_{19}\hat{h}_1 Q^{-v_1-1}, \\ |y_0 - \beta_1| < n(2n+1)\delta_0^{-n+1}\hat{h}_2 Q^{-v_2-1} = c_{19}\hat{h}_2 Q^{-v_2-1}. \end{cases}$$

For $Q > Q_0$, the roots α_1 and β_1 are real, as is easy to check.

Let $\Gamma = \{(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$ be a maximal system of real algebraic integer points such that

$$|\alpha_i - \alpha_j| > c_{19}\hat{h}_1 Q^{-v_1-1} \quad \text{or} \quad |\beta_i - \beta_j| > c_{19}\hat{h}_2 Q^{-v_2-1}, \qquad 1 \le i \ne j \le m.$$

By the same method as in the previous section, it follows that for any $(x_0, y_0) \in B^2$ there exists an algebraic integer point $(\alpha_i, \beta_i) \in \Gamma$ satisfying

$$|x_0 - \alpha_i| < 2c_{19}\hat{h}_1 Q^{-v_1-1}, \qquad |y_0 - \beta_i| < 2c_{19}\hat{h}_2 Q^{-v_2-1}.$$

This implies

$$B^2 \subset \bigcup_{i=1}^{m} \{(x,y) \in E : |x - \alpha_i| < 2c_{19}\hat{h}_1 Q^{-v_1-1},\ |y - \beta_i| < 2c_{19}\hat{h}_2 Q^{-v_2-1}\},$$

where

$$m > \tfrac{3}{64} \cdot c_{19}^{-2}\hat{h}_1^{-1}\hat{h}_2^{-1} Q^n \mu_2 E = c_7 Q_1^n \mu_2 E,$$

which finishes the proof.

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