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ON THE GLOBAL LIPSCHITZ CONTINUITY OF THE BERGMAN PROJECTION ON A CLASS OF CONVEX DOMAINS OF INFINITE TYPE IN $\mathbb{C}^2$

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Abstract. The main purpose of this paper is to prove the global Lipschitz continuity of the Bergman projection in a class of smoothly bounded, convex domains admitting maximal type F in $\mathbb{C}^2$. The maximal type F here is a geometric condition which includes all cases of finite type and many cases of infinite type in the sense of Range (1978). Let Ω be such a domain. We prove that the Bergman projection $\mathcal{P}$ maps continuously $\Lambda^{t^\alpha}(\Omega)$ to $\Lambda^{g_\alpha}(\Omega)$ for $0<\alpha\le 1$, where g_α is a function depending on F.

1. Introduction. Let Ω be a smooth, bounded domain in the complex Euclidean space $\mathbb{C}^n$. Let the class of square integrable functions in Ω with the Lebesgue measure dV for $\mathbb{C}^n$ denoted by $L^2(\Omega)$, and let $\mathcal{O}(\Omega)$ be the class of holomorphic functions in Ω. The Bergman projection $\mathcal{P}$ is the orthogonal projection $L^2(\Omega)$ onto the Bergman space $L^2(\Omega)\cap\mathcal{O}(\Omega)$. From functional analysis, the following facts are well-known in the theory of Bergman projections:

(1) $\mathcal{P}u(z)=u(z)$ for all $u\in L^2(\Omega)\cap\mathcal{O}(\Omega)$;
(2) $\mathcal{P}^*=\mathcal{P}$, where $\mathcal{P}^*$ is the dual operator in $L^2(\Omega)$;
(3) $\|\mathcal{P}u\|_{L^2(\Omega)}\le\|u\|_{L^2(\Omega)}$ for all $u\in L^2(\Omega)$;
(4) for all $u\in L^2(\Omega)$ and $z\in\Omega$,

$$\mathcal{P}[u](z)=\langle u,\overline{P(\cdot,z)}\rangle=\int_\Omega u(\zeta)P(\zeta,z)\,dV(\zeta);$$

(5) the Bergman kernel function $P(\zeta,z)$ is holomorphic with respect to $z\in\Omega$ and independent of u.

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The Bergman projection plays an important role in the theory of partial differential equations in several complex variables. In particular, let $\bar{\partial}^*$ be the Hilbert adjoint of $\bar{\partial}$ in $L^2(\Omega)$ and $\mathcal{N}$ be the $\bar{\partial}$-Neumann operator discovered by J. J. Kohn. If u is an arbitrary square integrable function in Ω, then

$$\mathcal{P}[u] = u - \bar{\partial}^*\mathcal{N}[\bar{\partial}u].$$

Note that Kohn's solution $\bar{\partial}^*\mathcal{N}$ to the $\bar{\partial}$ equation exists on all weakly pseudoconvex domains in $\mathbb{C}^n$ [2].

In this paper, we deal with the global Lipschitz continuity in a class of smoothly bounded, convex domains in $\mathbb{C}^2$. For this purpose, we recall its long and well-known history. For $0 < \alpha < 1$, let us define

$$\begin{aligned}&\Lambda^{t^\alpha}(\Omega)\\ &\quad = \left\{u \in L^\infty(\Omega) : \|u\|_{t^\alpha} := \|u\|_{L^\infty(\Omega)} + \sup_{z,z+h\in\Omega} \frac{|u(z+h)-u(z)|}{|h|^\alpha} < \infty\right\}.\end{aligned}$$

When Ω is a strongly pseudoconvex or strongly convex domain in $\mathbb{C}^n$ with smooth boundary $b\Omega$, the Bergman projection $\mathcal{P}$ maps continuously $\Lambda^{t^\alpha}(\Omega)$ to $\Lambda^{t^\alpha}(\Omega)$. This is a fundamental and significant result proved, applying the theory of singular integrals, by Phong and Stein [16], and by Ahern and Schneider [1] via the Boutet de Monvel expression. Applying the Cauchy–Fantappiè theory, Ligocka [14] showed that the Bergman projection $\mathcal{P}$ maps $\Lambda^{\alpha}(\Omega)$ to $\Lambda^{t^{\alpha/2}}(\Omega)$ continuously.

More generally, let $\Omega \subset \mathbb{C}^2$ be a weakly pseudoconvex domain of finite type $m \geq 2$ in the sense of commutators. Nagel et al. [15] proved that $\mathcal{P} : \Lambda^{t^\alpha}(\Omega) \to \Lambda^{t^\alpha}(\Omega)$ continuously. They introduced a class of singular integral operators associated with non-isotropic distances, named non-isotropic smooth operators. The result is that $\mathcal{P}$ is a non-isotropic smooth operator of order zero. A typical domain here is

$$\Omega^m = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_2) > P(z_1)\},$$

where P is a subharmonic, non-harmonic polynomial in the complex plane.

Recently, Halfpap et al. [4] have described singularities of the Bergman kernel function associated to weakly pseudoconvex domains of infinite type in $\mathbb{C}^2$. A special model in this work is

$$\Omega^\infty = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_2) > \exp(-|\mathrm{Re}(z_1)|^{-a})\} \quad \text{for } a \geq 1.$$

Lanzani and Stein [13] have recently studied the Bergman operator on domains with minimal smoothness. Khanh and Raich [10] have also established local Lipschitz estimates for the Bergman projection on a class of weakly pseudoconvex domains admitting a good dilation and satisfying the Bell–Ligocka condition.

In the present paper, the main goal is to study the global Lipschitz continuity of the Bergman projection in a class of smoothly bounded, convex domains of infinite type in $\mathbb{C}^2$. In particular, we have:

THEOREM 1.1 (Main Theorem). *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, convex domain. Assume that Ω admits maximal type F at all boundary points, for some function F (see Definition 2.3). Then for every $0 < \alpha \leq 1$, the Bergman projection $\mathcal{P}$ maps continuously $\Lambda^{t^\alpha}(\Omega)$ into $\Lambda^{g_\alpha}(\Omega)$, where*

$$g_\alpha(d^{-1}) := \left(\int_0^d \frac{[\sqrt{F^*(t)}]^\alpha}{t}\, dt\right)^{-1},$$

and F^* is the inverse of F.

Here, $\Lambda^f(\Omega)$ is the f-Lipschitz space on Ω: for f an increasing function such that $\lim_{t\to\infty} f(t) = \infty$,

$$\Lambda^f(\Omega) = \Big\{u \in L^\infty(\Omega) : \|u\|_f := \|u\|_{L^\infty(\Omega)} + \sup_{z,z+h\in\Omega} f(|h|^{-1})|u(z+h)-u(z)| < \infty\Big\}.$$

It is clear that if $f(t) = t^\alpha$ for $0 < \alpha < 1$, the space $\Lambda^f(\Omega)$ coincides with the classical Lipschitz space of order α. For convenience, we also recall

$$\Lambda^t(\Omega) := \mathrm{Lip}^1(\Omega) = \Big\{u \in L^\infty(\Omega) : \|u\|_{\mathrm{Lip}^1(\Omega)} := \|u\|_{L^\infty(\Omega)} + \sup_{\zeta,z\in\Omega} \frac{|u(\zeta) - u(z)|}{|\zeta - z|} < \infty\Big\}.$$

EXAMPLE 1.1. Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, convex domain of finite type $2m$ in the sense of Range, where $m \geq 1$ is an integer. Then, for all $0 < \alpha \leq 1$, the Bergman projection maps continuously $\Lambda^{t^\alpha}(\Omega)$ into $\Lambda^{t^{\alpha/(2m)}}(\Omega)$.

EXAMPLE 1.2. Let

$$\Omega = \Big\{(z_1, z_2) \in \mathbb{C}^2 : \rho(z) := \exp(1 + 2/s) \cdot \exp\Big(\frac{-1}{|z_1|^s}\Big) + |z_2|^2 - 1 < 0\Big\}$$

for some $0 < s < 1/2$. Since $F(t) = \exp(-1/32t^s)$, we have

$$g_\alpha(t) = \frac{1024^s(\alpha - 2s)}{2s}(|\ln t|)^{\alpha/(2s)-1},$$

for $2s < \alpha < 1$. Thus, for all $2s < \alpha \leq 1$, the Bergman projection maps continuously $\Lambda^{t^\alpha}(\Omega)$ into $\Lambda^{g_\alpha}(\Omega)$.

The paper is organized as follows. In Section 2, we briefly recall needed notions from Cauchy–Fantappiè theory and the geometric notion of maximal type F. Section 3 is devoted to the proof of the Main Theorem 1.1. The proof is based on Kerzman–Stein–Ligocka's setup [8, 14].

2. Preliminaries

2.1. On a Cauchy–Fantappiè form. In this subsection, we recall one of Cauchy–Fantappiè forms which is applied to construct an integral representation for the Bergman projection. For more details, the reader is referred to [19] or [13].

Let Ω be a bounded convex domain in $\mathbb{C}^n$ $(n \geq 2)$ with smooth boundary $b\Omega$. Let ρ be a defining function for Ω so that $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ and $b\Omega = \{z \in \mathbb{C}^n : \rho(z) = 0\}$, $\nabla\rho \neq 0$ on $b\Omega$, and $\nabla\rho \perp b\Omega$. For $z_j = x_j + \sqrt{-1}\, x_{n+j}$, $j = 1, \ldots, n$, the convexity means

$$\sum_{i,j=1}^{2n} \frac{\partial^2 \rho}{\partial x_i \partial x_j}(x) a_i a_j \geq 0 \quad \text{on } b\Omega,$$

for every nonzero $a = (a_1, \ldots, a_{2n}) \in \mathbb{R}^{2n}$ with $\sum_{j=1}^{2n} a_j \frac{\partial \rho}{\partial x_j}(x) = 0$ on $b\Omega$. For $\zeta \in \bar{\Omega}$ and $z \in \Omega$ define

$$\Phi(\zeta, z) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j). \tag{2.1}$$

The convexity of Ω implies

$$\operatorname{Re}\Big(\sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j)\Big) \neq 0$$

and so $\Phi(\zeta, z) \neq 0$ as well, for all $\zeta \in b\Omega$ and $z \in \Omega$. Moreover, we also have:

LEMMA 2.1. *There are positive constants δ, c such that for all $\zeta \in b\Omega$:*

(1) *Φ is of class C^1 in (ζ, z);*
(2) *$\Phi(\zeta, \cdot)$ is holomorphic in $z \in B(\zeta, \delta)$;*
(3) *for some $A > 0$, $|\Phi(\zeta, z)| \geq A$ for all $z \in \Omega$ and $|z - \zeta| \geq c/2$.*

DEFINITION 2.2. Under the above notation, let

$$C(\zeta, z) = \frac{1}{2\pi i}\left[\sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j}(\zeta) d\zeta_j\right] \frac{1}{\Phi(\zeta, z)} \quad \text{for } \zeta \in b\Omega, z \in \Omega,$$

be a $(1,0)$-form in ζ-variables. The Cauchy kernel on Ω is defined to be the *Cauchy–Fantappiè* $(n, n-1)$*-form* in ζ-variables for $C(\zeta, z)$, that is

$$\Omega_0(C(\zeta, z)) = C(\zeta, z) \wedge (\bar{\partial}_\zeta C(\zeta, z))^{n-1},$$

where $(\bar{\partial}_\zeta C(\zeta, z))^{n-1}$ is the $(n-1)$-wedge product $\bar{\partial}_\zeta C(\zeta, z) \wedge \cdots \wedge \bar{\partial}_\zeta C(\zeta, z)$.

More explicitly, we have

$$\begin{aligned}(2.2)\quad \Omega_0(C(\zeta,z)) &= C(\zeta,z)\wedge(\bar\partial_\zeta C(\zeta,z))^{n-1}\\ &= \frac{\sum_{j=1}^n \frac{\partial\rho}{\partial\zeta_j}(\zeta)d\zeta_j}{\Phi(\zeta,z)}\wedge\bar\partial_\zeta\bigg(\frac{\sum_{j=1}^n \frac{\partial\rho}{\partial\zeta_j}(\zeta)d\zeta_j}{\Phi(\zeta,z)}\bigg)^{n-1}\\ &= \frac{\big(\sum_{j=1}^n \frac{\partial\rho}{\partial\zeta_j}(\zeta)d\zeta_j\big)\wedge\big(\sum_{j,k=1}^n \frac{\partial^2\rho}{\partial\bar\zeta_k\partial\zeta_j}(\zeta)d\bar\zeta_k\wedge d\zeta_j\big)^{n-1}}{\Phi^n(\zeta,z)}\\ &= \sum_{\substack{\text{finite sum over } I_0\\ |I_0|=n-1}}\frac{A_{I_0}(\zeta)}{\Phi^n(\zeta,z)}d\zeta_1\wedge\cdots\wedge d\zeta_n\wedge d\bar\zeta_{I_0},\end{aligned}$$

where I_0 denotes an increasing multi-index of order $n-1$, and $A_{I_0}(\zeta)$ is a polynomial involving first and second ζ-derivatives of ρ. Since Ω is smooth, $A_{I_0}(\zeta)$ is smooth in ζ on a neighborhood of $\bar\Omega$, for all I_0 in the finite sum.

In the case $n=2$, the Cauchy kernel is

$$\Omega_0(C(\zeta,z)) = C(\zeta,z)\wedge\bar\partial_\zeta C(\zeta,z) = \sum_{\substack{\text{finite sum}\\ \text{over a single index } I_0\\ d\bar\zeta_{I_0}\in\{d\bar\zeta_1,d\bar\zeta_2\}}}\frac{A_{I_0}(\zeta)}{\Phi^2(\zeta,z)}d\zeta_1\wedge d\zeta_2\wedge d\bar\zeta_{I_0}.$$

A fundamental property of the Cauchy–Fantappiè form is: if u is holomorphic on a neighborhood of $\bar\Omega$, then

$$u(z) = \int_{b\Omega} u(\zeta)\Omega_0(C(\zeta,z))\quad\text{for } z\in\Omega.$$

It is clear that this definition is a natural extension of the Cauchy kernel in $\mathbb{C}$. In particular,

$$C(\zeta,z) = \frac{1}{2\pi i}\frac{\frac{\partial\rho}{\partial\zeta}(\zeta)d\zeta}{\frac{\partial\rho}{\partial\zeta}(\zeta)(\zeta-z)} = \frac{1}{2\pi i}\frac{d\zeta}{\zeta-z},$$

and so

$$\Omega_0(C(\zeta,z)) = \frac{1}{2\pi i}\frac{d\zeta}{\zeta-z}.$$

2.2. Maximal type F. The geometric notion of maximal type F is a natural extension of the notion of finite type in the sense of Range. Moreover, it also covers many convex domains of infinite type in which the classical methods (e.g., by Henkin [5, 6], Range [19], Shaw [21, 22], etc.) cannot be applied. Maximal type F was introduced in [3] to obtain L^p and Lipschitz estimates for the $\bar\partial_b$-equation on pseudoconvex boundaries of infinite type in $\mathbb{C}^2$.

DEFINITION 2.3. Let $F:[0,\infty)\to[0,\infty)$ be a smooth, increasing function such that

- $F(0)=0$;
- $\int_0^\delta |\ln F(r^2)|dr < \infty$ for some small $\delta > 0$;
- $F(r)/r$ is increasing.

Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded, convex domain. Then Ω is said to *admit maximal type* F at $P \in b\Omega$ if there are positive constants c, c' such that, for all $\zeta \in b\Omega \cap B(P, c')$,

$$\rho(z) \gtrsim F(|z-\zeta|^2) \quad \text{for all } z \in B(\zeta, c) \text{ with } \Phi(\zeta, z) = 0.$$

Here and in what follows, $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant, and $\approx$ means the combination of $\lesssim$ and $\gtrsim$.

Now, the following examples show that the class of convex domains admiting maximal type F includes many convex domains of finite type and infinite type in the sense of Range.

Example 2.1. (1) Let Ω be a strongly pseudoconvex domain with strictly plurisubharmonic defining function ρ. Then

$$-\operatorname{Re}\Phi(\zeta, z) \geq \rho(\zeta) - \rho(z) + \lambda_0|\zeta - z|^2$$

for $|\zeta - z|$ and $|\rho(\zeta)|$ small, and $\lambda_0 > 0$. Hence, when $\zeta \in b\Omega \cap \{|\zeta - z| < c\}$ and $\Phi(\zeta, z) = 0$, we have

$$\rho(z) \gtrsim F(|z-\zeta|^2)$$

with $F(t) = t$. So, in this case Ω admits maximal type F.

(2) Let $\Omega \subset \mathbb{C}^n$ be convex of strict type $m(p)$ at every point $p \in b\Omega$, as defined in [11], and generalized by Range [17, 18] and Shaw [21]. From this definition, the mapping $p \mapsto m(p)$ is upper semicontinuous. Hence, $m_0 := \sup_{p\in b\Omega} m(p) < \infty$.

From [21, Theorem 3.1], we have

$$\rho(z) \gtrsim F(|\zeta - z|^2)$$

for all $\zeta \in b\Omega$ with $|z-\zeta| < c$ and $\Phi(\zeta, z) = 0$, where $F(t) = t^{m_0/2}$. Hence, Ω admits maximal type F.

A basic example is the complex ellipsoid

$$\Omega = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^{2m_1} + \cdots + |z_n|^{2m_n} < 1\}.$$

In this case, $F(t) = t^m$ for $m = \max\{m_1, \dots, m_n\}$.

(3) Let

$$\begin{aligned}\Omega^\infty = \{&(z_1, \dots, z_n) \in \mathbb{C}^n : \\ &\rho(z) := \exp(1 + 2/s) \cdot \exp(-1/|z_1|^s) + |z_2|^2 + \cdots + |z_n|^2 - 1 < 0\}.\end{aligned}$$

Since

$$\operatorname{Re}[\Phi(\zeta, z)] \lesssim -\rho(\zeta) + \rho(z) - \exp(1 + 2/s)\exp\left\{\frac{-1}{32|\zeta - z|^{2s}}\right\}$$

for $0 < s < 1/2$, Ω^∞ is a convex domain with maximal type $F(t) = \exp(-1/32t^s)$ [23].

The following result is the main contribution to our analysis.

Lemma 2.4. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, convex domain. Assume that Ω admits maximal type F at $P \in b\Omega$ for some function F. Then there is a positive constant c such that the support function $\Phi(\zeta, z)$ satisfies the estimate*

$$|\Phi(\zeta,z)| \gtrsim |\rho(z)| + |\operatorname{Im}\Phi(\zeta,z)| + F(|z-\zeta|^2), \tag{2.3}$$

for every $\zeta \in b\Omega \cap B(P,c)$ and $z \in \overline{\Omega}$ with $|z-\zeta| < c$.

Proof. We use the technique of Range [17, 18]. Let δ, c, $\rho(z)$, and $\Phi(\zeta,z)$ be as above, and let $(w', w_n) = (w_1, \ldots, w_{n-1}, w_n)$. For any $\zeta \in b\Omega \cap B(P,\delta)$, we define the holomorphic map $\psi_\zeta : z \mapsto w = (w', w_n) = (z' - \zeta', \Phi(\zeta,z))$ for $z \in B(\zeta, c)$. The Jacobian matrix of ψ_ζ at ζ is unitary by Lemma 2.1(4). Hence, we can choose $c, \delta > 0$ sufficiently small so that $\psi_\zeta : B(\zeta,c) \to \psi_\zeta(B(\zeta,c)) \subset B(0,\delta)$ is biholomorphic. Then the inverse map ψ_ζ^{-1} exists and it can be assumed that its Jacobian matrix is uniformly bounded on $B(0,\delta)$. As a consequence, $|\psi_\zeta(z) - \psi_\zeta(Z)| \approx |z - Z|$ for all $z, Z \in B(\zeta, c)$. We define $\rho_\zeta(w', w_n) := \rho(\psi_\zeta^{-1}(w', w_n))$; then ρ_ζ is a defining function for $\psi_\zeta(\Omega \cap B(\zeta, c))$.

By Lemma 2.1(3) and maximal type F, after shrinking c, for some d small we obtain

$$\begin{aligned} \rho_\zeta(w',0) &> 0 \quad \text{for } 0 < |w'| < d, \\ \rho_\zeta(w',0) &\gtrsim F(|w'|^2) \quad \text{for } 0 \le |w'| < d. \end{aligned} \tag{2.4}$$

Therefore, by Taylor's Theorem, for any $|w| < d$, we have

$$\begin{aligned} \rho_\zeta(w', w_n) &= \rho_\zeta(w',0) + 2\operatorname{Re}\left(\frac{\partial \rho_\zeta}{\partial w_n}(w',0).w_n\right) + o(|w_n|) \\ &\ge 2\operatorname{Re} w_n + AF(|w'|^2) + o(1)|w_n|, \end{aligned} \tag{2.5}$$

where the last inequality follows from $\partial_w \rho_\zeta(0) = dw_n$ and $o(1) \to 0$ when $|w| \to 0$. Here, the convergence is uniform in the ζ-variables, since $o(1)$ in our case only depends on the modulus of continuity of the first order partial derivatives of $\rho_\zeta(w', w_n)$. So, let $0 < d^* < d$ be so small that $o(1)|w_n| \le |\operatorname{Re} w_n| + |\operatorname{Im} w_n|$ for every $|w| \le d^*$. Hence, the above inequality implies

$$-2\operatorname{Re} w_n + |\operatorname{Re} w_n| \ge \rho_\zeta(w', w_n) - |\operatorname{Im} w_n| + AF(|w'|^2) \quad \text{for } |w| < d^*.$$

That means

$$|\operatorname{Re} w_n| \gtrsim \rho_\zeta(w', w_n) - |\operatorname{Im} w_n| + AF(|w'|^2) \quad \text{for } |w| < d^*.$$

The last step is to convert $\rho_\zeta(w)$ to $\rho(z)$. To do this, we choose $c^* < c$ so small that $\psi_\zeta(B(\zeta, c^*)) \subset B(0, d^*)$. Then, using Taylor's formula and the fact that F is smooth, we have

$$F(|w'|^2) = F(|w|^2) + O(1)|w_n|^2,$$

so

$$|\mathrm{Re}\,\Phi(\zeta, z)| \gtrsim -\rho(z) - |\mathrm{Im}\,\Phi(\zeta, z)| + AF(|\zeta - z|^2).$$

Replacing the left hand side by $C|\Phi|$, for $C > 0$ large enough, yields

$$|\Phi(\zeta, z)| \gtrsim |\rho(z)| + |\mathrm{Im}\,\Phi(\zeta, z)| + F(|z - \zeta|^2).$$

This completes the proof. ∎

Remark 2.5. In [3], the uniform total pseudoconvexity is used to show the existence of the support function $\Phi(\zeta, z)$. Therefore, in the same manner but with more calculations applied in the z-variables, Main Theorem 1.1 and its corollaries are also true for any uniformly totally pseudoconvex domain of maximal type F, whose closure $\bar{\Omega}$ has a Stein neighborhood basis.

3. Proof of the main theorem. Kerzman and Stein [8] were the first to apply Cauchy–Fantappiè theory to describe the singularities of the Szegö projection (the boundary version of the Bergman projection) on strongly pseudoconvex domains. Then Ligocka [14] adapted this process to the Bergman projection. The following long proof is based on her work, but on convex domains of infinite type in the sense of Range.

Let u belong to $C^1(\bar{\Omega})$ and be holomorphic on Ω (i.e., $\bar{\partial} u = 0$ on Ω). By the Stokes Theorem, we have

$$u(z) = \int_{\Omega} u(\zeta)\bar{\partial}_\zeta(\Omega_0(C(\zeta, z))), \qquad z \in \Omega.$$

By the smoothness of $A_{I_0}(\zeta)$ and $\Phi(\zeta, z)$, the form $\bar{\partial}_\zeta(\Omega_0(C(\zeta, z)))$ is also smooth on $\bar{\Omega} \times \Omega$.

For $\delta > 0$, let $\Omega_\delta = \{z \in \mathbb{C}^n : \rho(z) < \delta\}$ and let P_z be the Hörmander solution operator to the $\bar{\partial}$-equation (see [7]) in the z-variables. Under this notation we can choose $\delta < c$ (c is the constant in Lemma 2.4) sufficiently small such that $\bar{\partial}_z\bar{\partial}_\zeta\Omega_0(C(\zeta, z))$ is smooth on $\bar{\Omega} \times \bar{\Omega}_\delta$

Definition 3.1. For $(\zeta, z) \in \bar{\Omega} \times \Omega_\delta$, define

$$\begin{aligned} Q(\zeta, z) &= -P_z(\bar{\partial}_z\bar{\partial}_\zeta\Omega_0(C(\zeta, z))), \\ G(\zeta, z) &= Q(\zeta, z) + \bar{\partial}_\zeta(\Omega_0(C(\zeta, z))). \end{aligned}$$

Lemma 3.2. *Let u be a holomorphic function defined in Ω_δ. Then*

$$u(z) = \int_{\Omega} u(\zeta)G(\zeta, z) \quad \textit{for } z \in \Omega.$$

Proof. Since

$$u(z)=\int_{\Omega} u(\zeta)\bar{\partial}_\zeta(\Omega_0(C(\zeta,z))),\quad z\in\Omega,$$

it is sufficient to prove that

$$\int_{\Omega} u(\zeta)P_z(\bar{\partial}_z\bar{\partial}_\zeta\Omega_0(C(\zeta,z)))=0.$$

Now, by the Stokes Theorem again,

$$\begin{aligned}\int_{\Omega} u(\zeta)P_z(\bar{\partial}_z\bar{\partial}_\zeta\Omega_0(C(\zeta,z))) &= P_z\Big(\int_{\Omega} u(\zeta)\bar{\partial}_z\bar{\partial}_\zeta\Omega_0(C(\zeta,z))\Big)\\ &= P_z\Big(\int_{\Omega} u(\zeta)\bar{\partial}_\zeta\bar{\partial}_z\Omega_0(C(\zeta,z))\Big)\\ &= P_z\Big(\int_{b\Omega} \bar{\partial}_z[\Omega_0(C(\zeta,z))]u(\zeta)\Big)=0,\end{aligned}$$

where the vanishing of the last term was proved in [8, (1.4.2)]. ■

LEMMA 3.3. *Let $\Omega\subset\mathbb{C}^2$ be a smoothly bounded, convex domain admitting maximal type F at all boundary points, for some function F. Define*

$$\mathcal{G}[u](z):=\int_{\Omega} u(\zeta)G(\zeta,z)\quad \textit{for } z\in\Omega.$$

Then $\mathcal{G}:L^2(\Omega)\to L^2(\Omega)\cap\mathcal{O}(\Omega)$ is a well-defined, continuous operator.

Proof. It is clear that $\mathcal{G}$ is well-defined and continuous on $\mathcal{O}(\Omega_\delta)$ and on $C_0^\infty(\Omega)$. Thus we can define its formal adjoint $\mathcal{G}^*$. In particular,

$$\mathcal{G}^*[u](z)=\int_{\Omega} u(\zeta)\overline{G(z,\zeta)}.$$

Let the difference of $\mathcal{G}$ and $\mathcal{G}^*$ be

$$\mathcal{B}[u](z)=\mathcal{G}^*[u](z)-\mathcal{G}[u](z)$$

with the kernel $B(\zeta,z)=\overline{G(z,\zeta)}-G(\zeta,z)$. We will now prove that $\mathcal{B}:L^1(\Omega)\to L^1(\Omega)$ is bounded.

By the Tonelli Theorem, it suffices to prove that

$$\int_{\Omega}|\mathcal{B}[u](z)|\,dV(z)\le\iint_{\Omega\times\Omega}|B(\zeta,z)u(\zeta)|\,dV(\zeta,z)\lesssim\|u\|_{L^1(\Omega)}<\infty.$$

Note that the kernel $B(\zeta,z)$ is bounded from above by

$$\frac{|\zeta-z|}{|\Phi(\zeta,z)|^3}.$$

On the other hand, by Lemma 2.4, the proof of the L^1 continuity is completed if we have the estimate

$$\iint\limits_{(\Omega\cap B(0,c/2))^2} |B(\zeta,z)u(\zeta)|\,dV(\zeta,z) \lesssim \|u\|_{L^1(\Omega)}.$$

Consider the change of variables $(\alpha,w)=(\alpha_1,\alpha_2,w_1,w_2)=(\zeta_1,\zeta_2,z_1-\zeta_1,$ $\rho(z)+i\,\mathrm{Im}(\Phi(\zeta,z)))$ and its Jacobian

$$J=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & \cdot & \cdot & \frac{\partial\rho(z)}{\partial(\mathrm{Re}\,z_2)} & \frac{\partial\rho(z)}{\partial(\mathrm{Im}\,z_2)}\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{\partial\,\mathrm{Im}\,\Phi(\zeta,z)}{\partial(\mathrm{Re}\,z_2)} & \frac{\partial\,\mathrm{Im}\,\Phi(\zeta,z)}{\partial(\mathrm{Im}\,z_2)}
\end{pmatrix}$$

(the entries marked with $\cdot$ are irrelevant). Then

$$\det(J)=\frac{\partial\,\mathrm{Im}(\Phi(\zeta,z))}{\partial x_4}\,\frac{\partial\rho(z)}{\partial x_2}-\frac{\partial\,\mathrm{Im}(\Phi(\zeta,z))}{\partial x_2}\,\frac{\partial\rho(z)}{\partial x_4}.$$

Since $\rho(z)\neq 0$, we can find $\delta>0$ so small that $\partial\rho/\partial x_4$ dominates the other partial derivatives of ρ for $|z-\zeta|\le\delta$. As a consequence, we have $\det(J)\neq 0$ on $|\zeta-z|\le\delta$.

Now, let $\delta'>0$ depend on Ω, c, δ and ρ and $u\in L^1(\Omega)$. In the new variables, by Lemma 2.4, we have

$$\begin{aligned}
&\iint\limits_{(\Omega\cap B(0,c/2))^2} |B(\zeta,z)u(\zeta)|\,dV(\zeta,z)\\
&\qquad\lesssim \iint\limits_{(\Omega\cap B(0,\delta'))\times B(0,\delta')} \frac{|u(\alpha)|}{(|w_2|^2+F^2(|w_1|^2))|w_1|}\,dV(\alpha,w)\\
&\qquad\lesssim \|u\|_{L^1(\Omega)}\int_0^{\delta'}\!\!\int_0^{\delta'} \frac{r_1r_2}{(r_2^2+F^2(r_1^2))r_1}\,dr_2\,dr_1\\
&\qquad\lesssim \|u\|_{L^1(\Omega)}\int_0^{\delta'} \ln F(r_1^2)\,dr_1<\infty \quad \text{(by the hypothesis on } u \text{ and } F).
\end{aligned}$$

That means the operator $\mathcal{B}: L^1(\Omega)\to L^1(\Omega)$ is bounded and it extends to an operator mapping $L^2(\Omega)\to L^2(\Omega)$ continuously.

Let $u \in C_0^\infty(\Omega)$ be such that $\mathcal{G}[u] \in \mathcal{O}(\Omega_\delta)$. We get

$$\begin{aligned}\|\mathcal{G}[u]\|^2_{L^2(\Omega)} &= \langle \mathcal{G}[u], \mathcal{G}[u]\rangle = \langle u, (\mathcal{G}+\mathcal{B})\mathcal{G}[u]\rangle \\ &\le \|u\|_{L^2(\Omega)}\big(\|\mathcal{G}^2[u]\|_{L^2(\Omega)} + \|\mathcal{B}\|_{L^2(\Omega)\to L^2(\Omega)}\|\mathcal{G}[u]\|_{L^2(\Omega)}\big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{since } \mathcal{G}^2[u] = \mathcal{G}[u]) \\ &= \|u\|_{L^2(\Omega)}\big(\|\mathcal{G}[u]\|_{L^2(\Omega)} + \|\mathcal{B}\|_{L^2(\Omega)\to L^2(\Omega)}\|\mathcal{G}[u]\|_{L^2(\Omega)}\big) \\ &= \|u\|_{L^2(\Omega)}\|\mathcal{G}[u]\|_{L^2(\Omega)}(1 + \|\mathcal{B}\|_{L^2(\Omega)\to L^2(\Omega)}),\end{aligned}$$

and thus $\|\mathcal{G}[u]\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}(1 + \|\mathcal{B}\|_{L^2(\Omega)\to L^2(\Omega)})$.

Applying Hörmander's technique in L^2-estimates for $\bar\partial$ problem (see [7] or [2]), it is not hard to prove that the space of all holomorphic functions on Ω_δ is dense in $L^2(\Omega) \cap \mathcal{O}(\Omega)$ in the corresponding norm (on strictly convex domains, this density result is trivial). From the above inequality, $\mathcal{G}$ is a continuous operator on $L^2(\Omega) \cap \mathcal{O}(\Omega)$ and so it is equal to the identity on $L^2(\Omega) \cap \mathcal{O}(\Omega)$. Hence, $\mathcal{G}^2[u] = \mathcal{G}[u]$ for all $u \in C_0^\infty(\Omega)$ Therefore, $\mathcal{G} : L^2(\Omega) \to L^2(\Omega) \cap \mathcal{O}(\Omega)$ is a well-defined and continuous operator. This is our claim. ∎

Now let $u \in L^2(\Omega)$. Then the Bergman image $\mathcal{P}[u]$ is in $L^2(\Omega) \cap \mathcal{O}(\Omega)$. By definition, we have

$$\begin{aligned}\mathcal{P}[u](z) &= \int_\Omega \mathcal{P}[u](\zeta)G(\zeta,z) = \int_\Omega \mathcal{P}[u](\zeta)\overline{G(z,\zeta)} - \int_\Omega \mathcal{P}[u](\zeta)B(\zeta,z) \\ &= \int_\Omega u(\zeta)\overline{\mathcal{P}[G](z,\zeta)} - \int_\Omega \mathcal{P}[u](\zeta)B(\zeta,z) \quad (\text{since } \mathcal{P} \text{ is Hermitian}) \\ &= \langle u, \mathcal{P}[G](z,\cdot)\rangle_\Omega - \langle \mathcal{P}[u], \overline{B(\cdot,z)}\rangle_\Omega \\ &= \langle u, G(z,\cdot)\rangle_\Omega - \langle \mathcal{P}[u], \overline{B(\cdot,z)}\rangle_\Omega \\ &\qquad\qquad\qquad\qquad (\text{since } G(z,\zeta) \text{ is holomorphic in } \zeta\text{-variables}) \\ &= \mathcal{G}^*[u](z) - \mathcal{B}\circ\mathcal{P}[u](z),\end{aligned}$$

or for short $\mathcal{P} = \mathcal{G}^* - \mathcal{B}\circ\mathcal{P}$. This identity and the above technique are applied to deduce the following decomposition on convex domains of maximal type F:

THEOREM 3.4 (Ligocka's decomposition). *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, convex domain. Assume that Ω admits maximal type F at all boundary points for some function F. Then*

$$\mathcal{P}u(z) = \mathcal{G}(I-\mathcal{B})^{-1}[u](z) = (I+\mathcal{B})^{-1}\mathcal{G}^*[u](z) \quad \textit{for all } z \in \Omega.$$

Proof. The main step in this proof is that $\mathcal{B} : L^2(\Omega) \to L^2(\Omega)$ is compact. Indeed, the compactness of $\mathcal{B}$ and the equality $\mathcal{B}^* = -\mathcal{B}$ imply that the eigenvalues of $\mathcal{B}$ belong to $i\mathbb{R}$. Therefore, the kernels of the operators $I-\mathcal{B}$ and $I+\mathcal{B}$ are $\{0\}$. Then, as a consequence of the Fredholm theory of compact operators [20], $(I+\mathcal{B})^{-1}$ and $(I+\mathcal{B})^{-1}$ exist and are bounded on $L^2(\Omega)$.

By [19, Theorem in Appendix C], to prove the compactness of $\mathcal{B}$ it suffices to show that

$$\begin{aligned}\int_{\Omega}|B(\zeta,z)|\,dV(\zeta)&\lesssim 1 \quad \text{for all } z\in\Omega,\\ \int_{\Omega}|B(\zeta,z)|\,dV(z)&\lesssim 1 \quad \text{for all } z\in\Omega.\end{aligned}\tag{3.1}$$

Since $|\zeta-z|^2\lesssim|\Phi(\zeta,z)|$, we have

$$|\overline{G(z,\zeta)}-G(\zeta,z)|\lesssim|\Phi(\zeta,z)|^{-5/2}\quad \text{for } (\zeta,z)\in\Omega\times\Omega \text{ close to } b\Omega.\tag{3.2}$$

Therefore,

$$\int_{\zeta\in\Omega,\,\rho(\zeta)\ge-\delta}|B(\zeta,z)|\,dV(\zeta)\lesssim\int_{\zeta\in\Omega,\,\rho(\zeta)\ge-\delta}\frac{dV(\zeta)}{|\Phi(\zeta,z)|^{5/2}}\tag{3.3}$$

for all $z\in\Omega$. Let $(\alpha_1,\alpha_2,w_1,w_2)$ be the new variables as before. We have

$$\begin{aligned}\int_{\zeta\in\Omega,\,\rho(\zeta)\ge-\delta}|B(\zeta,z)|\,dV(\zeta)&\lesssim\int_{\Omega\cap B(0,\delta)}\frac{dV(w_1,w_2)}{(|\rho(z)|+|w_2|+F(|w_1|^2))^2|w_1|^{1/2}}\\ &\lesssim\int_{|(t_1,t_2,t_3,t_4)|\le\delta}\frac{dt_1\,dt_2\,dt_3\,dt_4}{(|\rho(z)|+|t_3|+|t_4|+F(t_1^2+t_2^2))^2|(t_1,t_2)|^{1/2}}\\ &\qquad(\text{where } w_1=t_1+\sqrt{-1}\,t_2, w_2=t_3+\sqrt{-1}\,t_4)\\ &\lesssim\int_{|(t_1,t_2,t_3)|\le\delta}\frac{dt_1\,dt_2\,dt_3}{(|\rho(z)|+|t_3|+F(t_1^2+t_2^2))|(t_1,t_2)|^{1/2}}\\ &\lesssim\int_0^\delta F(r^2)\,dr\lesssim 1\quad(\text{by the hypothesis on } F).\end{aligned}$$

Finally, the first inequality of (3.1) immediately implies the second since $B(z,\zeta)=\overline{-B(\zeta,z)}$. ∎

Lemma 3.5. *Let $\Omega\subset\mathbb{C}^2$ be a smoothly bounded, convex domain. Assume that Ω admits maximal type F at all boundary points for some function F. Then the operator $\mathcal{B}$ maps continuously $L^\infty(\Omega)$ into $\Lambda^f(\Omega)$, where*

$$f(d^{-1}):=\left(\int_0^d\frac{\sqrt{F^*(t)}}{t}\,dt\right)^{-1},$$

and F^* is the inverse of F.

As a consequence, $I+\mathcal{B}, I-\mathcal{B} : L^\infty(\Omega) \to L^\infty(\Omega)$ are invertible Fredholm operators [20, Chapter 4]. Before proving the lemma, we recall the General Hardy–Littlewood Lemma for $\Lambda^f(\Omega)$-continuity, which was established by Khanh [9].

Lemma 3.6 (General Hardy–Littlewood Lemma). *Let Ω be a smoothly bounded domain in $\mathbb{R}^n$ and let ρ be a defining function of Ω. Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $G(t)/t$ is decreasing and $\int_0^d (G(t)/t)\,dt < \infty$ for $d > 0$ small enough. If $u \in C^1(\Omega)$ is such that*

$$|\nabla u(x)| \lesssim G(|\rho(x)|)/|\rho(x)| \quad \textit{for every } x \in \Omega,$$

then

$$f(|x-y|^{-1})|u(x)-u(y)| < \infty$$

uniformly in $x, y \in \Omega$, $x \neq y$, and where $f(d^{-1}) := (\int_0^d (G(t)/t)\,dt)^{-1}$.

Remark 3.7. If $G(t) = t^\alpha$, for $0 < \alpha \le 1$, then $f(t) = \alpha t^\alpha$ and so Lemma 3.6 traces back to the classical Hardy–Littlewood Lemma [12].

Proof of Lemma 3.5. Let $u \in L^\infty(\Omega)$. Then the gradient of $\mathcal{B}[u]$ satisfies

$$|\nabla \mathcal{B}[u](z)| \lesssim \|u\|_{L^\infty(\Omega)} \int_\Omega \frac{|\zeta - z|}{|\Phi(\zeta,z)|^4}\,dV(\zeta).$$

For a small $\sigma > 0$ and $\sigma < c/12$ (c is the constant in Lemma 2.4), we choose a cutoff function $\psi \in C^\infty(\mathbb{C}^2 \times \mathbb{C}^2)$ such that $\psi(\zeta,z) = 1$ on the set $\{|\rho(z)| + |\mathrm{Im}\,\Phi(\zeta,z)| + F(|z-\zeta|^2) < \sigma/2\}$ and $\psi(\zeta,z) = 0$ on the set $\{|\rho(z)| + |\mathrm{Im}\,\Phi(\zeta,z)| + F(|z-\zeta|^2) > \sigma\}$.

Hence,

$$\begin{aligned}|\nabla \mathcal{B}[u](z)| &\lesssim \underbrace{\|u\|_{L^\infty(\Omega)} \int_\Omega |1-\psi(\zeta,z)| \frac{|\zeta - z|}{|\Phi(\zeta,z)|^4}\,dV(\zeta)}_{\text{non-singular by the construction of } \psi} \\ &\quad + \underbrace{\|u\|_{L^\infty(\Omega)} \int_\Omega |\psi(\zeta,z)| \frac{|\zeta - z|}{|\Phi(\zeta,z)|^4}\,dV(\zeta)}_{\text{singular for } \zeta \text{ near } z} \\ &\lesssim \|u\|_{L^\infty(\Omega)}\Big(1 + \int_\Omega |\psi(\zeta,z)| \frac{|\zeta - z|}{|\Phi(\zeta,z)|^4}\,dV(\zeta)\Big).\end{aligned}$$

To estimate the singular integral term in $(\ldots)$, for $0 < \delta' < \sigma$, by recalling the new variables $(\alpha_1, \alpha_2, w_1, w_2)$ as before (with a modification) and applying Lemma 2.4, we have

$$\begin{aligned}\int_{\Omega} |\psi(\zeta,z)|\frac{|\zeta-z|}{|\Phi(\zeta,z)|^4}\,dV(\zeta) &\lesssim \int_{\Omega\cap B(0,\delta')} \frac{dV(w_1,w_2)}{(|\rho(z)|+|w_2|+F(|w_1|^2)^3|w_1|}\\ &\lesssim \int_{|(t_1,t_2,t_3,t_4)|\le\delta'} \frac{dt_1\,dt_2\,dt_3\,dt_4}{(|\rho(z)|+|t_3|+|t_4|+F(t_1^2+t_2^2))^3|(t_1,t_2)|}\\ &\qquad\qquad (\text{where } w_1=t_1+\sqrt{-1}\,t_2, w_2=t_3+\sqrt{-1}\,t_4)\\ &\lesssim \int_{|(t_1,t_2)|\le\delta'} \frac{dt_1\,dt_2}{(|\rho(z)|+F(|t_1^2+t_2^2|))|(t_1,t_2)|}\\ &\lesssim \int_0^c \frac{dr}{|\rho(z)|+F(r^2)}.\end{aligned}$$

Now, for the last integral, we split it into two parts

$$\int_0^c \frac{dr}{|\rho(z)|+F(r^2)} = \underbrace{\int_0^{\sqrt{F^*(|\rho(z)|)}} \frac{dr}{|\rho(z)|+F(r^2)}}_{\text{easy part}} + \underbrace{\int_{\sqrt{F^*(|\rho(z)|)}}^{c} \frac{dr}{|\rho(z)|+F(r^2)}}_{\text{difficult part}}.$$

It is clear that the easy part is bounded from above by $\sqrt{F^*(|\rho(z)|)}/|\rho(z)|$.

For the "difficult" part, if $r\ge\sqrt{F^*(|\rho(z)|)}$, then

$$\frac{F(r^2)}{r^2} \ge \frac{F(F^*(|\rho(z)|))}{F^*(|\rho(z)|)} = \frac{|\rho(z)|}{|F^*(|\rho(z)|)|} \quad (\text{since } F \text{ is increasing}),$$

so

$$\frac{F(r^2)}{|\rho(z)|} \ge \frac{r^2}{F^*(|\rho(z)|)}.$$

Therefore,

$$\begin{aligned}\int_{\sqrt{F^*(|\rho(z)|)}}^{c} \frac{dr}{|\rho(z)|+F(r^2)} &\le \frac{1}{|\rho(z)|}\int_{\sqrt{F^*(|\rho(z)|)}}^{c} \frac{dr}{1+r^2/F^*(|\rho(z)|)}\\ &\le \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}\int_1^\infty \frac{dy}{1+y^2} = \frac{\pi}{4}\,\frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}.\end{aligned}$$

Thus,

$$\int_{\Omega} |\psi(\zeta,z)|\frac{|\zeta-z|}{|\Phi(\zeta,z)|^4}\,dV(\zeta) \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}.$$

Next the operator $\mathcal{B} : L^\infty(\Omega)\to\Lambda^f(\Omega)$ is continuous if the function $\sqrt{F^*(t)}/t$ satisfies all conditions in the General Hardy–Littlewood Lemma.

The fact that $\sqrt{F^*(t)}/t$ is decreasing is trivial.

Now, for some positive, sufficiently small d, by the change of variables $y=\sqrt{F^*(t)}$, we have

$$\begin{aligned}\int_0^d \frac{\sqrt{F^*(t)}}{t}\,dt &= \int_0^{\sqrt{F^*(d)}} y(\ln F(y^2))'\,dy \\ &= \sqrt{F^*(d)}\ln d - \lim_{t\to 0} t(\ln F(t^2)) - \underbrace{\int_0^{\sqrt{F^*(d)}} (\ln F(y^2))\,dy}_{\text{finite by hypothesis}}.\end{aligned}$$

Since $|\ln(F^*(t^2))|$ is decreasing when $0\le t\le\delta$, for $\delta>0$ small enough we have

$$|\ln F(\eta^2)|\eta \le \int_0^\eta |\ln F(t^2)|\,dt \le \int_0^\delta |\ln F(t^2)|\,dt < \infty$$

uniformly in $0\le\eta\le\delta$. Hence, $\sqrt{F^*(t)}|\ln t|<\infty$ for all $0\le t\le\sqrt{F^*(\delta)}$, and $\lim_{t\to 0} t|\ln F(t^2)|=0$. These results imply

$$\int_0^d \frac{\sqrt{F^*(t)}}{t}\,dt<\infty,$$

as desired. Hence, the proof of Lemma 3.5 is complete. ■

LEMMA 3.8. *Let $\Omega\subset\mathbb{C}^2$ be a smoothly bounded, convex domain. Assume that Ω admits maximal type F at all boundary points for some function F. Then the operators $\mathcal{G}$ and $\mathcal{G}^*$ map $\Lambda^{t^\alpha}(\Omega)\to\Lambda^{g_\alpha}(\Omega)$ continuously for $0<\alpha\le 1$, where*

$$g_\alpha(d^{-1}) := \Big(\int_0^d \frac{[\sqrt{F^*(t)}]^\alpha}{t}\,dt\Big)^{-1}.$$

Proof. Since $\int_\Omega G(\zeta,z)\,dV(\zeta)=1$, for $u\in\Lambda^{t^\alpha}(\Omega)$ (with $0<\alpha\le 1$) we have

$$(I-\mathcal{G})[u](z)=\int_\Omega (u(z)-u(\zeta))G(\zeta,z).$$

Hence, in the distributional sense, for $j=1,2$, we get

$$\frac{\partial u}{\partial z_j}(z)-\frac{\partial\mathcal{G}[u]}{\partial z_j} = \underbrace{\int_\Omega \frac{\partial u}{\partial z_j}(z)G(\zeta,z)}_{=\frac{\partial u}{\partial z_j}(z)} - \int_\Omega (u(z)-u(\zeta))\frac{\partial G}{\partial z_j}(\zeta,z).$$

That means

$$\nabla_z\mathcal{G}[u](z)=\int_\Omega (u(z)-u(\zeta))\nabla_z G(\zeta,z).$$

By recalling the cutoff function $\psi(\zeta, z)$ as before, we have

$$\nabla_z \mathcal{G}[u](z) = \underbrace{\int\limits_{\Omega} (1-\psi(\zeta,z))(u(z)-u(\zeta))\nabla_z G(\zeta,z)}_{\text{non-singular part}} + \underbrace{\int\limits_{\Omega} \psi(\zeta,z)(u(z)-u(\zeta))\nabla_z G(\zeta,z)}_{\text{singular part}}. \tag{3.4}$$

Now define

$$\mathcal{Q}[u](z) = \int\limits_{\Omega} u(\zeta)\psi(\zeta,z)G(\zeta,z).$$

Then

$$u(z)h(z,z) - \mathcal{Q}[u](z) = \int\limits_{\Omega} [\psi(z,z)u(z) - \psi(\zeta,z)u(\zeta)]G(\zeta,z). \tag{3.5}$$

Similarly, we get

$$\nabla_z \mathcal{Q}[u](z) = \underbrace{\int\limits_{\Omega} [\psi(z,z)u(z) - \psi(\zeta,z)u(\zeta)]\nabla_z G(\zeta,z)}_{= \text{ the singular part of } \nabla_z \mathcal{G}} \tag{3.6}$$

and

$$\nabla_{\bar{z}} \mathcal{Q}[u](z) = \int\limits_{\Omega} \nabla_{\bar{z}}\psi(\zeta,z).u(\zeta)G(\zeta,z). \tag{3.7}$$

Since $u \in \Lambda^{t^\alpha}(\Omega)$, in the new variables $w_1 = t_1 + \sqrt{-1}\,t_2, w_2 = t_3 + \sqrt{-1}\,t_4$ we get

$$\begin{aligned}
|\nabla_z \mathcal{Q}[u](z)| &\lesssim \|u\|_{\Lambda^{t^\alpha}(\Omega)} \int\limits_{\Omega\cap B(0,\delta')} \frac{|w_1|^\alpha}{(|\rho(z)| + |w_2| + F(|w_1|^2))^4}\, dV(w_1,w_2) \\
&\lesssim \|u\|_{\Lambda^{t^\alpha}(\Omega)} \int\limits_{|(t_1,t_2,t_3,t_4)|\le\delta'} \frac{|(t_1,t_2)|^\alpha\, dt_1\, dt_2\, dt_3\, dt_4}{(|\rho(z)| + |t_3| + |t_4| + F(|(t_1,t_2)|))^4} \\
&\lesssim \int\limits_0^{\delta'} \frac{r^{\alpha-1}}{|\rho(z)| + F(r^2)}\, dr.
\end{aligned}$$

Applying the same calculations in the case $\alpha = 1$ ($g_1 = f$) to the right hand integral, together with the property that

$$\int\limits_1^\infty \frac{1}{(1+y^2)y^{1-\alpha}} < \infty,$$

we also obtain

$$|\nabla_z \mathcal{Q}[u](z)| \lesssim \left[\sqrt{F^*(|\rho(z)|)}\right]^\alpha / |\rho(z)|.$$

On the other hand,

$$|\nabla_{\bar{z}}\mathcal{Q}[u](z)| \lesssim \|u\|_{\Lambda^{t^\alpha}(\Omega)} \int\limits_{\Omega} \frac{dV(\zeta)}{|\Phi(\zeta,z)|^3}$$
$$\lesssim (\ln(|\rho(z)|))^2 \lesssim \big[\sqrt{F^*(|\rho(z)|)}\big]^\alpha/|\rho(z)|.$$

For $0 < y < 1$, we have $1/y^{1-\alpha} > 1$. By the same reasoning as in the case $\alpha = 1$, it is not hard to show that the function $\big[\sqrt{F^*(t)}\big]^\alpha/t$ satisfies all conditions in the General Hardy–Littlewood Lemma. Thus these estimates imply that $\mathcal{Q}$ maps continuously $\Lambda^{t^\alpha}(\Omega)$ to $\Lambda^{g_\alpha}(\Omega)$, and hence so $\mathcal{G}$ does as well. By duality, we have the same assertion for $\mathcal{G}^*$, which completes the proof. ■

Since $g_\alpha \le f$, Lemma 3.5 implies that the operators $I + \mathcal{B}, I - \mathcal{B} : \Lambda^{g_\alpha}(\Omega) \to \Lambda^{g_\alpha}(\Omega)$ are bounded. Thus, by the Ascoli–Arzelà Theorem, the embedding $\Lambda^{g_\alpha}(\Omega) \to L^\infty(\Omega)$ is compact and thus

$$\mathcal{B} : \Lambda^{g_\alpha}(\Omega) \to \Lambda^{g_\alpha}(\Omega)$$

is also compact. Again, by the Fredholm property, the fact that the kernel of $I - \mathcal{B}$ on $\Lambda^{g_\alpha}(\Omega)$ is exactly $\{0\}$ implies $(I - \mathcal{B})^{-1} : \Lambda^{g_\alpha}(\Omega) \to \Lambda^{g_\alpha}(\Omega)$ exists and is bounded, and similarly for $(I + \mathcal{B})^{-1}$. Therefore, Lemma 3.8 and Ligocka's Decomposition 3.4 show that the Bergman projection maps $\Lambda^{t^\alpha}(\Omega) \to \Lambda^{g_\alpha}(\Omega)$ continuously for any $0 < \alpha \le 1$.

We end the paper with another corollary of the Main Theorem.

COROLLARY 3.9. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, convex domain. Assume that Ω admits maximal type F at all boundary points for some function F. Then, for each $z_0 \in \Omega$, the Bergman kernel function $P(\zeta, z_0)$ belongs to $\Lambda^f(\Omega)$, where*

$$f(d^{-1}) := \bigg(\int\limits_0^d \frac{\sqrt{F^*(t)}}{t}\,dt\bigg)^{-1},$$

and F^* is the inverse of F.

For example:

- If $F(t) = t^m$, then $f(t) = t^{1/m}$.
- If $F(t) = \exp\big(\frac{-1}{32t^s}\big)$, then $f(t) = \frac{1024^s(1-2s)}{2s}(|\ln t|)^{1/(2s)-1}$.

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