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RANDOM GROUPS ARE NOT LEFT-ORDERABLE

BY

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Abstract. We prove that random groups in the Gromov density model at any density d have with overwhelming probability no non-trivial left-orderable quotients. In particular, random groups at densities $d < 1/2$ are not left-orderable.

1. Introduction. We work in the density model for random groups introduced by Gromov.

DEFINITION 1.1 ([5, Section 9.B], [8, Definition 7]). Let F_n be the free group on $n \geq 2$ generators $a_1, \ldots, a_n$. For any integer L let $R_L \subset F_n$ be the set of reduced words of length L in these generators.

Let $d \in (0,1)$. A *random set of relators at density d and length L* is a sequence of $\lfloor (2n-1)^{dL} \rfloor$ elements of R_L, picked independently and uniformly at random from all elements of R_L.

A *random group at density d and length L* is the group G presented by $\langle S \mid R \rangle$, where $S = \{a_1, \ldots, a_n\}$ and R is a random set of relators at density d and length L.

The relators in R_L are not assumed to be cyclically reduced.

Of particular interest in the study of random groups are the properties occurring *with overwhelming probability.*

DEFINITION 1.2 ([5, Section 9.B], [8, Definition 7]). Let $I \subset \mathbb{N}_+$ be infinite. We say that a property of random sets of relators, or of random groups, occurs *with I-overwhelming probability* (briefly, *w. I-o.p.*) *at density d* if its probability of occurrence tends to 1 as $L \to \infty$, for $L \in I$ and fixed d. We omit "I-" if $I = \mathbb{N}_+$.

Basic characteristics of the model are given by the following phase transition theorem, due to Gromov.

THEOREM 1.3 ([5, Section 9.B], [7, Theorem 2]). *A random group is with overwhelming probability*

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- *trivial or* $\mathbb{Z}/2\mathbb{Z}$ *at density* $d > 1/2$,
- *infinite, hyperbolic and torsion-free at density* $d < 1/2$.

A number of interesting properties are known to hold for random groups w.o.p. at various densities [8, Section I.3].

In this paper we consider *left-orderability.*

DEFINITION 1.4. A group G is said to be *left-ordered* by $\leq$ if $\leq$ is a total order on G which is *left-invariant*: for all $g_1, g_2, h \in G$ the condition $g_1 \leq g_2$ implies $hg_1 \leq hg_2$.

Our main result is the following.

THEOREM 1.5. *Let* $d \in (0,1)$. *A random group in the Gromov density model at density* d *has w.o.p. no non-trivial left-orderable quotients.*

In conjunction with Theorem 1.3, this shows non-left-orderability of random groups below the critical density $d = 1/2$.

If G is a countable group (e.g. a quotient of a random group), then G is left-orderable if and only if it admits a faithful action on the real line by orientation-preserving homeomorphisms [3, Section 1.1.3]. Theorem 1.5 may thus be treated as a result connected to the Gromov conjecture that a random group should not have any smooth actions on any compact manifolds [4, Conjecture 4.22].

As a side note, Theorem 1.5 provides also an alternative way of showing that random groups are not free of rank ≥ 1 at any density, since free groups are left-orderable [1, Theorem 2.3.1]. The more usual proof of this proceeds by establishing that random groups have trivial abelianisations.

The main idea of our proof is to use the order on the given non-trivial quotient Q of the random group $G = \langle S \mid R \rangle$ as follows. We explicitly construct a high-density set P of words in F_n, representing strictly positive (in the sense of the order) elements of Q. It happens that for fixed d, the density of P exceeds $1 - d$ for n sufficiently large. By a well-known fact it thus contains w.o.p. a word w from the set R of relators, leading to a contradiction of the element corresponding to w in Q being both positive and trivial. Finally, we use the approach of [2] to increase the number of generators we work with and obtain the result for all $n \geq 2$.

The whole proof is phrased in the language of b-automata and associated groups, as introduced in [2], and follows a very similar framework. In the case of fixed d and sufficiently large n one can entirely avoid referring to [2] and provide a slightly shorter argument. It consists in considering sets $\mathcal{L}_{\varepsilon,i}$ from the proof of Lemma 3.10, proving they all intersect w.o.p. a random set of relators by the usual density argument, and then proceeding as in the proof of Proposition 3.9.

This paper is structured as follows. Section 2 deals with the basic properties of left-ordered groups. In Section 3 we introduce the notion of a b-automaton and its language and use them to give a proof of Theorem 1.5 for n sufficiently large. In Section 4 we use the concept of associated groups to generalise the argument to all $n \geq 2$. In the Appendix we reprove a well-known generalisation of the fact that a random set of elements at density d intersects w.o.p. any fixed set of elements of density d' such that $d + d' > 1$. The more general statement is that their intersection is roughly of density $d + d' - 1$ if $d < 1/2$ [5, Section 9.A]. The assumption on d is not really restrictive, in view of Theorem 1.3. It comes from the fact that we define "a random set at density d" to be a tuple with possible repetitions. If, however, $d < 1/2$, then there are w.o.p. no such repetitions and the counting is easier.

2. Left orders. Let G be a group left-ordered by $\leq$. The symbols $<$ and $>$ are the usual shorthands. By e we denote the neutral element of G. The following remarks are easily obtained from Definition 1.4.

REMARK 2.1. Any non-empty product of elements strictly greater than e is itself strictly greater than e.

REMARK 2.2. For every $g \in G \setminus \{e\}$ one can choose a sign $\varepsilon \in \{-1, 1\}$ such that $g^{\varepsilon} > e$.

Those two quickly imply the following.

COROLLARY 2.3. *G is torsion-free.*

Moreover, by combining Remarks 2.1 and 2.2, we obtain Lemma 2.4, which will be used to construct high-density sets of words representing non-trivial elements.

LEMMA 2.4. *For any non-trivial $g_1, \ldots, g_n \in G$ there exists a sequence of signs $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ for which every non-empty product (possibly with repetitions) of elements of the form $g_i^{\varepsilon_i}$ is non-trivial.*

Proof. Choose $(\varepsilon_i)_{i=1}^{n}$ for which $g_i^{\varepsilon_i} > e$ for $i = 1, \ldots, n$. ∎

Lemma 2.4 is in fact equivalent to G being left-orderable [1, Theorem 7.1.1], but we will only need the implication we have proved.

3. Random groups with large number of generators. We begin by recalling the terminology and useful observations of [2, Section 2]. We denote by S a finite set, called the *alphabet*. We define S^{-1} to be the set of the formal inverses of elements of S, and denote $S^{\pm} = S \cup S^{-1}$. The elements of $S^{\pm}$ are called *letters*. By a *word over the alphabet S* we mean a finite sequence of letters. We denote $S = \{a_1, \ldots, a_n\}$, hence $n = |S|$. The set S is to be interpreted as the set of generators of F_n.

Definition 3.1 ([2, Definition 2.1]). A *basic automaton* (briefly a *b-automaton*) *over an alphabet* S *with transition data* $\{\sigma_s\}$ is a pair $(S, \{\sigma_s\})$, where $\{\sigma_s\}_{s\in\{\emptyset\}\cup S^\pm}$ is a family of subsets of $S^\pm$.

The *language* of a b-automaton with transition data $\{\sigma_s\}$ is the set of all non-empty words over S beginning with a letter in $\sigma_\emptyset$ and such that for any two consecutive letters s, s' we have $s' \in \sigma_s$.

We say that a b-automaton is λ-*large*, for some $\lambda \in (0, 1)$, if $\sigma_\emptyset \neq \emptyset$ and for each $s \in S^\pm$ we have $|\sigma_s| \geq 2\lambda n$.

Remark 3.2 ([2, Remark 2.2(i)]). There are exactly $2^{2n(2n+1)}$ b-automata over a fixed alphabet S of size n.

Remark 3.3 ([2, Remark 2.2(ii)]). If a b-automaton is λ-large, then its language contains at least $\lceil 2\lambda n\rceil^{L-1}$ words of length L and at least $(\lceil 2\lambda n\rceil - 1)^{L-1}$ reduced words of length L.

Definition 3.4 ([2, Definition 2.3]). Let $I \subset \mathbb{N}_+$ be infinite and let $\mathcal{L}$ be a set of reduced words over an alphabet S, containing for all but finitely many $L \in I$ at least ck^L words of length L, where $c > 0$ and $k > 1$. Then we say that the *I-growth rate of $\mathcal{L}$ is at least k*. Similarly, if $k > k'$, then we say that the *I-growth rate of $\mathcal{L}$ is greater than k'*.

It is convenient to extend the notion of density from Definition 1.1 in the following way.

Definition 3.5. Let $I \subset \mathbb{N}_+$ be infinite and let $\mathcal{L}$ be a set of reduced words over an alphabet S, containing for all but finitely many $L \in I$ at least $c(2n-1)^{dL}$ words of length L, where $c > 0$ and $d \in (0, 1)$. Then we say that the *I-density of $\mathcal{L}$ is at least d*.

Notions of density d and growth rate k of the set $\mathcal{L}$ are easily seen to be related by $k = (2n-1)^d$, i.e. for such k, d, with $d \in (0, 1)$, the set $\mathcal{L}$ has I-growth rate at least k if and only if it has I-density at least d.

The following is a well known fact in random groups. We reprove it in a stronger form in the Appendix.

Proposition 3.6 ([5, Section 9.A]). *Let $I \subset \mathbb{N}_+$ be infinite. Suppose $d, d' \in (0, 1)$ are such that $d + d' > 1$ and $R_f \subset F_n$ is a fixed set of relators in some fixed number n of generators, of I-density at least d'. Then w. I-o.p. a random set R of relators at density d intersects R_f.*

From this we get

Lemma 3.7 ([2, Lemma 2.4]). *Let $I \subset \mathbb{N}_+$ be infinite and let $\mathcal{L}$ be a set of reduced words over the alphabet S, of I-growth rate greater than $(2n-1)^{1-d}$ for some $d \in (0, 1)$. Then w. I-o.p. a random set of relators at density d intersects $\mathcal{L}$.*

We will be interested in the following consequence, proven in [2].

COROLLARY 3.8 ([2, Corollary 2.5]). *For given* $\lambda, d \in (0,1)$, *if* n *is sufficiently large, then w.o.p. a random set of relators at density* d *intersects the languages of all* λ*-large b-automata over the alphabet* S.

For a group G with presentation $G = \langle S \mid R\rangle$ and a word w over the alphabet S, we will denote by $\overline{w}$ the corresponding element of G.

To obtain Theorem 1.5 for n sufficiently large, we just need the following.

PROPOSITION 3.9. *Let* G *be a group with presentation* $G = \langle S \mid R\rangle$ *such that* R *intersects the languages of all* $1/2$*-large b-automata over* S. *Then* G *has no non-trivial left-orderable quotients.*

In order to prove Proposition 3.9, we use the following lemma, which is our main step towards exploiting the hypothetical left-orderability.

LEMMA 3.10. *Let* R *be a set of words over* S. *Assume* R *intersects the languages of all* $1/2$*-large b-automata over* S. *Then for every* $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1,1\}^n$ *and* $i \in \{1,\ldots,n\}$, *there exists a non-empty reduced word* $w \in R$, *consisting only of letters from the set* $\{a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}\}$, *with at least one occurrence of* $a_i^{\varepsilon_i}$.

Proof of Lemma 3.10. Consider a b-automaton $\mathbb{A}_{\varepsilon,i}$ over S with transition data $\sigma_\emptyset = \{a_i^{\varepsilon_i}\}$ and $\sigma_s = \{a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}\}$ for every $s \in S^\pm$. Every word in its language $\mathcal{L}_{\varepsilon,i}$ is reduced. Since $\mathbb{A}_{\varepsilon,i}$ is $1/2$-large, there exists $w \in \mathcal{L}_{\varepsilon,i} \cap R$. The word w starts with $a_i^{\varepsilon_i}$ and satisfies the desired conditions. ∎

Proof of Proposition 3.9. Suppose there exists a set of relators R', containing R and not necessarily finite, such that $Q = \langle S \mid R'\rangle$ is left-orderable and non-trivial. Let $a_{i_1}, \ldots, a_{i_m}$ be all those $a_j \in S$ such that $\overline{a_j} \in Q$ is non-trivial. There must be at least one, since Q is generated by elements of the form $\overline{a_j}$. By Lemma 2.4, we can find $\varepsilon_{i_1}, \ldots, \varepsilon_{i_m} \in \{-1,1\}$ such that every non-empty word consisting of letters from $\{a_{i_1}^{\varepsilon_{i_1}}, \ldots, a_{i_m}^{\varepsilon_{i_m}}\}$ represents a non-trivial element of Q. Note that those words are always reduced.

Now for $j \in \{1,\ldots,n\} \setminus \{i_1,\ldots,i_m\}$ choose $\varepsilon_j \in \{-1,1\}$ in an arbitrary way. We have thus defined a sequence $(\varepsilon_1,\ldots,\varepsilon_n) \in \{-1,1\}^n$. By Lemma 3.10 applied to this sequence and $i = i_1$, we obtain a word w which lies in R, so it represents the trivial element of Q, and consists of letters of the form $a_j^{\varepsilon_j}$ with at least one occurrence of $a_{i_1}^{\varepsilon_{i_1}}$. As $a_j^{\varepsilon_j}$ for $j \notin \{i_1,\ldots,i_m\}$ represents the trivial element, we can remove all occurrences of such letters from w, obtaining a word w_1, still representing the trivial element and consisting only of letters of the form $a_{i_j}^{\varepsilon_{i_j}}$. The word w_1 is, however, non-empty because of at least one occurrence of $a_{i_1}^{\varepsilon_{i_1}}$. We thus arrive at a contradiction with the earlier definition of $\varepsilon_{i_1}, \ldots, \varepsilon_{i_m}$. ∎

For fixed $d \in (0,1)$ and $\lambda = 1/2$ there is n_0 such that the conclusion of Corollary 3.8 holds for all $n \geq n_0$. For such n, Theorem 1.5 is now almost immediate.

Proof of Theorem 1.5 for $n \geq n_0$. A random group G at density d is w.o.p. presented by $\langle S \mid R\rangle$, where R intersects the languages of all 1/2-large b-automata over S, so, by Proposition 3.9, it has no non-trivial left-orderable quotients. ∎

4. Increasing the number of generators. We now generalise our partial proof of Theorem 1.5 to arbitrary values of $n \geq 2$. We closely follow the ideas of [2, Section 3].

We fix $n \geq 2$ and $d \in (0,1)$. We furthermore fix B to be a natural number that is sufficiently large with respect to n and d in a way to be specified later.

As before, S is the set of generators $\{a_1, \ldots, a_n\}$. Let $\widetilde{S} \subset F_n$ denote the set of reduced words of length B over the alphabet S. The involution on $\widetilde{S}$ mapping each word to its inverse does not have fixed points. Thus we can partition $\widetilde{S}$ into $\hat{S}$ and $\hat{S}^{-1}$. We introduce the notation $\hat{S}^{\pm} = \hat{S} \cup \hat{S}^{-1}$ in place of $\widetilde{S}$. Let $\hat{n} = |\hat{S}| = n(2n-1)^{B-1}$.

Furthermore, for $0 \leq P < B$ let $I_P \subset \mathbb{N}_+$ denote the set of those L that can be written as $L = B\hat{L} + P$ with $\hat{L} > 0$.

Definition 4.1 ([2, Definition 3.1]). Let r be a word of length $L \in I_0$ over the alphabet S. Divide the word r into $\hat{L}$ blocks of length B. This determines a new word $\hat{r}$ of length $\hat{L}$ over the alphabet $\hat{S}$, which we call the word *associated* to r.

Definition 4.2 ([2, Definition 3.2]). Given a set R of reduced relators over S of equal length $L \in I_P$, we define the *associated group* $\hat{G}$ in the following way.

If $P = 0$, then we consider the set $\hat{R}$ of relators associated to relators in R. We define $\hat{G}$ to be the group $\langle \hat{S} \mid \hat{R}\rangle$.

If $1 \leq P < B$, then we do the following construction. Suppose that $r_1, r_2 \in R$ are two relators of length L over S, satisfying $r_1 = q_1 v^{-1}$ and $r_2 = v q_2$ (we assume that q_1, q_2, v are reduced and there are no reductions between q_1 and v^{-1} or between v and q_2) for some word v over S of length P. We then obtain a (possibly non-reduced) word $q_1 q_2$ over S, of length $2B\hat{L}$, with $\overline{q_1 q_2} = e$ in $G = \langle S \mid R\rangle$. To this word we can associate, as before, a relator over $\hat{S}$, of length $2\hat{L}$ (possibly non-reduced), which we denote by $\hat{r}(r_1, r_2)$. We denote by $\hat{R}$ the set of all $\hat{r}(r_1, r_2)$ as above and we define $\hat{G} = \langle \hat{S} \mid \hat{R}\rangle$.

The main intuition here is that $\hat{R}$ obtained from a random set R of relators over S, at density d and length $L \in I_0$, is very similar to a random set of relators over $\hat{S}$ at the same density d and length L/B [2, Section 3].

By increasing B, the number $\hat{n}$ can be made arbitrarily large. We can thus have $\hat{n}$ large enough to obtain the conclusion of Corollary 3.8 for intersections of the languages of 1/2-large b-automata over $\hat{S}$ with random sets of relators at density d. We then use the following analogue of Proposition 3.9.

Proposition 4.3. *Suppose that $\hat{R}$, obtained as in Definition* 4.2 *from R being a set of reduced relators of the same length, intersects the languages of all* 1/2*-large b-automata over $\hat{S}$. Then $G = \langle S \mid R\rangle$ has no non-trivial left-orderable quotients.*

Proof. Suppose there exists a set of relators R', containing R, such that $Q = \langle S \mid R'\rangle$ is left-orderable and non-trivial. The construction of $\hat{G}$ was performed in such a way that by expanding elements of $\hat{S}$ into words over S we get a natural epimorphism $\phi : \hat{G} \twoheadrightarrow H$, where H is the subgroup of Q generated by the elements corresponding to the reduced words of length B over S.

We note that $H \subset Q$ is of finite index, since every element $g \in Q$ is of the form $g = \overline{w}$ for some reduced word w over S and we may write $w = uv$ with u of length at most B and v of length divisible by B. We thus have $\overline{v} \in H$, hence $g \in \overline{u}H$ and the index $[Q : H]$ is not greater than the number of possible values of $\overline{u}$, which is finite.

Moreover, H is non-trivial, because otherwise Q would be finite and non-trivial, hence not torsion-free, contradicting left-orderability (by Corollary 2.3).

Denote elements of $\hat{G}$, represented by single letters from $\hat{S}$, by $b_1, \dots, b_{\hat{n}}$. They generate $\hat{G}$, so H is generated by $\phi(b_1), \dots, \phi(b_{\hat{n}})$, not all of them being trivial. Let $\phi(b_{i_1}), \dots, \phi(b_{i_m})$ be all non-trivial elements of the form $\phi(b_j)$. The subgroup H is left-orderable, so, by Lemma 2.4, there exist $\varepsilon_{i_1}, \dots, \varepsilon_{i_m} \in \{-1, 1\}$ such that every non-empty product of elements of the form $\phi(b_{i_j})^{\varepsilon_{i_j}}$ is non-trivial. For $i \in \{1, \dots, \hat{n}\} \setminus \{i_1, \dots, i_m\}$ we choose $\varepsilon_i \in \{-1, 1\}$ in an arbitrary way.

Fix $i = i_1$. For this index i and the set $\hat{R}$ of words over $\hat{S}$ we apply Lemma 3.10 to conclude that there exists a product of elements of the form $b_j^{\varepsilon_j}$, with at least one occurrence of $b_{i_1}^{\varepsilon_{i_1}}$, which evaluates to the trivial element in $\hat{G} = \langle \hat{S} \mid \hat{R}\rangle$.

By evaluating ϕ on this product, we get a product of elements of the form $\phi(b_j)^{\varepsilon_j}$, with at least one occurrence of $\phi(b_{i_1})^{\varepsilon_{i_1}}$, which evaluates to the trivial element in H. Finally, by leaving the non-trivial factors only, we get a non-empty product of elements of the form $\phi(b_{i_j})^{\varepsilon_{i_j}}$, evaluating to the trivial element, which contradicts the definition of $\varepsilon_{i_1}, \dots, \varepsilon_{i_m}$. ∎

The last element of the proof of Theorem 1.5 is the following.

LEMMA 4.4 ([2, Section 3]). *If B is sufficiently large, then, in the Gromov density model with n generators, a random set R of relators at density d has w.o.p. the property that the set $\hat{R}$, obtained from R as in Definition* 4.2, *intersects the languages of all* 1/2*-large b-automata over $\hat{S}$.*

Assuming Lemma 4.4, the proof of Theorem 1.5 is straightforward.

Proof of Theorem 1.5. Let B be sufficiently large for the conclusion of Lemma 4.4 to hold. Then, by Proposition 4.3 and Lemma 4.4, a random group $G = \langle S \mid R \rangle$ in the Gromov density model has w.o.p. no non-trivial left-orderable quotients. ∎

The proof of Lemma 4.4 (in a slightly stronger form) is given in [2, Section 3] in the first five lines of the proof of [2, Theorem 1.5]. The hypothesis of [2, Proposition 2.6] for the group $\hat{G} = \langle \hat{S} \mid \hat{R} \rangle$, obtained from a random group $G = \langle S \mid R \rangle$, is checked there, which amounts to proving that $\hat{R}$, obtained from a random set R of relators in the Gromov model, intersects the languages of all 1/3-large b-automata over $\hat{S}$. It remains to note that all 1/2-large b-automata are, in particular, 1/3-large.

Appendix. Intersections of high-density sets. From now on, we denote by $I \subset \mathbb{N}_+$ a fixed infinite subset, and all limits as $L \to \infty$ are taken over $L \in I$. The main result of this appendix is the following.

PROPOSITION A.1. *Suppose that for each $L \in I$ a set R_L of size $c_L > 0$ with $a_L > 0$ elements distinguished is given. For fixed L pick uniformly and independently at random entries of a b_L-tuple ($b_L > 0$) from R_L and consider the random variable D_L equal to the number of distinguished entries of the resulting tuple. Assume that $a_L b_L / c_L \to \infty$ as $L \to \infty$. Then for every $\varepsilon > 0$,*

$$\lim_{L\to\infty} \mathbb{P}\bigg((1-\varepsilon)\frac{a_L b_L}{c_L} \le D_L \le (1+\varepsilon)\frac{a_L b_L}{c_L} \bigg) = 1.$$

Before proving Proposition A.1, let us use it to give a proof of Proposition 3.6 and its generalisation.

Proof of Proposition 3.6. Let c_L, for $L \in I$, denote the number of all reduced relators of length L over S, i.e. $c_L = |R_L| = 2n(2n-1)^{L-1}$. Moreover, let $a_L = |R_f \cap R_L|$ be the number of relators of length L we distinguish by wanting them to be selected in the random tuple. Let $b_L = \lfloor (2n-1)^{dL} \rfloor$. We assume $a_L \ge C(2n-1)^{d'L}$ for $L \in I$ sufficiently large and some $C > 0$. At length L, R is a tuple of b_L elements, chosen uniformly and independently at random from R_L. Let D_L be as in Proposition A.1. Note that $a_L b_L / c_L \to \infty$ as $L \to \infty$, since $d + d' > 1$. We may thus apply Proposition A.1 for any $\varepsilon > 0$ to see that a random set R of relators at density d and length L has w. I-o.p. at least $D_L \geq (1-\varepsilon)a_L b_L/c_L \geq K(2n-1)^{(d+d'-1)L}$ entries from R_f, for some $K > 0$. For L sufficiently large this clearly implies that R and R_f intersect. ∎

If we moreover assume that $d < 1/2$ and R_f is roughly (not just at least) of density d', then we can prove that the intersection is roughly of density $d + d' - 1$.

PROPOSITION A.2. *Suppose $d, d' \in (0,1)$ are such that $d + d' > 1$ and $d < 1/2$. Let $R_f \subset F_n$ be a fixed set of relators in some fixed number n of generators such that for some $C_1, C_2 > 0$ the inequalities*

$$C_1(2n-1)^{d'L} \leq |R_f \cap R_L| \leq C_2(2n-1)^{d'L}$$

hold for all sufficiently large $L \in I$. Then for some $K_1, K_2 > 0$ a random set R of relators at density d and length L satisfies w. I-o.p. the inequalities

$$K_1(2n-1)^{(d+d'-1)L} \leq |R_f \cap R| \leq K_2(2n-1)^{(d+d'-1)L},$$

where $|R_f \cap R|$ denotes the number of distinct entries of R, belonging to R_f.

Proof. We use the notation of the proof of Proposition 3.6. Analogously to that proof, for some $K_1, K_2 > 0$ we obtain

$$\text{(A.2)} \qquad K_1(2n-1)^{(d+d'-1)L} \leq D_L \leq K_2(2n-1)^{(d+d'-1)L},$$

occurring w. I-o.p.

Since $d < 1/2$, we have $b_L^2/c_L \to 0$ as $L \to \infty$.

Let us estimate the probability q_L that in the experiment defining D_L all elements of the resulting b_L-tuple are pairwise distinct. It is the same as the probability that every element of the tuple is different from the elements having smaller indices (we assume some fixed order on the tuple), so

$$q_L = 1\left(1 - \frac{1}{c_L}\right)\left(1 - \frac{2}{c_L}\right)\cdots\left(1 - \frac{b_L - 1}{c_L}\right) \geq \left(1 - \frac{b_L - 1}{c_L}\right)^{b_L}.$$

For $L \in I$ sufficiently large we have $b_L^2/c_L < 1$, so $b_L \leq b_L^2 < c_L$ and the number $x_L = -(b_L - 1)/c_L$ satisfies $x_L \geq -1$. This means that we can use Bernoulli's inequality to obtain

$$q_L \geq \left(1 + \left(-\frac{b_L - 1}{c_L}\right)\right)^{b_L} \geq 1 - b_L\frac{b_L - 1}{c_L}.$$

Obviously, $b_L(b_L - 1)/c_L \to 0$ as $L \to \infty$, because $b_L^2/c_L \to 0$ as $L \to \infty$. It follows that $q_L \to 1$ as $L \to \infty$, so w. I-o.p., D_L is the number of distinct entries of R belonging to R_f, which combined with (A.2) concludes the proof. ∎

For the proof of Proposition A.1 we will apply the following bound known as Chebyshev's inequality.

LEMMA A.3 ([6, Lemma 3.1]). *If ξ is a random variable with $\mathbb{E}\xi^2 < \infty$, then for every $\alpha > 0$,*

$$\mathbb{P}(|\xi - \mathbb{E}\xi| \geq \alpha) \leq \frac{\operatorname{Var}\xi}{\alpha^2}.$$

Proof of Proposition A.1. Fix $L \in I$. For $i = 1, \ldots, b_L$ denote by $X_i^{(L)}$ the random variable equal to 1 if the ith element of the random tuple considered is distinguished, and equal to 0 otherwise.

Observe that the variables $(X_i^{(L)})_i$ are independent and we have the equalities $\mathbb{P}(X_i^{(L)} = 1) = a_L/c_L = 1 - \mathbb{P}(X_i^{(L)} = 0)$, so $\mathbb{E}X_i^{(L)} = a_L/c_L$. Next we obtain $\operatorname{Var} X_i^{(L)} = (a_L/c_L)(1 - a_L/c_L)$. Since $D_L = \sum_{i=1}^{b_L} X_i^{(L)}$, we deduce that $\mathbb{E}D_L = a_L b_L/c_L$ and $\operatorname{Var} D_L = (a_L b_L/c_L)(1-a_L/c_L)$. Fix $\varepsilon > 0$. Now we apply Lemma A.3 for $\xi = D_L$ and $\alpha = \varepsilon\mathbb{E}D_L$, obtaining

$$\begin{aligned}\mathbb{P}\left(\left|D_L - \frac{a_L b_L}{c_L}\right| \geq \varepsilon\frac{a_L b_L}{c_L}\right) &= \mathbb{P}(|D_L - \mathbb{E}D_L| \geq \varepsilon\mathbb{E}D_L)\\ &\leq \frac{\operatorname{Var} D_L}{(\varepsilon\mathbb{E}D_L)^2} = \frac{1 - a_L/c_L}{\varepsilon^2 a_L b_L/c_L} \leq \frac{1}{\varepsilon^2 a_L b_L/c_L} \to 0\end{aligned}$$

as $L \to \infty$, since we have assumed that $a_L b_L/c_L \to \infty$. ∎

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