

On the structure of bounded smooth measures associated with a quasi-regular Dirichlet form

by

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Summary. We consider a quasi-regular Dirichlet form. We show that a bounded signed measure charges no set of zero capacity associated with the form if and only if the measure can be decomposed into the sum of an integrable function and a bounded linear functional on the domain of the form. The decomposition allows one to describe explicitly the set of bounded measures charging no sets of zero capacity for interesting classes of Dirichlet forms. By way of illustration, some examples are given.

1. Introduction. Boccardo, Gallouët and Orsina [1] have shown that if $D \subset \mathbb{R}^d$ is an open bounded set and μ is a bounded (signed) Borel measure on D then μ charges no set of zero Newtonian capacity if and only if μ can be decomposed into the sum of an integrable function and an element of the dual space $H^{-1}(D)$ of the Sobolev space $H_0^1(D)$. If we denote by \mathcal{M}_b the space of all bounded Borel measures on D with bounded total variation, and by $\mathcal{M}_{0,b}$ its subset consisting of all measures charging no set of zero capacity, then the decomposition of [1] can be stated succinctly as

$$(1.1) \quad \mathcal{M}_{0,b}(D) = L^1(D; dx) + H^{-1}(D) \cap \mathcal{M}_b(D).$$

The decomposition (1.1) when combined with the analogue of the Lebesgue decomposition theorem saying that each bounded Borel measure on D can be uniquely decomposed into the absolutely continuous and the singular part with respect to the capacity (see Fukushima, Sato and Taniguchi [9]) gives a complete description of bounded Borel measures on D .

2010 *Mathematics Subject Classification*: Primary 31C25; Secondary 46E99, 60J45.

Key words and phrases: Dirichlet form, smooth measure.

Received 22 March 2017.

Published online 14 July 2017.

In the language of Dirichlet forms the decomposition (1.1) says that if $\mu \in \mathcal{M}_b(D)$, then its variation $|\mu|$ is smooth with respect to the capacity associated with the classical Dirichlet form if and only if μ admits a decomposition into an integrable function and an element of the dual space of the domain of the form. The decomposition (1.1) is interesting in its own right and together with the decomposition of [9] proved to be useful in investigating elliptic equations involving local operators and measure data (see, e.g., [1, 2, 4, 14]; see also [6] for a parabolic version of (1.1) and its applications to parabolic equations). This and the fact that the decomposition in [9] is proved in the setting of general Dirichlet forms motivated us to ask whether (1.1) can also be generalized to the case of bounded smooth measures with respect to a general Dirichlet form.

The answer to this question is “yes”. Let E be a metrizable Lusin space, m a positive σ -finite measure with full support on the σ -field of Borel subsets of E , and $(\mathcal{E}, D(\mathcal{E}))$ a quasi-regular Dirichlet form on $L^2(E; m)$. Our main result says that

$$(1.2) \quad \mathcal{M}_{0,b}(E) = L^1(E; m) + D(\mathcal{E})^* \cap \mathcal{M}_b(E),$$

i.e., if μ is a bounded Borel measure on E then $|\mu|$ is smooth with respect to the capacity determined by $(\mathcal{E}, D(\mathcal{E}))$ if and only if μ admits a decomposition of the form

$$(1.3) \quad \mu = f \cdot m + \nu,$$

where $f \in L^1(E; m)$, $\nu \in D(\mathcal{E})^* \cap \mathcal{M}_b(E)$ and $D(\mathcal{E})^*$ is the dual space of $D(\mathcal{E})$ equipped with the inner product $\tilde{\mathcal{E}}_1$ (for notation see Section 2). Moreover, if $(\mathcal{E}, D(\mathcal{E}))$ is transient, then (1.2) holds with $D(\mathcal{E})^*$ replaced by the dual \mathcal{F}_e^* of the extended Dirichlet space $(\mathcal{F}_e, \tilde{\mathcal{E}})$. We also provide a simple example showing that in general in (1.2) one cannot replace $D(\mathcal{E})^* \cap \mathcal{M}_b(E)$ by $(S_0 - S_0) \cap \mathcal{M}_b(E)$, where S_0 is the set of \mathcal{E} -smooth measures on E of finite energy (see Section 2), or equivalently the set of all positive elements of $D(\mathcal{E})^*$.

For many interesting classes of forms, one can describe the structure of the spaces $D(\mathcal{E})^*$ and \mathcal{F}_e^* . Consequently, for such classes the decomposition (1.2) gives an explicit description of the set of bounded smooth measures. In Section 4, we provide some examples to illustrate how (1.2) works in practice.

2. Preliminaries. Throughout, we assume that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E; m)$ (see [3, 12, 13] for the definitions). For $\alpha \geq 0$ we set $\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v)$ for $u, v \in D(\mathcal{E})$, where (\cdot, \cdot) stands for the usual inner product in $L^2(E; m)$. By $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ we denote the symmetric part of $(\mathcal{E}, D(\mathcal{E}))$ defined as $\tilde{\mathcal{E}}(u, v) = \frac{1}{2}(\mathcal{E}(u, v) + \mathcal{E}(v, u))$ for $u, v \in D(\mathcal{E})$.

We denote by \mathcal{F}_e the extended Dirichlet space associated with the symmetric Dirichlet form $(\tilde{\mathcal{E}}, D(\mathcal{E}))$. For $u \in \mathcal{F}_e$ we set $\mathcal{E}(u, u) = \lim_{n \rightarrow \infty} \mathcal{E}(u_n, u_n)$, where $\{u_n\}$ is an approximating sequence for u (see [8, Theorem 1.5.2]).

If $(\mathcal{E}, D(\mathcal{E}))$ is transient then by [8, Lemma 1.5.5], $(\mathcal{F}_e, \tilde{\mathcal{E}})$ is a Hilbert space. Note also that if $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form, then by [13, Proposition IV.3.3] each element $u \in D(\mathcal{E})$ admits a quasi-continuous m -version denoted by \tilde{u} , and that \tilde{u} is \mathcal{E} -q.e. unique for every $u \in D(\mathcal{E})$. If moreover $(\mathcal{E}, D(\mathcal{E}))$ is transient, then the last statement holds true for $D(\mathcal{E})$ replaced by \mathcal{F}_e (see Remark 2.2).

Recall that a positive measure μ on $\mathcal{B}(E)$ is said to be \mathcal{E} -smooth ($\mu \in S$ in notation) if $\mu(B) = 0$ for all \mathcal{E} -exceptional sets $B \in \mathcal{B}(E)$ and there exists an \mathcal{E} -nest $\{F_k\}_{k \in \mathbb{N}}$ of compact sets such that $\mu(F_k) < \infty$ for $k \in \mathbb{N}$.

A measure $\mu \in S$ is said to be of *finite energy integral* ($\mu \in S_0$ in notation) if there is $c > 0$ such that

$$(2.1) \quad \int_E |\tilde{u}(x)| \mu(dx) \leq c \mathcal{E}_1(u, u)^{1/2}, \quad u \in D(\mathcal{E}).$$

If additionally $(\mathcal{E}, D(\mathcal{E}))$ is transient, then $\mu \in S$ is said to be of *finite 0-order energy integral* ($\mu \in S_0^{(0)}$ in notation) if there is $c > 0$ such that

$$\int_E |\tilde{u}(x)| \mu(dx) \leq c \mathcal{E}(u, u)^{1/2}, \quad u \in \mathcal{F}_e.$$

If $(\mathcal{E}, D(\mathcal{E}))$ is regular and E is a locally compact separable metric space, then the notion of smooth measures defined above coincides with that in [8]. Moreover, if μ is a positive Radon measure on E such that (2.1) is satisfied for all $v \in C_0(E) \cap D(\mathcal{E})$, then μ charges no \mathcal{E} -exceptional set (see [11, Remark A.2]) and hence $\mu \in S_0$.

In the next section in the proof of our main theorem we will need the lemma given below. It follows from the corresponding result for regular forms by the so-called transfer method (see [3, 13]) and is perhaps known, but we could not find a proper reference.

LEMMA 2.1. *Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient. If $\mu \in S$, then there is a nest $\{F_n\}$ such that $\mathbf{1}_{F_n} \cdot \mu \in S_0^{(0)}$ for each $n \in \mathbb{N}$.*

Proof. Let $(\mathcal{E}^\#, D(\mathcal{E}^\#))$ denote the regular extension of $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ specified by [13, Theorem VI.1.2] and let $i : E \rightarrow E^\#$ denote the inclusion map. Then $(\mathcal{E}^\#, D(\mathcal{E}^\#))$ is transient, and by [13, Lemma IV.4.5 and Corollary VI.1.4], $\mu^\# = \mu \circ i^{-1}$ is a smooth measure on $\mathcal{B}(E^\#)$. Therefore, by the 0-order version of [8, Theorem 2.2.4] (see [8, remark following Corollary 2.2.2]), there exists an $\mathcal{E}^\#$ -nest $\{F_k\}$ on $E^\#$ such that $\mu_k^\# \equiv \mathbf{1}_{F_k} \cdot \mu^\# \in S_0^{(0)}(E^\#)$ for $k \geq 1$. Let $\{E_k\}$ be an \mathcal{E} -nest of [13, Theorem VI.1.2] and

let $F'_k = F_k \cap E_k$ for $k \in \mathbb{N}$. By [13, Corollary VI.1.4], $\{F'_k\}$ is an \mathcal{E} -nest on E . Set $\mu_k = \mathbf{1}_{F'_k} \cdot \mu$. We are going to show that $\mu_k \in S_0^{(0)}$, i.e for any nonnegative $u \in \mathcal{F}_e$,

$$(2.2) \quad \langle \mu_k, \tilde{u} \rangle \leq c\mathcal{E}(u, u)^{1/2}$$

for some $c > 0$. To this end, let us consider an approximating sequence $\{u_n\}$ for u and extend u, u_n to functions $u^\#, u_n^\#$ on $E^\#$ by setting $u^\#(x) = u_n^\#(x) = 0$ for $x \in E^\# \setminus E$. Then $\mathcal{E}^\#(u_n^\# - u_l^\#, u_n^\# - u_l^\#) = \mathcal{E}(u_n - u_l, u_n - u_l)$ for $n, l \in \mathbb{N}$, so $\{u_n^\#\}$ is an $\mathcal{E}^\#$ -Cauchy sequence. Moreover, as $m^\#(E^\# \setminus E) = 0$, $u_n^\# \rightarrow u^\#$ $m^\#$ -a.e. Consequently, $\{u_n^\#\}$ is an $\mathcal{E}^\#$ -approximating sequence for $u^\#$. It follows that $u^\#$ belongs to the extended space $\mathcal{F}_e^\#$ for $\mathcal{E}^\#$ and $\mathcal{E}(u, u) = \mathcal{E}^\#(u^\#, u^\#)$. Since $\tilde{u}^\#|_E$ is an m -version of u and by [13, Corollary VI.1.4] the function $\tilde{u}^\#|_E$ is \mathcal{E} -quasi-continuous, we have $\tilde{u} = \tilde{u}^\#|_E$ \mathcal{E} -q.e. From this and the fact that $\mu_k^\# = \mu_k$ on E it follows that

$$\langle \mu_k, \tilde{u} \rangle = \langle \mu_k, \tilde{u}^\#|_E \rangle = \langle \mu_k^\#, \tilde{u}^\# \rangle \leq c\mathcal{E}^\#(u^\#, u^\#)^{1/2},$$

which gives (2.2). ■

REMARK 2.2. Note that the argument following (2.2) shows that each $u \in \mathcal{F}_e$ admits an \mathcal{E} -quasi-continuous modification.

Recall that by [13, Theorem IV.3.5] there exists an m -tight special standard Markov process $\mathbb{X} = (X_t, P_x)$ properly associated with $(\mathcal{E}, D(\mathcal{E}))$. By [13, Proposition IV.2.8], this means that for all $\alpha > 0$ and $f \in L^2(E; m)$ the resolvent $(R_\alpha)_{\alpha>0}$ of \mathbb{X} defined as

$$R_\alpha f(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t) dt, \quad x \in E, \alpha > 0, f \in \mathcal{B}^+(E),$$

(E_x stands for the expectation with respect to P_x) is an \mathcal{E} -quasi-continuous version of $G_\alpha f$, where $(G_\alpha)_{\alpha>0}$ is the resolvent associated with $(\mathcal{E}, D(\mathcal{E}))$.

3. Decomposition of bounded smooth measures. Let \mathcal{M}_b denote the set of all Borel measures μ on E such that $|\mu|(E) < \infty$, where $|\mu|$ is the total variation of μ . We denote by $\mathcal{M}_{0,b}$ the set of all measures $\mu \in \mathcal{M}_b$ such that $|\mu| \in S$, and by $\mathcal{M}_{0,b}^+$ the subset of $\mathcal{M}_{0,b}$ consisting of all positive measures. Since $|\mu| \in S$ if and only if μ can be expressed as $\mu = \mu^+ - \mu^-$ with $\mu^+, \mu^- \in S$, we have $\mathcal{M}_{0,b} = (S - S) \cap \mathcal{M}_b$. To shorten notation, given a measure μ on E and a function $u : E \rightarrow \mathbb{R}$, we write

$$\langle \mu, u \rangle = \int_E u(x) \mu(dx),$$

whenever the integral is well defined.

In what follows we consider $D(\mathcal{E})$ (resp. \mathcal{F}_e) equipped with the scalar product $\tilde{\mathcal{E}}_1(\cdot, \cdot)$ (resp. $\tilde{\mathcal{E}}(\cdot, \cdot)$). We denote by $D(\mathcal{E})^*$ (resp. \mathcal{F}_e^*) the dual space of $D(\mathcal{E})$ (resp. \mathcal{F}_e^*).

PROPOSITION 3.1. *If $\mu \in D(\mathcal{E})^* \cap \mathcal{M}_b$, then $|\mu| \in S$.*

Proof. Since the notions of the spaces $D(\mathcal{E})^*$ and S only depend on the symmetric part of the form, we may and do assume that $(\mathcal{E}, D(\mathcal{E}))$ is symmetric. Let us define $(\mathcal{E}^\#, D(\mathcal{E}^\#))$, $E^\#, i$ as in the proof of Lemma 2.1 and set $\mu^\# = \mu \circ i^{-1}$. Then $\mu^\#$ is a bounded Borel measure on $\mathcal{E}^\#$ and $\mu^\# \in D(\mathcal{E}^\#)^*$. Moreover, by [13, Corollary VI.1.4], $|\mu| \in S$ if and only if $\mu^\#$ is smooth with respect to $(\mathcal{E}^\#, D(\mathcal{E}^\#))$. Therefore, without loss of generality, we may and do assume that E is a locally compact separable metric space, m is a positive Radon measure on E with $\text{supp}[m] = E$, and $(\mathcal{E}, D(\mathcal{E}))$ is a regular form on $L^2(E; m)$.

Let $E = E^+ \cup E^-$ be the Hahn decomposition of E (for the measure μ), and B a Borel subset of E such that $\text{Cap}(B) = 0$. We may assume that $B \subset E^+$. For every $\varepsilon > 0$ there exists a compact set $K \subset B$ and an open set U such that $B \subset U \subset E$ and $|\mu|(U \setminus K) \leq \varepsilon$. Let $(\mathcal{E}_U, D(\mathcal{E}_U))$ denote the part of $(\mathcal{E}, D(\mathcal{E}))$ on U . Let $\eta \in D(\mathcal{E}_U) \cap C_0(U)$ be such that $\eta \geq \mathbf{1}_K$ and set $\bar{\eta} = (\eta \vee 0) \wedge 1$. Then $\bar{\eta} \in D(\mathcal{E}_U) \cap C_0(U)$, $\bar{\eta} \geq \mathbf{1}_K$ and

$$\mu^+(K) \leq \int_K \bar{\eta}(x) \mu(dx) = \int_E \bar{\eta}(x) \mu(dx) - \int_{E \setminus K} \bar{\eta}(x) \mu(dx) \leq \|\mu\|_{D(\mathcal{E})^*} \|\bar{\eta}\|_{\mathcal{E}} + \varepsilon.$$

Since $\|\bar{\eta}\|_{\mathcal{E}} = \|\bar{\eta}\|_{\mathcal{E}_U} \leq \|\eta\|_{\mathcal{E}_U}$, we have

$$(3.1) \quad \mu^+(K) \leq \|\mu\|_{D(\mathcal{E})^*} \|\eta\|_{\mathcal{E}_U} + \varepsilon.$$

Let Cap (resp. Cap_U) denote the capacity associated with $(\mathcal{E}, D(\mathcal{E}))$ (resp. $(\mathcal{E}_U, D(\mathcal{E}_U))$) (see [8, Section 2.1]). Since $D(\mathcal{E}_U) \cap C_0(U)$ is a special standard core of \mathcal{E}_U , (3.1) and [8, Lemma 2.2.7] imply that

$$\mu^+(K) \leq \text{Cap}_U(K) \cdot \|\mu\|_{D(\mathcal{E})^*} + \varepsilon \leq \varepsilon,$$

the last inequality being a consequence of the fact that if $\text{Cap}(K) = 0$, then $\text{Cap}_U(K) = 0$, which follows from [13, Exercise III.2.10, Theorem III.2.11(ii), Theorem IV.5.29(i)]. Hence $\mu^+(B) \leq 2\varepsilon$ for $\varepsilon > 0$, which shows that $\mu^+ \in S$. In much the same way we show that $\mu^- \in S$. Thus $|\mu| \in S$, and the proof is complete. ■

THEOREM 3.2. *Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient and let $\mu \in \mathcal{M}_b$. Then $\mu \in \mathcal{M}_{0,b}$ if and only if there exist $f \in L^1(E; m)$ and $\nu \in \mathcal{F}_e^* \cap \mathcal{M}_b$ such that*

$$(3.2) \quad \mu = f \cdot m + \nu,$$

i.e. for every bounded $u \in \mathcal{F}_e$,

$$(3.3) \quad \langle \mu, \tilde{u} \rangle = \langle f, u \rangle + \langle \nu, \tilde{u} \rangle.$$

Proof. Since the notions of the spaces $S, \mathcal{F}_e, \mathcal{F}_e^*$ only depend on $(\tilde{\mathcal{E}}, D(\mathcal{E}))$, we may and do assume that $(\mathcal{E}, D(\mathcal{E}))$ itself is symmetric.

First assume that $\mu \in \mathcal{M}_{0,b}$. Without loss of generality we may assume that μ is positive. By Lemma 2.1 there exists a nest $\{F_n\}$ such that $\mathbf{1}_{F_n} \cdot \mu \in S_0^{(0)}$. Clearly, $\mu_n \equiv \mathbf{1}_{F_{n+1} \setminus F_n} \cdot \mu \in S_0^{(0)}$, and since $(\bigcup_{n=1}^{\infty} F_n)^c$ is exceptional and μ is smooth, $\mu = \sum_{n=1}^{\infty} \mu_n$. Let \mathbb{X} be an m -tight special standard Markov process properly associated with $(\mathcal{E}, D(\mathcal{E}))$, $(R_\alpha)_{\alpha>0}$ be the resolvent of \mathbb{X} , and let A^{μ_n} be a positive continuous additive functional of \mathbb{X} associated with μ_n in the Revuz sense (see [13, Theorem VI.2.4]). For $\alpha > 0$ set

$$\mu_n^\alpha = (\alpha U_{A^{\mu_n}}^\alpha \mathbf{1}) \cdot m,$$

where

$$U_{A^{\mu_n}}^\alpha \mathbf{1}(x) = E_x \int_0^\infty e^{-\alpha t} dA_t^{\mu_n}, \quad x \in E.$$

By [11, Proposition A.7], $U_{A^{\mu_n}}^\alpha \mathbf{1}$ is an \mathcal{E} -quasi-continuous version of the α -potential $U_\alpha \mu_n$ of μ_n . From this and [11, Theorem A.8(iv)] it follows that

$$(3.4) \quad \langle \mu_n^\alpha, u \rangle = \alpha \langle u, U_{A^{\mu_n}}^\alpha \mathbf{1} \rangle = \alpha \langle \mu_n, R_\alpha u \rangle$$

for every nonnegative Borel measurable u . From (3.4) one can deduce that

$$(3.5) \quad \langle \mu_n^\alpha, u \rangle = \langle \mu_n, \alpha R_\alpha u \rangle$$

for $u \in \mathcal{F}$. Let $u \in \mathcal{F}_e$ and let $\{u_k\} \subset D(\mathcal{E})$ be an approximating sequence for u . Then

$$(3.6) \quad \langle \mu_n^\alpha, u_k \rangle = \mathcal{E}(U \mu_n^\alpha, u_k) \leq \mathcal{E}(U \mu_n, U \mu_n)^{1/2} \mathcal{E}(u_k, u_k)^{1/2}$$

for every $k \in \mathbb{N}$ because by (3.5),

$$\begin{aligned} \langle \mu_n^\alpha, u_k \rangle &= \mathcal{E}(U \mu_n, \alpha R_\alpha u_k) \leq \mathcal{E}(U \mu_n, U \mu_n)^{1/2} \mathcal{E}(\alpha R_\alpha u_k, \alpha R_\alpha u_k)^{1/2} \\ &\leq \mathcal{E}(U \mu_n, U \mu_n)^{1/2} \mathcal{E}(u_k, u_k)^{1/2}. \end{aligned}$$

Letting $k \rightarrow \infty$ in (3.6), we get

$$(3.7) \quad \langle \mu_n^\alpha, u \rangle \leq \mathcal{E}(U \mu_n, U \mu_n)^{1/2} \mathcal{E}(u, u)^{1/2}.$$

Thus $\mu_n^\alpha \in S_0^{(0)}$. Given $\gamma \in S_0^{(0)}$, let T_γ be the bounded linear operator on \mathcal{F}_e defined as

$$T_\gamma(u) = \langle \gamma, \tilde{u} \rangle = \mathcal{E}(U \gamma, u).$$

From (3.7) it follows that for every $n \in \mathbb{N}$,

$$\sup_{\alpha>0} \mathcal{E}(U \mu_n^\alpha, U \mu_n^\alpha)^{1/2} \leq \sup_{\alpha>0} \|T_{\mu_n^\alpha}\| \leq \mathcal{E}(U \mu_n, U \mu_n)^{1/2} < \infty,$$

where $\|T_{\mu_n^\alpha}\|$ stands for the operator norm of $T_{\mu_n^\alpha}$. By the above and the Banach–Saks theorem, for every $n \in \mathbb{N}$ we can choose a sequence $\{\alpha_l^n\}$ such

that $\alpha_l^n \rightarrow \infty$ as $l \rightarrow \infty$, and the sequence $\{U(\gamma_k(\mu_n))\}$, where

$$\gamma_k(\mu_n) = f_k(\mu_n) \cdot m, \quad f_k(\mu_n) = \frac{1}{k} \sum_{l=1}^k \alpha_l^n U_{A^{\mu_n}}^{\alpha_l^n} \mathbf{1},$$

is \mathcal{E} -convergent to some $g \in \mathcal{F}_e$ as $k \rightarrow \infty$. Equivalently, $\|T_{\gamma_k(\mu_n)} - T\| \rightarrow 0$ as $k \rightarrow \infty$, where $T(u) = \mathcal{E}(g, u)$ for $u \in \mathcal{F}_e$. On the other hand, by (3.5) and [13, Theorem I.2.13], for every $u \in \mathcal{F}_e$,

$$T_{\mu_n^\alpha}(u) = \langle \mu_n, \alpha R_\alpha u \rangle = \mathcal{E}(U\mu_n, \alpha R_\alpha u) \rightarrow \mathcal{E}(U\mu_n, u) = T_{\mu_n}(u)$$

as $\alpha \rightarrow \infty$. It follows that in fact $T = T_{\mu_n}$. We can therefore find a subsequence $\{k_n\}$ such that

$$(3.8) \quad \|T_{\gamma_{k_n}(\mu_n)} - T_{\mu_n}\| \leq 2^{-n}$$

for every $n \in \mathbb{N}$. Set

$$f = \sum_{n=1}^{\infty} f_{k_n}(\mu_n), \quad \nu = \sum_{n=1}^{\infty} (\mu_n - \gamma_{k_n}(\mu_n)).$$

Then

$$\mu = \sum_{n=1}^{\infty} \mu_n = f \cdot m + \nu.$$

Since m is σ -finite, there exists a sequence $\{U_l\}$ of Borel subsets of E such that $\bigcup_{l=1}^{\infty} U_l = E$, $U_l \subset U_{l+1}$ and $m(U_l) < \infty$ for all $l \in \mathbb{N}$. By (3.4),

$$(\alpha U_{A^{\mu_n}}^\alpha \mathbf{1}, \mathbf{1}_{U_l}) \leq \langle \mu_n, \alpha R_\alpha \mathbf{1} \rangle \leq \langle \mu_n, \mathbf{1} \rangle$$

for every $\alpha > 0$. Therefore, for every $l \in \mathbb{N}$ we have

$$(f, \mathbf{1}_{U_l}) \leq \sum_{n=1}^{\infty} (f_{k_n}(\mu_n), \mathbf{1}_{U_l}) \leq \sum_{n=1}^{\infty} \langle \mu_n, \mathbf{1} \rangle \leq \sum_{n=1}^{\infty} \|\mu_n\|_{\text{TV}} = \|\mu\|_{\text{TV}}.$$

Hence $\|f\|_{L^1(E; m)} < \infty$ by the monotone convergence theorem. It follows in particular that $\nu = \mu - f \cdot m \in \mathcal{M}_b$. On the other hand, by (3.8), $\nu \in \mathcal{F}_e^*$, which proves that μ is of the form (3.2).

Now, suppose that μ is given by the right-hand side of (3.2). Since $f \cdot m \in \mathcal{M}_{0,b}$, we have only to prove that if $\mu \in \mathcal{F}_e^* \cap \mathcal{M}_b$ then $|\mu| \in \mathcal{S}$. But this follows from Proposition 3.1 since $\mathcal{F}_e^* \subset D(\mathcal{E})^*$. ■

COROLLARY 3.3. *Let $\mu \in \mathcal{M}_b$. Then $\mu \in \mathcal{M}_{0,b}$ if and only if there exist $f \in L^1(E; m)$ and $\nu \in D(\mathcal{E})^*$ such that (3.3) holds true for every bounded $u \in D(\mathcal{E})$.*

Proof. The Dirichlet form $(\mathcal{E}_1, D(\mathcal{E}))$ is transient, quasi-regular and its extended Dirichlet space is $(D(\mathcal{E}), \tilde{\mathcal{E}}_1)$. Moreover, $|\mu|$ is smooth with respect to $(\mathcal{E}, D(\mathcal{E}))$ if and only if it is smooth with respect to $(\mathcal{E}_1, D(\mathcal{E}))$. Therefore the corollary follows from Theorem 3.2 applied to the form $(\mathcal{E}_1, D(\mathcal{E}))$. ■

COROLLARY 3.4. *Let $\mu \in \mathcal{M}_b$. Then $\mu \in \mathcal{M}_{0,b}$ if and only if there exist $f \in L^1(E; m)$ and $v \in D(\mathcal{E})$ such that for every bounded $u \in D(\mathcal{E})$,*

$$(3.9) \quad \langle \mu, \tilde{u} \rangle = (f, u) + \mathcal{E}_1(v, u).$$

Proof. Follows immediately from Corollary 3.3 and the Lax–Milgram theorem. ■

REMARK 3.5. (i) The decomposition (3.2) is not unique because $L^1(E; m) \cap \mathcal{F}_e^* \neq \emptyset$.

(ii) In general, $D(\mathcal{E})^* \cap \mathcal{M}_b$ in the decomposition (1.2) cannot be replaced by $(S_0 - S_0) \cap \mathcal{M}_b$. To see this, let us consider the classical Dirichlet form (see Example 4.1 in the next section) with $D = B(0, 1) \subset \mathbb{R}^7$, where $B(0, r)$ denotes the open ball with radius $r > 0$ and center 0. Let σ_{a_n} denote the surface measure on $\partial B(0, a_n)$ with $a_n = n^{-1/4}$, and let

$$\mu = \sum_{n=1}^{\infty} \sigma_{a_n}.$$

From [8, Example 5.2.2] it follows that $\sigma_{a_n} \in S$ for each $n \in \mathbb{N}$. Hence $\mu \in S$. Moreover, $\mu(D) = c \sum_{n=1}^{\infty} a_n^6 < \infty$, so $\mu \in \mathcal{M}_{0,b}^+$. Let

$$(3.10) \quad \mu(dx) = f dx + \nu(dx)$$

be a decomposition of μ as in Theorem 3.2. Then $\nu \notin S_0 - S_0$. To show this, let us denote by μ_s and ν_s the singular parts (with respect to the Lebesgue measure) of μ and ν , respectively, and observe that $\mu = \mu_s = \nu_s$. Suppose that $\nu \in S_0 - S_0$. Then $\nu^+, \nu^- \in S_0$, and hence $\nu_s^+, \nu_s^- \in S_0$ because $\nu_s^+ \leq \nu^+$ and $\nu_s^- \leq \nu^-$. Consequently, $\mu_s = \nu_s^+ - \nu_s^- \in S_0 - S_0$, which implies that $\mu \in S_0$ since μ is positive. On the other hand, if we set $u(x) = |x|^{-2} - 1$, $x \in D$, then $u \in H_0^1(D)$ and

$$\langle \mu, u \rangle = \sum_{n=1}^{\infty} \int_{\partial B(0, a_n)} u(x) \sigma_{a_n}(dx) = c \sum_{n=1}^{\infty} (a_n^4 - a_n^6) = \infty,$$

which is a contradiction, because by [8, Theorem 2.2.2], if $\mu \in S_0$, then quasi-continuous elements of $H_0^1(D)$ are integrable with respect to μ .

(iii) From the proof of Theorem 3.2 it follows that if $\mu \in \mathcal{M}_{0,b}^+$, then the L^1 part f of its decomposition can be chosen to be positive. The example given above shows that in general this is not true for ν , because if ν in (3.10) were positive, we would have $\nu \in S_0$ and hence $\mu \in S_0$.

(iv) By the definition of S_0 , $(S_0 - S_0) \cap \mathcal{M}_b \subset D(\mathcal{E})^* \cap \mathcal{M}_b$. The opposite inclusion is false and ν of (3.10) can serve as a counterexample. Below we give an explicit construction of another counterexample. Let D , a_n , σ_{a_n} be as in (ii), and let $b_n = \left(\frac{n+(1/2)}{n(n+1)}\right)^{1/4}$, so that $a_1 > b_1 > a_2 > b_2 > \dots$. Let $\nu_{a_n}(x)$ and $\nu_{b_n}(x)$ denote the first components of the outer normal vectors

to $\partial B(0, a_n)$ and $\partial B(0, b_n)$ at x . Set

$$\mu(dx) = \sum_{n=1}^{\infty} (\nu_{a_n}(x) \sigma_{a_n}(dx) - \nu_{b_n}(x) \sigma_{b_n}(dx)).$$

Then $\mu \in \mathcal{M}_{0,b}$ and for every $\eta \in H_0^1(D)$ we have

$$\begin{aligned} \langle \mu, \eta \rangle &= \sum_{n=1}^{\infty} \left(\int_{\partial B(0, a_n)} \eta(x) \nu_{a_n}(x) \sigma_{a_n}(dx) - \int_{\partial B(0, b_n)} \eta(x) \nu_{b_n}(x) \sigma_{b_n}(dx) \right) \\ &= \sum_{n=1}^{\infty} \int_{B(0, a_n) \setminus B(0, b_n)} \frac{\partial \eta}{\partial x_1}(x) dx \leq C \|\eta\|_{H_0^1(D)}. \end{aligned}$$

Hence $\mu \in H^{-1}(D)$. But $|\mu| \notin H^{-1}(D)$ because if it were true, the series

$$\sum_{n=1}^{\infty} \int_{\partial B(0, a_n)} u(x) \sigma_{a_n}(dx) = c \sum_{n=1}^{\infty} a_n^4$$

would be convergent.

4. Examples. In this section, we apply Theorem 3.2 to give an explicit description of the set $\mathcal{M}_{0,b}$ for some classes of regular local forms, regular nonlocal forms and quasi-regular forms.

EXAMPLE 4.1 (Classical Dirichlet form). Let $D \subset \mathbb{R}^d$ be a bounded domain. Consider the classical form

$$\mathbb{D}(u, v) = \frac{1}{2} \int_D \langle \nabla u, \nabla v \rangle_{\mathbb{R}^d} dx, \quad u, v \in D(\mathcal{E}) = H_0^1(D).$$

It is known that the form $(\mathbb{D}, H_0^1(D))$ is a transient regular Dirichlet form on $L^2(D; dx)$ (see [8, Example 1.5.1]). If $\mu \in \mathcal{M}_b^+$, then $\mu \in S$ if and only if μ charges no set of Newtonian capacity zero. By Poincaré's inequality, the norms determined by \mathbb{D} and \mathbb{D}_1 are equivalent (of course, the norm determined by \mathbb{D}_1 is the usual norm in the Sobolev space $H_0^1(D)$). As a consequence, $\mathcal{F}_e = H_0^1(D)$ and hence $\mathcal{F}_e^* = H^{-1}(D)$. From Theorem 3.2 and the well known characterization of the space $H^{-1}(D)$ it follows that if $\mu \in \mathcal{M}_b$, then $\mu \in \mathcal{M}_{0,b}$ if and only if

$$(4.1) \quad \mu = f^0 - \operatorname{div} F$$

for some $f^0 \in L^1(D; dx)$ and $F = (F^1, \dots, F^d)$ such that $F^i \in L^2(D; dx)$, $i = 1, \dots, d$. We see that in the case of the classical form the decomposition of Theorem 3.2 reduces to the decomposition proved in [1, Theorem 2.1].

One can easily check that (4.1) holds for more general (possibly non-symmetric) regular Dirichlet forms defined by [13, (2.17), Section II.2] with coefficients satisfying the assumptions of [13, Proposition II.2.11].

EXAMPLE 4.2 (Regular nonlocal Dirichlet forms). Let $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous negative definite function, let $s \in \mathbb{R}$, and let

$$H^{\psi,s} = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + \psi(\xi))^s |\hat{u}(\xi)|^2 d\xi < \infty \right\},$$

where $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions and \hat{u} is the Fourier transform of u . Note that $H^{\psi,1} = \{u \in L^2(\mathbb{R}^d; dx) : \int_{\mathbb{R}^d} \psi(\xi) |\hat{u}(\xi)|^2 d\xi < \infty\}$. Consider the form

$$\Psi(u, v) = \int_{\mathbb{R}^d} \psi(\xi) \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi, \quad u, v \in H^{\psi,1}.$$

It is known (see [8, Example 1.4.1]) that $(\Psi, H^{\psi,1})$ is a symmetric regular Dirichlet form on $L^2(\mathbb{R}^d; dx)$.

By [10, Theorem 3.10.11], the dual space of $H^{\psi,1}$ is the space $H^{\psi,-1}$ in the sense that for every $\nu \in (H^{\psi,1})^*$ there is $v \in H^{\psi,-1}$ such that $\hat{\nu}\hat{u}$ may be interpreted as an element of $L^1(\mathbb{R}^d; dx)$ and the value of ν on $u \in H^{\psi,1}$ is equal to $\int_{\mathbb{R}^d} \hat{\nu}(x) \hat{u}(x) dx$. Therefore from Corollary 3.3 it follows that if $\mu \in \mathcal{M}_{0,b}$, then there exist $f \in L^1(\mathbb{R}^d; dx)$ and $v \in H^{\psi,-1}$ such that for every bounded $u \in H^{\psi,1}$,

$$\langle \mu, \tilde{u} \rangle = (f, u) + \int_{\mathbb{R}^d} \hat{v}(x) \hat{u}(x) dx.$$

Since $H^{\psi,1} = H^1(\mathbb{R}^d)$ for $\psi(\xi) = |\xi|^2$, the above decomposition may be viewed as a generalization of (4.1).

EXAMPLE 4.3 (Gradient Dirichlet forms on infinite-dimensional spaces). Let H be a separable real Hilbert space and let A be a self-adjoint operator A such that $\langle Ax, x \rangle_H \leq -\omega |x|_H^2$ for $x \in D(A)$, with some $\omega > 0$ and A^{-1} of trace class. Let $Q_\infty = -\frac{1}{2}A^{-1}$, and let γ denote the Gaussian measure on H with mean 0 and covariance operator Q_∞ . We consider the form

$$\mathcal{E}(u, v) = \frac{1}{2} \int_H \langle \nabla u, \nabla v \rangle_H \gamma(dx), \quad u, v \in \mathcal{FC}_b^\infty,$$

where \mathcal{FC}_b^∞ is the space of finitely based smooth bounded functions on H (see [13, Section II.3]) and ∇ is the H -gradient defined for $u \in \mathcal{FC}_b^\infty$ as the unique element of H such that $\langle \nabla u(x), h \rangle_H = \frac{\partial u}{\partial h}(x)$ for $x \in H$. By [13, Proposition II.3.8], the form $(\mathcal{E}, \mathcal{FC}_b^\infty)$ is closable, and its closure, which we denote by $(\mathcal{E}, W^{1,2}(H))$, is a symmetric Dirichlet form. Moreover, by the results of [13, Section IV.4], it is quasi-regular.

By Corollary 3.4, if $\mu \in \mathcal{M}_b$, then $\mu \in \mathcal{M}_{0,b}$ if and only if there exist $f \in L^1(H; \gamma)$ and $v \in W^{1,2}(H)$ such that for every bounded $u \in W^{1,2}(H)$,

$$(4.2) \quad \langle \mu, \tilde{u} \rangle = \int_H f u \gamma(dx) + \mathcal{E}_1(v, u).$$

In fact, $\mu \in \mathcal{M}_{0,b}$ if and only if for some $f^0 \in L^1(H; \gamma)$ and $F \in L^2(H; \gamma)$ the measure μ can be written in the form

$$(4.3) \quad \mu = f^0 - \operatorname{div}_\gamma F,$$

similar to (4.1). Indeed, if $\mu \in \mathcal{M}_{0,b}$, then by (4.2),

$$(4.4) \quad \langle \mu, \tilde{u} \rangle = \int_H f^0 u \gamma(dx) + \frac{1}{2} \int_H \langle F, \nabla u \rangle_H \gamma(dx)$$

with $f^0 = v + f \in L^1(H; \gamma)$ and $F = \nabla v \in L^2(H; \gamma)$. On the other hand, if $F \in C_b^1(H; H)$ has finite divergence with respect to γ (see [5, Section 11.1] for the definition), then by [5, Lemma 11.1.9],

$$(4.5) \quad \int_H \langle F, \nabla u \rangle_H \gamma(dx) = - \int_H (\operatorname{div}_\gamma F) u \gamma(dx),$$

where $\operatorname{div}_\gamma F(x) = \operatorname{div} F(x) - \langle Q_\infty^{-1} x, F(x) \rangle_H$, $x \in H$, which when combined with (4.4) makes it legitimate to write μ in the form (4.3). Conversely, if $\mu \in \mathcal{M}_b$ and μ is of the form (4.3) for some $f^0 \in L^1(H; \gamma)$ and $F \in L^2(H; \gamma)$, then $\operatorname{div}_\gamma F \in \mathcal{M}_b$ and by (4.5),

$$|\langle \operatorname{div}_\gamma F, u \rangle| \leq \|F\|_{L^2(H; \gamma)} \|\nabla u\|_{L^2(H; \gamma)} \leq C \mathcal{E}(u, u)^{1/2}$$

for all bounded $u \in W^{1,2}(H)$. That $\operatorname{div}_\gamma F$ is smooth now follows from [8, Lemma 2.2.3].

The assertion that $\mu \in \mathcal{M}_{0,b}$ if and only if μ has a decomposition (4.3) holds true for forms more general than those considered in Example 4.3. In fact, slightly modifying the argument in Example 4.3, one can show that it holds for the form which is the closure of the form defined by [7, (20)] if [7, Hypotheses 3.1 and 3.2] (the latter with $R = R^* > 0$ such that R^{-1} is bounded) are satisfied.

Acknowledgements. The first author was supported by Polish National Science Centre (grant no. 2012/07/D/ST1/02107).

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