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Hardy spaces for ball quasi-Banach function spaces

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Contents

Abstract

This article unifies the theory for Hardy spaces built on Banach lattices on $\mathbb{R}^n$ satisfying certain weak conditions on indicator functions of balls. The authors introduce a new family of function spaces, named the ball quasi-Banach function spaces, to define Hardy type spaces. The ones in this article extend classical Hardy spaces and include various known function spaces, for example, Hardy–Lorentz spaces, Hardy–Herz spaces, Hardy–Orlicz spaces, Hardy–Morrey spaces, Musielak–Orlicz–Hardy spaces, variable Hardy spaces and variable Hardy–Morrey spaces. Among them, Hardy–Herz spaces are shown to naturally arise in the context of any function spaces above. The example of Hardy–Morrey spaces shows that the absolute continuity of the quasi-norm is not necessary, which is used to guarantee the density of the set of functions having compact supports in Hardy spaces for ball quasi-Banach function spaces, but the decomposition result on these Hardy-type spaces never requires this absolute continuity of the quasi-norm. Moreover, via assuming that the powered Hardy–Littlewood maximal operator satisfies certain Fefferman–Stein vector-valued maximal inequality as well as it is bounded on the associate space, the atomic characterizations of Hardy type spaces are obtained. Although the results are based on the rather abstract theory of function spaces, they improve and extend the results for Orlicz spaces and Musielak–Orlicz spaces. Moreover, local Hardy type spaces and Hardy type spaces associated with operators in this setting are also studied.

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1. Introduction

One of the techniques in harmonic analysis is to replace the function with the grand maximal function. We apply this approach to investigate the properties of many function spaces. Among them, we show that the Herz space $K_{p,q}^{-n/p}(\mathbb{R}^n)$, defined by (2.21) below, plays a crucial role. We unify the results on Hardy spaces built on general function spaces by proposing the notion of ball quasi-Banach spaces in Definition 2.2. Our main contribution in this article is to establish the theory without assuming the absolute continuity of the quasi-norm (see Definition 2.5).

Recall that Stein and Weiss [122] as well as Fefferman and Stein [36] introduced classical Hardy spaces. The family of classical Hardy spaces naturally generalizes and substitutes the one of Lebesgue spaces. Moreover, the works in [36, 122] also inspire many new ideas for the theory of function spaces. The characterizations of classical Hardy spaces show the connections among some important notions in harmonic analysis, such as harmonic functions, the Hardy–Littlewood maximal function (see (2.5)) and the square function (see Definition 3.16). One of the prominent examples is the boundary value of various harmonic functions.

As we have mentioned, Fefferman and Stein introduced an important notion on Hardy spaces, the grand maximal function (see (2.13)), which plays a key role in [36]. It is worth noticing that it is one of the convenient ways to handle operators acting on function spaces appearing in partial differential equations or potential analysis, whenever these function spaces possess the grand maximal function characterization. Morrey spaces [65], Orlicz spaces [102] and Lorentz spaces are examples of such spaces. Most recently, Herz spaces, variable Lebesgue spaces [28, 101] and Musielak–Orlicz spaces are some other important examples (see Section 7).

There are several extensions of classical Hardy spaces to those spaces built on general Banach lattices via various maximal functions. A series of studies on Hardy type spaces motivates us to establish a unified theory for Hardy spaces built on general function spaces. Orlicz spaces, Lorentz spaces and variable Lebesgue spaces are quasi-Banach function spaces; they are often ingredients of Hardy type spaces. We refer the reader to [10, Chapter 1] and [26, Section 2.10.3] for more details on quasi-Banach function spaces. However, Morrey spaces and weighted Lebesgue spaces are not necessarily quasi-Banach function spaces (see [117] for Morrey spaces, and Subsection 7.1 for weighted Lebesgue spaces). Therefore, the notion of quasi-Banach function spaces is restrictive. We aim to extend it further so that Morrey spaces are included in this generalized framework. This new notion is a little more restrictive than the one of Banach lattices.

We find that the difficulty lies in the role of Lebesgue measurable sets appearing in the definition of quasi-Banach function spaces. Roughly speaking, in the example of Morrey spaces, the characteristic function of Lebesgue measurable sets is not necessarily in the associate space (Köthe dual) of Morrey spaces. Therefore, to overcome this difficulty, we introduce a new family of function spaces, the ball quasi-Banach function spaces. Their definition is similar to that of quasi-Banach function spaces. All we have to do is to replace Lebesgue measurable sets with balls in $\mathbb{R}^n$. The precise definition is given in Definition 2.2.

For a ball quasi-Banach function space X, we actually introduce the Hardy space $H_X(\mathbb{R}^n)$ via the grand maximal functions (see Subsection 2.3). One of the methods to define $H_X(\mathbb{R}^n)$ employed in this article is the maximal operator of Peetre type appearing in [36, Lemma 1]. A reduction to the classical grand maximal operator given by (2.12) is to require the boundedness of the Hardy–Littlewood maximal function or at least its weak variants. When it comes to local Hardy spaces, we can replace the assumption of the boundedness of the Hardy–Littlewood maximal function with the weaker one of the translation operator (see (5.6) for the details). Whenever the Hardy–Littlewood maximal function is bounded on the p-convexification of X, then different choices of admissible functions to define $H_X(\mathbb{R}^n)$ yield equivalent quasi-norms (see Definition 2.6). That is, several different types of quasi-norms defined in terms of different maximal functions are equivalent. We give the precise statement of this result in Theorem 3.1.

Although we introduce the Hardy space $H_X(\mathbb{R}^n)$ via the grand maximal function, we also characterize $H_X(\mathbb{R}^n)$ by using the Lusin-area function (see Theorem 3.21 for the details). Indeed, there exist several different approaches to the characterization of general function spaces in terms of the Lusin-area function (see, for example, [48, 50, 86, 90]).

Furthermore, one of the major breakthroughs in the theory of classical Hardy spaces is atomic characterization. Coifman and Latter found an atomic characterization of the classical Hardy space [23, 82]. One of the big advantages of atomic characterizations is the separation of the quality of functions (atoms) and the quantity of functions (sequence norms). However, there does not exist a unified theory for this direction of research: the study of the generalized function space and the grand maximal operator. Another aim of this article is to unify the existing theories by means of ball quasi-Banach function spaces so as to include Morrey spaces, which are not necessarily quasi-Banach function spaces. More precisely, we establish an atomic characterization for the Hardy space $H_X(\mathbb{R}^n)$ in Theorems 3.6 and 3.7. Moreover, we find that these atomic characterizations rely on the Fefferman–Stein vector-valued maximal inequality and its boundedness on the associate space of the powered Hardy–Littlewood maximal operator (see (2.8) and (2.9)), although we do not have to depend on this inequality to a large extent, as is seen from Lemma 2.13 below. We have already demonstrated the relation between atomic decompositions and the Fefferman–Stein vector-valued maximal inequality in [58, 101, 102].

In addition, as the example of Hardy–Morrey spaces shows, we still have another difficulty regarding the convergence of the atomic decomposition. It arises from the failure of the absolute continuity of quasi-norms (see Definition 2.5 or [10, Chapter 1, Definition 3.1] for the definition of the absolute continuity of quasi-norms). Comparing with [121], we cannot use the same argument as in [121, Section 2.3.2]. We do not impose any assumption on the (quasi-)norm of the ball (quasi-)Banach function spaces in our main results (see Theorems 2.9 and 2.21). The Herz space $K_{p,q}^{-n/p}(\mathbb{R}^n)$, with some $p,\ q \in (0,1]$, removes the obstacle arising from the failure of the absolute continuity of quasi-norms, which is also a ball quasi-Banach function space. It appears naturally from the standing assumption. With this Herz space, it turns out that there is no need to assume that the (quasi-)norm is absolutely continuous. It is known that there exist many attempts to replace the functions in X with their grand maximal functions in many concrete spaces. In the present article, what is different from the cases of Orlicz–Hardy spaces [102], variable Hardy spaces [28, 101, 115], Hardy–Morrey spaces [65, 71] and variable Hardy–Morrey spaces [58] is that we need to extract some concrete and quantitative information from the vector-valued inequality (2.8) (see Lemma 2.14 and (2.26) for more details). More precisely, as we have mentioned, our key space is the Herz space $K_{p,q}^{-n/p}(\mathbb{R}^n)$ with $p, q \in (0,1)$. In a word, we can embed X continuously into $K_{p,q}^{-n/p}(\mathbb{R}^n)$, and moreover $K_{p,q}^{-n/p}(\mathbb{R}^n)$ does not contain the constant function 1, which is crucial to the application of Whitney's decomposition theorem in the proof of Proposition 4.9.

The layout of the remainder of this article is as follows.

Section 2 contains the definitions of the ball (quasi-)Banach function spaces as well as of the corresponding Hardy spaces. In Proposition 2.3, we show that ball Banach spaces are closed under taking the associate spaces. Moreover, we also formulate an atomic characterization for the ball (quasi-)Banach function spaces in Theorems 2.9 and 2.21. Section 3 consists of some fundamental properties of the Hardy type space $H_X(\mathbb{R}^n)$. In Subsection 3.4, we formulate our results in full generality. The additional assumption of the absolute continuity of the (quasi-)norms guarantees the density of the set of all elements which are represented by $L^\infty(\mathbb{R}^n)$ functions in Hardy type spaces (see Corollary 3.11). We postpone the proofs of the results of Sections 2 and 3 until Section 4; more precisely, we prove Theorems 2.9 and 2.21 and the results of Section 3 together with some related assertions in Section 4. Whenever we are not able to use the absolute continuity of the quasi-norms of the function spaces, we need to prove some truncation results (see Lemma 4.4 and Corollary 4.6 for the details). In particular, we consider the Herz space $K_{\theta+\varepsilon(X),s}^{-n/[\theta+\varepsilon(X)]}(\mathbb{R}^n)$ in Lemma 2.14 and use it to obtain the convergence of the atomic decomposition as we mentioned above.

Based on the model case obtained in Sections 2 through 4, we consider some applications of our results in Sections 5 and 6. More precisely, we deal with the local Hardy type space $h_X(\mathbb{R}^n)$ in Section 5, and also show that our techniques are applicable to the setting of the Hardy type space associated with operators in Section 6.

Moreover, our results complement and reinforce those obtained in [85, 102, 130, 132]. We provide many examples in Section 7, where we discuss the relation between the existing results and the results obtained in this article. In particular, the results for Orlicz spaces (see Theorem 7.5) and Musielak–Orlicz spaces (see Theorem 7.14) are new.

Furthermore, we make some conventions on notation. Throughout the whole article, we always denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. We also use $C_{(\gamma,\beta,\dots)}$ to denote a *positive constant* depending on the indicated parameters γ, β, ... The *symbol* $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. When we need to emphasize or keep in mind that the positive constant C does depend on the indicated parameters α, β, γ, ... :

- instead of $A \lesssim B$, we write $A \lesssim_{\alpha,\beta,\gamma,\dots} B$;
- instead of $A \gtrsim B$, we write $A \gtrsim_{\alpha,\beta,\gamma,\dots} B$;
- instead of $A \sim B$, we write $A \sim_{\alpha,\beta,\gamma,\dots} B$.

The symbol $\lfloor s \rfloor$ for $s \in \mathbb{R}$ denotes the maximal integer not greater than s. For any given normed spaces $\mathcal{A}$ and $\mathcal{B}$ with the corresponding norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$, the symbol $\mathcal{A} \hookrightarrow \mathcal{B}$ means that if $f \in \mathcal{A}$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{A}}$. For each cube $Q := Q(x_Q, l_Q) \subset \mathbb{R}^n$, with center $x_Q \in \mathbb{R}^n$ and side-length $l_Q \in (0,\infty)$, and $\alpha \in (0,\infty)$, let $\alpha Q := Q(x_Q, \alpha l_Q)$. Denote by $\mathcal{Q}$ the *set* of all cubes having their edges parallel to the coordinate axes. For any subset E of $\mathbb{R}^n$, we denote by $E^{\complement}$ the set $\mathbb{R}^n \setminus E$, and by χ_E its characteristic function. We also let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any $\theta := (\theta_1, \dots, \theta_n) \in \mathbb{Z}_+^n$, let $|\theta| := \theta_1 + \cdots + \theta_n$. Furthermore, for any cube Q in $\mathbb{R}^n$ and $j \in \mathbb{Z}_+$, let $S_j(Q) := (2^{j+1}Q) \setminus (2^j Q)$ with $j \in \mathbb{N}$ and $S_0(Q) := 2Q$. Finally, for any $q \in [1,\infty]$, we denote by q' its *conjugate exponent*, with $1/q + 1/q' = 1$.

2. Definitions and preliminaries

In this section, we present some definitions and preliminary facts. As we have mentioned in Section 1, we encounter certain problems in quasi-Banach spaces. One of them is that the proof of the boundedness of operators becomes more and more complicated because we need to handle more and more delicate quasi-Banach function spaces; it seems that the proof of the boundedness of operators requires a remedy in each case. Another problem is that Morrey spaces and some other related spaces are not quasi-Banach function spaces. These are the motivations for us to study ball quasi-Banach function spaces.

2.1. Ball quasi-Banach function spaces. In this subsection, we give the definition of the ball quasi-Banach function space. We use function spaces to describe the quantity and the quality of functions. Among many function spaces, Banach function spaces are used to describe the quantity of functions. Let us first recall the definition of Banach function spaces from [10, Chapter 1, Definitions 1.1 and 1.3].

Denote by $\mathcal{M}$ the set of all measurable functions on $\mathbb{R}^n$.

DEFINITION 2.1. A Banach space $Y \subset \mathcal{M}$ is called a *Banach function space* if it satisfies

(i) $\|f\|_Y = 0$ implies that $f = 0$ almost everywhere;
(ii) $|g| \leq |f|$ almost everywhere implies that $\|g\|_Y \leq \|f\|_Y$;
(iii) $0 \leq f_m \uparrow f$ almost everywhere implies that $\|f_m\|_Y \uparrow \|f\|_Y$;
(iv) $\chi_E \in Y$ for any measurable set $E \subset \mathbb{R}^n$ with finite measure;

(v) for any measurable set $E \subset \mathbb{R}^n$ with finite measure, there exists a positive constant $C_{(E)}$, depending on E, such that, for all $f \in Y$,

$$\int_E |f(x)|\,dx \le C_{(E)}\|f\|_Y. \tag{2.1}$$

However, it is worth pointing out that condition (2.1) is too restrictive. Indeed, the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ (see, for example, [100] or (7.1) below for its definition) with $1 \le q < p < \infty$ violates (2.1) (see, for example, [117]). Thus, although Morrey spaces and related function spaces are important to describe the quality of functions, they are not Banach function spaces in general. Moreover, Lebesgue spaces $L^p(\mathbb{R}^n)$ with $p \in (0,1)$ are not Banach spaces.

With this in mind, we propose the following notion of ball quasi-Banach function spaces. For $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, let $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$ and

$$\mathbb{B} := \{B(x,r) : x \in \mathbb{R}^n \text{ and } r \in (0,\infty)\}. \tag{2.2}$$

We now extend the notion of Banach function spaces as follows.

Definition 2.2. A quasi-Banach space $X \subset \mathcal{M}$ is called a *ball quasi-Banach function space* if it satisfies

(i) $\|f\|_X = 0$ implies that $f = 0$ almost everywhere;
(ii) $|g| \le |f|$ almost everywhere implies that $\|g\|_X \le \|f\|_X$;
(iii) $0 \le f_m \uparrow f$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$;
(iv) $B \in \mathbb{B}$ implies that $\chi_B \in X$, where $\mathbb{B}$ is as in (2.2).

Moreover, a ball quasi-Banach function space X is called a *ball Banach function space* if the norm of X satisfies the triangle inequality: for all $f, g \in X$,

$$\|f+g\|_X \le \|f\|_X + \|g\|_X,$$

and, for any $B \in \mathbb{B}$, there exists a positive constant $C_{(B)}$, depending on B, such that, for all $f \in X$,

$$\int_B |f(x)|\,dx \le C_{(B)}\|f\|_X. \tag{2.3}$$

Obviously, every Banach function space is a ball Banach function space. On the other hand, the family of ball Banach function spaces includes Morrey type spaces, which are not necessarily Banach function spaces (see [117] for the details).

For any ball Banach function space X, the *associate space* (*Köthe dual*) X' is defined by

$$X' := \left\{f \in \mathcal{M} : \|f\|_{X'} := \sup\{\|fg\|_{L^1(\mathbb{R}^n)} : g \in X,\ \|g\|_X = 1\} < \infty\right\}$$

(see [10, Chapter 1, Section 2] for the details). For any ball Banach function space X, we prove that X' is also a ball Banach function space:

Proposition 2.3. *Let X be a ball Banach function space. Then its associate X' is also a ball Banach function space.*

Proof. We need to show (i) through (iv) of Definition 2.2 and (2.3) with X replaced by X'. We concentrate on (iii) since the other assertions are easily proved.

Indeed, by Lebesgue's differentiation theorem and (2.3), we conclude that (i) of Definition 2.2 with X replaced by X' holds true. From Definition 2.2 and the definition of X', it follows that (ii) of Definition 2.2 with X replaced by X' holds true. By (2.3) again, we obtain (iv) of Definition 2.2 with X replaced by X'. Moreover, from the fact that $\chi_B \in X$ for all $B \in \mathbb{B}$ and the definition of X', we deduce that (2.3) with X replaced by X' holds true.

Finally, we prove (iii) of Definition 2.2 with X replaced by X'. Let $\{f_n\}_{n\in\mathbb{N}} \subset X'$, let $f \in X'$ satisfy $0 \le f_n \uparrow f$ almost everywhere, and let $A \in (0, \|f\|_{X'})$. Then, by the definition of $\|f\|_{X'}$, there exists a real-valued function $g \in X$ with $\|g\|_X = 1$ such that

$$A < \int_{\mathbb{R}^n} f(x)g(x)\,dx \le \int_{\mathbb{R}^n} f(x)|g(x)|\,dx,$$

which, combined with the monotone convergence theorem, implies that there exists $N \in \mathbb{N}$ such that

$$A < \int_{\mathbb{R}^n} f_N(x)|g(x)|\,dx.$$

Thus, $\|f_N\|_{X'} > A$, which together with the arbitrariness of $A \in (0, \|f\|_{X'})$ implies (iii) of Definition 2.2 with X replaced by X'. This finishes the proof of Proposition 2.3. ■

The notion of ball quasi-Banach spaces extends the one of quasi-Banach spaces, which we recall now.

Definition 2.4. A ball quasi-Banach function space $Y \subset \mathcal{M}$ is called a *quasi-Banach function space* if, for all measurable sets $E \subset \mathbb{R}^n$ with finite measure, $\chi_E \in Y$.

A typical example of all of these notions is, needless to say, the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0, \infty]$.

We summarize the above notions in the table:

	Triangle inequality	Quasi-triangle inequality
(2.1)	Banach function space	quasi-Banach function space
(2.3)	ball Banach function space	ball quasi-Banach function space

As an analogy of the absolute continuity of the quasi-norm of quasi-Banach spaces, we introduce the following notion.

Definition 2.5. A ball quasi-Banach function space X is said to have an *absolutely continuous quasi-norm* if $\|\chi_{E_j}\|_X \downarrow 0$ whenever $\{E_j\}_{j=1}^\infty$ is a sequence of measurable sets that satisfies $E_j \supset E_{j+1}$ for all $j \in \mathbb{N}$ and $\bigcap_{j=1}^\infty E_j = \emptyset$.

For example, the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0, \infty)$ has an absolutely continuous quasi-norm. On the other hand, the space $L^\infty(\mathbb{R}^n)$ and the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ with $1 \le q < p < \infty$ do not have an absolutely continuous norm.

Now we recall the notions of convexity and concavity of ball quasi-Banach spaces.

Definition 2.6 ([106, Chapter 2], [91, Definition 1.d.3]). Let X be a ball quasi-Banach function space and $p \in (0, \infty)$.

(i) The *p-convexification* X^p of X is defined by setting $X^p := \{f \in \mathcal{M} : |f|^p \in X\}$ equipped with the quasi-norm $\|f\|_{X^p} := \|\,|f|^p\|_X^{1/p}$.

(ii) The space X is said to be *p-convex* if there exists a positive constant C such that, for any $\{f_j\}_{j\in\mathbb{N}}\subset X^{1/p}$,

$$\left\|\sum_{j=1}^\infty |f_j|\right\|_{X^{1/p}}\le C\sum_{j=1}^\infty \|f_j\|_{X^{1/p}}.$$

In particular, when $C=1$, X is said to be *strictly p-convex.*

(iii) The space X is said to be *p-concave* if there exists a positive constant C such that, for any $\{f_j\}_{j\in\mathbb{N}}\subset X^{1/p}$,

$$\sum_{j=1}^\infty \|f_j\|_{X^{1/p}}\le C\left\|\sum_{j=1}^\infty |f_j|\right\|_{X^{1/p}}.$$

In particular, when $C=1$, X is said to be *strictly p-concave.*

For example, it is easy to see that, for any $p\in[1,\infty)$, $(L^1(\mathbb{R}^n))^p=L^p(\mathbb{R}^n)$ with equivalent norms; here and hereafter, for any $p\in(0,\infty)$, $L^p(\mathbb{R}^n)$ denotes the space of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(\mathbb{R}^n)}:=\left[\int_{\mathbb{R}^n}|f(x)|^p\,dx\right]^{1/p}<\infty.$$

2.2. Convolution estimates. In what follows, for any $L\in\mathbb{Z}_+$, $\mathcal{P}_L$ denotes the set of all polynomials on $\mathbb{R}^n$ of degree no more than L; for any $a\in L^1(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n}(1+|x|)^L|a(x)|\,dx<\infty,$$

we write $a\perp\mathcal{P}_L$ if

$$\int_{\mathbb{R}^n}a(x)x^\alpha\,dx=0$$

for all $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|\le L$; the symbol $C^L(\mathbb{R}^n)$ denotes the set of all functions on $\mathbb{R}^n$ which have continuous derivatives up to order L. The next lemma is our starting point here; its proof can be found in [43, p. 466].

LEMMA 2.7. *Let $y,z\in\mathbb{R}^n$, $\nu,\mu\in\mathbb{Z}$ with $\nu\ge\mu$, $M\in(0,\infty)$, $L\in\mathbb{N}$ and $N\in(M+L+n,\infty)$. Suppose that φ is a $C^L(\mathbb{R}^n)$-function such that, for all $x\in\mathbb{R}^n$,*

$$|\nabla^L\varphi(x)|\le\frac{2^{\mu(n+L)}}{(1+2^\mu|x-y|)^M},$$

where ∇^L denotes the gradient operator of order L, namely, $\nabla^L:=\nabla(\nabla^{L-1})$ and ∇^0 denotes the identity operator. Assume, in addition, that ψ is a measurable function such that $\psi\perp\mathcal{P}_{L-1}$ and, for all $x\in\mathbb{R}^n$,

$$|\psi(x)|\le\frac{2^{\nu n}}{(1+2^\nu|x-z|)^N}.$$

Then there exists a positive constant C, independent of y, z, φ and ψ, such that

$$\left|\int_{\mathbb{R}^n}\varphi(x)\psi(x)\,dx\right|\le C\frac{2^{\mu n-(\nu-\mu)L}}{(1+2^\mu|y-z|)^M}.$$

Denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^n)$ its *dual space* (namely, the *space of all tempered distributions*). Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be the *Fourier transform* and its *inverse*, respectively: for any $f \in \mathcal{S}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$,

$$\mathcal{F}f(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx \quad \text{and} \quad \mathcal{F}^{-1}f(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{ix\cdot\xi}\, dx.$$

By [125, (2.66)] and the argument in [125, proof of Theorem 2.8], we have the following estimate, the details being omitted here. In what follows, let $\vec{0}_n$ denote the origin of $\mathbb{R}^n$.

Lemma 2.8. *Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy*

$$\mathcal{F}\Phi(\vec{0}_n) \neq 0 \tag{2.4}$$

and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then, for all $t \in [1,2]$, $0 < b \le N$, $r \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\sup_{y\in\mathbb{R}^n} \frac{|\Phi_t * f(y)|^r}{(1+|x-y|)^{br}} \le C \sum_{k=0}^{\infty} 2^{k(n-Nr)} \int_{\mathbb{R}^n} \frac{|\Phi_{2^{-k}t} * f(y)|^r}{(1+|x-y|)^{br}}\, dy,$$

where, for all $t \in (0,\infty)$ and $y \in \mathbb{R}^n$, $\Phi_t(y) := t^{-n}\Phi(y/t)$, and the positive constant C is independent of Φ, f, x and t, but depends on r.

Denote by $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the set of all locally integrable functions on $\mathbb{R}^n$. Recall that the *Hardy–Littlewood maximal operator M* is defined by setting, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$Mf(x) := \sup_{r\in(0,\infty)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy. \tag{2.5}$$

We now present a statement on atomic decompositions for ball Banach function spaces, whose proof is given in Subsection 4.1.

Theorem 2.9 (Reconstruction). *Let $r \in (1,\infty]$ and X be a ball Banach function space such that the M in (2.5) is bounded on $(X')^{1/r'}$. Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, $\{a_j\}_{j=1}^\infty \subset L^r(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ are such that, for any $j \in \mathbb{N}$,*

$$\|a_j\|_{L^r(\mathbb{R}^n)} \le \frac{|Q_j|^{1/r}}{\|\chi_{Q_j}\|_X}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^\infty \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j} \right\|_X < \infty. \tag{2.6}$$

Then $f := \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and has the property that there exists a positive constant C, independent of f and depending on X and r, such that

$$\|f\|_X \le C \left\| \sum_{j=1}^\infty \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j} \right\|_X. \tag{2.7}$$

For any $\theta \in (0,\infty)$, the *powered Hardy–Littlewood maximal operator $M^{(\theta)}$* is defined by setting, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M^{(\theta)}(f)(x) := \{M(|f|^\theta)(x)\}^{1/\theta}.$$

In order to obtain a variant of Theorem 2.9 on any given ball quasi-Banach function space X, we need the following additional assumption: For some $\theta, s \in (0,1]$, there exists a positive constant C such that, for any $\{f_j\}_{j=1}^\infty \subset \mathcal{M}$,

$$\left\|\left\{\sum_{j=1}^{\infty}[M^{(\theta)}(f_j)]^s\right\}^{1/s}\right\|_X \le C\left\|\left\{\sum_{j=1}^{\infty}|f_j|^s\right\}^{1/s}\right\|_X. \tag{2.8}$$

We point out that it is crucial to use (2.8) to establish an atomic characterization of the Hardy spaces studied in this article. The inequality (2.8) is called the Fefferman–Stein vector-valued maximal inequality, and its version with $X := L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$, $\theta = 1$ and $s \in (1,\infty]$ was originally established by Fefferman and Stein [37, Theorem 1]. Observe that, by [37, Theorem 1], we know that (2.8) also holds true when $\theta, s \in (0,1]$, $\theta < s$, $X := L^p(\mathbb{R}^n)$ and $p \in (\theta,\infty)$.

Using Theorem 2.9, we obtain its variant on strictly s-convex ball quasi-Banach function spaces as follows.

Theorem 2.10 (Reconstruction). *Let $s \in (0,1]$ and $q \in (1,\infty]$. Assume that X is a strictly s-convex ball quasi-Banach function space satisfying* (2.8) *for some $\theta \in (0,1]$ and that, for all $f \in (X^{1/s})'$,*

$$\|M^{((q/s)')}(f)\|_{(X^{1/s})'} \le C\|f\|_{(X^{1/s})'}, \tag{2.9}$$

where C is a positive constant independent of f. Let $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, $\{a_j\}_{j=1}^\infty \subset L^q(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ be such that, for any $j \in \mathbb{N}$,

$$\|a_j\|_{L^q(\mathbb{R}^n)} \le \frac{|Q_j|^{1/q}}{\|\chi_{Q_j}\|_X}, \quad \operatorname{supp}(a_j) \subset Q_j$$

and

$$\left\|\left\{\sum_{j=1}^{\infty}\left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s \chi_{Q_j}\right\}^{1/s}\right\|_X < \infty.$$

Then $f := \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and has the property that there exists a positive constant C, independent of f, such that

$$\|f\|_X \le C\left\|\left\{\sum_{j=1}^{\infty}\left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s \chi_{Q_j}\right\}^{1/s}\right\|_X.$$

Theorem 2.10 follows directly from Theorem 2.9 together with

$$\left\|\sum_{j=1}^{\infty}\lambda_j|a_j|\right\|_X \le \left\|\left\{\sum_{j=1}^{\infty}(\lambda_j|a_j|)^s\right\}^{1/s}\right\|_X.$$

Moreover, we can also consider the case when a_j is not compactly supported. In this case, we can extend Theorem 2.10 as follows.

Theorem 2.11 (Reconstruction). *Let $\tau \in (0,\infty)$, $\theta \in (0,1]$ and $q \in (1,\infty]$ be such that*

$$\tau > n(1/\theta - 1/q). \tag{2.10}$$

Assume that X is a strictly s-convex ball quasi-Banach function space satisfying (2.8) *for some $s \in (0,1]$ and* (2.9). *Let $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, $\{m_j\}_{j=1}^\infty \subset L^q(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ be* *such that, for any* $j \in \mathbb{N}$ *and* $k \in \mathbb{Z}_+$,

$$\|\chi_{S_k(Q_j)} m_j\|_{L^q(\mathbb{R}^n)} \le 2^{-\tau k} \frac{|Q_j|^{1/q}}{\|\chi_{Q_j}\|_X}$$

and

$$\left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X < \infty.$$

Then $f := \sum_{j=1}^{\infty} \lambda_j m_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$ *and has the property that there exists a positive constant* C*, independent of* f*, such that*

$$\|f\|_X \le C \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X.$$

The proof of Theorem 2.11 is also given in Subsection 4.1.

2.3. Grand maximal operators. In this subsection, we recall the definitions of various maximal functions. To formulate our results, we first recall the following fundamental notion.

Topologize the space $\mathcal{S}(\mathbb{R}^n)$ by norms $\{p_N\}_{N\in\mathbb{N}}$ given by setting, for any $N \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$p_N(\varphi) := \sum_{\alpha\in\mathbb{Z}_+^n,\, |\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)|. \tag{2.11}$$

Then, for any $N \in \mathbb{N}$, define $\mathcal{F}_N := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}$. We also endow $\mathcal{S}'(\mathbb{R}^n)$ with the weak-$*$ topology.

Definition 2.12. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $N \in \mathbb{N}$, $a, b \in (0,\infty)$, $\Phi \in \mathcal{S}(\mathbb{R}^n)$ and $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$.

(i) The *radial maximal function* $M(f,\Phi)$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$M(f,\Phi)(x) := \sup_{t\in(0,\infty)} |(\Phi_t * f)(x)|. \tag{2.12}$$

(ii) The *grand maximal function* $\mathcal{M}_N(f)$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}_N(f)(x) := \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(y)| : t \in (0,\infty),\ |x-y| < t,\ \psi \in \mathcal{F}_N\}. \tag{2.13}$$

(iii) The *non-tangential maximal function* $M_a^*(f,\Phi)$, with aperture $a \in (0,\infty)$, is defined by setting, for all $x \in \mathbb{R}^n$,

$$M_a^*(f,\Phi)(x) := \sup_{t\in(0,\infty)} \left\{ \sup_{y\in\mathbb{R}^n,\, |y-x|<at} |\Phi_t * f(y)| \right\}.$$

(iv) The *maximal function* $M_b^{**}(f,\Phi)$ *of Peetre type* is defined by setting, for all $x \in \mathbb{R}^n$,

$$M_b^{**}(f,\Phi)(x) := \sup_{(y,t)\in\mathbb{R}^{n+1}_+} \frac{|(\Phi_t * f)(x-y)|}{(1+t^{-1}|y|)^b}. \tag{2.14}$$

(v) The *grand maximal function* $\mathcal{M}_{b,N}^{**}(f)$ *of Peetre type* is defined by setting, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}_{b,N}^{**}(f)(x) := \sup_{\psi\in\mathcal{F}_N}\left\{\sup_{(y,t)\in\mathbb{R}_+^{n+1}} \frac{|(\psi_t * f)(x-y)|}{(1+t^{-1}|y|)^b}\right\}.$$

The following lemma is the key to the definition of Hardy type spaces in this article.

Lemma 2.13. *Let N be a large positive integer, $b \in [n+1,\infty)$, and $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (2.4). Then there exists a positive constant C such that, for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$\mathcal{M}_{b,N}^{**}(f)(x) \le CM_b^{**}(f,\Phi)(x). \tag{2.15}$$

Proof. Denote by δ the Dirac delta function at zero. Let $L := \lfloor 2b+1 \rfloor$. Following the argument in [111, Theorem 1.6], we see that there exist $\Theta, \rho \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\delta = \Theta * \Phi + \sum_{l=1}^{\infty} [2^{ln}\rho(2^l\cdot)] * [2^{ln}\Phi(2^l\cdot) - 2^{(l-1)n}\Phi(2^{l-1}\cdot)] \tag{2.16}$$

and $\rho \perp \mathcal{P}_L$.

Fix $t \in (0,\infty)$ and $\kappa \in \mathcal{S}(\mathbb{R}^n)$ such that $p_N(\kappa) \le 1$. Then, by (2.16), we find that, for all $x, y \in \mathbb{R}^n$,

$$\begin{aligned}\kappa_t * f(x-y) &= \kappa_t * \Theta_t * \Phi_t * f(x-y)\\ &\quad + \sum_{l=1}^{\infty} \kappa_t * \rho_{2^{-l}t} * [\Phi_{2^{-l}t} - \Phi_{2^{-l+1}t}] * f(x-y).\end{aligned} \tag{2.17}$$

Since N is a large positive integer and $\rho \perp \mathcal{P}_L$, it follows from Lemma 2.7 that, for all $x \in \mathbb{R}^n$,

$$|\kappa * \Theta(x)| \lesssim \frac{1}{(1+|x|)^{b+n+1}}$$

and

$$|\kappa * [2^{ln}\rho(2^l\cdot)](x)| \lesssim \frac{2^{-lL}}{(1+|x|)^{b+n+1}},$$

which, combined with $b \ge n+1$, implies that

$$\begin{aligned}&\frac{|\kappa_t * \rho_{2^{-l}t} * [\Phi_{2^{-l}t} - \Phi_{2^{-l+1}t}] * f(x-y)|}{(1+t^{-1}|y|)^b}\\
&\quad\lesssim 2^{-lL}\int_{\mathbb{R}^n} \frac{|\Phi_{2^{-l}t} * f(x-y-z)| + |\Phi_{2^{-l+1}t} * f(x-y-z)|}{t^n(1+t^{-1}|y|)^b(1+t^{-1}|z|)^{b+n+1}}\,dz\\
&\quad\lesssim 2^{-l(L-2b)}\int_{\mathbb{R}^n} \frac{|\Phi_{2^{-l}t} * f(x-y-z)| + |\Phi_{2^{-l+1}t} * f(x-y-z)|}{t^n(1+2^lt^{-1}|y|)^b(1+2^lt^{-1}|z|)^b(1+t^{-1}|z|)^{n+1}}\,dz\\
&\quad\lesssim 2^{-l(L-2b)}\int_{\mathbb{R}^n} \frac{|\Phi_{2^{-l}t} * f(x-y-z)| + |\Phi_{2^{-l+1}t} * f(x-y-z)|}{t^n(1+2^lt^{-1}|y+z|)^b(1+t^{-1}|z|)^{n+1}}\,dz\\
&\quad\lesssim 2^{-l(L-2b)} \sup_{0<v\le t,\, w\in\mathbb{R}^n} \frac{|\Phi_v * f(x-w)|}{(1+v^{-1}|w|)^b}\int_{\mathbb{R}^n} \frac{1}{t^n(1+t^{-1}|z|)^{n+1}}\,dz.\end{aligned}$$

From this, we conclude that, for all $x, y \in \mathbb{R}^n$,

$$\frac{|\kappa_t * \rho_{2^{-l}t} * [\Phi_{2^{-l}t} - \Phi_{2^{-l+1}t}] * f(x-y)|}{(1+t^{-1}|y|)^b} \lesssim 2^{-l(L-2b)} \sup_{0<v\le t,\, w\in\mathbb{R}^n} \frac{|\Phi_v * f(x-w)|}{(1+v^{-1}|w|)^b}. \tag{2.18}$$

Furthermore, similarly to (2.18), we know that, for all $x, y \in \mathbb{R}^n$,

$$\frac{|\kappa_t * \Theta_t * \Phi_t * f(x-y)|}{(1+t^{-1}|y|)^b} \lesssim \sup_{0<v\le t,\, w\in\mathbb{R}^n} \frac{|\Phi_v * f(x-w)|}{(1+v^{-1}|w|)^b}. \tag{2.19}$$

Therefore, it follows from (2.17)–(2.19), the definition of $M_b^{**}(f,\Phi)$ and $L > 2b$ that, for any $x, y \in \mathbb{R}^n$,

$$\frac{|\kappa_t * f(x-y)|}{(1+t^{-1}|y|)^b} \lesssim \sum_{l=0}^{\infty} 2^{-l(L-2b)} M_b^{**}(f,\Phi)(x) \sim M_b^{**}(f,\Phi)(x),$$

which, together with the arbitrariness of $\kappa \in \mathcal{F}_N$ and $x \in \mathbb{R}^n$, implies that (2.15) holds true. This finishes the proof of Lemma 2.13. ∎

2.4. Maximal estimates. In this subsection, we discuss what our standing assumption (2.8) implies. Let w be a *weight*, that is, w is a measurable function on $\mathbb{R}^n$ such that $0 < w < \infty$ almost everywhere. The *weighted Lebesgue space* $L^q_w(\mathbb{R}^n)$ with $q \in (0,\infty)$ is defined by setting

$$L^q_w(\mathbb{R}^n) := \left\{ f \in \mathcal{M} : \|f\|_{L^q_w(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |f(x)|^q w(x)\,dx \right]^{1/q} < \infty \right\}. \tag{2.20}$$

Moreover, for $p, q \in (0,\infty)$ and $a \in \mathbb{R}$, the *Herz space* $K^a_{p,q}(\mathbb{R}^n)$ is defined by setting

$$K^a_{p,q}(\mathbb{R}^n) := \{ f \in \mathcal{M} : \|f\|_{K^a_{p,q}(\mathbb{R}^n)} < \infty \},$$

where

$$\|f\|_{K^a_{p,q}(\mathbb{R}^n)} := \|\chi_{Q(\vec{0}_n,2)} f\|_{L^p(\mathbb{R}^n)} + \left\{ \sum_{j=1}^{\infty} [2^{aj} \|\chi_{S_j(Q(\vec{0}_n,1))} f\|_{L^p(\mathbb{R}^n)}]^q \right\}^{1/q}. \tag{2.21}$$

For the spaces $K^a_{p,q}(\mathbb{R}^n)$ and X, we have the following significant conclusion.

LEMMA 2.14. *Let X be a ball quasi-Banach function space satisfying* (2.8) *for some $\theta, s \in (0,1]$ with $\theta < s$. Then there exist positive constants C and $\varepsilon(X)$, depending on X, such that, for all $f \in X$,*

$$\|f\|_{K^{-n/[\theta+\varepsilon(X)]}_{\theta+\varepsilon(X),\,s}(\mathbb{R}^n)} \le C\|f\|_X. \tag{2.22}$$

Proof. As was proved in [83, p. 366], (2.8) implies that there exists a positive constant $\varepsilon(X)$, depending on X, such that, for any $\{f_j\}_{j=1}^{\infty} \subset X$,

$$\left\| \left\{ \sum_{j=1}^{\infty} [M^{(\theta+\varepsilon(X))}(f_j)]^s \right\}^{1/s} \right\|_X \lesssim \left\| \left\{ \sum_{j=1}^{\infty} |f_j|^s \right\}^{1/s} \right\|_X. \tag{2.23}$$

Moreover, by the definition of the powered Hardy–Littlewood maximal operator, we conclude that, for all $x \in \mathbb{R}^n$,

$$\Big\{\sum_{j=1}^{\infty}[M^{(\theta+\varepsilon(X))}(\chi_{S_j(Q(\vec{0}_n,1))}f)(x)]^s\Big\}^{1/s} \gtrsim \|\chi_{\mathbb{R}^n\setminus Q(\vec{0}_n,2)}f\|_{K_{\theta+\varepsilon(X),\,s}^{-n/[\theta+\varepsilon(X)]}(\mathbb{R}^n)}\chi_{Q(\vec{0}_n,1)}(x).$$

Likewise, we also know that, for all $x \in \mathbb{R}^n$,

$$M^{(\theta+\varepsilon(X))}(\chi_{Q(\vec{0}_n,2)}f)(x) \gtrsim \|f\chi_{Q(\vec{0}_n,2)}\|_{K_{\theta+\varepsilon(X),\,s}^{-n/[\theta+\varepsilon(X)]}(\mathbb{R}^n)}\chi_{Q(\vec{0}_n,1)}(x).$$

Thus, we conclude that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}&\|f\|_{K_{\theta+\varepsilon(X),\,s}^{-n/[\theta+\varepsilon(X)]}(\mathbb{R}^n)}\chi_{Q(\vec{0}_n,1)}(x)\\&\quad\lesssim M^{(\theta+\varepsilon(X))}(\chi_{Q(\vec{0}_n,2)}f)(x)+\Big\{\sum_{j=1}^{\infty}[M^{(\theta+\varepsilon(X))}(\chi_{S_j(Q(\vec{0}_n,1))}f)(x)]^s\Big\}^{1/s}.\end{aligned}$$

Since X is a ball quasi-Banach function space, it follows that $\chi_{Q(\vec{0}_n,1)} \in X$ and

$$\begin{aligned}&\|f\|_{K_{\theta+\varepsilon(X),\,s}^{-n/[\theta+\varepsilon(X)]}(\mathbb{R}^n)}\\&\quad\lesssim \|M^{(\theta+\varepsilon(X))}(\chi_{Q(\vec{0}_n,2)}f)\|_X+\Big\|\Big\{\sum_{j=1}^{\infty}[M^{(\theta+\varepsilon(X))}(\chi_{S_j(Q(\vec{0}_n,1))}f)]^s\Big\}^{1/s}\Big\|_X,\end{aligned}$$

which, together with (2.23), implies (2.22). This finishes the proof of Lemma 2.14. ■

When the Hardy–Littlewood maximal operator is bounded on X, we have the following estimates which are used when we relate the space X to the Hardy type space $H_X(\mathbb{R}^n)$.

Lemma 2.15. *Assume that X is a ball quasi-Banach function space on which the Hardy–Littlewood maximal operator M is bounded.*

(i) *There exists a positive constant C such that, for all $R \in (1,\infty)$ and $f \in X$,*

$$\frac{1}{|Q(\vec{0}_n,R)|}\int_{Q(\vec{0}_n,R)}|f(y)|\,dy \le C\|f\|_X. \tag{2.24}$$

(ii) *There exist a constant $\eta \in (1,\infty)$, depending on X, as well as a positive constant C, depending on X and η, such that, for all $f \in X$,*

$$\|M^{(\eta)}(f)\|_X \le C\|f\|_X. \tag{2.25}$$

(iii) *Let η be as in (2.25). Then there exist a large positive constant N and a positive constant C such that, for all $f \in X$,*

$$\|f\|_{L^\eta((1+|\cdot|)^{-N})} \le C\|f\|_X.$$

Proof. We first prove (i). By the definition of M, we find that, for any $R \in (1,\infty)$,

$$\frac{1}{|Q(\vec{0}_n,R)|}\int_{Q(\vec{0}_n,R)}|f(y)|\,dy \le \inf_{z\in Q(\vec{0}_n,1)} Mf(z),$$

which, combined with the assumption that M is bounded on X, implies (2.24). The conclusion of (ii) is well known (see, for example, [84]). The proof of (iii) is similar to that of Lemma 2.14, the details being omitted here. This finishes the proof of Lemma 2.15. ∎

In connection with Lemma 2.14, we have the following important estimate.

LEMMA 2.16. *Let $p, s \in (0,\infty)$. Then there exists a positive constant C, depending on p and s, such that, for all $L \in \mathbb{N}$,*

$$C^{-1}\sqrt[s]{L} \le \|\chi_{[0,2^L]^n}\|_{K_{p,\,s}^{-n/p}(\mathbb{R}^n)} \le C\sqrt[s]{L}. \tag{2.26}$$

Proof. The proof is a direct calculation, the details being omitted here. ∎

2.5. Boyd indices and the Hardy–Littlewood maximal function. In this subsection, we first extend the notion of Boyd indices to ball Banach function spaces. We then apply them to establish the Fefferman–Stein vector-valued inequalities and a decomposition theorem on ball Banach function spaces.

DEFINITION 2.17. Let Y be a ball Banach function space. Denote by M the Hardy–Littlewood maximal function in (2.5), and by Y' the associate space of Y. The *lower generalized Boyd index* l_Y and the *upper generalized Boyd index* u_Y of Y are, respectively, defined by

$$l_Y := \sup(\{l \in (1,\infty) : M \text{ is bounded on } Y^{1/l}\} \cup \{1\}) \tag{2.27}$$

and

$$u_Y := \inf(\{u \in (1,\infty) : M \text{ is bounded on } (Y')^{1-1/u}\} \cup \{\infty\}). \tag{2.28}$$

Recall that we have the following Fefferman–Stein vector-valued maximal inequalities for Y with $1 < l_Y \le u_Y < \infty$.

LEMMA 2.18 ([53, Theorem 3.3]). *Let $p, q \in (1,\infty)$. Assume that Y is a Banach function space with $1 < l_Y \le u_Y < \infty$. Then there exists a positive constant C such that, for all $\{f_j\}_{j=1}^\infty \subset \mathcal{M}$,*

$$\Big\|\Big\{\sum_{j=1}^\infty (Mf_j)^q\Big\}^{1/q}\Big\|_{Y^p} \le C\Big\|\Big\{\sum_{j=1}^\infty |f_j|^q\Big\}^{1/q}\Big\|_{Y^p}.$$

We improve this theorem in two different directions: $p = 1$ or $l_Y = 1$.

THEOREM 2.19 (The case $p = 1$). *Let Y be a ball Banach function space. If $1 < l_Y \le u_Y < \infty$ and $q \in (1,\infty]$, then there exists a positive constant C such that, for all $\{f_j\}_{j=1}^\infty \subset \mathcal{M}$,*

$$\Big\|\Big\{\sum_{j=1}^\infty (Mf_j)^q\Big\}^{1/q}\Big\|_{Y} \le C\Big\|\Big\{\sum_{j=1}^\infty |f_j|^q\Big\}^{1/q}\Big\|_{Y}.$$

THEOREM 2.20 (The case $l_Y = 1$). *Let Y be a ball Banach function space. If $u_Y \in [1,\infty)$, $p \in (1,\infty)$ and $q \in (1,\infty]$, then there exists a positive constant C such that, for all $\{f_j\}_{j=1}^\infty \subset \mathcal{M}$,*

$$\Big\|\Big\{\sum_{j=1}^\infty (Mf_j)^q\Big\}^{1/q}\Big\|_{Y^p} \le C\Big\|\Big\{\sum_{j=1}^\infty |f_j|^q\Big\}^{1/q}\Big\|_{Y^p}.$$

We prove Theorems 2.19 and 2.20 in Subsection 4.1.

Now we state the following decomposition result on the functions in ball quasi-Banach function spaces, in which we answer the following question: given a function f in X, how good functions can we use to express f? In what follows, let $\theta \in (0,1]$ be the constant from (2.8) and

$$d_X := \lfloor n(1/\theta - 1) \rfloor. \tag{2.29}$$

THEOREM 2.21 (Decomposition). *Let X be a ball quasi-Banach function space satisfying (2.8) for some $\theta, s \in (0,1]$. Assume that $l_X \in (1,\infty]$ and $f \in X$. Then there exists a triplet $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$ and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$, for any $j \in \mathbb{N}$, $|a_j| \le \|\chi_{Q_j}\|_X^{-1} \chi_{Q_j}$ almost everywhere, $a_j \perp \mathcal{P}_d$ and*

$$\left\| \left\{ \sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X \le C \|f\|_X,$$

where $d \in \mathbb{N}$, $d \ge d_X$ with d_X as in (2.29), *and the positive constant C depends on s, but is independent of f.*

Notice that Theorem 2.21 dates back to [70], where Janson and Jones proved Theorem 2.21 for the Lebesgue space $L^p(\mathbb{R}^n)$ from the viewpoint of martingales. The proof of Theorem 2.21 is given in Subsection 4.3.

2.6. Hardy type spaces. With (2.14) in mind, we now introduce the Hardy type space associated with X, which is denoted by $H_X(\mathbb{R}^n)$.

DEFINITION 2.22. Let X be a ball quasi-Banach function space. Then the *Hardy space* $H_X(\mathbb{R}^n)$ associated with X is defined as

$$H_X(\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H_X(\mathbb{R}^n)} := \|M_b^{**}(f,\Phi)\|_X < \infty \},$$

where $M_b^{**}(f,\Phi)$ is as in (2.14) with b sufficiently large and $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} \Phi(x)\, dx \neq 0. \tag{2.30}$$

When $X := L^p(\mathbb{R}^n)$ with $p \in (0,1]$, the Hardy type space $H_X(\mathbb{R}^n)$ agrees with the classical Hardy space $H^p(\mathbb{R}^n)$ (see, for example, [121]).

Later we prove that

$$\|M_b^{**}(f,\Phi)\|_X \sim \|M_{b_1}^{**}(f,\Psi)\|_X \tag{2.31}$$

as long as b, b_1 are large positive real numbers and $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (2.30) (see Theorem 3.1 below for the details), where the implicit positive constants are independent of f.

2.7. Calderón–Zygmund decompositions. As we have demonstrated in the proof of the atomic decomposition for classical Hardy spaces, we need to use the Calderón–Zygmund decomposition to break down functions or distributions into atoms. Even though the Calderón–Zygmund decomposition is well known, we recall it for completeness (see, for example, [121] for the details). In what follows, $C_c^\infty(\mathbb{R}^n)$ denotes the set of all infinite differentiable functions with compact supports.

Lemma 2.23. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $d \in \mathbb{Z}_+$. For any $j \in \mathbb{Z}$, let*

$$\mathcal{O}_j := \{y \in \mathbb{R}^n : \mathcal{M}_N(f)(y) > 2^j\}, \tag{2.32}$$

where $\mathcal{M}_N(f)$ is as in Definition 2.12 with $N \in \mathbb{N}$ large enough. Then:

(i) *For any $j \in \mathbb{Z}$, there exist a set K_j of indices and a family $\{Q_{j,k}\}_{k\in K_j}$ of closed cubes with disjoint interiors such that*

$$\mathcal{O}_j = \bigcup_{k\in K_j} Q_{j,k}.$$

Moreover, there exists a positive constant D such that, for any $j \in \mathbb{Z}$,

$$\chi_{\mathcal{O}_j} \le \sum_{k\in K_j} \chi_{Q_{j,k}} \le \sum_{k\in K_j} \chi_{5Q_{j,k}} \le D\chi_{\mathcal{O}_j}. \tag{2.33}$$

(ii) *There exist distributions $\{g_j\}_{j\in\mathbb{Z}}$ and $\{b_j\}_{j\in\mathbb{Z}}$ such that, for each $j \in \mathbb{Z}$, $f = g_j + b_j$ in $\mathcal{S}'(\mathbb{R}^n)$.*

(iii) *For any $j \in \mathbb{Z}$, the distribution g_j is such that, for any $x \in \mathbb{R}^n$,*

$$\mathcal{M}_N(g_j)(x) \lesssim \mathcal{M}_N(f)(x)\chi_{{\mathcal{O}_j}^c}(x) + \sum_{k\in K_j} \frac{2^j \ell_{j,k}^{n+d+1}}{(\ell_{j,k} + |x - x_{j,k}|)^{n+d+1}}, \tag{2.34}$$

where the implicit positive constant is independent of f and g_j. Here and hereafter, for each $j\in\mathbb{Z}$ and $k\in K_j$, $x_{j,k}$ and $\ell_{j,k}$ denote the center and the side-length of $Q_{j,k}$, respectively. Furthermore, if $f \in L^1_{\rm loc}(\mathbb{R}^n)$, then, for any $j \in \mathbb{Z}$, $g_j \in L^\infty(\mathbb{R}^n)$ and $\|g_j\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-j}$ with the implicit positive constant independent of j.

(iv) *If $f \in H_X(\mathbb{R}^n)$, then, for any $j \in \mathbb{Z}$, $b_j = \sum_{k\in K_j} b_{j,k}$ in $\mathcal{S}'(\mathbb{R}^n)$ and, for each $k \in K_j$, $b_{j,k} := (f - c_{j,k})\eta_{j,k}$, where $\{\eta_{j,k}\}_{k\in K_j}$ is a partition of unity with respect to $\{Q_{j,k}\}_{k\in K_j}$, namely, for each $k \in K_j$, $\eta_{j,k} \in C_c^\infty(\mathbb{R}^n)$, $\mathrm{supp}(\eta_{j,k}) \subset Q_{j,k}$, $0 \le \eta_{j,k} \le 1$, and*

$$\sum_{k\in K_j} \eta_{j,k} = \chi_{\mathcal{O}_j},$$

$c_{j,k} \in \mathcal{P}_d$ is a polynomial satisfying, for any $q \in \mathcal{P}_d$,

$$\langle f - c_{j,k}, q\eta_{j,k}\rangle = 0.$$

Moreover, for any $j \in \mathbb{Z}$, $k \in K_j$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}_N(b_{j,k})(x) \lesssim \mathcal{M}_N(f)(x)\chi_{Q_{j,k}}(x) + \frac{2^j \ell_{j,k}^{n+d+1}}{(\ell_{j,k} + |x - x_{j,k}|)^{n+d+1}}, \tag{2.35}$$

where the implicit positive constant is independent of f, k and j.

(v) *For any $s \in (0,\infty)$ and $x \in \mathbb{R}^n$,*

$$\sum_{j=-\infty}^{\infty} [2^j\chi_{\mathcal{O}_j}(x)]^s \sim \Bigg[\sum_{j=-\infty}^{\infty} 2^j\chi_{\mathcal{O}_j}(x)\Bigg]^s \sim \Bigg[\sum_{j=-\infty}^{\infty} 2^j\chi_{\mathcal{O}_j\setminus\mathcal{O}_{j+1}}(x)\Bigg]^s, \tag{2.36}$$

where the equivalent positive constants are independent of f.

Lemma 2.24. *Let X be a ball quasi-Banach function space satisfying* (2.8) *for some $\theta, s \in (0,1]$ and $\{Q_\lambda\}_{\lambda\in\Lambda}$ a family of cubes having the bounded overlapping property. Assume, in addition, that $d \in \mathbb{Z}_+$ with $d \geq \lfloor n(1/\theta - 1)\rfloor$ and*

$$\chi_O \in X, \quad \text{where} \quad O := \bigcup_{\lambda\in\Lambda} Q_\lambda.$$

Then

$$\sum_{\lambda\in\Lambda} \frac{[\ell(Q_\lambda)]^{n+d+1}}{[\ell(Q_\lambda)]^{n+d+1} + |\cdot - c(Q_\lambda)|^{n+d+1}} \in L^1_{\mathrm{loc}}(\mathbb{R}^n);$$

here and hereafter, for each $\lambda \in \Lambda$, $\ell(Q_\lambda)$ and $c(Q_\lambda)$ denote the side-length and the center of the cube Q_λ, respectively.

Proof. Let $K \subset \mathbb{R}^n$ be a compact set of the form $K := Q(\vec{0}_n, 2R)$, where $R \in (0, \infty)$. Define

$$\begin{aligned}
\Lambda_1 &:= \left\{\lambda \in \Lambda : \tfrac{1}{2}Q_\lambda \subset Q(\vec{0}_n, 20R)\right\},\\
\Lambda_2 &:= \left\{\lambda \in \Lambda \setminus \Lambda_1 : \tfrac{1}{2}Q_\lambda \cap Q(\vec{0}_n, 2R) \neq \emptyset\right\},\\
\Lambda_3 &:= \{\lambda \in \Lambda \setminus (\Lambda_1 \cup \Lambda_2) : \ell(Q_\lambda) \leq R\},\\
\Lambda_4 &:= \{\lambda \in \Lambda \setminus (\Lambda_1 \cup \Lambda_2) : \ell(Q_\lambda) > R\}.
\end{aligned}$$

By the definition of Λ_2, we know that, for any $\lambda \in \Lambda_2$, $Q(\vec{0}_n, 2R) \subset Q_\lambda$, which, together with the bounded overlapping property of $\{Q_\lambda\}_{\lambda\in\Lambda}$, implies that Λ_2 is a finite set. From this and the definition of Λ_1, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \sum_{\lambda\in\Lambda_1\cup\Lambda_2} \frac{[\ell(Q_\lambda)]^{n+d+1}}{[\ell(Q_\lambda)]^{n+d+1} + |x - c(Q_\lambda)|^{n+d+1}}\,dx &\sim \sum_{\lambda\in\Lambda_1} |Q_\lambda| + \sum_{\lambda\in\Lambda_2} |Q_\lambda|\\
&\sim R^n + \sum_{\lambda\in\Lambda_2} |Q_\lambda| < \infty. \qquad (2.37)
\end{aligned}$$

Moreover, by the definition of Λ_3, for any $\lambda \in \Lambda_3$ and $x \in Q_\lambda$, we have $|x| \sim |c(Q_\lambda)|$, $|c(Q_\lambda)| > 8R$ and $\ell(Q_\lambda) \leq R$, which further implies that

$$\begin{aligned}
&\int_{Q(\vec{0}_n,2R)} \sum_{\lambda\in\Lambda_3} \frac{[\ell(Q_\lambda)]^{n+d+1}}{[\ell(Q_\lambda)]^{n+d+1} + |x - c(Q_\lambda)|^{n+d+1}}\,dx\\
&\quad\lesssim \int_{Q(\vec{0}_n,2R)} \sum_{\lambda\in\Lambda_3} \frac{R^{d+1}[\ell(Q_\lambda)]^n}{R^{n+d+1} + |x - c(Q_\lambda)|^{n+d+1}}\,dx\\
&\quad\lesssim \sum_{\lambda\in\Lambda_3} \frac{R^{n+d+1}|Q_\lambda|}{R^{n+d+1} + |c(Q_\lambda)|^{n+d+1}}\\
&\quad\lesssim \sum_{\lambda\in\Lambda_3} \int_{Q_\lambda} \frac{R^{n+d+1}}{R^{n+d+1} + |x|^{n+d+1}}\,dx\\
&\quad\lesssim \int_{\mathbb{R}^n} \frac{R^{n+d+1}}{R^{n+d+1} + |x|^{n+d+1}}\,dx < \infty. \qquad (2.38)
\end{aligned}$$

Finally, we handle Λ_4. From the definition of Λ_4, we deduce that, for any $\lambda \in \Lambda_4$ and

$x \in Q(\vec{0}_n, 2R)$,

$$|x - c(Q_\lambda)| \sim |c(Q_\lambda)|, \tag{2.39}$$

which implies that

$$\int_{Q(\vec{0}_n,2R)} \sum_{\lambda\in\Lambda_4} \frac{[\ell(Q_\lambda)]^{n+d+1}}{[\ell(Q_\lambda)]^{n+d+1} + |x - c(Q_\lambda)|^{n+d+1}}\, dx \sim \sum_{\lambda\in\Lambda_4} \frac{[\ell(Q_\lambda)]^{n+d+1} R^n}{|c(Q_\lambda)|^{n+d+1}}. \tag{2.40}$$

Furthermore, by (2.39), we know that, for any $x \in Q(\vec{0}_n, 2R)$,

$$\left[\frac{\ell(Q_\lambda)}{|c(Q_\lambda)|}\right]^{n+d+1} \lesssim \left[\frac{\ell(Q_\lambda)}{|x - c(Q_\lambda)|}\right]^{n+d+1} \lesssim [M(\chi_{Q_\lambda})(x)]^{(n+d+1)/n} \lesssim [M^{(\theta)}(\chi_{Q_\lambda})(x)]^s,$$

which, combined with (2.8), further implies that

$$\begin{aligned}\left\{\sum_{\lambda\in\Lambda_4} \left[\frac{\ell(Q_\lambda)}{|c(Q_\lambda)|}\right]^{n+d+1}\right\}^{1/s} &\lesssim \left\|\left\{\sum_{\lambda\in\Lambda_4} \left[\frac{\ell(Q_\lambda)}{|c(Q_\lambda)|}\right]^{n+d+1}\right\}^{1/s} \chi_{Q(\vec{0}_n,2R)}\right\|_X \\ &\lesssim \left\|\left\{\sum_{\lambda\in\Lambda_4} [M^{(\theta)}(\chi_{Q_\lambda})]^s\right\}^{1/s}\right\|_X \\ &\lesssim \left\|\sum_{\lambda\in\Lambda_4} \chi_{Q_\lambda}\right\|_X \lesssim \|\chi_O\|_X < \infty.\end{aligned}$$

From this, (2.37), (2.38), (2.40) and the arbitrariness of the compact set $K \subset \mathbb{R}^n$, it follows that the conclusion of Lemma 2.24 holds true. ∎

3. Fundamental properties of $H_X(\mathbb{R}^n)$

In this section, we present the main results for the Hardy type space $H_X(\mathbb{R}^n)$. More precisely, we characterize the space $H_X(\mathbb{R}^n)$ via various maximal functions, atoms, molecules, Poisson integrals and Lusin-area functions, while the proofs of these results are postponed until Section 4.

3.1. Characterizations by means of various maximal operators. In this subsection, we present maximal function characterizations of $H_X(\mathbb{R}^n)$. More precisely, we compare the quantities

$$\|M(f,\Phi)\|_X, \quad \|M_a^*(f,\Phi)\|_X, \quad \|\mathcal{M}_N(f)\|_X, \quad \|M_b^{**}(f,\Phi)\|_X, \quad \|\mathcal{M}_{b,N}^{**}(f)\|_X$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$.

Recall that $\|f\|_{H_X(\mathbb{R}^n)} := \|M_b^{**}(f,\Phi)\|_X$ (see Definition 2.22). The main theorem of this subsection is as follows.

THEOREM 3.1. *Let $a, b \in (0,\infty)$, let X be a ball quasi-Banach function space and let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \Phi(x)\, dx \neq 0$.*

(i) *Let $N \geq \lfloor b+2 \rfloor$ be an integer. Then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\|M(f,\Phi)\|_X \lesssim \|M_a^*(f,\Phi)\|_X \lesssim \|M_b^{**}(f,\Phi)\|_X, \tag{3.1}$$

$$\|M(f,\Phi)\|_X \lesssim \|\mathcal{M}_N(f)\|_X \leq \|\mathcal{M}_{\lfloor b+2 \rfloor}(f)\|_X \lesssim \|M_b^{**}(f,\Phi)\|_X, \tag{3.2}$$

$$\|M_b^{**}(f,\Phi)\|_X \sim \|\mathcal{M}_{b,N}^{**}(f)\|_X, \tag{3.3}$$

where the implicit positive constants are independent of f.

(ii) *Let $r \in (0,\infty)$. Assume that $b \in (n/r,\infty)$ and M is bounded on $X^{1/r}$. Then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\|M_b^{**}(f,\Phi)\|_X \lesssim \|M(f,\Phi)\|_X, \tag{3.4}$$

where the implicit positive constant is independent of f. In particular, when $N \geq \lfloor b+2 \rfloor$, if one of the quantities

$$\|M(f,\Phi)\|_X, \quad \|M_a^*(f,\Phi)\|_X, \quad \|\mathcal{M}_N(f)\|_X, \quad \|M_b^{**}(f,\Phi)\|_X, \ \|\mathcal{M}_{b,N}^{**}(f)\|_X$$

is finite, then the others are also finite and mutually equivalent with the implicit positive constants independent of f.

The proof of Theorem 3.1 is given in Subsection 4.2.

We learn much more than (2.31) from Theorem 3.1. Denote by $e^{t\Delta}f$ the heat extension of $f \in \mathcal{S}'(\mathbb{R}^n)$ for $t \in (0,\infty)$, namely, for any $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$e^{t\Delta}f(x) := \left\langle f, \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-\cdot|^2}{4t}\right)\right\rangle.$$

Then, as a corollary of Theorem 3.1, we immediately obtain the following conclusion, the details being omitted.

Corollary 3.2. *Let X be a ball quasi-Banach function space such that M is bounded on X^r for some $r \in (0,\infty)$. Then, for any $f \in H_X(\mathbb{R}^n)$,*

$$\|f\|_{H_X(\mathbb{R}^n)} \sim \left\| \sup_{t\in(0,\infty)} |e^{t\Delta}f| \right\|_X,$$

where the implicit positive constants are independent of f.

Thanks to Corollary 3.2, we can define the Hardy type space $H_X(\mathbb{R}^n)$ to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm $\|\sup_{t\in(0,\infty)} |e^{t\Delta}f|\,\|_X$ is finite.

3.2. Poisson integral characterization. In this subsection, we give a characterization of the space $H_X(\mathbb{R}^n)$ by means of the Poisson integral.

Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to be a *bounded tempered distribution* if, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\varphi * f \in L^\infty(\mathbb{R}^n)$. Moreover, for any bounded tempered distribution f, the *Poisson semigroup* of f is defined by setting, for any $t \in (0,\infty)$,

$$P_tf := e^{-t\sqrt{-\Delta}}f := \mathcal{F}^{-1}(e^{-t|\cdot|}\mathcal{F}f)$$

(see, for example, [121, p. 89] for the details). Then we have the following characterization for the space $H_X(\mathbb{R}^n)$.

THEOREM 3.3. *Let X be a ball quasi-Banach function space such that M is bounded on X^r for some $r \in (0,\infty)$ and there exists a positive constant C satisfying*

$$\inf_{x\in\mathbb{R}^n} \|\chi_{B(x,1)}\|_X \geq C. \tag{3.5}$$

Assume that $f \in \mathcal{S}'(\mathbb{R}^n)$. Then the following are equivalent:

(i) $f \in H_X(\mathbb{R}^n)$;
(ii) f *is a bounded distribution and* $\sup_{t\in(0,\infty)} |P_t * f| \in X$.

The proof of Theorem 3.3 is given in Subsection 4.2.

3.3. Relation between X and $H_X(\mathbb{R}^n)$. In this subsection, we discuss the relation between the spaces X and $H_X(\mathbb{R}^n)$. More precisely, we have the following assertion.

THEOREM 3.4. *Let X be a ball quasi-Banach function space with $l_X \in (1,\infty]$. Then:*

(i) $X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, *the space X embeds continuously into $\mathcal{S}'(\mathbb{R}^n)$.*
(ii) *If $f \in X$, then $f \in H_X(\mathbb{R}^n)$.*
(iii) *If $f \in H_X(\mathbb{R}^n)$, then there exists a locally integrable function $g \in X$ such that g represents f, which means that $f = g$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\|g\|_{H_X(\mathbb{R}^n)} = \|f\|_{H_X(\mathbb{R}^n)}$.*

Theorem 3.4 is a bridge connecting X and $H_X(\mathbb{R}^n)$, which generalizes the classical result that $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with $p \in (1,\infty)$. The proof of Theorem 3.4 is given in Subsection 4.3.

3.4. Atomic characterization. In this subsection, we present an atomic decomposition of $H_X(\mathbb{R}^n)$. We first give the notion of atoms.

DEFINITION 3.5. Let X be a ball quasi-Banach function space satisfying (2.8) and let $q \in [1,\infty]$. Assume that $d \in \mathbb{Z}_+$ satisfies $d \geq d_X$, where d_X is as in (2.29). Then the function a is called an (X, q, d)-*atom* if there exists $Q \in \mathcal{Q}$ such that $\mathrm{supp}(a) \subset Q$,

$$\|a\|_{L^q(\mathbb{R}^n)} \leq \frac{|Q|^{1/q}}{\|\chi_Q\|_X} \tag{3.6}$$

and $a \perp \mathcal{P}_d$.

We use the $L^q(\mathbb{R}^n)$-norm in the size condition (3.6) for atoms. Notice that, for the classical Hardy spaces, we also have atoms with size conditions defined in terms of Banach function spaces. The reader is referred to [49, 51, 52, 68] for details.

We now state an atomic characterization for the space $H_X(\mathbb{R}^n)$: Similar to Theorems 2.9 and 2.21, there are atomic and molecular decompositions for $H_X(\mathbb{R}^n)$. We first present a reconstruction theorem.

THEOREM 3.6 (Reconstruction). *Let $s \in (0,1]$, $q \in (1,\infty]$ and d_X be as in* (2.29)*. Assume that X is a strictly s-convex ball quasi-Banach function space satisfying* (2.8) *for some $\theta \in (0,1]$ and, for any $f \in \mathcal{M}$,*

$$\|M^{((q/s)')}(f)\|_{(X^{1/s})'} \lesssim \|f\|_{(X^{1/s})'}, \tag{3.7}$$

where the implicit positive constant is independent of f. Let $\{a_j\}_{j=1}^\infty$ be a sequence of $(X,\, q,\, d_X)$-atoms, supported on the cubes $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ be such *that*

$$\left\|\left\{\sum_{j=1}^{\infty}\left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s\chi_{Q_j}\right\}^{1/s}\right\|_X<\infty. \tag{3.8}$$

Then the series

$$f:=\sum_{j=1}^{\infty}\lambda_j a_j \tag{3.9}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$*,* $f\in H_X(\mathbb{R}^n)$ *and*

$$\|f\|_{H_X(\mathbb{R}^n)}\lesssim\left\|\left\{\sum_{j=1}^{\infty}\left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s\chi_{Q_j}\right\}^{1/s}\right\|_X,$$

where the implicit positive constant is independent of f*.*

The proof of Theorem 3.6 is given in Subsection 4.3. Theorem 3.6 can be seen as a counterpart of Theorem 2.9. Now we formulate a decomposition theorem.

Theorem 3.7 (Decomposition). *Let* X *be a ball quasi-Banach function space satisfying* (2.8) *for some* $\theta,s\in(0,1]$*,* $d\geq d_X$ *be a fixed integer, where* d_X *is as in* (2.29)*, and* $f\in H_X(\mathbb{R}^n)$*. Then there exist a sequence* $\{a_j\}_{j=1}^{\infty}$ *of* (X,∞,d)*-atoms, supported on the cubes* $\{Q_j\}_{j=1}^{\infty}\subset\mathcal{Q}$*, and a sequence* $\{\lambda_j\}_{j=1}^{\infty}\subset[0,\infty)$ *such that*

$$f=\sum_{j=1}^{\infty}\lambda_j a_j\quad\text{in }\mathcal{S}'(\mathbb{R}^n) \tag{3.10}$$

and

$$\left\|\left\{\sum_{j=1}^{\infty}\left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s\chi_{Q_j}\right\}^{1/s}\right\|_X\lesssim_s\|f\|_{H_X(\mathbb{R}^n)},$$

where the implicit positive constant is independent of f*, but depends on* s*.*

The proof of Theorem 3.7 is given in Subsection 4.3. Now we consider the molecular decomposition of $H_X(\mathbb{R}^n)$. We begin with the notion of molecules.

Definition 3.8. Let X be a ball quasi-Banach function space satisfying (2.8) for some $\theta,s\in(0,1]$, $q\in[1,\infty]$, $d\in\mathbb{N}$ satisfy $d\geq d_X$, with d_X as in (2.29), and $\tau\in(0,\infty)$. A measurable function m on $\mathbb{R}^n$ is called an (X,q,d,τ)-*molecule* centered at a cube $Q\in\mathcal{Q}$ if, for any $j\in\mathbb{Z}_+$,

$$\|\chi_{S_j(Q)}m\|_{L^q(\mathbb{R}^n)}\leq 2^{-\tau j}\frac{|Q|^{1/q}}{\|\chi_Q\|_X}$$

and $m\perp\mathcal{P}_d$. In analogy, one defines an (X,q,d,τ)-*molecule* centered at a ball B.

It is easy to see that, for any (X,q,d)-atom α with $q\in[1,\infty]$ and $d\in\mathbb{N}$, α is also an (X,q,d,τ)-molecule for any $\tau\in(0,\infty)$. With this in mind, let us formulate a molecular characterization of the space $H_X(\mathbb{R}^n)$ as follows.

Theorem 3.9 (Molecular characterization). *Assume that* X *is a strictly* s*-convex ball quasi-Banach function space satisfying* (2.8) *for some* $\theta,s\in(0,1]$ *and* (3.7)*. Let* $q\in(1,\infty]$ *and* $\tau\in(0,\infty)$ *satisfy*

$$\tau>n(1/\theta-1/q).$$

Then $f \in H_X(\mathbb{R}^n)$ *if and only if there exist a sequence* $\{m_j\}_{j=1}^\infty$ *of* $(X,\, q,\, d_X,\, \tau)$*-molecules, centered at the cubes* $\{Q_j\}_{j=1}^\infty \in \mathcal{Q}$*, and* $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ *satisfying* (3.8) *such that*

$$f = \sum_{j=1}^\infty \lambda_j m_j \quad \textit{in } \mathcal{S}'(\mathbb{R}^n).$$

Moreover,

$$\|f\|_{H_X(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X,$$

where the implicit positive constants are independent of f*.*

The proof of Theorem 3.9 is presented in Subsection 4.3. We have some standard applications to atomic and molecular characterizations of mapping properties of some singular integral operators. We refer the reader to [102, Section 5] for details.

The above atomic and molecular characterization theorems for $H_X(\mathbb{R}^n)$ extend and unify those for classical Hardy spaces [23, 82].

Before we go further, a helpful remark may be in order.

Remark 3.10. Let $s \in (0,1]$ and X be a strictly s-convex ball quasi-Banach function space satisfying (2.8) for some $\theta \in (0,1]$.

(i) For any cube $Q \subset \mathbb{R}^n$, there exists a dyadic cube $\widetilde{Q}$ such that $|\widetilde{Q}| \le |Q|$ and $Q \subset 6\widetilde{Q}$. Then, by (2.8), we find that $\|\chi_Q\|_X \sim \|\chi_{\widetilde{Q}}\|_X$ with the equivalent positive constants independent of Q and $\widetilde{Q}$.
(ii) In Theorem 3.7, observing that there exists a dyadic cube R_j such that $R_j \supset Q_j$ and $\ell(R_j) \sim \ell(Q_j)$ as in (i) for each Q_j with $j \in \mathbb{N}$, with the implicit positive constants are independent of j, we may assume that, for any $j \in \mathbb{N}$, Q_j is a dyadic cube.
(iii) Denote by $\mathcal{D}$ the *set of all dyadic cubes* in $\mathbb{R}^n$. Since $s \in (0,1]$, one can rephrase the conclusion of Theorem 3.7 as follows: for any $f \in H_X(\mathbb{R}^n)$, there exist a sequence $\{a_j\}_{j=1}^\infty$ of (X,∞,d)-atoms, supported on the dyadic cubes $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}$, and a sequence $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\left\| \left\{ \sum_{j=1}^\infty (\lambda_j/\|\chi_{Q_j}\|_X)^s \chi_{Q_j} \right\}^{1/s} \right\|_X \lesssim_s \|f\|_{H_X(\mathbb{R}^n)},$$

where the implicit positive constant is independent of f, but depends on s.

When X has an absolutely continuous quasi-norm, we find that the atomic decompositions in (3.9) and (3.10) converge in $H_X(\mathbb{R}^n)$ (see Corollary 3.11 below for the details).

Corollary 3.11. *Let* X *be as in Theorem* 3.6*. Assume further that* X *has an absolutely continuous quasi-norm. Then:*

(i) $H_X(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *is dense in* $H_X(\mathbb{R}^n)$*.*
(ii) *The convergences of* (3.9) *and* (3.10) *hold true in* $H_X(\mathbb{R}^n)$*.*

Corollary 3.11 is proved in Subsection 4.3.

REMARK 3.12. Let X be as in Corollary 3.11, and let $f \in H_X(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^\infty$ and $\{a_j\}_{j=1}^\infty$ be as in Theorem 3.7. For any $N \in \mathbb{N}$, let $f_N := \sum_{j=1}^N \lambda_j a_j$. Then, for any $N \in \mathbb{N}$, $f_N \in L^2(\mathbb{R}^n)$, which, together with Corollary 3.11(ii), implies that $\|f - f_N\|_{H_X(\mathbb{R}^n)} \to 0$ as $N \to \infty$. Thus, $L^2(\mathbb{R}^n) \cap H_X(\mathbb{R}^n)$ is dense in $H_X(\mathbb{R}^n)$.

3.5. Boyd indices and Hardy spaces. In this subsection, we rephrase our results in terms of Boyd indices, as a direct consequence of Theorem 2.9; the details are omitted.

THEOREM 3.13. *Let Y be a ball Banach function space satisfying $u_Y \in [1,\infty)$, let $\{\lambda_k\}_{k\in\mathbb{N}} \subset [0,\infty)$ and let $\{Q_k\}_{k\in\mathbb{N}} \subset \mathcal{Q}$ be a sequence of cubes. Then, for any $q \in (u_Y,\infty]$ and sequence $\{b_k\}_{k\in\mathbb{N}} \subset L^q(\mathbb{R}^n)$ such that, for any $k \in \mathbb{N}$,* $\operatorname{supp}(b_k) \subset Q_k$ *and*

$$\|b_k\|_{L^q(\mathbb{R}^n)} \le \frac{|Q_k|^{1/q}}{\|\chi_{Q_k}\|_Y},$$

we have

$$\Big\|\sum_{k\in\mathbb{N}} \lambda_k b_k\Big\|_Y \lesssim \Big\|\sum_{k\in\mathbb{N}} \frac{\lambda_k}{\|\chi_{Q_k}\|_Y}\chi_{Q_k}\Big\|_Y,$$

where the implicit positive constant is independent of $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$.

Notice that, in Theorem 3.13, no moment condition on $\{b_k\}_{k\in\mathbb{N}}$ is necessary. Moreover, we generalize Theorem 3.13 to the case when $X := Y^p$ with $p \in (0,1]$ as follows.

THEOREM 3.14. *Let Y be a ball Banach function space satisfying $u_Y \in [1,\infty)$, and let $s \in (0,1]$, $q \in (\max\{1, su_Y\},\infty]$, $d \in \mathbb{N} \cap (\lfloor n/s - n\rfloor,\infty)$ and $X := Y^s$. Then, for any sequence $\{a_j\}_{j\in\mathbb{N}}$ of (X,q,d)-atoms, supported on the cubes $\{Q_j\}_{j\in\mathbb{N}} \subset \mathcal{Q}$, and $\{\lambda_j\}_{j\in\mathbb{N}} \subset [0,\infty)$ satisfying*

$$\left\|\left\{\sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s \chi_{Q_j}\right\}^{1/s}\right\|_X < \infty,$$

the series $f := \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$, $f \in H_X(\mathbb{R}^n)$ and

$$\|f\|_{H_X(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s \chi_{Q_j}\right\}^{1/s}\right\|_X,$$

where the implicit positive constant is independent of f.

THEOREM 3.15. *Let X be a ball Banach function space with $1 < l_X \le u_X < \infty$, and let $f \in X$ and $L \in \mathbb{N}$. Then there exists a triplet $\{\lambda_j\}_{j\in\mathbb{N}} \subset [0,\infty)$, $\{Q_j\}_{j\in\mathbb{N}} \subset \mathcal{Q}$ and $\{a_j\}_{j\in\mathbb{N}} \subset L^\infty(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and, for some $s \in (0,1]$,*

(i) $|a_j| \le \chi_{Q_j}/\|\chi_{Q_j}\|_X$,
(ii) $a_j \perp \mathcal{P}_L$,
(iii) $\left\|\left\{\sum_{j=1}^\infty (\lambda_j/\|\chi_{Q_j}\|_X)^s \chi_{Q_j}\right\}^{1/s}\right\|_X \lesssim_s \|f\|_X$,

where the implicit positive constant is independent of f, but depends on s.

The proofs of Theorems 3.14 and 3.15 are given in Subsection 4.3.

3.6. Characterizations by means of Lusin-area functions. As Peetre established in [109], Triebel–Lizorkin spaces cover classical Hardy spaces as a special case. Indeed, in addition to the study of Hardy spaces built on general function spaces, there are some general approaches to the study of Triebel–Lizorkin spaces built on general function spaces (see, for example, [48, 50, 86, 90]). We seek the relation between Hardy spaces built on general function spaces X and Triebel–Lizorkin spaces built on general function spaces X.

In this subsection, we characterize the Hardy space $H_X(\mathbb{R}^n)$ by means of the Lusin-area function. We begin by introducing the tent space associated with the ball quasi-Banach function space.

Definition 3.16. For $x \in \mathbb{R}^n$, let $\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$, which is called the *cone* of aperture 1 with vertex $x \in \mathbb{R}^n$.

For a closed set $F \subset \mathbb{R}^n$, denote by $\mathcal{R}(F)$ the union of all cones with vertices in F,

$$\mathcal{R}(F) := \bigcup_{x\in F} \Gamma(x).$$

For an open set $O \subset \mathbb{R}^n$, define the tent $\widehat{O}$ over O by

$$\widehat{O} := \{(x,t) \in \mathbb{R}^{n+1}_+ : B(x,t) \subset O\}.$$

It is easy to see that $\widehat{O} = [\mathcal{R}(O^{\complement})]^{\complement}$.

Let $g : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ be a measurable function. Then the *Lusin-area function* of g is defined by setting, for all $x \in \mathbb{R}^n$,

$$\mathcal{A}(g)(x) := \left\{ \int_{\Gamma(x)} |g(y,t)|^2 \, \frac{dy\,dt}{t^{n+1}} \right\}^{1/2},$$

where $\Gamma(x)$ for all $x \in \mathbb{R}^n$ is as in Definition 3.16. Coifman et al. [24] introduced the tent space $T^p_2(\mathbb{R}^{n+1}_+)$ for any $p \in (0,\infty)$. Recall that a measurable function g is said to belong to the *tent space* $T^p_2(\mathbb{R}^{n+1}_+)$, with $p \in (0,\infty)$, if $\|g\|_{T^p_2(\mathbb{R}^{n+1}_+)} := \|\mathcal{A}(g)\|_{L^p(\mathbb{R}^n)} < \infty$.

For a given ball quasi-Banach function space X, the *X-tent space* $T_X(\mathbb{R}^{n+1}_+)$ is defined to be the set of all measurable functions $g : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ for which the quasi-norm $\|g\|_{T_X(\mathbb{R}^{n+1}_+)} := \|\mathcal{A}(g)\|_X$ is finite.

Definition 3.17. Let $p \in (1,\infty)$. A measurable function $a : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ is said to be a *(T_X, p)-atom* if there exists a ball $B \subset \mathbb{R}^n$ such that

(i) $\operatorname{supp}(a) \subset \widehat{B}$,
(ii) $\|a\|_{T^p_2(\mathbb{R}^{n+1}_+)} \le |B|^{1/p}/\|\chi_B\|_X$.

Furthermore, if a is a (T_X, p)-atom for all $p \in (1,\infty)$, then a is called a *(T_X,∞)-atom.*

By the definition of (T_X, p)-atoms, the following conclusion holds true, the details being omitted.

Lemma 3.18. *Let $p \in (1,\infty)$. Then, for any (T_X, p)-atom a supported on $\widehat{B}$, $\mathcal{A}(a)$ is supported on B and $\|\mathcal{A}(a)\|_{L^p(\mathbb{R}^n)} \le |B|^{1/p}\|\chi_B\|_X^{-1}$.*

We write

$$\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}) := \left\|\left\{\sum_{j=1}^{\infty}\left(\frac{\lambda_j}{\|\chi_{B_j}\|_X}\right)^s \chi_{B_j}\right\}^{1/s}\right\|_X$$

or

$$\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}) := \left\|\left\{\sum_{j=1}^{\infty}\left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s \chi_{Q_j}\right\}^{1/s}\right\|_X,$$

where $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$ is a sequence, $\{\alpha_j\}_{j=1}^{\infty}$ a sequence of atoms (resp. molecules) supported on (resp. centered at) balls $\{B_j\}_{j=1}^{\infty}$ or cubes $\{Q_j\}_{j=1}^{\infty}$, and $s \in (0,1]$. The main result on $T_X(\mathbb{R}^{n+1}_+)$ is as follows.

Theorem 3.19. *Let $f : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ be a measurable function. Assume that X satisfies* (2.8) *for some $\theta, s \in (0,1]$, with $\theta < s$, and* (2.9) *for some $q \in (1,\infty)$. Then $f \in T_X(\mathbb{R}^{n+1}_+)$ if and only if there exist a sequence $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$ and a sequence $\{a_j\}_{j=1}^{\infty}$ of (T_X,∞)-atoms such that, for almost every $(x,t) \in \mathbb{R}^{n+1}_+$,*

$$f(x,t) = \sum_{j=1}^{\infty}\lambda_j a_j(x,t), \quad |f(x,t)| = \sum_{j=1}^{\infty}\lambda_j |a_j(x,t)| \tag{3.11}$$

and

$$\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}) < \infty, \tag{3.12}$$

where, for each $j \in \mathbb{N}$, $\widehat{B_j}$ appears in the support of a_j. Moreover,

$$\|f\|_{T_X(\mathbb{R}^{n+1}_+)} \sim_s \Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}),$$

where the implicit positive constants are independent of f, but depend on s.

To relate Hardy spaces built on ball quasi-Banach function spaces to the square functions, we recall a notion of distributions vanishing weakly at infinity. For any $t \in (0,\infty)$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, let

$$\psi(tD)(f) := \mathcal{F}^{-1}[\psi(t\,\cdot)\mathcal{F}f]. \tag{3.13}$$

Recall that a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity* if $\lim_{t\downarrow 0}\psi(tD)f = 0$ in $\mathcal{S}'(\mathbb{R}^n)$ for all $\psi \in \mathcal{S}(\mathbb{R}^n)$.

Example 3.20. Let $p \in (1,\infty)$. Then, for any $f \in L^p(\mathbb{R}^n)$, f vanishes weakly at infinity. Indeed, let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\chi_{B(\vec{0}_n,4)\setminus B(\vec{0}_n,2)} \le \varphi \le \chi_{B(\vec{0}_n,8)\setminus B(\vec{0}_n,1)}$. By the well-known g-function characterization of $L^p(\mathbb{R}^n)$ (see, for example, [44]), we know that, for all $f \in L^p(\mathbb{R}^n)$,

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{j=-\infty}^{\infty}|\varphi(2^{-j}D)(f)|^2\right\}^{1/2}\right\|_{L^p(\mathbb{R}^n)},$$

which, combined with the well-known fact that, for any $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\|\left\{\sum_{j=-\infty}^{\infty}|\psi(tD)\varphi(2^{-j}D)(f)|^2\right\}^{1/2}\right\|_{L^p(\mathbb{R}^n)} \to 0$$

as $t \downarrow 0$, implies that f vanishes weakly at infinity.

Theorem 3.21 (Lusin-area function characterization). *Assume that X is a ball quasi-Banach function space satisfying* (2.8) *for some $\theta, s \in (0,1]$ with $\theta < s$, and* (2.9) *for some $q \in (1,\infty)$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that*

$$\chi_{B(\vec{0}_n,4)\backslash B(\vec{0}_n,2)} \le \varphi \le \chi_{B(\vec{0}_n,8)\backslash B(\vec{0}_n,1)}.$$

Then $f \in H_X(\mathbb{R}^n)$ if and only if f vanishes weakly at infinity and

$$\left\| \left\{ \int_{\Gamma(\cdot)} |\varphi(tD)(f)(y)|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2} \right\|_X < \infty,$$

where $\Gamma(\cdot)$ is as in Definition 3.16. *Moreover,*

$$\|f\|_{H_X(\mathbb{R}^n)} \sim \left\| \left\{ \int_{\Gamma(\cdot)} |\varphi(tD)(f)(y)|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2} \right\|_X,$$

where the implicit positive constants are independent of f.

The proofs of Theorems 3.19 and 3.21 are presented in Subsection 4.4.

4. Proofs of main results

In this section, we give the proofs of the main results stated in Sections 2 and 3. The proofs of the equivalent characterizations of $H_X(\mathbb{R}^n)$ via various maximal functions, atoms, molecules and Lusin-area functions are presented in Subsections 4.1 through 4.4. To be precise, in Subsection 4.1, we prove Theorems 2.19, 2.20, 2.9, 2.11 and 3.4; in Subsection 4.2, the proofs of Theorems 3.1 and 3.3 are given; in Subsection 4.3, we show Theorems 3.6, 3.7, 3.9 and 3.13 and Corollary 3.11; finally, in Subsection 4.4, the proofs of Theorems 3.19 and 3.21 are presented.

4.1. The maximal estimates. We begin with the proof of Theorem 2.19.

Proof of Theorem 2.19. Going through the same argument as in [27, proof of Theorem 4.10], we prove Theorem 2.19. For completeness, we supply the details.

Recall that a function w on $\mathbb{R}^n$ is called an $A_1(\mathbb{R}^n)$-*weight* if there exists a positive constant C such that $Mw(x) \le Cw(x)$ for almost every $x \in \mathbb{R}^n$. Moreover, a non-negative function w on $\mathbb{R}^n$ is called an $A_2(\mathbb{R}^n)$-*weight* if

$$\sup_{Q\subset\mathbb{R}^n} \left\{ \frac{1}{|Q|} \int_Q w(x)\,dx \right\} \left\{ \frac{1}{|Q|} \int_Q [w(y)]^{-1}\,dy \right\} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. It is well known that $w \in A_2(\mathbb{R}^n)$ if and only if $w = w_1 w_2^{-1}$ for some $w_1, w_2 \in A_1(\mathbb{R}^n)$ (see, for example, [78, Theorem, p. 511]).

Thanks to Definition 2.2(iii), to prove Theorem 2.19, it suffices to show that there exists a positive constant C such that, for any $N \in \mathbb{N}$,

$$\left\| \left\{ \sum_{j=1}^N (Mf_j)^q \right\}^{1/q} \right\|_Y \le C \left\| \left\{ \sum_{j=1}^N |f_j|^q \right\}^{1/q} \right\|_Y, \tag{4.1}$$

where M denotes the Hardy–Littlewood maximal function in (2.5). For any $N\in\mathbb{N}$, let

$$\mathfrak{F}:=\Big\{\sum_{j=1}^N|f_j|^q\Big\}^{1/q},\quad \mathfrak{G}:=\Big\{\sum_{j=1}^N(Mf_j)^q\Big\}^{1/q},\quad h:=\frac{1}{\|\mathfrak{F}\|_Y}\mathfrak{F}+\frac{1}{\|\mathfrak{G}\|_Y}\mathfrak{G}.$$

Assume that $g\in Y'$ is a non-negative function. Define

$$G:=\sum_{l=0}^\infty\frac{M^lg}{2^l(\|M\|_{Y'\to Y'})^l},\qquad H:=\sum_{l=0}^\infty\frac{M^lh}{2^l(\|M\|_{Y\to Y})^l},$$

where, for any $l\in\mathbb{N}$, $M^l:=M(M^{l-1})$ and M^0 denotes the operator given by $M^0h(x):=|h(x)|$ for all $h\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$. Here and hereafter, $\|M\|_{Y'\to Y'}$ and $\|M\|_{Y\to Y}$ denote the operator norms of M from Y' to Y', respectively, from Y to Y. Then

$$g\le G,\quad h\le H,\quad G\in A_1(\mathbb{R}^n),\qquad H\in A_1(\mathbb{R}^n)\tag{4.2}$$

(see, for example, [27, Chapter 4]). By (4.2), Hölder's inequality and the weighted norm inequality for $A_2(\mathbb{R}^n)$-weights (see, for example, [2]), we conclude that

$$\begin{aligned}\|\mathfrak{G}g\|_{L^1(\mathbb{R}^n)}&\le\|\mathfrak{G}G\|_{L^1(\mathbb{R}^n)}\\&\le\Big\{\int_{\mathbb{R}^n}[\mathfrak{G}(x)]^2[H(x)]^{-1}G(x)\,dx\Big\}^{1/2}\|HG\|^{1/2}_{L^1(\mathbb{R}^n)}\\&\lesssim_{\|M\|_{Y\to Y},\|M\|_{Y'\to Y'}}\Big\{\int_{\mathbb{R}^n}[\mathfrak{F}(x)]^2[H(x)]^{-1}G(x)\,dx\Big\}^{1/2}\|HG\|^{1/2}_{L^1(\mathbb{R}^n)},\end{aligned}$$

which, combined with the facts that $\|G\|_{Y'}\le2\|g\|_{Y'}$ and $\|H\|_Y\le2\|h\|_Y\le2$, further implies that

$$\|\mathfrak{G}g\|_{L^1(\mathbb{R}^n)}\lesssim_{\|M\|_{Y\to Y},\|M\|_{Y'\to Y'}}\|\mathfrak{F}^2H^{-1}\|_Y^{1/2}\|g\|_{Y'}.\tag{4.3}$$

Moreover, from the definition of h and (4.2), it follows that $\mathfrak{F}\le\|\mathfrak{F}\|_Yh\le\|\mathfrak{F}\|_YH$, which implies that $\mathfrak{F}^2H^{-1}\le\|\mathfrak{F}\|_Y\mathfrak{F}$. From this and (4.3), we find that

$$\|\mathfrak{G}g\|_{L^1(\mathbb{R}^n)}\lesssim_{\|M\|_{Y\to Y},\|M\|_{Y'\to Y'}}\|\mathfrak{F}\|_Y\|g\|_{Y'},$$

which implies (4.1). This finishes the proof of Theorem 2.19. ■

Proof of Theorem 2.20. The proof is similar to that of [27, Theorem 4.6]. For completeness, we give it here. Similarly to the proof of Theorem 2.19, it suffices to show that there exists a positive constant C such that, for any $N\in\mathbb{N}$,

$$\Big\|\Big\{\sum_{j=1}^N(Mf_j)^q\Big\}^{p/q}\Big\|_Y\le C\Big\|\Big\{\sum_{j=1}^N|f_j|^q\Big\}^{p/q}\Big\|_Y,\tag{4.4}$$

where M denotes the Hardy–Littlewood maximal function in (2.5).

Let $N\in\mathbb{N}$, let $g\in Y'$ be non-negative and set

$$\mathrm{I}:=\int_{\mathbb{R}^n}g(x)\Big\{\sum_{j=1}^N[Mf_j(x)]^q\Big\}^{p/q}\,dx.$$

For any $x\in\mathbb{R}^n$, define

$$G(x):=\sum_{l=0}^\infty\frac{M^lg(x)}{2^l(\|M\|_{Y'\to Y'})^l}.$$

Then $g \le G$ and $G \in A_1(\mathbb{R}^n)$, which further implies that

$$\mathrm{I} \le \int_{\mathbb{R}^n} G(x) \Big\{ \sum_{j=1}^N [Mf_j(x)]^q \Big\}^{p/q} \, dx.$$

From this, $G \in A_1(\mathbb{R}^n)$ and the weighted Fefferman–Stein vector-valued maximal inequality (see, for example, [2]), we conclude that

$$\mathrm{I} \lesssim \int_{\mathbb{R}^n} G(x) \Big\{ \sum_{j=1}^N |f_j(x)|^q \Big\}^{p/q} \, dx. \tag{4.5}$$

By the definition of G, we find that $\|G\|_{Y'} \le 2\|g\|_{Y'}$, which, together with (4.5), implies that

$$\mathrm{I} \lesssim \Big\| \Big\{ \sum_{j=1}^N |f_j|^q \Big\}^{p/q} \Big\|_Y \|g\|_{Y'}.$$

From this, we know that (4.4) holds true. This finishes the proof of Theorem 2.20. ∎

Proof of Theorem 2.9. We argue by duality. Let the notation be as in Theorem 2.9. Without loss of generality, we may assume that, for all $j \in \mathbb{N}$, a_j is non-negative; otherwise, we consider $a_j^+ := \max\{a_j, 0\}$, respectively, $a_j^- := \max\{-a_j, 0\}$. For any $N \in \mathbb{N}$, let $f_N := \sum_{j=1}^N \lambda_j a_j$. Assume that $g \in X'$ is non-negative and $\|g\|_{X'} = 1$. Then, by Hölder's inequality,

$$\|f_N g\|_{L^1(\mathbb{R}^n)} = \sum_{j=1}^N \lambda_j \int_{Q_j} a_j(x) g(x) \, dx \le \sum_{j=1}^N \lambda_j \|a_j\|_{L^r(Q_j)} \|g\|_{L^{r'}(Q_j)}. \tag{4.6}$$

Moreover, from the definition of $M^{(r')}$, it follows that, for any $j \in \mathbb{N}$ and $x \in Q_j$,

$$M^{(r')}(g)(x) \ge \frac{1}{|Q_j|^{1/r'}} \|g\|_{L^{r'}(Q_j)},$$

which, combined with (4.6) and (2.6), further implies that

$$\begin{aligned}\|f_N g\|_{L^1(\mathbb{R}^n)} &\le \sum_{j=1}^N \int_{\mathbb{R}^n} M^{(r')}(g)(x) \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j}(x) \, dx \\ &= \int_{\mathbb{R}^n} M^{(r')}(g)(x) \Big\{ \sum_{j=1}^N \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j}(x) \Big\} \, dx. \end{aligned} \tag{4.7}$$

Furthermore, by the assumption that M is bounded on $(X')^{1/r'}$, we know that $M^{(r')}$ is bounded on X', which, together with (4.7) and $\|g\|_{X'} = 1$, implies that

$$\|f_N g\|_{L^1(\mathbb{R}^n)} \le \|M^{(r')}(g)\|_{X'} \Big\| \sum_{j=1}^N \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j} \Big\|_X \lesssim \Big\| \sum_{j=1}^N \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j} \Big\|_X.$$

From this and the arbitrariness of $N \in \mathbb{N}$ and g, together with Definition 2.2(iii), it follows that

$$\|f\|_X = \lim_{N\to\infty} \|f_N\|_X \lesssim \lim_{N\to\infty} \Big\| \sum_{j=1}^N \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j} \Big\|_X \lesssim \Big\| \sum_{j=1}^\infty \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j} \Big\|_X,$$

which implies (2.7).

Finally, we prove that $f=\sum_{j=1}^{\infty}\lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$. Let $\psi\in\mathcal{S}(\mathbb{R}^n)$. Then, by the definition of the operator $M^{(r')}$, we find that $M^{(r')}(\chi_{Q(\vec{0}_n,1)})\gtrsim|\psi|$, which, combined with (4.7) and $\chi_{Q(\vec{0}_n,1)}\in X'$, further implies that, for any $N\in\mathbb{N}$,

$$\begin{aligned}\|f_N\psi\|_{L^1(\mathbb{R}^n)}&\lesssim\|M^{(r')}(\chi_{Q(\vec{0}_n,1)})\|_{X'}\left\|\sum_{j=1}^{N}\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\chi_{Q_j}\right\|_X\\&\lesssim\|\chi_{Q(\vec{0}_n,1)}\|_{X'}\left\|\sum_{j=1}^{N}\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\chi_{Q_j}\right\|_X\sim\left\|\sum_{j=1}^{\infty}\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\chi_{Q_j}\right\|_X<\infty.\end{aligned}$$

From this, the monotone convergence theorem and the arbitrariness of $\psi\in\mathcal{S}(\mathbb{R}^n)$, we deduce that

$$f=\sum_{j=1}^{\infty}\lambda_j a_j\quad\text{in }\mathcal{S}'(\mathbb{R}^n).\tag{4.8}$$

This finishes the proof of Theorem 2.9. ■

Proof of Theorem 2.11. Let the notation be as in Theorem 2.11. First, by Theorem 2.10,

$$\left\|\sum_{j=1}^{\infty}\lambda_j\chi_{S_0(Q_j)}m_j\right\|_X\lesssim\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}).$$

To finish the proof, it remains to show that

$$\left\|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\lambda_j\chi_{S_k(Q_j)}m_j\right\|_X\lesssim\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}).\tag{4.9}$$

For any $j,k\in\mathbb{N}$, let $\mu_{j,k}:=\lambda_j2^{-(\tau+n/q)k}\|\chi_{2^{k+1}Q_j}\|_X\|\chi_{Q_j}\|_X^{-1}$ and

$$a_{j,k}:=2^{(\tau+n/q)k}\|\chi_{Q_j}\|_X\|\chi_{2^{k+1}Q_j}\|_X^{-1}\chi_{S_k(Q_j)}m_j.$$

Then $\mu_{j,k}a_{j,k}=\lambda_j\chi_{S_k(Q_j)}m_j$, $\operatorname{supp}(a_{j,k})\subset2^{k+1}Q_j$, and by the assumption that

$$\|\chi_{S_k(Q_j)}m_j\|_{L^q(\mathbb{R}^n)}\le2^{-\tau k}|Q_j|^{1/q}\|\chi_{Q_j}\|_X^{-1},$$

we further know that $\|a_{j,k}\|_{L^q(\mathbb{R}^n)}\le|2^{k+1}Q_j|^{1/q}\|\chi_{2^{k+1}Q_j}\|_X^{-1}$. Thus, applying Theorem 2.10 to $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mu_{j,k}a_{j,k}$, we conclude that

$$\left\|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\lambda_j\chi_{S_k(Q_j)}m_j\right\|_X\lesssim\left\|\left\{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{2^{-(\tau+n/q)k}\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s\chi_{2^{k+1}Q_j}\right\}^{1/s}\right\|_X.\tag{4.10}$$

Furthermore, it is easy to see that, for any $j\in\mathbb{N}$, $\chi_{2^{k+1}Q_j}\lesssim2^{kn/\theta}M^{(\theta)}(\chi_{Q_j})$, which, together with (2.8), (2.10) and (4.10), further implies that

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\lambda_j\chi_{S_k(Q_j)}m_j\right\|_X&\lesssim\left\|\left\{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{2^{-(\tau+n/q-n/\theta)k}\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s\chi_{Q_j}\right\}^{1/s}\right\|_X\\&\sim\left\|\left\{\sum_{j=1}^{\infty}\left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s\chi_{Q_j}\right\}^{1/s}\right\|_X.\end{aligned}$$

From this, we deduce (4.9). Moreover, similarly to the proof of (4.8), we conclude that $f=\sum_{j=1}^{\infty}\lambda_j m_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$. This finishes the proof of Theorem 2.11. ■

Proof of Theorem 3.4. We first show (i). Let $f \in X$. By the assumption $l_X \in (1,\infty]$, the Hardy–Littlewood maximal operator M in (2.5) is bounded on X. Then, from Lemma 2.15(i), it follows that, for any $R \in (1,\infty)$,

$$\int_{Q(\vec{0}_n,R)} |f(y)|\,dy \lesssim R^n,$$

which implies that $f \in \mathcal{S}'(\mathbb{R}^n)$, and hence completes the proof of (i).

Now we prove (ii). Let $f \in X$. By (i), we know that $f \in \mathcal{S}'(\mathbb{R}^n)$. Thus, for any $t \in (0,\infty)$, $e^{t\Delta} f$ makes sense and is a smooth function on $\mathbb{R}^n$. Then, from basic calculus, we deduce that $\sup_{t\in(0,\infty)} |e^{t\Delta} f| \le Mf$ (see, for example, [32, Proposition 2.7]), which, combined with the fact that M is bounded on X and Corollary 3.2, implies that f is in $H_X(\mathbb{R}^n)$, and hence completes the proof of (ii).

Finally, we prove (iii). Let $\eta \in (1,\infty)$ and $N \ge 1$ be as in Lemma 2.15(iii), and $f \in H_X(\mathbb{R}^n)$. Then, by Corollary 3.2, $\{e^{t\Delta} f\}_{t>0}$ is a bounded set of X. According to Lemma 2.15(iii), X embeds continuously into the space $L^\eta((1+|\cdot|)^{-N})$, which, together with Banach–Alaoglu's theorem (see, for example, [99, p. 229, Theorem 2.6.18]), implies that there exists a sequence $\{t_j\}_{j=1}^\infty$ decreasing to 0 such that $e^{t_j\Delta} f$ converges to a function $F \in L^\eta((1+|\cdot|)^{-N})$ as $j \to \infty$. Since $L^\eta((1+|\cdot|)^{-N}) \subset \mathcal{S}'(\mathbb{R}^n)$, it follows that $F \in \mathcal{S}'(\mathbb{R}^n)$. Thus, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \varphi(x) F(x)\,dx\right| &= \lim_{j\to\infty} \left|\int_{\mathbb{R}^n} \varphi(x) e^{t_j\Delta} f(x)\,dx\right| \\ &\le \int_{\mathbb{R}^n} |\varphi(x)| \sup_{t\in(0,\infty)} |e^{t\Delta} f(x)|\,dx, \end{aligned} \tag{4.11}$$

which, combined with the arbitrariness of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and Lebesgue's differentiation theorem, further implies that, for almost every $x \in \mathbb{R}^n$,

$$|F(x)| \le \sup_{t\in(0,\infty)} |e^{t\Delta} f(x)|.$$

From this, it follows that $F \in X$.

Now we show that F represents f. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Notice that

$$\mathcal{S}(\mathbb{R}^n) \subset \left(L^\eta((1+|\cdot|)^{-N})\right)' = L^{\eta'}((1+|\cdot|)^{N})$$

and, for any $h \in \mathcal{S}'(\mathbb{R}^n)$,

$$\langle h, \varphi\rangle = \lim_{j\to\infty} \langle e^{t_j\Delta} h, \varphi\rangle,$$

which, together with (4.11), implies that

$$\langle f, \varphi\rangle = \lim_{j\to\infty} \int_{\mathbb{R}^n} \varphi(x) e^{t_j\Delta} f(x)\,dx = \int_{\mathbb{R}^n} F(x)\varphi(x)\,dx.$$

Thus, F represents f. This finishes the proof of Theorem 3.4. ■

4.2. Fundamental properties of Hardy spaces. In this subsection, we give the proofs of Theorems 3.1 and 3.3.

Proof of Theorem 3.1. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We first prove (i). From Definition 2.12, it follows that, for all $x \in \mathbb{R}^n$,

$$M(f,\Phi)(x) \le M_a^*(f,\Phi)(x) \lesssim M_b^{**}(f,\Phi)(x),$$

which implies (3.1).

Moreover, by Definition 2.12 again, we find that, for all $x \in \mathbb{R}^n$,

$$M(f,\Phi)(x) \lesssim \mathcal{M}_N(f)(x) \lesssim \mathcal{M}_{\lfloor b+2\rfloor}(f)(x). \tag{4.12}$$

Furthermore, from [44, proof of Theorem 6.4.4(d)], we deduce that $\mathcal{M}_{\lfloor b+2\rfloor}(f)(x) \lesssim M_b^{**}(f,\Phi)(x)$ for all $x \in \mathbb{R}^n$, which, together with (4.12), implies (3.2).

It is easy to see that, for all $x \in \mathbb{R}^n$, $M_b^{**}(f,\Phi)(x) \lesssim \mathcal{M}_{b,\,N}^{**}(f)(x)$, which, combined with Lemma 2.13, implies (3.3). This finishes the proof of (i).

Now we prove (ii). Applying Lemma 2.8 with r, b and N satisfying $r=\theta$, $Nr>n$ and $(b-N)r>n$, we know that, for all $x \in \mathbb{R}^n$,

$$M_b^{**}(f,\Phi)(x) \lesssim_r M^{(r)}\Big[\sup_{t\in(0,\infty)} |\Phi_t * f|\Big](x) \sim M^{(r)}(M(f,\Phi))(x)$$

(see [101, Lemma 3.2] for the details), which, together with the assumption that M is bounded on $X^{1/r}$, further implies that

$$\|M_b^{**}(f,\Phi)\|_X \lesssim \|M^{(r)}(M(f,\Phi))\|_X \lesssim \|M(f,\Phi)\|_X.$$

Thus, (3.4) holds true. This finishes the proof of Theorem 3.1. ■

Proof of Theorem 3.3. We first prove (i)⇒(ii). Let $f \in H_X(\mathbb{R}^n)$. Then $\|\mathcal{M}_N(f)\|_X<\infty$ for some $N\in\mathbb{N}$. It is easy to see that, for any fixed $\varphi\in\mathcal{S}(\mathbb{R}^n)$, there exists a positive constant D_φ such that $D_\varphi\varphi\in\mathcal{F}_N$. Therefore, $M_1^*(f,D_\varphi\varphi)\le\mathcal{M}_N(f)$, which, together with (3.5), further implies that, for all $x\in\mathbb{R}^n$,

$$\begin{aligned} D_\varphi|(\varphi*f)(x)| &\le \inf_{|y-x|<1} M_1^*(f,D_\varphi\varphi)(y) \\ &\le \frac{\|\chi_{B(x,1)}M_1^*(f,D_\varphi\varphi)\|_X}{\|\chi_{B(x,1)}\|_X} \lesssim \|\mathcal{M}_N(f)\|_X<\infty. \end{aligned}$$

This guarantees that f is a bounded tempered distribution. Moreover, from the proof of [44, Theorem 6.4.4(e)], we deduce that, for any $N\in\mathbb{N}$ and $x\in\mathbb{R}^n$, $\sup_{t\in(0,\infty)}|P_t*f(x)|\lesssim \mathcal{M}_Nf(x)$, which implies that $\sup_{t\in(0,\infty)}|P_t*f|\in X$. Thus, (ii) holds true.

Now we prove (ii)⇒(i). Let f be a bounded tempered distribution and

$$\sup_{t\in(0,\infty)}|P_t*f|\in X.$$

Then, by an argument as in [121, p. 99], we know that, for some $\Phi_0\in\mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\Phi_0(x)\,dx\neq0$ and any $x\in\mathbb{R}^n$,

$$M(f,\,\Phi_0)(x)\lesssim \sup_{t\in(0,\infty)}|P_t*f(x)|,$$

which, together with $\sup_{t\in(0,\infty)}|P_t*f|\in X$ and Theorem 3.1(ii), implies (i). This finishes the proof of Theorem 3.3. ■

4.3. Proofs of theorems on atomic characterizations of $H_X(\mathbb{R}^n)$. In this subsection, we give the proofs of Theorems 3.6, 3.9, 3.7, 2.21, 3.14 and 3.15 in this order.

Proof of Theorem 3.6. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\mathrm{supp}(\phi) \subset Q(\vec{0}_n, 1)$ and $\int_{\mathbb{R}^n} \phi(x)\,dx \neq 0$. Assume that θ is as in (2.8). It follows from the assumption that X satisfies (2.8) that M is bounded on $X^{1/\theta}$, which, combined with Theorem 3.1(ii), implies that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\|M_b^{**}(f, \Phi)\|_X \sim \|M(f,\, \phi)\|_X,$$

where b and Φ are as in Definition 2.22. Thus, to prove Theorem 3.6, it suffices to show

$$\left\|M\Big(\sum_{j=1}^{\infty} \lambda_j a_j,\, \phi\Big)\right\|_X \lesssim \left\|\left\{\sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s \chi_{Q_j}\right\}^{1/s}\right\|_X. \tag{4.13}$$

Let a be an $(X,\, q,\, d_X)$-atom supported on the cube $Q := Q(x_Q, \ell_Q)$ with $x_Q \in \mathbb{R}^n$ and $\ell_Q \in (0, \infty)$. Then, for any $x \in 2Q$,

$$M(a,\, \phi)(x) \lesssim Ma(x). \tag{4.14}$$

Moreover, repeating the proof of [101, (4.4)] with $\|\chi_Q\|_{L^{p(\cdot)}}$ replaced by $\|\chi_Q\|_X$, we find that, for any $x \in \mathbb{R}^n \setminus 2Q$,

$$M(a,\, \phi)(x) \lesssim \left(\frac{\ell_Q}{|x - x_Q|}\right)^{n+d_X+1} \|\chi_Q\|_X^{-1} \lesssim \frac{1}{\|\chi_Q\|_X} M^{(\theta)}(\chi_Q)(x), \tag{4.15}$$

which, together with (4.14), implies that, for any $x \in \mathbb{R}^n$,

$$M(a,\, \phi)(x) \lesssim \chi_{2Q}(x) Ma(x) + \frac{1}{\|\chi_Q\|_X} M^{(\theta)}(\chi_Q)(x). \tag{4.16}$$

By the boundedness of M on $L^q(\mathbb{R}^n)$ with $q \in (1, \infty]$, we know that

$$\|\chi_{2Q} Ma\|_{L^q(\mathbb{R}^n)} \lesssim |Q|^{1/q}/\|\chi_Q\|_X$$

or, equivalently,

$$\|(\chi_{2Q} Ma)^s\|_{L^{q/s}(\mathbb{R}^n)} \lesssim (|Q|^{1/q}/\|\chi_Q\|_X)^s,$$

which, combined with (3.7) and Theorem 2.9, implies that

$$\begin{aligned}
\left\|\left\{\sum_{j=1}^{\infty} (\lambda_j \chi_{2Q_j} Ma_j)^s\right\}^{1/s}\right\|_X &= \left\{\left\|\sum_{j=1}^{\infty} (\lambda_j \chi_{2Q_j} Ma_j)^s\right\|_{X^{1/s}}\right\}^{1/s} \\
&\lesssim \left\{\left\|\sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X}\right)^s \chi_{Q_j}\right\|_{X^{1/s}}\right\}^{1/s} \\
&\sim \Lambda(\{\lambda_j a_j\}_{j\in\mathbb{N}}).
\end{aligned}$$

From this, (4.16) and (2.8), we deduce that

$$\begin{aligned}
&\Big\|M\Big(\sum_{j=1}^{\infty}\lambda_j a_j,\phi\Big)\Big\|_X\\
&\qquad\le \Big\|\sum_{j=1}^{\infty}\lambda_j M(a_j,\phi)\Big\|_X\\
&\qquad\lesssim \Big\|\Big\{\sum_{j=1}^{\infty}(\lambda_j\chi_{2Q_j}Ma_j)^s\Big\}^{1/s}\Big\|_X+\Big\|\Big\{\sum_{j=1}^{\infty}\Big[\frac{\lambda_j}{\|\chi_{Q_j}\|_X}M^{(\theta)}(\chi_{Q_j})\Big]^s\Big\}^{1/s}\Big\|_X\\
&\qquad\lesssim \Lambda(\{\lambda_j a_j\}_{j\in\mathbb{N}}),
\end{aligned}$$

which implies (4.13). This finishes the proof of Theorem 3.6. ∎

To show Theorem 3.9, we need Lemma 4.1 below.

LEMMA 4.1. *Let X be as in Theorem 3.9, $q\in(1,\infty]$, d_X be as in (2.29), and $\tau\in(0,\infty)$ satisfy $\tau>n(1/\theta-1/q)$. Assume that $\Phi_0\in\mathcal{S}(\mathbb{R}^n)$ satisfies $\chi_{Q(\vec{0}_n,1)}\le\Phi_0\le\chi_{Q(\vec{0}_n,2)}$ and m is an $(X,\,q,\,d_X,\,\tau)$-molecule centered at the cube $Q:=Q(x_Q,\ell_Q)$ with $x_Q\in\mathbb{R}^n$ and $\ell_Q\in(0,\infty)$. Then, for all $x\in\mathbb{R}^n$,*

$$\begin{aligned}
M(\Phi_0,\,m)(x)&\lesssim \chi_{3Q}(x)Mm(x)+\sum_{k=1}^{\infty}\chi_{3^{k+1}Q\setminus 3^kQ}(x)M(\chi_{3^{k+2}Q\setminus 3^{k-1}Q}m)(x)\\
&\quad+\frac{1}{\|\chi_Q\|_X}M^{(\theta)}(\chi_Q)(x),
\end{aligned}$$

where θ is as in (2.8) and the implicit positive constant is independent of m and x.

Proof. When $x\in 3Q$, it is easy to see that

$$M(\Phi_0,\,m)(x)\lesssim Mm(x) \tag{4.17}$$

(see, for example, [32, Proposition 2.7]).

Now we assume that $x\in 3^{k+1}Q\setminus 3^kQ$ with $k\in\mathbb{N}$. To prove Lemma 4.1, we need to show that

$$M(\Phi_0,\,m)(x)\lesssim M(\chi_{3^{k+2}Q\setminus 3^{k-1}Q}m)(x)+\frac{1}{\|\chi_Q\|_X}M^{(\theta)}(\chi_Q)(x). \tag{4.18}$$

When $t\in(0,\ell_Q]$, it is easy to see that

$$|(\Phi_0)_t*m(x)|\lesssim M(\chi_{3^{k+2}Q\setminus 3^{k-1}Q}m)(x). \tag{4.19}$$

When $t\in(\ell_Q,\infty)$, repeating the proof of [101, (5.2)] with $\|\chi_Q\|_{L^{p(\cdot)}}$ replaced by $\|\chi_Q\|_X$, we obtain

$$|(\Phi_0)_t*m(x)|\lesssim\frac{1}{\|\chi_Q\|_X}M^{(\theta)}(\chi_Q)(x),$$

which, together with (4.19), implies (4.18). Then, by (4.17) and (4.18), we conclude that Lemma 4.1 holds true. ∎

Now we prove Theorem 3.9 using Lemma 4.1 above and Theorem 3.7, which we admit now.

Proof of Theorem 3.9. Via Theorem 3.7 and the fact that, for any (X,q,d)-atom α, α is also an (X,q,d,τ)-molecule, we find that the necessity part of Theorem 3.9 holds true. The proof of the sufficiency part is similar to that of Theorem 3.6. More precisely, replacing (4.14) and (4.16) by Lemma 4.1, respectively, Theorem 2.9 by Theorem 2.11, then repeating the proof of Theorem 3.6, we complete the proof of Theorem 3.9, the details being omitted. ■

We move on to the proof of Theorem 3.7. The proof is made up of Lemmas 4.2, 4.4, and 4.5, Proposition 4.3 and Corollary 4.6. The idea is taken from the book of Stein [121, pp. 101–112]. Following Stein, we consider the Calderón–Zygmund decomposition $\{g_j\}_{j=-\infty}^{\infty}$ of f. It is known that g_j can be easily handled but f is difficult despite the fact that one has the decomposition in Lemma 4.2(iii) below. Lemma 4.2 concerns the decomposition basically based on [121, pp. 107–109]. Among other things, it matters that we cannot use the absolute continuity of the quasi-norm of X in the proof of Lemma 4.2(iii). To overcome this absence of absolute continuity, we first establish Lemma 4.4 below. On the other hand, Lemma 4.5 below is obtained from the standing assumption (2.8), which will justify our strategy: g_j is a locally integrable function. Finally, applying the technique described in [121, pp. 109–111], we obtain the decomposition with the help of the fact that g_j is a locally integrable function.

LEMMA 4.2. *Let $f \in H_X(\mathbb{R}^n)$, $\{K_j\}_{j\in\mathbb{Z}}$, $\{Q_{j,k}\}_{j\in\mathbb{Z},\,k\in K_j}$, $\{g_j\}_{j\in\mathbb{Z}}$ and $\{b_j\}_{j\in\mathbb{Z}}$ be as in Lemma 2.23. Assume that s is as in Theorem 3.7. Then:*

(i) *For all $j \in \mathbb{Z}$,*

$$\left\|\left\{\sum_{k\in K_j} (2^j \chi_{Q_{j,k}})^s\right\}^{1/s}\right\|_X \le \|f\|_{H_X(\mathbb{R}^n)}. \tag{4.20}$$

(ii) *Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $p_N(\Phi) \le 1$, where N is as in Theorem 3.1 and p_N as in (2.11). Then there exists a positive constant C, independent of f, such that, for all $j \in \mathbb{Z}$,*

$$|\langle b_j, \Phi\rangle| \le C\|f\|_{H_X(\mathbb{R}^n)}. \tag{4.21}$$

(iii) *In the topology of $\mathcal{S}'(\mathbb{R}^n)$, $g_j \to 0$ as $j \to -\infty$ and $b_j \to 0$ as $j \to \infty$. In particular,*

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j) \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Proof. We first prove (i). For any $j \in \mathbb{Z}$, let $\mathcal{O}_j$ be as in Lemma 2.23. By the fact that $\{Q_{j,k}\}_{k\in K_j}$ is the Whitney decomposition of $\mathcal{O}_j$, we find that

$$\left\|\left\{\sum_{k\in K_j} (2^j \chi_{Q_{j,k}})^s\right\}^{1/s}\right\|_X = \|2^j \chi_{\mathcal{O}_j}\|_X \le \|f\|_{H_X(\mathbb{R}^n)},$$

which shows (i).

Now we show (ii). From the definition of the grand maximal operator $\mathcal{M}_N$ in (2.13), it follows that, for any $j \in \mathbb{Z}$,

$$|\langle b_j, \Phi\rangle| \lesssim \inf_{x\in Q(\vec{0}_n,2)} \mathcal{M}_N(b_j)(x) \lesssim \|\mathcal{M}_N(b_j)\|_X \|\chi_{Q(\vec{0}_n,2)}\|_X^{-1}, \tag{4.22}$$

where the implicit positive constant is independent of Φ. Moreover, from $d \geq d_X$ and (2.29), we conclude that, for any $j \in \mathbb{Z}$, $k \in K_j$ and $x \in \mathbb{R}^n$,

$$\frac{\ell_{j,k}^{n+d+1}}{(\ell_{j,k} + |x - x_{j,k}|)^{n+d+1}} \leq \left(\frac{\ell_{j,k}}{\ell_{j,k} + |x - x_{j,k}|} \right)^{n/\theta} \lesssim M^{(\theta)}(\chi_{Q_{j,k}})(x), \tag{4.23}$$

where, for each j and k, $\ell_{j,k}$ and $x_{j,k}$ denote, respectively, the side-length and the center of $Q_{j,k}$; together with (2.35) and (4.22), this implies that, for any $j \in \mathbb{Z}$,

$$|\langle b_j, \Phi\rangle| \lesssim \|\mathcal{M}_N(f)\|_X + \left\| \sum_{k \in K_j} 2^j M^{(\theta)}(\chi_{Q_{j,k}}) \right\|_X.$$

From this and $s \in (0, 1]$, we deduce that, for any $j \in \mathbb{Z}$,

$$|\langle b_j, \Phi\rangle| \lesssim \|\mathcal{M}_N(f)\|_X + \left\| \left\{ \sum_{k \in K_j} [2^j M^{(\theta)}(\chi_{Q_{j,k}})]^s \right\}^{1/s} \right\|_X,$$

which, combined with (2.8) and (4.20), implies that, for any $j \in \mathbb{Z}$,

$$|\langle b_j, \Phi\rangle| \lesssim \|\mathcal{M}_N(f)\|_X + \left\| \left\{ \sum_{k \in K_j} (2^j \chi_{Q_{j,k}})^s \right\}^{1/s} \right\|_X \lesssim \|f\|_{H_X(\mathbb{R}^n)}.$$

This shows (ii).

Finally, we prove (iii). It suffices to prove that $b_j \to 0$ as $j \to \infty$ and $g_j \to 0$ as $j \to -\infty$. We concentrate on the former since the latter is similar.

Fix $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\chi_{Q(\vec{0}_n, 2)} \leq \psi \leq \chi_{Q(\vec{0}_n, 4)}$. For any $l \in \mathbb{N}$, let

$$\psi_{(l)}(\cdot) := \psi(2^{-l} \cdot) - \psi(2^{-l+1} \cdot),$$

and hence $\mathrm{supp}(\psi_{(l)}) \subset \overline{Q(\vec{0}_n, 2^{l+2}) \setminus Q(\vec{0}_n, 2^l)}$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. To prove that $b_j \to 0$ as $j \to \infty$, it suffices to show that

$$\langle b_j, \varphi\rangle = \langle b_j, \psi\varphi\rangle + \sum_{l=1}^{\infty} \langle b_j, \psi_{(l)}\varphi\rangle \to 0 \tag{4.24}$$

as $j \to \infty$. By $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and the definition of $\psi_{(l)}$, we find that, for all $l \in \mathbb{N}$,

$$p_N(\psi_{(l)}\varphi) \lesssim 2^{-l}. \tag{4.25}$$

Via (4.21) and (4.25), to prove (4.24), we only need to show that, for any given $l \in \mathbb{N}$,

$$\lim_{j \to \infty} \langle b_j, \psi(2^{-l} \cdot)\varphi\rangle = 0. \tag{4.26}$$

For any $j \in \mathbb{Z}$ and $l \in \mathbb{N}$, let $K_j(l) := \{k \in K_j : \mathrm{supp}(b_{j,k}) \cap Q(\vec{0}_n, 2^{l+2}) \neq \emptyset\}$, where K_j is as in Lemma 2.23. Notice that

$$p_N(\psi(2^{-l} \cdot)\varphi) \lesssim p_N(\varphi)$$

with the implicit positive constant independent of l, which, together with (2.35) and

(4.23), further implies that, for all $x \in Q(\vec{0}_n, 2)$,

$$|\langle b_j, \psi(2^{-l}\cdot)\varphi\rangle| \lesssim \sum_{k\in K_j(l)} \mathcal{M}_N(b_{j,k})(x)$$
$$\lesssim \mathcal{M}_N(f)(x)\chi_{\mathcal{O}_j}(x) + \sum_{k\in K_j(l)} 2^j M^{(\theta)}(\chi_{Q_{j,k}})(x). \quad (4.27)$$

Assume that there exists $k \in K_j(l)$ such that $|Q_{j,k}| > 2^{l+1}$. Then, by (2.33) and the fact that $Q_{j,k} \cap Q(\vec{0}_n, 2^{l+2}) \neq \emptyset$, we know that $Q(\vec{0}_n, 2) \subset Q(\vec{0}_n, 2^{l+2}) \subset D\mathcal{O}_j$, where the positive constant D is as in (2.33), which further implies that, for any $x \in Q(\vec{0}_n, 2)$,

$$|\langle b_j, \psi(2^{-l}\cdot)\varphi\rangle| \lesssim \chi_{D\mathcal{O}_j}(x)\mathcal{M}_N(f)(x).$$

From this and (4.27), it follows that, for any $x \in Q(\vec{0}_n, 2)$,

$$|\langle b_j, \psi(2^{-l}\cdot)\varphi\rangle| \lesssim \chi_{D\mathcal{O}_j}(x)\mathcal{M}_N(f)(x) + \sum_{k\in K_j(l),\, |Q_{j,k}|\le 2^{l+1}} 2^j M^{(\theta)}(\chi_{Q_{j,k}})(x). \quad (4.28)$$

Moreover, when there does not exist $k \in K_j(l)$ such that $|Q_{j,k}| > 2^{l+1}$, by (4.27), we conclude that (4.28) is trivial.

Let $\varepsilon \in (0, \infty)$. By (4.28), the Fefferman–Stein vector-valued maximal inequality for $L^{(\theta+\varepsilon)/\theta}(\mathbb{R}^n)$ (see [37, Theorem 1]) and (2.32), we know that

$$\begin{aligned}
&|\langle b_j, \psi(2^{-l}\cdot)\varphi\rangle| \\
&\quad\lesssim \|\chi_{D\mathcal{O}_j\cap Q(\vec{0}_n,2^{l+2})}\mathcal{M}_N(f)\|_{L^{\theta+\varepsilon}(\mathbb{R}^n)} \\
&\qquad + \Bigg\| \sum_{k\in K_j(l),\, |Q_{j,k}|\le 2^{l+1}} 2^j [M\chi_{Q_{j,k}}]^{1/\theta} \Bigg\|_{L^{\theta+\varepsilon}(\mathbb{R}^n)} \\
&\quad\lesssim \|\chi_{D\mathcal{O}_j\cap Q(\vec{0}_n,2^{l+2})}\mathcal{M}_N(f)\|_{L^{\theta+\varepsilon}(\mathbb{R}^n)} + \Bigg\| \sum_{k\in K_j(l),\, |Q_{j,k}|\le 2^{l+1}} 2^j \chi_{Q_{j,k}} \Bigg\|_{L^{\theta+\varepsilon}(\mathbb{R}^n)} \\
&\quad\lesssim \|\chi_{D\mathcal{O}_j\cap Q(\vec{0}_n,2^{l+3})}\mathcal{M}_N(f)\|_{L^{\theta+\varepsilon}(\mathbb{R}^n)},
\end{aligned}$$

which, together with the fact that $\mathcal{O}_j \downarrow \emptyset$ as $j \to \infty$, further implies that (4.26) holds true. This finishes the proof of Lemma 4.2. ∎

Proposition 4.3. *Assume that* $f \in H_X(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$. *Then there exists a decomposition as in* (3.10) *and, for any* $v \in (0, 1]$,

$$\left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^v \chi_{Q_j} \right\}^{1/v} \right\|_X \lesssim_v \|f\|_{H_X(\mathbb{R}^n)},$$

where the implicit positive constant is independent of f, *but depends on* v.

Proof. For each $j \in \mathbb{Z}$, let $\mathcal{O}_j$ be as in (2.32). Then, by Lemma 2.23, for any $j \in \mathbb{Z}$,

$$f = g_j + b_j, \quad b_j := \sum_{k\in K_j} b_{j,k}, \quad b_{j,k} := (f - c_{j,k})\eta_{j,k},$$

where K_j, $c_{j,k}$ and $\eta_{j,k}$ are as in Lemma 2.23.

From Lemma 4.2(iii), it follows that

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j)$$

with the series converging in $\mathcal{S}'(\mathbb{R}^n)$. Let $\{Q_{j,k}\}_{j\in\mathbb{Z},\,k\in K_j}$ be as in Lemma 2.23. Then the same argument as in [121, pp. 108–109] shows that, for any $j \in \mathbb{Z}$,

$$f = \sum_{j\in\mathbb{Z},\,k\in K_j} A_{j,k} \quad \text{and} \quad g_{j+1} - g_j = \sum_{k\in K_j} A_{j,k} \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where, for any $j \in \mathbb{Z}$ and $k \in K_j$, $A_{j,k}$ is such that $\operatorname{supp}(A_{j,k}) \subset Q_{j,k}$, there exists a positive constant C_0 such that

$$\|A_{j,k}\|_{L^\infty(\mathbb{R}^n)} \le C_0 2^j,$$

and $\int_{\mathbb{R}^n} A_{j,k}(x)x^\alpha\,dx = 0$ for every $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le d \in [d_X,\infty)\cap\mathbb{Z}_+$. For any $j \in \mathbb{Z}$ and $k \in K_j$, let

$$a_{j,k} := \frac{A_{j,k}}{C_0 2^j \|\chi_{Q_{j,k}}\|_X} \quad \text{and} \quad \kappa_{j,k} := C_0 2^j \|\chi_{Q_{j,k}}\|_X.$$

Then it is easy to see that, for any $j \in \mathbb{Z}$ and $k \in K_j$, $a_{j,k}$ is an (X,∞,d)-atom, and $f = \sum_{j\in\mathbb{Z},\,k\in K_j} \kappa_{j,k} a_{j,k}$ in $\mathcal{S}'(\mathbb{R}^n)$.

For any $v \in (0,1]$, let

$$\alpha_v := \left\|\left\{\sum_{j=-\infty}^{\infty}\sum_{k\in K_j}\left(\frac{\kappa_{j,k}}{\|\chi_{Q_{j,k}}\|_X}\right)^v \chi_{Q_{j,k}}\right\}^{1/v}\right\|_X.$$

Then, from (2.36), it follows that

$$\begin{aligned}\alpha_v &= C_0\left\|\left\{\sum_{j=-\infty}^{\infty}\sum_{k\in K_j} 2^{jv}\chi_{Q_{j,k}}\right\}^{1/v}\right\|_X \\ &\lesssim \left\|\left\{\sum_{j=-\infty}^{\infty}(2^j\chi_{\mathcal{O}_j\setminus\mathcal{O}_{j+1}})^v\right\}^{1/v}\right\|_X \\ &\sim \left\|\sum_{j=-\infty}^{\infty} 2^j\chi_{\mathcal{O}_j\setminus\mathcal{O}_{j+1}}\right\|_X,\end{aligned}$$

which, combined with the fact that $2^j < \mathcal{M}f(x)$ for any $x \in \mathcal{O}_j$ with $j \in \mathbb{Z}$, further implies that

$$\alpha_v \lesssim \left\|\sum_{j=-\infty}^{\infty}\chi_{\mathcal{O}_j\setminus\mathcal{O}_{j+1}}\mathcal{M}_N(f)\right\|_X \sim \|\mathcal{M}_N(f)\|_X \sim \|f\|_{H_X(\mathbb{R}^n)}.$$

This finishes the proof of Proposition 4.3. ■

To deal with the general case, we need the following lemma.

LEMMA 4.4. *Assume that X is as in Theorem 3.7 and $f \in H_X(\mathbb{R}^n)$. For any $j \in \mathbb{Z}$, let g_j be as in Lemma 2.23 with $d \in [d_X,\infty)\cap\mathbb{Z}_+$, where d_X is as in (2.29). Then, for any $j \in \mathbb{Z}$, we have $g_j \in H_X(\mathbb{R}^n)$ and*

$$\|g_j\|_{H_X(\mathbb{R}^n)} \lesssim \|f\|_{H_X(\mathbb{R}^n)}$$

with the implicit positive constant independent of f and j.

Proof. Let $\{K_j\}_{j\in\mathbb{Z}}$ be as in Lemma 2.23. For any $j\in\mathbb{Z}$ and $k\in K_j$, let $Q_{j,k}$ be as in Lemma 2.23. Denote by $x_{j,k}$ and $\ell_{j,k}$ the center and the side-length of $Q_{j,k}$, respectively. Then, by (4.23) and (2.34), for any $j\in\mathbb{Z}$,

$$\begin{aligned}\|\mathcal{M}_N(g_j)\|_X &\lesssim \|\mathcal{M}_N(f)\|_X + \Big\|2^j\sum_{k\in K_j}[M(\chi_{Q_{j,k}})]^{1/\theta}\Big\|_X\\ &\lesssim \|\mathcal{M}_N(f)\|_X + \Big\|\Big\{\sum_{k\in K_j}[M^{(\theta)}(2^j\chi_{Q_{j,k}})]^s\Big\}^{1/s}\Big\|_X.\end{aligned}$$

From this, (2.8) and (2.33), it follows that, for any $j\in\mathbb{Z}$,

$$\begin{aligned}\|\mathcal{M}_N(g_j)\|_X &\lesssim \|\mathcal{M}_N(f)\|_X + \Big\|\Big\{\sum_{k\in K_j}(2^j\chi_{Q_{j,k}})^s\Big\}^{1/s}\Big\|_X\\ &\lesssim \|\mathcal{M}_N(f)\|_X + \|2^j\chi_{\mathcal{O}_j}\|_X \lesssim \|\mathcal{M}_N(f)\|_X, \end{aligned}\tag{4.29}$$

which implies that $g_j\in H_X(\mathbb{R}^n)$ and $\|g_j\|_{H_X(\mathbb{R}^n)}\lesssim\|f\|_{H_X(\mathbb{R}^n)}$. This finishes the proof of Lemma 4.4. ■

Following the idea from [9, Lemma 7], we now prove that, for any $j\in\mathbb{Z}$, the function g_j is locally integrable.

LEMMA 4.5. *Let X be as in Theorem* 3.7 *and $f\in H_X(\mathbb{R}^n)$. Assume that $\{g_j\}_{j\in\mathbb{Z}}$ is as in Lemma* 2.23. *Then $g_j\in L^1_{\rm loc}(\mathbb{R}^n)$ for any $j\in\mathbb{Z}$.*

Proof. Let $\{K_j\}_{j\in\mathbb{Z}}$ be as in Lemma 2.23. For any $j\in\mathbb{Z}$ and $k\in K_j$, let $Q_{j,k}:=Q(x_{j,k},\ell_{j,k})$ be as in Lemma 2.23. From Lemma 2.24, it follows that

$$\sum_{k\in K_j}\frac{\ell_{j,k}^{n+d+1}}{(\ell_{j,k}+|\cdot-x_{j,k}|)^{n+d+1}}\in L^1_{\rm loc}(\mathbb{R}^n),\tag{4.30}$$

where d is as in (2.34). Then, by (4.30), (2.34) and the fact that $\chi_{\mathcal{O}_j^c}\mathcal{M}_N(f)\le 2^j$, we conclude that $\mathcal{M}_N(g_j)\in L^1_{\rm loc}(\mathbb{R}^n)$, which, together with an argument similar to that in [9, proof of Lemma 7], implies that $g_j\in L^1_{\rm loc}(\mathbb{R}^n)$. This finishes the proof of Lemma 4.5. ■

Combining Proposition 4.3, Remark 3.10 and Lemmas 4.4 and 4.5, we have the following corollary, the details being omitted.

COROLLARY 4.6. *Let X, d and s be as in Theorem* 3.7. *Assume that $f\in H_X(\mathbb{R}^n)$ and $\{g_j\}_{j\in\mathbb{Z}}$ is as in Lemma* 2.23. *Then, for any $j\in\mathbb{Z}$, g_j admits a decomposition*

$$g_j=\sum_{Q\in\mathcal{D}}\lambda_{j,Q}a_{j,Q}\quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where, for any $Q\in\mathcal{D}$, $a_{j,Q}$ is an (X,∞,d)-atom supported on the dyadic cube Q and

$$\Lambda(\{\lambda_{j,Q}a_{j,Q}\}_{Q\in\mathcal{D}})\lesssim\|f\|_{H_X(\mathbb{R}^n)}\tag{4.31}$$

with the implicit positive constant independent of f and j.

Via Corollary 4.6, we finish the proof of Theorem 3.7. To this end, we need the following notation: For any $L \in \mathbb{Z}$, let

$$\mathcal{D}_L := \{Q \in \mathcal{D} : |Q| = 2^{-Ln}\}, \quad \mathcal{D}_{\geq L} := \bigcup_{l \geq L} \mathcal{D}_l, \quad \mathcal{D}_{\leq L} := \bigcup_{l \leq L} \mathcal{D}_l.$$

Moreover, for any $Q \in \mathcal{D}$, denote by c_Q and ℓ_Q its center and its side-length, respectively.

Proof of Theorem 3.7. Let $\{\lambda_{j,Q}\}_{j\in\mathbb{Z},\, Q\in\mathcal{D}}$ and $\{a_{j,Q}\}_{j\in\mathbb{Z},\, Q\in\mathcal{D}}$ be as in Corollary 4.6. Then, by (4.31), we know that, for each $Q \in \mathcal{D}$, $\{\lambda_{j,Q}\}_{j\in\mathbb{Z}}$ are bounded, which, combined with the fact that the dual of $L^1(\mathbb{R}^n)$ is $L^\infty(\mathbb{R}^n)$, implies that, for each $Q \in \mathcal{D}$, there exists a subsequence $\{j_k\}_{k=1}^\infty$ such that $\lambda_{j_k,Q} \to \lambda_Q$ and $a_{j_k,Q} \to a_Q$ as $k \to \infty$, where the latter convergence is in the weak-$*$ topology of $L^\infty(\mathbb{R}^n)$. Then, for each $Q \in \mathcal{D}$, a_Q is an (X, ∞, d)-atom. From Fatou's property of X (see, for example, [10, Chapter 1, Lemma 1.5(ii)]), it follows that

$$\Lambda(\{\lambda_Q a_Q\}_{Q\in\mathcal{D}}) \leq \liminf_{k\to\infty} \Lambda(\{\lambda_{j_k,Q} a_{j_k,Q}\}_{Q\in\mathcal{D}}),$$

which, together with (4.31), implies that

$$\Lambda(\{\lambda_Q a_Q\}_{Q\in\mathcal{D}}) \lesssim \|f\|_{H_X(\mathbb{R}^n)} < \infty. \tag{4.32}$$

Since $|a_Q| \leq \frac{1}{\|\chi_Q\|_X}\chi_Q$ almost everywhere and $a_Q \perp \mathcal{P}_d$, similarly to the proof of (4.15), we find that, for any $x \in \mathbb{R}^n$,

$$\mathcal{M}_N(a_Q)(x) \lesssim \frac{1}{\|\chi_Q\|_X} M^{(\theta)}(\chi_Q)(x).$$

Let

$$g := \sum_{Q\in\mathcal{D}} \lambda_Q a_Q \in H_X(\mathbb{R}^n).$$

Once we show that $f = g$ in $\mathcal{S}'(\mathbb{R}^n)$, by (4.32) we know that the conclusion of Theorem 3.7 holds true for any $f \in H_X(\mathbb{R}^n)$.

To prove that $f = g$ in $\mathcal{S}'(\mathbb{R}^n)$, let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then, from Lemma 4.2(iii) and Corollary 4.6, it follows that

$$\langle f, \varphi\rangle = \lim_{k\to\infty} \langle g_{j_k}, \varphi\rangle = \lim_{k\to\infty} \sum_{Q\in\mathcal{D}} \lambda_{j_k,Q}\langle a_{j_k,Q}, \varphi\rangle. \tag{4.33}$$

On the other hand,

$$\langle g, \varphi\rangle = \sum_{Q\in\mathcal{D}} \lambda_Q \langle a_Q, \varphi\rangle. \tag{4.34}$$

In what follows, we denote $\{\lambda_{j_k,Q}\}_{k=1}^\infty$ and $\{a_{j_k,Q}\}_{k=1}^\infty$ simply by $\{\lambda_{j,Q}\}_{j=1}^\infty$ and $\{a_{j,Q}\}_{j=1}^\infty$. Fix $j \in \mathbb{N}$ and $L \in \mathbb{Z}_+$. By Lemma 2.7 and since $|Q| = 2^{-nL} \leq 1$ for all $Q \in \mathcal{D}_L$, we know that, for all $Q \in \mathcal{D}_L$,

$$\sum_{Q\in\mathcal{D}_L} |\lambda_{j,Q}\langle a_{j,Q}, \varphi\rangle| \lesssim \sum_{Q\in\mathcal{D}_L} \frac{\lambda_{j,Q}\ell_Q^{n+d+1}}{(1+|c_Q|)^{n/[\theta+\varepsilon(X)]}\|\chi_Q\|_X},$$

which, combined with (2.21) and $s \in (0,1]$, implies that

$$
\begin{aligned}
&\sum_{Q\in\mathcal{D}_L} |\lambda_{j,Q}\langle a_{j,Q},\varphi\rangle| \\
&\qquad \lesssim 2^{-L\{n+d+1-n/[\theta+\varepsilon(X)]\}} \left\| \sum_{Q\in\mathcal{D}_L} \frac{\lambda_{j,Q}}{\|\chi_Q\|_X}\chi_Q \right\|_{K^{-n/[\theta+\varepsilon(X)]}_{\theta+\varepsilon(X),\,s}(\mathbb{R}^n)} \\
&\qquad \lesssim 2^{-L\{n+d+1-n/[\theta+\varepsilon(X)]\}} \left\| \left\{ \sum_{Q\in\mathcal{D}_L} \left(\frac{\lambda_{j,Q}}{\|\chi_Q\|_X}\right)^s \chi_Q \right\}^{1/s} \right\|_{K^{-n/[\theta+\varepsilon(X)]}_{\theta+\varepsilon(X),s}(\mathbb{R}^n)}.
\end{aligned}
$$

From this, Lemma 2.14, (4.31) and the fact that

$$
n+d+1-\frac{n}{\theta+\varepsilon(X)} > n+d+1-\frac{n}{\theta} > 0,
$$

it follows that, for any $L \in \mathbb{Z}_+$ and $j \in \mathbb{N}$,

$$
\begin{aligned}
\sum_{Q\in\mathcal{D}_{\ge L}} |\lambda_{j,Q}\langle a_{j,Q},\varphi\rangle| &\lesssim \sum_{k=L}^{\infty} 2^{-k\{n+d+1-n/[\theta+\varepsilon(X)]\}} \|f\|_{H_X(\mathbb{R}^n)} \\
&\lesssim 2^{-L\{n+d+1-n/[\theta+\varepsilon(X)]\}} \|f\|_{H_X(\mathbb{R}^n)}. \qquad (4.35)
\end{aligned}
$$

Moreover, (2.26) implies that, for any $L, j \in \mathbb{N}$,

$$
\begin{aligned}
&\sum_{Q\in\mathcal{D}_{\le -L}} \chi_{5Q}(\vec{0}_n)|\lambda_{j,Q}\langle a_{j,Q},\varphi\rangle| \\
&\qquad\qquad \lesssim \sum_{Q\in\mathcal{D}_{\le -L}} \chi_{5Q}(\vec{0}_n)\frac{\lambda_{j,Q}}{\|\chi_Q\|_X} \\
&\qquad\qquad \lesssim \frac{1}{\sqrt[s]{L}} \left\| \sum_{Q\in\mathcal{D}_{\le -L}} \chi_{5Q}(\vec{0}_n)\frac{\lambda_{j,Q}}{\|\chi_Q\|_X}\chi_{[0,2^L]^n} \right\|_{K^{-n/[\theta+\varepsilon(X)]}_{\theta+\varepsilon(X),\,s}(\mathbb{R}^n)},
\end{aligned}
$$

which, together with Lemma 2.14, implies that

$$
\sum_{Q\in\mathcal{D}_{\le -L}} \chi_{5Q}(\vec{0}_n)|\lambda_{j,Q}\langle a_{j,Q},\varphi\rangle| \lesssim \frac{1}{\sqrt[s]{L}} \left\| \sum_{Q\in\mathcal{D}_{\le -L}} \chi_{5Q}(\vec{0}_n)\frac{\lambda_{j,Q}}{\|\chi_Q\|_X}\chi_{[0,2^L]^n} \right\|_X.
$$

From this, (2.8) and (4.31), we conclude that, for any $L, j \in \mathbb{N}$,

$$
\begin{aligned}
&\sum_{Q\in\mathcal{D}_{\le -L}} \chi_{5Q}(\vec{0}_n)|\lambda_{j,Q}\langle a_{j,Q},\varphi\rangle| \\
&\qquad\qquad \lesssim \frac{1}{\sqrt[s]{L}} \left\| \left\{ \sum_{Q\in\mathcal{D}_{\le -L}} \chi_{5Q}(\vec{0}_n)\left[\frac{\lambda_{j,Q}}{\|\chi_Q\|_X}M^{(\theta)}(\chi_Q)\right]^s \right\}^{1/s} \right\|_X \\
&\qquad\qquad \lesssim \frac{1}{\sqrt[s]{L}} \left\| \left\{ \sum_{Q\in\mathcal{D}_{\le -L}} \chi_{5Q}(\vec{0}_n)\left(\frac{\lambda_{j,Q}}{\|\chi_Q\|_X}\right)^s \chi_Q \right\}^{1/s} \right\|_X \\
&\qquad\qquad \lesssim \frac{1}{\sqrt[s]{L}} \|f\|_{H_X(\mathbb{R}^n)}.
\end{aligned}
$$

Thus, for any $L, j \in \mathbb{N}$,

$$
\sum_{Q\in\mathcal{D}_{\le -L}} \chi_{5Q}(\vec{0}_n)|\lambda_{j,Q}\langle a_{j,Q},\varphi\rangle| \lesssim \frac{1}{\sqrt[s]{L}} \|f\|_{H_X(\mathbb{R}^n)}. \qquad (4.36)
$$

Moreover, from $\varphi\in\mathcal{S}(\mathbb{R}^n)$, we deduce that, for any $L,j\in\mathbb{N}$,

$$\begin{aligned}\sum_{Q\in\mathcal{D}_{\le -L}}&\chi_{\mathbb{R}^n\setminus 5Q}(\vec{0}_n)|\lambda_{j,Q}\langle a_{j,Q},\varphi\rangle|\\ &\lesssim\sum_{Q\in\mathcal{D}_{\le -L}}\frac{\lambda_{j,Q}}{\|\chi_Q\|_X}\chi_{\mathbb{R}^n\setminus 5Q}(\vec{0}_n)\int_Q(1+|x|)^{-(2n+d+1)}\,dx\\ &\lesssim\sum_{Q\in\mathcal{D}_{\le -L}}\frac{\lambda_{j,Q}\ell_Q^{n+d+1}}{(1+|c_Q|)^{2n+d+1}\|\chi_Q\|_X}.\end{aligned}$$

Using this estimate, similarly to the proof of (4.36), we find that, for any $L,j\in\mathbb{N}$,

$$\begin{aligned}\sum_{Q\in\mathcal{D}_{\le -L}}&\chi_{\mathbb{R}^n\setminus 5Q}(\vec{0}_n)|\lambda_{j,Q}\langle a_{j,Q},\varphi\rangle|\\ &\lesssim 2^{-Ln}\left\|\sum_{Q\in\mathcal{D}_{\le -L}}\frac{\lambda_{j,Q}\ell_Q^{n+d+1}}{(1+|c_Q|)^{n+d+1}\|\chi_Q\|_X}\chi_{[0,2]^n}\right\|_{K_{\theta+\varepsilon(X),\,s}^{-n/[\theta+\varepsilon(X)]}(\mathbb{R}^n)}\\ &\lesssim 2^{-Ln}\left\|\sum_{Q\in\mathcal{D}_{\le -L}}\frac{\lambda_{j,Q}}{\|\chi_Q\|_X}M^{(\theta)}(\chi_Q)\right\|_X\\ &\lesssim 2^{-Ln}\left\|\left\{\sum_{Q\in\mathcal{D}_{\le -L}}\left[\frac{\lambda_{j,Q}}{\|\chi_Q\|_X}M^{(\theta)}(\chi_Q)\right]^s\right\}^{1/s}\right\|_X\\ &\lesssim 2^{-Ln}\left\|\left\{\sum_{Q\in\mathcal{D}_{\le -L}}\left(\frac{\lambda_{j,Q}}{\|\chi_Q\|_X}\right)^s\chi_Q\right\}^{1/s}\right\|_X\\ &\lesssim 2^{-Ln}\|f\|_{H_X(\mathbb{R}^n)}.\end{aligned}\tag{4.37}$$

Moreover, from Fatou's lemma, we deduce that (4.35) through (4.37) hold true for λ_Q and a_Q. Thus, by (4.35), (2.26), (4.36) and (4.37), we find that, for any $L,j\in\mathbb{N}$,

$$\sum_{Q\in(\mathcal{D}_{\ge L}\cup\mathcal{D}_{\le -L})}|\langle\lambda_{j,Q}a_{j,Q}-\lambda_Qa_Q,\varphi\rangle|\lesssim\frac{1}{\sqrt[s]{L}}\|f\|_{H_X(\mathbb{R}^n)}.\tag{4.38}$$

Moreover, from (4.35) with $L=0$, and (4.36) and (4.37) with $L=1$, we further deduce that, for any $j\in\mathbb{N}$,

$$\sum_{Q\in\mathcal{D}}|\langle\lambda_{j,Q}a_{j,Q}-\lambda_Qa_Q,\varphi\rangle|\lesssim\|f\|_{H_X(\mathbb{R}^n)}.\tag{4.39}$$

It then follows from (4.38) that, for any given $\epsilon\in(0,\infty)$, there exists $L_0\in\mathbb{N}$, depending on ϵ, such that, for any $j\in\mathbb{N}$,

$$\sum_{Q\in(\mathcal{D}_{\ge L_0}\cup\mathcal{D}_{\le -L_0})}|\langle\lambda_{j,Q}a_{j,Q}-\lambda_Qa_Q,\varphi\rangle|<\frac{\epsilon}{3}.\tag{4.40}$$

Moreover, by (4.39), for any $j\in\mathbb{N}$,

$$\sum_{k=-(L_0-1)}^{L_0-1}\sum_{Q\in\mathcal{D}_k}|\langle\lambda_{j,Q}a_{j,Q}-\lambda_Qa_Q,\varphi\rangle|\lesssim\|f\|_{H_X(\mathbb{R}^n)},$$

which further implies that, for any given $\epsilon \in (0,\infty)$ as in (4.40), there exists $R_0 \in (0,\infty)$, depending on ϵ, such that, for any $j \in \mathbb{N}$,

$$\sum_{k=-(L_0-1)}^{L_0-1} \sum_{Q\in\mathcal{D}_k,\, Q\not\subset Q(\vec{0}_n,R_0)} |\langle \lambda_{j,Q}a_{j,Q} - \lambda_Q a_Q, \varphi\rangle| < \frac{\epsilon}{3}. \tag{4.41}$$

Furthermore, since

$$\{Q \in \mathcal{D} : Q \in \mathcal{D}_k,\ -(L_0-1) \le k \le L_0 - 1,\ Q \subset Q(\vec{0}_n, R_0)\}$$

is a finite set and $\lambda_{j,Q} \to \lambda_Q$ and $a_{j,Q} \to a_Q$ as $j \to \infty$, we deduce that, for any $\epsilon \in (0,\infty)$ as in (4.40), there exists $j_0 \in \mathbb{N}$ such that, for any $j \in \mathbb{N}$ with $j \ge j_0$,

$$\sum_{k=-(L_0-1)}^{L_0-1} \sum_{Q\in\mathcal{D}_k,\, Q\subset Q(\vec{0}_n,R)} |\langle \lambda_{j,Q}a_{j,Q} - \lambda_Q a_Q, \varphi\rangle| < \frac{\epsilon}{3},$$

which, combined with (4.40) and (4.41), further implies that, for any $\epsilon \in (0,\infty)$ as in (4.40), there exists $j_0 \in \mathbb{N}$ such that, for any $j \in \mathbb{N}$ with $j \ge j_0$,

$$\sum_{Q\in\mathcal{D}} |\langle \lambda_{j,Q}a_{j,Q} - \lambda_Q a_Q, \varphi\rangle| < \epsilon,$$

and hence

$$\lim_{j\to\infty} \sum_{Q\in\mathcal{D}} |\langle \lambda_{j,Q}a_{j,Q} - \lambda_Q a_Q, \varphi\rangle| = 0.$$

From this, (4.33) and (4.34), we conclude that $f = g$ in $\mathcal{S}'(\mathbb{R}^n)$. This finishes the proof of Theorem 3.7. ∎

Proof of Corollary 3.11. Let $f \in H_X(\mathbb{R}^n)$, $\{b_j\}_{j\in\mathbb{Z}}$ and $\{\mathcal{O}_j\}_{j\in\mathbb{Z}}$ be as in Lemma 2.23. Then, via (2.8), (2.33) and (2.35), similarly to the proof of (4.29), we conclude that, for any $j \in \mathbb{Z}$,

$$\|\mathcal{M}_N(b_j)\|_X \lesssim \|\chi_{\mathcal{O}_j}\mathcal{M}_N(f)\|_X,$$

which, together with the absolute continuity of the quasi-norm of X, implies that $b_j \to 0$ in $H_X(\mathbb{R}^n)$ as $j \to \infty$. From this and the fact that, for any $j \in \mathbb{Z}$, $f = g_j + b_j$ in $\mathcal{S}'(\mathbb{R}^n)$, we deduce that $g_j \to f$ in $H_X(\mathbb{R}^n)$ as $j \to \infty$. Moreover, thanks to Lemma 4.5, we know that $g_j \in L^1_{\rm loc}(\mathbb{R}^n)$ for each $j \in \mathbb{Z}$. Thus, to prove Corollary 3.11, it suffices to show that $L^\infty(\mathbb{R}^n) \cap H_X(\mathbb{R}^n)$ is dense in $L^1_{\rm loc}(\mathbb{R}^n) \cap H_X(\mathbb{R}^n)$.

Let $g \in L^1_{\rm loc}(\mathbb{R}^n) \cap H_X(\mathbb{R}^n)$. Then, by Lemma 2.23, we know that there exist sequences $\{\widetilde{g}_j\}_{j\in\mathbb{Z}}$ and $\{\widetilde{b}_j\}_{j\in\mathbb{Z}}$ of functions such that, for any $j \in \mathbb{Z}$, $g = \widetilde{g}_j + \widetilde{b}_j$ in $\mathcal{S}'(\mathbb{R}^n)$, $\widetilde{g}_j \in L^\infty(\mathbb{R}^n) \cap H_X(\mathbb{R}^n)$ and $\widetilde{g}_j \to g$ in $H_X(\mathbb{R}^n)$ as $j \to \infty$. Thus, $L^\infty(\mathbb{R}^n) \cap H_X(\mathbb{R}^n)$ is dense in $L^1_{\rm loc}(\mathbb{R}^n) \cap H_X(\mathbb{R}^n)$. This finishes the proof of Corollary 3.11. ∎

Proof of Theorem 2.21. Theorem 2.21 is just a corollary of Theorems 3.4 and 3.7. ∎

Proof of Theorem 3.14. We will use Theorem 3.6. It suffices to show that all the assumptions of Theorem 3.6 hold true in this case. Namely, we only need to prove that (2.8) and (3.7) hold true.

Let $\rho\in(1,\infty)$ and define θ by $s=\theta\rho$. Assume that $\{f_j\}_{j\in\mathbb{N}}\subset\mathcal{M}$. Then

$$\begin{aligned}
&\left\|\left\{\sum_{j=1}^{\infty}[M^{(\theta)}(f_j)]^s\right\}^{1/s}\right\|_X^s\\
&\qquad=\left\|\sum_{j=1}^{\infty}[M(|f_j|^{\theta})]^{\rho}\right\|_Y\\
&\qquad=\sup\left\{\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}[M(|f_j|^{\theta})(x)]^{\rho}\right)g(x)\,dx:g\in Y',\ \|g\|_{Y'}=1\right\}. \qquad (4.42)
\end{aligned}$$

By [42, Chapter II, Theorem 2.12], for any $p\in(1,\infty)$ and non-negative measurable function ϕ,

$$\int_{\mathbb{R}^n}[Mf(x)]^p\phi(x)\,dx\lesssim\int_{\mathbb{R}^n}|f(x)|^pM\phi(x)\,dx,$$

which, together with (4.42), implies that

$$\begin{aligned}
&\left\|\left\{\sum_{j=1}^{\infty}[M^{(\theta)}(f_j)]^s\right\}^{1/s}\right\|_X^s\\
&\qquad\lesssim\sup\left\{\int_{\mathbb{R}^n}\left[\sum_{j=1}^{\infty}|f_j(x)|^s\right]Mg(x)\,dx:g\in Y',\ \|g\|_{Y'}=1\right\}\\
&\qquad\lesssim\left\|\sum_{j=1}^{\infty}|f_j|^s\right\|_Y\sup\{\|Mg\|_{Y'}:g\in Y',\ \|g\|_{Y'}=1\}\\
&\qquad\sim\left\|\left\{\sum_{j=1}^{\infty}|f_j|^s\right\}^{1/s}\right\|_X^s\sup\{\|Mg\|_{Y'}:g\in Y',\ \|g\|_{Y'}=1\}. \qquad (4.43)
\end{aligned}$$

Moreover, from $u_Y<\infty$, it follows that $\|Mg\|_{Y'}\lesssim\|g\|_{Y'}$ for any $g\in Y'$, which, combined with (4.43), implies that

$$\left\|\left\{\sum_{j=1}^{\infty}[M^{(\theta)}(f_j)]^s\right\}^{1/s}\right\|_X\lesssim\left\|\left\{\sum_{j=1}^{\infty}|f_j|^s\right\}^{1/s}\right\|_X.$$

Thus, (2.8) holds true in this case.

Furthermore, from $u_Y<\infty$ again, we conclude that, for any $v\in(u_Y,\infty)$, $M^{(v')}$ is bounded on Y', which implies that (3.7) holds true for any $q\in(\max\{1,su_Y\},\infty]$. This finishes the proof of Theorem 3.14. ■

Proof of Theorem 3.15. Let $f\in X$. By the assumption $l_X\in(1,\infty)$, we know that M is bounded on X, which, together with Theorem 3.4, implies that $f\in H_X(\mathbb{R}^n)$ with $\|f\|_{H_X(\mathbb{R}^n)}\sim\|f\|_X$. Moreover, an argument as in the proof of Theorem 3.14 indicates that (2.8) holds true. By monotonicity, we may assume that $L\geq d=d_X$. Then we obtain the desired result via Theorem 3.7. ■

4.4. Proofs of theorems related to tent spaces. In this subsection, we give the proofs of Theorems 3.19 and 3.21.

To show Theorem 3.19, we first introduce some notation. Let $f \in T_X(\mathbb{R}^{n+1}_+)$. For each $k \in \mathbb{Z}$, let

$$O_k := \{x \in \mathbb{R}^n : \mathcal{A}(f)(x) > 2^k\} \quad \text{and} \quad F_k := [\mathbb{R}^n \setminus O_k].$$

Moreover, for any given $\gamma \in (0,1)$ and $k \in \mathbb{Z}$, define

$$(O_k)^*_\gamma := \{x \in \mathbb{R}^n : M(\chi_{O_k})(x) > 1-\gamma\} \quad \text{and} \quad (F_k)^*_\gamma := [\mathbb{R}^n \setminus O^*_k]. \tag{4.44}$$

Since $\mathcal{A}(f)(x) < \infty$ for almost every $x \in \mathbb{R}^n$, it follows that $f \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$. Therefore, the set of all Lebesgue points of f is almost equal to $\mathbb{R}^{n+1}_+$ except for a Lebesgue measurable set of measure zero.

Lemma 4.7. *Let $f \in T_X(\mathbb{R}^{n+1}_+)$ and $\gamma \in (0,1)$. Then*

$$\mathrm{supp}(f) \subset \bigcup_{k\in\mathbb{Z}} \widehat{(O_k)^*_\gamma} \cup E, \tag{4.45}$$

where, for each $k \in \mathbb{Z}$, $(O_k)^*_\gamma$ is as in (4.44) and $E \subset \mathbb{R}^{n+1}_+$ satisfies $\int_E \frac{dy\,dt}{t} = 0$.

Proof. For each $k \in \mathbb{Z}$, denote $(O_k)^*_\gamma$ simply by O^*_k. Let $(x,t) \in \mathbb{R}^{n+1}_+$ be a Lebesgue point of f and $(x,t) \notin \bigcup_{k\in\mathbb{Z}} \widehat{O^*_k}$. Then, as $(x,t) \notin \bigcup_{k\in\mathbb{Z}} \widehat{O^*_k}$, there exists a sequence $\{y_k\}_{k\in\mathbb{Z}} \subset B(x,t)$ such that $y_k \notin O^*_k$ for each k, which, together with (4.44), implies that $M\chi_{O_k}(y_k) \le 1-\gamma$ for each $k \in \mathbb{Z}$. From this, we further deduce that $|B(x,t) \cap O_k| \le (1-\gamma)|B(x,t)|$, and hence

$$|B(x,t) \cap \{z \in \mathbb{R}^n : \mathcal{A}(f)(z) \le 2^k\}| \ge \gamma |B(x,t)|.$$

Letting $k \to -\infty$, we see that $|B(x,t) \cap \{z \in \mathbb{R}^n : \mathcal{A}(f)(z) = 0\}| \ge \gamma|B(x,t)|$. Therefore, since $\gamma \in (0,1)$, there exists $y \in B(x,t)$ such that $\mathcal{A}(f)(y) = 0$. From this and the definition of $\mathcal{A}(f)$, we find that $f = 0$ almost everywhere in $\Gamma(y)$, which, combined with Lebesgue's differentiation theorem, implies that $f(x,t) = 0$. Since almost every $(x,t) \in \mathbb{R}^{n+1}_+$ is a Lebesgue point of f, we deduce that (4.45) holds true, which completes the proof of Lemma 4.7. ∎

To prove Theorem 3.19, we also need the following estimate; see [24, Lemma 2] for its proof.

Lemma 4.8. *There exist positive constants $\gamma \in (0,1)$ and $C_{(\gamma)}$ such that, for any closed subset F of $\mathbb{R}^n$ whose complement has finite measure, and any non-negative measurable function H on $\mathbb{R}^{n+1}_+$,*

$$\int_{\mathcal{R}(F^*_\gamma)} H(y,t) t^n\,dy\,dt \le C_{(\gamma)} \int_F \left\{ \int_{\Gamma(x)} H(y,t)\,dy\,dt \right\} dx,$$

where $F^*_\gamma := [\{x \in \mathbb{R}^n : M(\chi_{F^\complement})(x) > 1-\gamma\}]^\complement$.

Proposition 4.9. *Let X be a ball quasi-Banach function space satisfying (2.8) for some $\theta, s \in (0,1]$, with $\theta < s$, and $f \in T_X(\mathbb{R}^{n+1}_+)$. Then there exist a sequence $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ and a sequence $\{a_j\}_{j=1}^\infty$ of (T_X,∞)-atoms such that, for almost every $(x,t) \in \mathbb{R}^{n+1}_+$,*

$$f(x,t) = \sum_{j=1}^\infty \lambda_j a_j(x,t), \qquad |f(x,t)| = \sum_{j=1}^\infty \lambda_j |a_j(x,t)|$$

and

$$\Lambda(\{\lambda_j a_j\}_{j=1}^\infty) \lesssim \|f\|_{T_X(\mathbb{R}^{n+1}_+)},$$

where, for each $j \in \mathbb{N}$, $\widehat{B_j}$ appears in the support of a_j and the implicit positive constant is independent of f.

Proof. Let $\gamma \in (0,1)$ be as in Lemma 4.8. For each $k \in \mathbb{Z}$, denote $(O_k)^*_\gamma$ simply by O^*_k. Since X satisfies (2.8), and by Lemma 2.14, we find that $1 \notin K^{-n/[\theta+\varepsilon(X)]}_{\theta+\varepsilon(X),\,s}(\mathbb{R}^n)$ and hence $1 \notin X$, which, together with $f \in T_X(\mathbb{R}^{n+1}_+)$, implies that, for each $j \in \mathbb{Z}$, O^*_k is a proper subset of $\mathbb{R}^n$. Since O^*_k is open, from the well-known Whitney decomposition theorem (see, for example, [120, p. 167]), it follows that there exist a set $\{I_k\}_{k\in\mathbb{Z}}$ of indices and a family $\{Q_{k,j}\}_{j\in I_k}$ of closed cubes with disjoint interiors such that

$$O^*_k = \bigcup_{j\in I_k} Q_{k,j}$$

and

$$\sqrt{n}\,\ell_{k,j} \le \operatorname{dist}(Q_{k,j}, (O^*_k)^\complement) \le 4\sqrt{n}\ell_{k,j},$$

where, for each k and j, $\ell_{k,j}$ denotes the side-length of $Q_{j,k}$ and

$$\operatorname{dist}(Q_{k,j}, (O^*_k)^\complement) := \inf\{|x-y| : x \in Q_{k,j},\, y \in (O^*_k)^\complement\}.$$

For any k and j, let $B_{k,j}$ be the ball with the same center as $Q_{k,j}$ and with radius $\frac{11\sqrt{n}}{2}\ell_{k,j}$. Moreover, for each k and j, let

$$A_{k,j} := \widehat{B_{k,j}} \cap [Q_{k,j} \times (0,\infty)] \cap \widehat{O^*_k} \cap (\widehat{O^*_{k+1}})^\complement,$$
$$a_{k,j} := 2^{-k}\|\chi_{B_{k,j}}\|_X^{-1} f\chi_{A_{k,j}}, \qquad \lambda_{k,j} := 2^k\|\chi_{B_{k,j}}\|_X.$$

Notice that

$$\{[Q_{k,j}\times(0,\infty)] \cap (\widehat{O^*_k} \setminus \widehat{O^*_{k+1}})\} \subset \widehat{B_{k,j}},$$

which, combined with (4.45), implies that

$$f = \sum_{k\in\mathbb{Z}}\sum_{j\in I_k} \lambda_{k,j}a_{k,j} \quad \text{and} \quad |f| = \sum_{k\in\mathbb{Z}}\sum_{j\in I_k} \lambda_{k,j}|a_{k,j}|$$

almost everywhere on $\mathbb{R}^{n+1}_+$.

We now show that, for each $k \in \mathbb{Z}$ and $j \in I_k$, $a_{k,j}$ is a (T_X,∞)-atom supported in $\widehat{B_{k,j}}$ up to a harmless constant multiple. Let $p \in (1,\infty)$ and $h \in T_2^{p'}(\mathbb{R}^{n+1}_+)$ with $\|h\|_{T_2^{p'}(\mathbb{R}^{n+1}_+)} \le 1$. Since $A_{k,j} \subset (\widehat{O^*_{k+1}})^\complement = F^*_{k+1}$, from Lemma 4.8 and Hölder's inequality, we deduce that

$$\begin{aligned}
|\langle a_{k,j}, h\rangle| &:= \left|\int_{\mathbb{R}^{n+1}_+} a_{k,j}(y,t)\chi_{A_{k,j}}(y,t)h(y,t)\,\frac{dy\,dt}{t}\right| \\
&\lesssim \int_{F_{k+1}}\int_{\Gamma(x)} |a_{k,j}(y,t)h(y,t)|\,\frac{dy\,dt}{t^{n+1}}\,dx \\
&\lesssim \int_{(O_{k+1})^\complement} \mathcal{A}(a_{k,j})(x)\mathcal{A}(h)(x)\,dx \\
&\lesssim 2^{-k}\|\chi_{B_{k,j}}\|_X^{-1}\left\{\int_{B_{k,j}\cap(O_{k+1})^\complement}[\mathcal{A}(f)(x)]^p\,dx\right\}^{1/p}\|h\|_{T_2^{p'}(\mathbb{R}^{n+1}_+)} \\
&\lesssim |B_{k,j}|^{1/p}\|\chi_{B_{k,j}}\|_X^{-1},
\end{aligned}$$

which, together with $(T_2^p(\mathbb{R}_+^{n+1}))^* = T_2^{p'}(\mathbb{R}_+^{n+1})$ (see [24, Theorem 2]), where $(T_2^p(\mathbb{R}_+^{n+1}))^*$ denotes the *dual space* of $T_2^p(\mathbb{R}_+^{n+1})$ and $1/p+1/p'=1$, implies that

$$\|a_{k,j}\|_{T_2^p(\mathbb{R}_+^{n+1})} \lesssim |B_{k,j}|^{1/p}\|\chi_{B_{k,j}}\|_X^{-1}.$$

Thus, $a_{k,j}$ is a (T_X,p)-atom supported in $\widehat{B_{k,j}}$ up to a constant multiple for all $p\in(1,\infty)$, and hence a (T_X,∞)-atom up to a constant multiple.

For each $k\in\mathbb{Z}$, we know that

$$\begin{aligned}\sum_{j\in I_k}(2^k\chi_{Q_{k,j}})^s &\lesssim 2^{ks}\chi_{O_k^*}\\ &\sim 2^{ks}\chi_{\{x\in\mathbb{R}^n:\,M(\chi_{O_k})(x)>1-\gamma\}}\\ &\sim 2^{ks}\chi_{\{x\in\mathbb{R}^n:\,M^{(\theta)}(\chi_{O_k})(x)>\sqrt[\theta]{1-\gamma}\}},\end{aligned}$$

which, combined with (2.8), further implies that

$$\begin{aligned}\left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{j\in I_k}(2^k\chi_{Q_{k,j}})^s\right\}^{1/s}\right\|_X &\lesssim \left\|\left[\sum_{k\in\mathbb{Z}}2^{ks}\chi_{\{x\in\mathbb{R}^n:\,M^{(\theta)}(\chi_{O_k})(x)>\sqrt[\theta]{1-\gamma}\}}\right]^{1/s}\right\|_X\\ &\lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}[2^kM^{(\theta)}(\chi_{O_k})]^s\right\}^{1/s}\right\|_X\\ &\lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}2^{ks}\chi_{O_k}\right\}^{1/s}\right\|_X.\end{aligned}$$

From this and (2.8), we further deduce that

$$\begin{aligned}\left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{j\in I_k}(2^k\chi_{B_{k,j}})^s\right\}^{1/s}\right\|_X &\lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{j\in I_k}[2^kM^{(\theta)}(\chi_{Q_{j,k}})]^s\right\}^{1/s}\right\|_X\\ &\lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{j\in I_k}(2^k\chi_{Q_{j,k}})^s\right\}^{1/s}\right\|_X\\ &\lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}2^{ks}\chi_{O_k}\right\}^{1/s}\right\|_X. \end{aligned}\tag{4.46}$$

Moreover, there exists a positive constant $\widetilde{C}$ such that, for all $m\in\mathbb{N}$,

$$\begin{aligned}\mathrm{H} &:= \left\|\left\{\sum_{k\in\mathbb{Z}}2^{ks}\chi_{O_k}\right\}^{1/s}\right\|_X\\ &\le \widetilde{C}2^{1/s}\left\|\left\{\sum_{k\in\mathbb{Z}}2^{ks}\chi_{O_k\setminus O_{k+m}}\right\}^{1/s}\right\|_X + \widetilde{C}2^{1/s}\left\|\left\{\sum_{k\in\mathbb{Z}}2^{ks}\chi_{O_{k+m}}\right\}^{1/s}\right\|_X\\ &= \widetilde{C}2^{1/s}\left\|\left\{\sum_{k\in\mathbb{Z}}2^{ks}\chi_{O_k\setminus O_{k+m}}\right\}^{1/s}\right\|_X + \widetilde{C}2^{-(m-1/s)}\mathrm{H}. \end{aligned}\tag{4.47}$$

Take $m_0\in\mathbb{N}$ such that $\widetilde{C}2^{-(m_0-1/s)}\le 1/2$, where $\widetilde{C}$ is as in (4.47). Then, by (4.47) again,

$$\mathrm{H} \lesssim \left\|\left\{\sum_{k\in\mathbb{Z}}2^{ks}\chi_{O_k\setminus O_{k+m_0}}\right\}^{1/s}\right\|_X \lesssim \|\mathcal{A}(f)\|_X, \tag{4.48}$$

which, together with (4.46) and the definition of $\lambda_{j,k}$, implies that

$$\Lambda(\{\lambda_{j,k}a_{j,k}\}_{k\in\mathbb{Z},\, j\in I_k}) \lesssim \|f\|_{T_X(\mathbb{R}^{n+1}_+)}.$$

This finishes the proof of Proposition 4.9. ■

Proof of Theorem 3.19. Assume first that there exist $\{\lambda_j\}_{j\in\mathbb{N}} \subset [0,\infty)$ and a sequence $\{a_j\}_{j\in\mathbb{N}}$ of (T_X,∞)-atoms such that (3.11) and (3.12) hold true. For any $N \in \mathbb{N}$ and $(x,t) \in \mathbb{R}^{n+1}_+$, let

$$f_N(x,t) := \sum_{j=1}^{N} \lambda_j a_j(x,t).$$

Then, from $s \in (0,1]$, Lemma 3.18 and Theorem 2.10, it follows that, for all $N \in \mathbb{N}$,

$$\begin{aligned}\|\mathcal{A}(f_N)\|_X &\le \Big\| \sum_{j=1}^{N} \lambda_j \mathcal{A}(a_j) \Big\|_X \le \Big\| \Big\{ \sum_{j=1}^{\infty} [\lambda_j \mathcal{A}(a_j)]^s \Big\}^{1/s} \Big\|_X \\ &\lesssim \Lambda(\{\lambda_j \alpha_j\}_{j\in\mathbb{N}}) < \infty. \end{aligned} \tag{4.49}$$

By the monotone convergence theorem, for almost every $x \in \mathbb{R}^n$,

$$\mathcal{A}(f)(x) \le \liminf_{N\to\infty} \mathcal{A}(f_N)(x),$$

which, combined with (4.49), implies that

$$\|\mathcal{A}(f)\|_X \le \liminf_{N\to\infty} \|\mathcal{A}(f_N)\|_X \lesssim \Lambda(\{\lambda_j \alpha_j\}_{j\in\mathbb{N}}).$$

Conversely, let $f \in T_X(\mathbb{R}^{n+1}_+)$. We obtain the desired conclusion via Proposition 4.9. This finishes the proof of Theorem 3.19. ■

Proof of Theorem 3.21. Let $f \in H_X(\mathbb{R}^n)$. Then, by Theorem 3.7, we know that

$$f = \sum_{Q\in\mathcal{D}} \lambda_Q a_Q \quad \text{in } H_X(\mathbb{R}^n), \tag{4.50}$$

where, for each $Q \in \mathcal{D}$, a_Q is an (X,∞,d)-atom supported on Q, $d \in [d_X,\infty) \cap \mathbb{Z}_+$ and $\{\lambda_Q\}_{Q\in\mathcal{D}} \subset [0,\infty)$ satisfies

$$\Lambda(\{\lambda_Q \alpha_Q\}_{Q\in\mathcal{D}}) \lesssim_s \|f\|_{H_X(\mathbb{R}^n)}, \tag{4.51}$$

where $s \in (0,1]$ is as in Theorem 3.7.

Take $\varphi,\psi \in \mathcal{S}(\mathbb{R}^n)$. Then, for any $t \in (0,\infty)$ and a finite set $\mathfrak{F} \subset \mathcal{D}$,

$$\begin{aligned}\int_{\mathbb{R}^n} \psi(tD)(f)(x)\varphi(x)\,dx &= \sum_{Q\in\mathfrak{F}} \lambda_Q \int_{\mathbb{R}^n} \psi(tD)(a_Q)(x)\varphi(x)\,dx \\ &\quad + \sum_{Q\in\mathcal{D}\setminus\mathfrak{F}} \lambda_Q \int_{\mathbb{R}^n} \psi(tD)(a_Q)(x)\varphi(x)\,dx. \end{aligned} \tag{4.52}$$

Notice that $a_Q \in L^2(\mathbb{R}^n)$ for each $Q \in \mathcal{D}$, which implies that a_Q vanishes weakly at infinity. Thus, for any given $\varepsilon \in (0,\infty)$ and a finite set $\mathfrak{F} \subset \mathcal{D}$, there exists $t_{\varepsilon,\mathfrak{F}} \in (0,\infty)$ such that, for any $t \in (0,t_{\varepsilon,\mathfrak{F}})$,

$$\Big| \sum_{Q\in\mathfrak{F}} \lambda_Q \int_{\mathbb{R}^n} \psi(tD)(a_Q)(x)\varphi(x)\,dx \Big| < \varepsilon. \tag{4.53}$$

Moreover, for any $Q \in \mathcal{D}$, using the fact that $|a_Q| \le \|\chi_Q\|_X^{-1}\chi_Q$ almost everywhere and $a_Q \perp \mathcal{P}_d$, similarly to the proof of (4.16), we conclude that, for any $t, t_1 \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$|e^{-t_1\Delta}\psi(tD)(a_Q)(x)| \lesssim \frac{1}{\|\chi_Q\|_X} M^{(\theta)}(\chi_Q)(x),$$

which, together with the fact that $H_X(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, Corollary 3.2, $s \in (0,1]$ and (2.8), implies that

$$\begin{aligned}
\left|\sum_{Q\in\mathcal{D}\setminus\mathfrak{F}} \lambda_Q \int_{\mathbb{R}^n} \psi(tD)(a_Q)(x)\varphi(x)\,dx\right| &\lesssim \left\|\sum_{Q\in\mathcal{D}\setminus\mathfrak{F}} \lambda_Q \psi(tD)(a_Q)\right\|_{H_X(\mathbb{R}^n)} \\
&\sim \left\|\sup_{t_1\in(0,\infty)}\left|\sum_{Q\in\mathcal{D}\setminus\mathfrak{F}} \lambda_Q e^{-t_1\Delta}\psi(tD)(a_Q)\right|\right\|_X \\
&\lesssim \left\|\sum_{Q\in\mathcal{D}\setminus\mathfrak{F}} \frac{\lambda_Q}{\|\chi_Q\|_X} M^{(\theta)}(\chi_Q)\right\|_X \\
&\lesssim \left\|\left\{\sum_{Q\in\mathcal{D}\setminus\mathfrak{F}} \left[\frac{\lambda_Q}{\|\chi_Q\|_X} M^{(\theta)}(\chi_Q)\right]^s\right\}^{1/s}\right\|_X \\
&\lesssim \left\|\left\{\sum_{Q\in\mathcal{D}\setminus\mathfrak{F}} \left(\frac{\lambda_Q}{\|\chi_Q\|_X}\right)^s \chi_Q\right\}^{1/s}\right\|_X.
\end{aligned}$$

From this and the fact that

$$\left\|\left\{\sum_{Q\in\mathcal{D}} \left(\frac{\lambda_Q}{\|\chi_Q\|_X}\right)^s \chi_Q\right\}^{1/s}\right\|_X < \infty,$$

we deduce that, for $\varepsilon \in (0,\infty)$ as in (4.53), there exists a finite set $\mathfrak{F} \subset \mathcal{D}$ such that, for any $t \in (0,\infty)$,

$$\left|\sum_{Q\in\mathcal{D}\setminus\mathfrak{F}} \lambda_Q \int_{\mathbb{R}^n} \psi(tD)(a_Q)(x)\varphi(x)\,dx\right| < \varepsilon,$$

which, combined with (4.52) and (4.53), further implies that f vanishes weakly at infinity.

For any $g \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$S(g)(x) := \left\{\int_{\Gamma(x)} |\varphi(tD)(g)(y)|^2 \,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}.$$

Now we prove that $S(f) \in X$ and $\|S(f)\|_X \lesssim \|f\|_{H_X(\mathbb{R}^n)}$. By (4.50), for any $x \in \mathbb{R}^n$,

$$S(f)(x) \le \sum_{Q\in\mathcal{D}} \lambda_Q S(a_Q)(x). \tag{4.54}$$

Let $q \in (1,\infty)$ be as in Theorem 2.10. Since S is bounded on $L^q(\mathbb{R}^n)$, for any $Q \in \mathcal{D}$ we have

$$\|\chi_{2Q} S(a_Q)\|_{L^q(\mathbb{R}^n)} \lesssim \|a_Q\|_{L^q(\mathbb{R}^n)} \lesssim |Q|^{1/q}\|\chi_Q\|_X^{-1},$$

which, together with Theorem 2.10, implies that

$$\Big\| \sum_{Q\in\mathcal{D}} \lambda_Q \chi_{2Q} S(a_Q) \Big\|_X \lesssim \Big\| \Big\{ \sum_{Q\in\mathcal{D}} \Big(\frac{\lambda_Q}{\|\chi_Q\|_X} \Big)^s \chi_Q \Big\}^{1/s} \Big\|_X. \tag{4.55}$$

Moreover, similarly to the proof of (4.15), for any $Q \in \mathcal{D}$ and $x \in \mathbb{R}^n$,

$$\chi_{\mathbb{R}^n\setminus 2Q}(x) S(a_Q)(x) \lesssim \frac{1}{\|\chi_Q\|_X} M^{(\theta)}(\chi_Q)(x),$$

which, combined with (2.8) and $s \in (0,1]$, implies that

$$\Big\| \sum_{Q\in\mathcal{D}} \lambda_Q \chi_{\mathbb{R}^n\setminus 2Q} S(a_Q) \Big\|_X \lesssim \Big\| \Big\{ \sum_{Q\in\mathcal{D}} \Big(\frac{\lambda_Q}{\|\chi_Q\|_X} \Big)^s \chi_Q \Big\}^{1/s} \Big\|_X.$$

From this, (4.54), (4.55) and (4.51), we know that

$$\|S(f)\|_X \le \Big\| \sum_{Q\in\mathcal{D}} \lambda_Q S(a_Q) \Big\|_X \lesssim \Big\| \Big\{ \sum_{Q\in\mathcal{D}} \Big(\frac{\lambda_Q}{\|\chi_Q\|_X} \Big)^s \chi_Q \Big\}^{1/s} \Big\|_X \lesssim \|f\|_{H_X(\mathbb{R}^n)}.$$

Therefore, $S(f) \in X$ and $\|S(f)\|_X \lesssim \|f\|_{H_X(\mathbb{R}^n)}$, which completes the proof of the necessity of Theorem 3.21.

Conversely, assume that $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishes weakly at infinity and $S(f) \in X$. For any $(x,t) \in \mathbb{R}^{n+1}_+$, let $F(x,t) := \varphi(tD)(f)(x)$. Then $F \in T_X(\mathbb{R}^{n+1}_+)$. Thanks to Theorem 3.19, we know that there exist a sequence $\{a_j\}_{j=1}^\infty$ of (T_X,∞)-atoms and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ such that, for almost every $(x,t) \in \mathbb{R}^{n+1}_+$,

$$\varphi(tD)(f)(x) = \sum_{j=1}^\infty \lambda_j a_j(x,t), \quad |\varphi(tD)(f)(x)| = \sum_{j=1}^\infty \lambda_j |a_j(x,t)| \tag{4.56}$$

and, for some $s \in (0,1]$,

$$\Big\| \Big\{ \sum_{j=1}^\infty \Big(\frac{\lambda_j}{\|\chi_{B_j}\|_X} \Big)^s \chi_{B_j} \Big\}^{1/s} \Big\|_X \lesssim_s \|F\|_{T_X(\mathbb{R}^{n+1}_+)} \sim \|S(f)\|_X, \tag{4.57}$$

where, for each $j \in \mathbb{N}$, $\widehat{B_j}$ appears in the support of a_j.

Moreover, since f vanishes weakly at infinity, there exists $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\vec{0}_n \notin \operatorname{supp}(\mathcal{F}\psi)$ such that

$$f = \int_0^\infty \psi(tD)\varphi(tD)(f) \, \frac{dt}{t}$$

in $\mathcal{S}'(\mathbb{R}^n)$ (see, for example, [44, Section 6.4]). For any $j \in \mathbb{N}$, let

$$A_j := \int_0^\infty \psi(tD)[a_j(\cdot,t)] \, \frac{dt}{t}.$$

Using the expression (4.56), we formally obtain

$$f = \sum_{j=1}^\infty \lambda_j A_j. \tag{4.58}$$

Let $q \in (1,\infty)$, $d \in [d_X,\infty) \cap \mathbb{Z}_+$ and $\tau \in (0,\infty)$ satisfy $\tau > n(1/\theta - 1/q)$. Then an argument similar to that in [64, proof of Lemma 4.8] shows that, for any $j \in \mathbb{N}$, A_j is an (X, q, d, τ)-molecule up to a harmless constant multiple. From this, (4.57), (4.58) and Theorem 3.9, we conclude that $f \in H_X(\mathbb{R}^n)$ and

$$\|f\|_{H_X(\mathbb{R}^n)} \lesssim \Lambda(\{\lambda_j A_j\}_{j\in\mathbb{N}}) \lesssim \|S(f)\|_X.$$

Finally, we establish (4.58). Let

$$g := \sum_{j=1}^{\infty} \lambda_j A_j.$$

Then, from (4.57), the fact that, for any $j \in \mathbb{N}$, A_j is an (X, q, d, τ)-molecule up to a constant multiple, and Theorem 3.9, we deduce that $g \in H_X(\mathbb{R}^n)$, and hence g vanishes weakly at infinity. Thus, to show (4.58), it suffices to prove that, for any $t_1 \in (0, \infty)$, $\varphi(t_1 D)(f) = \varphi(t_1 D)(g)$.

Fix $t_1 \in (0, \infty)$. By the fact that $\varphi(t_1 D)$ is bounded on $\mathcal{S}'(\mathbb{R}^n)$, we conclude that

$$\varphi(t_1 D)(f) = \int_0^\infty \varphi(t_1 D)\psi(tD)\varphi(tD)(f)\,\frac{dt}{t}. \tag{4.59}$$

Moreover, from the definition of $\varphi(tD)$ and a direct calculation, it follows that, for any $t \in (0, \infty)$,

$$\varphi(t_1 D)\psi(tD)\varphi(tD)(f) = \mathcal{F}^{-1}\big(\varphi(t_1\,\cdot)\psi(t\,\cdot)\varphi(t\,\cdot)\mathcal{F}(f)\big),$$

which, together with (4.59) and the fact that

$$\chi_{B(\vec{0}_n,4)\setminus B(\vec{0}_n,2)} \leq \varphi \leq \chi_{B(\vec{0}_n,8)\setminus B(\vec{0}_n,1)},$$

further implies that

$$\varphi(t_1 D)(f) = \int_{t_1/8}^{8t_1} \varphi(t_1 D)\psi(tD)\varphi(tD)(f)\,\frac{dt}{t}.$$

Furthermore, by the fact that f is a band-limited distribution, we find that there exists $N \in \mathbb{N}$ such that, for any $t \in [t_1/8, 8t_1]$ and $x \in \mathbb{R}^n$,

$$|\varphi(tD)(f)(x)| \lesssim (1 + |x|)^N,$$

which, combined with (4.56), implies that, for all $t \in [t_1/8, 8t_1]$ and $x \in \mathbb{R}^n$,

$$\sum_{j=1}^{\infty} \lambda_j |a_j(x,t)| = |\varphi(tD)(f)(x)| \lesssim (1 + |x|)^N. \tag{4.60}$$

Thanks to (4.60), (4.56) and Lebesgue's convergence theorem, we conclude that

$$\varphi(t_1 D)(f) = \sum_{j=1}^{\infty} \lambda_j \int_{t_1/8}^{8t_1} \varphi(t_1 D)\psi(tD)[a_j(\cdot, t)]\,\frac{dt}{t}.$$

From this and the arbitrariness of $t_1 \in (0, \infty)$, we deduce that (4.58) holds true. This finishes the proof of the sufficiency of Theorem 3.21, and hence the whole proof of Theorem 3.21. ∎

5. Local Hardy type spaces

In this section, we introduce the local Hardy type space $h_X(\mathbb{R}^n)$, associated with the ball quasi-Banach space X, and obtain several maximal function characterizations of it. Moreover, we also establish the relation between the local Hardy type space $h_X(\mathbb{R}^n)$ and the Hardy type space $H_X(\mathbb{R}^n)$.

5.1. Definition of local Hardy type spaces. So far, we have introduced the Hardy type space $H_X(\mathbb{R}^n)$ in Definition 2.12 and we have been investigating its properties. Keeping in mind Definition 2.12 and the works [45, 46], we now introduce the local Hardy type space $h_X(\mathbb{R}^n)$ via first giving the following local version of Definition 2.12.

Definition 5.1. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $N \in \mathbb{N}$, $a, b \in (0,\infty)$ and $\Phi \in \mathcal{S}(\mathbb{R}^n)$.

(i) The *local radial maximal function* $m(f,\Phi)$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$m(f,\Phi)(x) := \sup_{t\in(0,1)} |(\Phi_t * f)(x)|.$$

(ii) The *local grand maximal function* $m_N(f)$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$m_N(f)(x) := \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(y)| : t \in (0,1),\ |x-y| < t,\ \psi \in \mathcal{F}_N\}.$$

(iii) The *local non-tangential maximal function* $m_a^*(f,\Phi)$, with aperture $a \in (0,\infty)$, is defined by setting, for all $x \in \mathbb{R}^n$,

$$m_a^*(f,\Phi)(x) := \sup_{t\in(0,1)} \Big\{ \sup_{y\in\mathbb{R}^n,\, |y-x|<at} |\Phi_t * f(y)| \Big\}.$$

(iv) The *local maximal function* $m_b^{**}(f,\Phi)$ *of Peetre type* is defined by setting, for all $x \in \mathbb{R}^n$,

$$m_b^{**}(f,\Phi)(x) := \sup_{(y,t)\in\mathbb{R}^n\times(0,1)} \frac{|(\Phi_t * f)(x-y)|}{(1+t^{-1}|y|)^b}. \tag{5.1}$$

(v) The *local grand maximal function* $m_{b,N}^{**}(f)$ *of Peetre type* is defined by setting, for all $x \in \mathbb{R}^n$,

$$m_{b,N}^{**}(f)(x) := \sup_{\psi\in\mathcal{F}_N} \Big\{ \sup_{(y,t)\in\mathbb{R}^n\times(0,1)} \frac{|(\psi_t * f)(x-y)|}{(1+t^{-1}|y|)^b} \Big\}.$$

Definition 5.2. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \Phi(x)\,dx \neq 0$. Then the *local Hardy type space* $h_X(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $\|f\|_{h_X(\mathbb{R}^n)} := \|m_b^{**}(f,\Phi)\|_X < \infty$, where $m_b^{**}(f,\Phi)$ is as in (5.1) with b sufficiently large.

Then we have the following estimates, which further give several maximal function characterizations of the space $h_X(\mathbb{R}^n)$.

Theorem 5.3. *Let a, $b \in (0,\infty)$, X be a ball quasi-Banach function space and $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \Phi(x)\,dx \neq 0$.*

(i) *Let $N \geq \lfloor b+2 \rfloor$ be an integer. Then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\|m(f,\Phi)\|_X \lesssim \|m_a^*(f,\Phi)\|_X \lesssim \|m_b^{**}(f,\Phi)\|_X, \tag{5.2}$$

$$\|m(f,\Phi)\|_X \lesssim \|m_{\lfloor b+2 \rfloor}(f)\|_X \lesssim \|m_N(f)\|_X \lesssim \|m_b^{**}(f,\Phi)\|_X, \tag{5.3}$$

$$\|m_b^{**}(f,\Phi)\|_X \sim \|m_{b,\,N}^{**}(f)\|_X, \tag{5.4}$$

where the implicit positive constants are independent of f.

(ii) *Let $r, b, A \in (0,\infty)$ satisfy*

$$(b-A)r > n. \tag{5.5}$$

If X is strictly r-convex and, for all $f \in X$,

$$\left\| \left\{ \int_{[0,1]^n} |f(\cdot - z)|^r \, dz \right\}^{1/r} \right\|_X \lesssim (1+|z|)^A \|f\|_X, \tag{5.6}$$

where the implicit positive constant is independent of f, then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\|m_{b,\,N}^{**}(f)\|_X \lesssim \|m(f,\Phi)\|_X,$$

where the implicit positive constant is independent of f. In particular, when $N \in \mathbb{N} \cap [\lfloor b+2 \rfloor, \infty)$, and one of the quantities

$$\|m(f,\Phi)\|_X, \quad \|m_a^*(f,\Phi)\|_X, \quad \|m_N(f)\|_X, \quad \|m_b^{**}(f,\Phi)\|_X, \quad \|m_{b,\,N}^{**}(f)\|_X$$

is finite, then the other quantities are also finite and mutually equivalent with the implicit positive constants independent of f.

Remark 5.4. We refer the reader to [69, 90, 110] for the assumption (5.6).

Proof of Theorem 5.3. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We first prove (i). The proofs of (5.2) and (5.3) are similar to those of (3.1) and (3.2), the details being omitted here.

From (2.18) and (2.19), we deduce that, for all $x \in \mathbb{R}^n$,

$$m_{b,\,N}^{**}(f)(x) \lesssim m_b^{**}(f,\Phi)(x),$$

which, combined with the fact that $m_b^{**}(f,\Phi)(x) \lesssim m_{b,\,N}^{**}(f)(x)$ for all $x \in \mathbb{R}^n$, implies that (5.4) holds true. This finishes the proof of (i).

Now we prove (ii). Let $b, N \in (0,\infty)$ satisfy

$$Nr > n \quad \text{and} \quad (b-A)r > n. \tag{5.7}$$

Using Lemma 2.8 with a dilation, we conclude that, for any $m \in \mathbb{N}$, $t \in [1,2]$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} \sup_{y \in \mathbb{R}^n} \frac{|\Phi_{2^{-m}t} * f(y)|^r}{(1+2^m|x-y|)^{br}} &\lesssim_r \sum_{k=0}^{\infty} 2^{k(n-Nr)} \int_{\mathbb{R}^n} \frac{|\Phi_{2^{-k-m}t} * f(y)|^r}{(1+2^m|x-y|)^{br}} \, dy \\ &\lesssim_r \sum_{k=0}^{\infty} 2^{k(n-Nr)} \int_{\mathbb{R}^n} \frac{|\Phi_{2^{-k-m}t} * f(y)|^r}{(1+|x-y|)^{br}} \, dy, \end{aligned}$$

which, since $m \in \mathbb{N}$ and $t \in [1,2]$ are arbitrary, further implies that, for any $x \in \mathbb{R}^n$,

$$m_b^{**}(f,\Phi)(x) \lesssim_r \left\{\sum_{k=0}^{\infty} 2^{k(n-Nr)} \int_{\mathbb{R}^n} \frac{[m(f,\Phi)(x-y)]^r}{(1+|y|)^{br}}\,dy\right\}^{1/r}$$
$$\sim \left\{\sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m\in\mathbb{Z}^n} \int_{m+[0,1]^n} \frac{[m(f,\Phi)(x-y)]^r}{(1+|m|)^{br}}\,dy\right\}^{1/r}.$$

From this and the assumption that X is strictly r-convex, it follows that

$$\begin{aligned}\|m_b^{**}(f,\Phi)\|_X &\lesssim_r \left\|\left\{\sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m\in\mathbb{Z}^n} \int_{m+[0,1]^n} \frac{[m(f,\Phi)(\cdot-y)]^r}{(1+|m|)^{br}}\,dy\right\}^{1/r}\right\|_X \\ &\lesssim_r \left\|\sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m\in\mathbb{Z}^n} \int_{m+[0,1]^n} \frac{[m(f,\Phi)(\cdot-y)]^r}{(1+|m|)^{br}}\,dy\right\|_{X^{1/r}}^{1/r} \\ &\lesssim_r \left\{\sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m\in\mathbb{Z}^n} \left\|\int_{m+[0,1]^n} \frac{[m(f,\Phi)(\cdot-y)]^r}{(1+|m|)^{br}}\,dy\right\|_{X^{1/r}}\right\}^{1/r} \\ &\sim_r \left\{\sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m\in\mathbb{Z}^n} \left\|\int_{m+[0,1]^n} \frac{m(f,\Phi)(\cdot-y)}{(1+|m|)^{b}}\,dy\right\|_X^r\right\}^{1/r},\end{aligned}$$

which, combined with (5.5)–(5.7), further implies that

$$\begin{aligned}\|m_b^{**}(f,\Phi)\|_X &\lesssim \|m(f,\Phi)\|_X \left\{\sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^{(b-A)r}}\right\}^{1/r} \\ &\sim \|m(f,\Phi)\|_X.\end{aligned}$$

This finishes the proof of (ii), and hence of Theorem 5.3. ∎

5.2. Relation between local Hardy spaces and Hardy spaces. In this subsection, we establish the relation between the spaces $h_X(\mathbb{R}^n)$ and $H_X(\mathbb{R}^n)$.

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $b \in (0,\infty)$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$\psi_b^*(D)(f)(x) := \sup_{y\in\mathbb{R}^n} \frac{|\psi(D)(f)(y)|}{1+|x-y|^b},$$

where $\psi(D)$ is as in (3.13). Then we have the following equivalence.

LEMMA 5.5. *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\chi_{Q(\vec{0}_n,2)} \le \psi \le \chi_{Q(\vec{0}_n,4)}$. Then, for any $b \in (1,\infty)$ sufficiently large and $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\|f\|_{h_X(\mathbb{R}^n)} \sim \|\psi_b^*(D)(f)\|_X + \|(1-\psi(D))f\|_{H_X(\mathbb{R}^n)}, \tag{5.8}$$

where the implicit positive constants are independent of f.

Proof. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We first claim that, for any $x \in \mathbb{R}^n$,

$$m_b^{**}(f,\mathcal{F}^{-1}\psi)(x) \sim \psi_b^*(D)f(x) + M_b^{**}\big((1-\psi(D))f,\mathcal{F}^{-1}\psi\big)(x). \tag{5.9}$$

Once (5.9) holds true, we know that

$$\begin{aligned}\|m_b^{**}(f,\mathcal{F}^{-1}\psi)\|_X &\sim \|\psi_b^*(D)(f)+M_b^{**}((1-\psi(D))f,\mathcal{F}^{-1}\psi)\|_X\\ &\lesssim \|\psi_b^*(D)(f)\|_X+\|M_b^{**}((1-\psi(D))f,\mathcal{F}^{-1}\psi)\|_X\\ &\lesssim \|\psi_b^*(D)(f)+M_b^{**}((1-\psi(D))f,\mathcal{F}^{-1}\psi)\|_X\\ &\sim \|m_b^{**}(f,\mathcal{F}^{-1}\psi)\|_X,\end{aligned}$$

which implies that (5.8) holds true.

Now we prove (5.9). Via $f=\psi(D)f+(1-\psi(D))f$ and the fact that

$$m_b^{**}\big((1-\psi(D))f,\mathcal{F}^{-1}\psi\big)\le M_b^{**}\big((1-\psi(D))f,\mathcal{F}^{-1}\psi\big),$$

to prove $\lesssim$ in (5.9), it suffices to show that, for any $x\in\mathbb{R}^n$,

$$\sup_{t\in(0,1),\,y\in\mathbb{R}^n}\frac{|\psi(tD)\psi(D)(f)(y)|}{(1+t^{-1}|x-y|)^b}\lesssim \psi_b^*(D)(f)(x). \tag{5.10}$$

It is easy to see that, for any $t\in(0,1]$ and $x,y,z\in\mathbb{R}^n$,

$$\begin{aligned}\frac{|\psi(D)(f)(y-z)|}{(1+t^{-1}|z-y-x|)^b} &= \frac{t^b|\psi(D)(f)(y-z)|}{(t+|z-y-x|)^b}\\ &\le \frac{|\psi(D)(f)(y-z)|}{(1+|z-y-x|)^b},\end{aligned}$$

which implies that, for any $t\in(0,1)$ and $x,y\in\mathbb{R}^n$,

$$\begin{aligned}\frac{|\psi(tD)\psi(D)(f)(y)|}{(1+t^{-1}|x-y|)^b} &\le \int_{\mathbb{R}^n}|(\mathcal{F}^{-1}\psi)_{t^{-1}}(z)|\frac{|\psi(D)(f)(y-z)|}{(1+t^{-1}|x-y|)^b}\,dz\\ &\le \int_{\mathbb{R}^n}|(\mathcal{F}^{-1}\psi)_{t^{-1}}(z)|(1+t^{-1}|z|)^b\frac{|\psi(D)(f)(y-z)|}{(1+t^{-1}|z-y-x|)^b}\,dz\\ &\le \psi_b^*(D)(f)(x)\int_{\mathbb{R}^n}|t^{-n}\mathcal{F}^{-1}\psi(t^{-1}z)|(1+t^{-1}|z|)^b\,dz\\ &\sim \psi_b^*(D)(f)(x).\end{aligned}$$

From this, we conclude that (5.10) holds true. Thus, we obtain $\lesssim$ in (5.9).

Finally, we prove $\gtrsim$ in (5.9). To this end, it suffices to prove that, for any $x\in\mathbb{R}^n$,

$$\sup_{t\in(0,\infty),\,y\in\mathbb{R}^n}\frac{|\psi(tD)[(1-\psi(D))f](y)|}{(1+t^{-1}|x-y|)^b}\lesssim m_b^{**}(f,\mathcal{F}^{-1}\psi)(x). \tag{5.11}$$

From the assumption that $\chi_{Q(\vec{0}_n,2)}\le\psi\le\chi_{Q(\vec{0}_n,4)}$, we deduce that, for any $t\in(4,\infty)$ and $\xi\in\mathbb{R}^n$, $\psi(t\xi)[1-\psi(\xi)]=0$. Therefore, (5.11) is equivalent to the estimate, for any $x\in\mathbb{R}^n$,

$$\sup_{(y,t)\in\mathbb{R}^n\times(0,4)}\frac{|\psi(tD)[(1-\psi(D))f](y)|}{(1+t^{-1}|x-y|)^b}\lesssim m_b^{**}(f,\mathcal{F}^{-1}\psi)(x). \tag{5.12}$$

However, by (5.10), we find that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}\sup_{(y,t)\in\mathbb{R}^n\times(0,4)} \frac{|\psi(tD)[(1-\psi(D))f](y)|}{(1+t^{-1}|x-y|)^b} &\le \sup_{(y,t)\in\mathbb{R}^n\times(0,4)} \frac{|\psi(tD)\psi(D)(f)(y)|}{(1+t^{-1}|x-y|)^b} \\ &\quad + \sup_{(y,t)\in\mathbb{R}^n\times(0,4)} \frac{|\psi(tD)(f)(y)|}{(1+t^{-1}|x-y|)^b} \\ &\lesssim m_b^{**}(f,\mathcal{F}^{-1}\psi(4\cdot))(x) \lesssim m_b^{**}(f,\mathcal{F}^{-1}\psi)(x),\end{aligned}$$

which implies that (5.12) holds true. This finishes the proof of $\gtrsim$ in (5.9), and hence of Lemma 5.5. ∎

6. Hardy type spaces associated with operators

In this section, we study Hardy type spaces associated with operators. More precisely, we introduce the Hardy type space via the Lusin-area function associated with operators, and then establish its molecular and atomic characterizations.

6.1. Definition of Hardy type spaces associated with operators. We first recall some notation and terminology.

For $\theta \in [0, \pi)$, the *open* and *closed sectors*, S_θ^0 and S_θ, of angle θ in the complex plane $\mathbb{C}$ are defined, respectively, by setting

$$S_\theta^0 := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\} \quad \text{and} \quad S_\theta := \{z \in \mathbb{C} : |\arg z| \le \theta\}.$$

Let $\omega \in [0, \pi)$. A closed operator T in $L^2(\mathbb{R}^n)$ is said to be of *type* ω if

(i) the *spectrum* of T, $\sigma(T)$, is contained in S_ω;

(ii) for each $\theta \in (\omega, \pi)$, there exists a positive constant C such that, for all $z \in \mathbb{C} \setminus S_\theta$,

$$\|(T - zI)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C|z|^{-1};$$

here and hereafter, for any normed linear space $\mathcal{H}$, $\|S\|_{\mathcal{L}(\mathcal{H})}$ denotes the *operator norm* of the linear operator $S : \mathcal{H} \to \mathcal{H}$.

For $\mu \in (0, \pi)$ and $\sigma, \tau \in (0, \infty)$, let

$$\begin{aligned}H(S_\mu^0) &:= \{f : f \text{ is a holomorphic function on } S_\mu^0\}, \\ H_\infty(S_\mu^0) &:= \{f \in H(S_\mu^0) : \|f\|_{L^\infty(S_\mu^0)} < \infty\}\end{aligned}$$

and

$$\begin{aligned}\Psi_{\sigma,\tau}(S_\mu^0) := \big\{f \in H(S_\mu^0) : &\text{ there exists a positive constant } C \text{ such that,} \\ &\text{for all } \xi \in S_\mu^0,\ |f(\xi)| \le C \inf\{|\xi|^\sigma, |\xi|^{-\tau}\}\big\}.\end{aligned}$$

It is known that every one-to-one operator T of type ω in $L^2(\mathbb{R}^n)$ has a unique holomorphic functional calculus (see, for example, [98]). More precisely, let T be a one-to-one operator of type ω with $\omega \in [0, \pi)$, and let $\mu \in (\omega, \pi)$, $\sigma, \tau \in (0, \infty)$ and $f \in \Psi_{\sigma,\tau}(S_\mu^0)$. The operator $f(T)$ can be defined by the H_∞-functional calculus in the following way:

$$f(T) := \frac{1}{2\pi i} \int_\Gamma (\xi I - T)^{-1} f(\xi)\, d\xi, \tag{6.1}$$

where

$$\Gamma := \{re^{i\nu} : \infty > r > 0\} \cup \{re^{-i\nu} : 0 < r < \infty\},$$

with $\nu \in (\omega, \mu)$, is a curve consisting of two rays parameterized anti-clockwise. It is known that $f(T)$ in (6.1) is independent of the choice of $\nu \in (\omega, \mu)$ and the integral in (6.1) is absolutely convergent in $\|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$ (see [98]).

In what follows, we *always assume* $\omega \in [0, \pi/2)$. Then it follows from [47, Proposition 7.1.1] that, for every operator T of type ω in $L^2(\mathbb{R}^n)$, $-T$ generates a holomorphic C_0-semigroup $\{e^{-zT}\}_{z\in S^0_{\pi/2-\omega}}$ on the open sector $S^0_{\pi/2-\omega}$ such that $\|e^{-zT}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le 1$ for all $z \in S^0_{\pi/2-\omega}$, and moreover every non-negative self-adjoint operator is of type 0.

Let $\Psi(S^0_\mu) := \bigcup_{\sigma,\tau>0} \Psi_{\sigma,\tau}(S^0_\mu)$. It is well known that the above holomorphic functional calculus initially defined on $\Psi(S^0_\mu)$ can be extended to $H_\infty(S^0_\mu)$ via a limit process (see [98]). Recall that, for any $\mu \in (0, \pi)$, the operator T is said to have a *bounded* $H_\infty(S^0_\mu)$*-functional calculus* in the Hilbert space $\mathcal{H}$ if there exists a positive constant C such that $\|\psi(T)\|_{\mathcal{L}(\mathcal{H})} \le C\|\psi\|_{L^\infty(S^0_\mu)}$ for all $\psi \in H_\infty(S^0_\mu)$; and T is said to have a *bounded* H_∞*-functional calculus* in the Hilbert space $\mathcal{H}$ if there exists $\mu \in (0, \pi)$ such that T has a bounded $H_\infty(S^0_\mu)$-functional calculus.

For any given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, each ball $B \subset \mathbb{R}^n$ and $j \in \mathbb{Z}_+$, let

$$\fint_{S_j(B)} |f(x)|\,dx := \frac{1}{|2^jB|}\int_{S_j(B)} |f(x)|\,dx.$$

Now we recall the notion of L^p-L^q off-diagonal estimates on balls, which was first introduced in [8].

Definition 6.1. Let $m \in \mathbb{N}$, $p, q \in [1,\infty]$ with $p \le q$, and $\{A_t\}_{t>0}$ be a family of linear operators. Let $\Upsilon(s) := \max\{s, s^{-1}\}$ for any $s \in (0,\infty)$. The family $\{A_t\}_{t>0}$ is said to satisfy L^p-L^q *off-diagonal estimates on balls of order* m, written $\{A_t\}_{t>0} \in \mathcal{O}_m(L^p\text{-}L^q)$, if there exist constants $\theta_1, \theta_2 \in [0,\infty)$ and $C, c \in (0,\infty)$ such that, for all $t \in (0,\infty)$ and all balls $B := B(x_B, r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, and all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\left\{\fint_B |A_t(\chi_B f)(x)|^q\,dx\right\}^{1/q} \le C\left[\Upsilon\left(\frac{r_B}{t^{1/[2m]}}\right)\right]^{\theta_2}\left\{\fint_B |f(x)|^p\,dx\right\}^{1/p} \tag{6.2}$$

and, for all $j \in \mathbb{N}$ with $j \ge 2$,

$$\left\{\fint_{S_j(B)} |A_t(\chi_B f)(x)|^q\,dx\right\}^{1/q} \le C2^{j\theta_1}\left[\Upsilon\left(\frac{2^jr_B}{t^{1/[2m]}}\right)\right]^{\theta_2} e^{-c\frac{(2^jr_B)^{2m/(2m-1)}}{t^{1/(2m-1)}}}\left\{\fint_B |f(x)|^p\,dx\right\}^{1/p}$$

and

$$\left\{\fint_B |A_t(\chi_{S_j(B)} f)(x)|^q\,dx\right\}^{1/q} \le C2^{j\theta_1}\left[\Upsilon\left(\frac{2^jr_B}{t^{1/[2m]}}\right)\right]^{\theta_2} e^{-c\frac{(2^jr_B)^{2m/(2m-1)}}{t^{1/(2m-1)}}}\left\{\fint_{S_j(B)} |f(x)|^p\,dx\right\}^{1/p}.$$

Similarly to the comments after [8, Definition 2.1], we have the following properties of $\mathcal{O}_m(L^p\text{-}L^q)$.

Remark 6.2. Let $m \in \mathbb{N}$ and $p, q \in [1, \infty]$ with $p \le q$.

(i) From Hölder's inequality, we deduce that, for any $p \le p_1 \le q_1 \le q$,

$$\mathcal{O}_m(L^p\text{-}L^q) \subset \mathcal{O}_m(L^{p_1}\text{-}L^{q_1}).$$

(ii) Similarly to [8, Proposition 2.2], we conclude that $\{A_t\}_{t>0} \in \mathcal{O}_m(L^1\text{-}L^\infty)$ if and only if the associated kernel p_t of A_t satisfies the *Gaussian upper bound estimates*, namely, there exist positive constants C and c such that, for all $t \in (0, \infty)$ and $x, y \in \mathbb{R}^n$,

$$|p_t(x, y)| \le \frac{C}{t^{n/(2m)}} \exp\left\{-c\frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}}\right\}.$$

(iii) $\{A_t\}_{t>0} \in \mathcal{O}_m(L^p\text{-}L^q)$ if and only if $\{A_t^*\}_{t>0} \in \mathcal{O}_m(L^{q'}\text{-}L^{p'})$.

Now we postulate the following two assumptions on the operator L, which are used throughout this section.

Assumption 6.3. *L is a one-to-one linear operator of type ω in $L^2(\mathbb{R}^n)$ with $\omega \in [0, \pi/2)$, has dense range in $L^2(\mathbb{R}^n)$ and a bounded H_∞-calculus in $L^2(\mathbb{R}^n)$.*

Assumption 6.4. *Let $m \in \mathbb{N}$. There exist $p_L \in [1, 2)$ and $q_L \in (2, \infty]$, depending on L, such that the family $\{(tL)^k e^{-tL}\}_{t>0}$, with $k \in \mathbb{Z}_+$, satisfies the* reinforced (p_L, q_L, m) off-diagonal estimates on balls, *namely, $\{(tL)^k e^{-tL}\}_{t>0} \in \mathcal{O}_m(L^p\text{-}L^q)$ for all $p, q \in (p_L, q_L)$ with $p \le q$.*

Remark 6.5. Let p_L and q_L be as in Assumption 6.4 and $m, k \in \mathbb{N}$. Denote by L^* the adjoint operator of L in $L^2(\mathbb{R}^n)$.

(i) If $(tL)^k e^{-tL}$ satisfies the reinforced (p_L, q_L, m) off-diagonal estimates on balls, then $(tL^*)^k e^{-tL^*}$ also satisfies the reinforced (q_L', p_L', m) off-diagonal estimates on balls.

(ii) Here we present some examples of operators satisfying Assumptions 6.3 and 6.4:

(a) the *second-order divergence form elliptic operator* $\operatorname{div}(A\nabla)$ interpreted in the usual weak sense via a sesquilinear form, where A is an $n \times n$ matrix with entries $\{a_{j,k}\}_{j,k=1}^n \subset L^\infty(\mathbb{R}^n, \mathbb{C})$ satisfying the ellipticity conditions, namely, there exist constants $0 < \lambda_A \le \Lambda_A < \infty$ such that, for any $\xi, \zeta \in \mathbb{C}^n$ and almost every $x \in \mathbb{R}^n$,

$$\lambda_A|\xi|^2 \le \operatorname{Re}(\langle A(x)\xi, \xi\rangle), \qquad |\langle A(x)\xi, \zeta\rangle| \le \Lambda_A|\xi|\,|\zeta|$$

(see, for example, [62]);

(b) the $2m$*-order homogeneous divergence form elliptic operator*

$$(-1)^m \sum_{|\alpha|=m=|\beta|} \partial^\beta(a_{\alpha,\beta}\partial^\alpha)$$

interpreted in the usual weak sense via a sesquilinear form, with complex bounded measurable coefficients $a_{\alpha,\beta}$ for all multi-indices α and β with $|\alpha| = m = |\beta|$ (see, for example, [11, 18, 19]);

(c) the *Schrödinger operator* $-\Delta + V$ on $\mathbb{R}^n$ with the non-negative potential $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (see, for example, [61, 75]);
(d) the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$ with the suitable real potential V as in [4];
(e) the non-negative self-adjoint operators satisfying *Gaussian upper bounds estimates*, namely, there exist positive constants C and c such that, for all $t \in (0,\infty)$ and $x, y \in \mathbb{R}^n$,

$$|p_t(x,y)| \le \frac{C}{t^{n/(2m)}} \exp\left\{-c\frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}}\right\},$$

where p_t is the associated kernel of e^{-tL}.

Let L satisfy Assumptions 6.3 and 6.4, and $m \in \mathbb{N}$ be as in (6.2). For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *Lusin-area function* $S_L(f)$, associated with L, is defined by setting

$$S_L(f)(x) := \left\{\int_{\Gamma(x)} |t^{2m} L e^{-t^{2m}L}(f)(y)|^2 \, \frac{dy\,dt}{t^{n+1}}\right\}^{1/2},$$

where $\Gamma(x)$ for all $x \in \mathbb{R}^n$ is as in Definition 3.16. By [14, Theorem 2.13], we know that, for any $p \in (p_L, q_L)$, where p_L and q_L are as in Assumption 6.4, there exists a positive constant $C_{(p)}$, depending on p, such that, for all $f \in L^p(\mathbb{R}^n)$,

$$\|S_L(f)\|_{L^p(\mathbb{R}^n)} \le C_{(p)} \|f\|_{L^p(\mathbb{R}^n)}. \tag{6.3}$$

Let L^* denote the adjoint of L in $L^2(\mathbb{R}^n)$. For any $\ell \in \mathbb{N}$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *Lusin-area function* $S_{L^*,\ell}(f)$ is defined by setting

$$S_{L^*,\ell}(f)(x) := \left\{\int_0^\infty \int_{B(x,t)} |(t^{2m} L^*)^\ell e^{-t^{2m}L^*}(f)(y)|^2 \, \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}.$$

Then it follows from [14, Theorem 2.13] that $S_{L^*,\ell}$ is bounded on $L^q(\mathbb{R}^n)$ for any $\ell \in \mathbb{N}$ and $q \in (q_L', p_L')$.

Now we introduce the Hardy type space $H_{X,L}(\mathbb{R}^n)$ associated with L via the Lusin-area function of L.

Definition 6.6. Let X be a ball quasi-Banach space and the operator L satisfy Assumptions 6.3 and 6.4. Then the *Hardy type space* $H_{X,L}(\mathbb{R}^n)$, associated with L, is defined as the completion of the set

$$H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : \|S_L(f)\|_X < \infty\}$$

with respect to the quasi-norm $\|f\|_{H_{X,L}(\mathbb{R}^n)} := \|S_L(f)\|_X$.

We point out that, via the Lusin-area function characterization of $H_X(\mathbb{R}^n)$ and Remark 3.12, the space $H_{X,L}(\mathbb{R}^n)$ coincides with $H_X(\mathbb{R}^n)$ when $L := -\Delta$ under the additional assumption that X has an absolutely continuous quasi-norm.

6.2. Molecular characterization of $H_{X,L}(\mathbb{R}^n)$. In this subsection, we establish the molecular characterization of the Hardy type space $H_{X,L}(\mathbb{R}^n)$. We begin with the definition of molecules associated with the operator L as follows.

Definition 6.7. Let X be a ball quasi-Banach function space and L satisfy Assumptions 6.3 and 6.4, and let p_L and q_L be as in Assumption 6.4. Assume that $q \in (p_L, q_L)$, $M \in \mathbb{N}$ and $\epsilon \in (0, \infty)$. Denote by $\mathcal{R}(L^M)$ the range of L^M.

(i) A function $\alpha \in L^q(\mathbb{R}^n)$ is called an $(X, q, M, \epsilon)_L$-*molecule* associated with the ball $B := B(x_B, r_B) \subset \mathbb{R}^n$, with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, if $\alpha \in \mathcal{R}(L^M)$ and, for each $k \in \{0, \dots, M\}$ and $j \in \mathbb{Z}_+$,

$$\|(r_B^{-2m} L^{-1})^k \alpha\|_{L^q(S_j(B))} \le 2^{-j\epsilon} |2^j B|^{1/q} \|\chi_B\|_X^{-1}.$$

Moreover, if α is an $(X, q, M, \epsilon)_L$-molecule for all $q \in (p_L, q_L)$, then α is called an $(X, M, \epsilon)_L$-*molecule.*

(ii) For $f \in L^2(\mathbb{R}^n)$, the equality $f = \sum_{j=1}^{\infty} \lambda_j \alpha_j$ is called a *molecular* (X, q, M, ϵ)-*representation* of f if, for each $j \in \mathbb{N}$, α_j is an $(X, q, M, \epsilon)_L$-molecule associated with the ball $B_j \subset \mathbb{R}^n$, the summation converges in $L^2(\mathbb{R}^n)$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset [0, \infty)$ satisfies

$$\Lambda(\{\lambda_j \alpha_j\}_{j\in\mathbb{N}}) := \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{B_j}\|_X} \right)^s \chi_{B_j} \right\}^{1/s} \right\|_X < \infty, \tag{6.4}$$

where $s \in (0, 1]$ is as in (2.8). Let

$$\widetilde{H}_{X,L}^{M,q,\epsilon}(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : f \text{ has a molecular } (X, q, M, \epsilon)\text{-representation}\}$$

equipped with the quasi-norm $\|\cdot\|_{H_{X,L}^{M,q,\epsilon}(\mathbb{R}^n)}$ given by setting, for all $f \in \widetilde{H}_{X,L}^{M,q,\epsilon}(\mathbb{R}^n)$,

$$\|f\|_{H_{X,L}^{M,q,\epsilon}(\mathbb{R}^n)} := \inf \Big\{ \Lambda(\{\lambda_j \alpha_j\}_{j\in\mathbb{N}}) : f = \sum_{j=1}^{\infty} \lambda_j \alpha_j$$
$$\text{is a molecular } (X, q, M, \epsilon)\text{-representation} \Big\},$$

where $\Lambda(\{\lambda_j \alpha_j\}_{j\in\mathbb{N}})$ is as in (6.4) and the infimum is taken over all molecular (X, q, M, ϵ)-representations of f as above.

The *molecular Hardy type space* $H_{X,L}^{M,q,\epsilon}(\mathbb{R}^n)$ is then defined as the completion of $\widetilde{H}_{X,L}^{M,q,\epsilon}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H_{X,L}^{M,q,\epsilon}(\mathbb{R}^n)}$.

Let $T_X^c(\mathbb{R}_+^{n+1})$ and $T_2^{p,c}(\mathbb{R}_+^{n+1})$ with $p \in (0, \infty)$ denote the sets of all functions in $T_X(\mathbb{R}_+^{n+1})$, respectively, $T_2^p(\mathbb{R}_+^{n+1})$ with bounded support in $\mathbb{R}_+^{n+1}$. Here and hereafter, a function f on $\mathbb{R}_+^{n+1}$ is said to have *bounded support* in $\mathbb{R}_+^{n+1}$ if there exist a ball $B \subset \mathbb{R}^n$ and $T \in (0, \infty)$ such that $\operatorname{supp}(f) \subset B \times (0, T)$. Let $L_c^2(\mathbb{R}_+^{n+1})$ be the set of all functions $f \in L^2(\mathbb{R}_+^{n+1})$ with bounded support in $\mathbb{R}_+^{n+1}$, $M \in \mathbb{N}$ and $M > \frac{n}{2m}\left[\frac{\theta_1}{n} + \frac{1}{\theta}\right]$, where m, θ_1 and θ are as in Definition 6.1 and (2.8), respectively. For any $f \in L_c^2(\mathbb{R}_+^{n+1})$ and $x \in \mathbb{R}^n$, define

$$\pi_{L,M}(f)(x) := C_{(m,M)} \int_0^{\infty} (t^{2m} L)^{M+1} e^{-t^{2m} L}(f(\cdot, t))(x) \, \frac{dt}{t}, \tag{6.5}$$

where $C_{(m,M)}$ is a positive constant such that

$$C_{(m,M)} \int_0^{\infty} t^{2m(M+2)} e^{-2t^{2m}} \, \frac{dt}{t} = 1. \tag{6.6}$$

For the operator $\pi_{L,M}$, we have the following boundedness.

PROPOSITION 6.8. *Let L satisfy Assumptions* 6.3 *and* 6.4*, $\pi_{L,M}$ be as in* (6.5)*, and X a ball quasi-Banach function space satisfying* (2.8) *for some $\theta, s \in (0,1]$ with $\theta < s$, and* (2.9) *for some $q \in [2, q_L)$, where q_L is as in Assumption* 6.4*. Assume that X has an absolutely continuous quasi-norm. Then*

(i) *the operator $\pi_{L,M}$, initially defined on the space $T_2^{p,c}(\mathbb{R}^{n+1}_+)$ with $p \in (p_L, q_L)$, extends to a bounded linear operator from $T_2^p(\mathbb{R}^{n+1}_+)$ to $L^p(\mathbb{R}^n)$;*
(ii) *the operator $\pi_{L,M}$, initially defined on the space $T_X^c(\mathbb{R}^{n+1}_+)$, extends to a bounded linear operator from $T_X(\mathbb{R}^{n+1}_+)$ to $H_{X,L}(\mathbb{R}^n)$.*

We point out that we use Proposition 6.8(i) to show Proposition 6.8(ii). To show Proposition 6.8, we need the following assertion for tent spaces.

LEMMA 6.9. *Let $p \in (0,\infty)$ and X be a ball quasi-Banach function space satisfying* (2.8) *for some $\theta, s \in (0,1]$ with $\theta < s$. Then the following statements hold true.*

(i) *If $f \in T_2^p(\mathbb{R}^{n+1}_+)$, then the decomposition* (3.11) *holds true in $T_2^p(\mathbb{R}^{n+1}_+)$.*
(ii) *$T_X^c(\mathbb{R}^{n+1}_+) \subset T_2^{2,c}(\mathbb{R}^{n+1}_+)$ as sets.*

Proof. The proof of (i) is similar to that of [72, Proposition 3.1], the details being omitted here.

Now we show (ii). Let $f \in T_X^c(\mathbb{R}^{n+1}_+)$. Then there exists a ball $B_0 \subset \mathbb{R}^n$ such that $\mathrm{supp}(\mathcal{A}(f)) \subset B_0$. It is known that, for any $p \in (0,\infty)$,

$$T_2^{p,c}(\mathbb{R}^{n+1}_+) \subset T_2^{2,c}(\mathbb{R}^{n+1}_+)$$

as sets (see, for example, [24, p. 306, (1.3)]). Thus, to prove $T_X^c(\mathbb{R}^{n+1}_+) \subset T_2^{2,c}(\mathbb{R}^{n+1}_+)$, it suffices to show that $T_X^c(\mathbb{R}^{n+1}_+) \subset T_2^{p,c}(\mathbb{R}^{n+1}_+)$ for some $p \in (0,\infty)$. Let θ, s and $\varepsilon(X)$ be as in Lemma 2.14. From $\mathrm{supp}(\mathcal{A}(f)) \subset B_0$, it follows that there exists $N_0 \in \mathbb{N}$ such that $B_0 \subset Q(\vec{0}_n, 2^{N_0})$, which implies that

$$\begin{aligned}
&\|\mathcal{A}(f)\|_{K^{-n/[\theta+\varepsilon(X)]}_{\theta+\varepsilon(X),s}(\mathbb{R}^n)}\\
&\quad\geq \|\mathcal{A}(f)\|_{L^{\theta+\varepsilon(X)}(Q(\vec{0}_n,2))} + \left\{\sum_{j=1}^{N_0}[2^{-nj/\{\theta+\varepsilon(X)\}}\|\mathcal{A}(f)\|_{L^{\theta+\varepsilon(X)}(S_j(Q(\vec{0}_n,1)))}]^s\right\}^{1/s}\\
&\quad\sim \|\mathcal{A}(f)\|_{L^{\theta+\varepsilon(X)}(Q(\vec{0}_n,2))} + \sum_{j=1}^{N_0} 2^{-nj/\{\theta+\varepsilon(X)\}}\|\mathcal{A}(f)\|_{L^{\theta+\varepsilon(X)}(S_j(Q(\vec{0}_n,1)))}\\
&\quad\sim \|\mathcal{A}(f)\|_{L^{\theta+\varepsilon(X)}(\mathbb{R}^n)}.
\end{aligned}$$

From this and Lemma 2.14, we conclude that $\mathcal{A}(f) \in L^{\theta+\varepsilon(X)}(\mathbb{R}^n)$, which further implies that $f \in T_2^{\theta+\varepsilon(X),c}(\mathbb{R}^{n+1}_+)$. This finishes the proof of (ii), and hence of Lemma 6.9. ∎

Moreover, to show Proposition 6.8(ii), we also need the following conclusion.

LEMMA 6.10. *Let L satisfy Assumptions* 6.3 *and* 6.4*, $\pi_{L,M}$ be as in* (6.5) *and X as in Proposition* 6.8*. Then, for any (T_X,∞)-atom A and some $\epsilon \in (n/\theta,\infty)$, $\alpha := \pi_{L,M}(A)$ is an $(X,M,\epsilon)_L$-molecule, up to a harmless constant multiple, associated with the ball B, where $\widehat{B}$ appears in the support of A.*

Proof. We may assume that θ_1 in Definition 6.1 satisfies $\theta_1 > n$ by replacing θ_1 with $\max\{\theta_1, n+1\}$ if necessary.

Assume that A is a (T_X,∞)-atom associated with the ball $B := B(x_B, r_B)$ for some $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, and $q \in (p_L, q_L)$. Since, for $q \in (p_L, 2)$, each $(X, 2, M, \epsilon)_L$-molecule is also an $(X, q, M, \epsilon)_L$-molecule, to prove the above claim it suffices to show that, for any $q \in [2, q_L)$, $\alpha := \pi_{L,M}(A)$ is an $(X,\ q,\ M,\ \epsilon)_L$-molecule, up to a harmless constant multiple, associated with B.

Let $q \in [2, q_L)$. When $k \in \{0, \dots, 4\}$, by Proposition 6.8(i) we know that

$$\begin{aligned}\|\alpha\|_{L^q(S_k(B))} &= \|\pi_{L,M}(A)\|_{L^q(S_k(B))} \lesssim \|A\|_{T_2^q(\mathbb{R}^{n+1}_+)} \\ &\lesssim |B|^{1/q}\|\chi_B\|_X^{-1} \sim 2^{-\epsilon k}|2^k B|^{1/q}\|\chi_B\|_X^{-1}. \end{aligned} \tag{6.7}$$

When $k \in \mathbb{N}$ with $k \geq 5$, we dualize $\|\pi_{L,M}(A)\|_{L^q(S_k(B))}$; take $h \in L^{q'}(\mathbb{R}^n)$ satisfying $\|h\|_{L^{q'}(\mathbb{R}^n)} \leq 1$ and $\mathrm{supp}(h) \subset S_k(B)$. Then, from Hölder's inequality and $q' \in (q_L', 2]$, it follows that

$$\begin{aligned} |\langle \pi_{L,M}(A), h\rangle| &\leq \int_{\mathbb{R}^n}\int_0^\infty |A(x,t)(t^{2m}L^*)^{M+1}e^{-t^{2m}L^*}(h)(x)|\,\frac{dt}{t}\,dx \\ &= \int_{\mathbb{R}^n}\int_0^\infty |A(x,t)\chi_{\widehat{B}}(x,t)(t^{2m}L^*)^{M+1}e^{-t^{2m}L^*}(h)(x)|\,\frac{dt}{t}\,dx \\ &\leq \|\mathcal{A}(A)\|_{L^q(\mathbb{R}^n)}\left\|\mathcal{A}\big(\chi_{\widehat{B}}(t^{2m}L^*)^{M+1}e^{-t^{2m}L^*}(h)\big)\right\|_{L^{q'}(\mathbb{R}^n)} \\ &\lesssim \|A\|_{T_2^q(\mathbb{R}^{n+1}_+)}|B|^{1/q'-1/2} \\ &\quad\times\left\{\int_{\widehat{B}}|(t^{2m}L^*)^{M+1}e^{-t^{2m}L^*}(h)(x,t)|^2\,\frac{dx\,dt}{t}\right\}^{1/2}. \end{aligned} \tag{6.8}$$

Observe that

$$\Upsilon(v)\exp(-v^{\frac{2m}{2m-1}}) \lesssim v^{\epsilon-n+\theta_1}$$

for $v \geq 1$. Combining this inequality with the fact that $\theta_1 > n$, Assumption 6.4 and Remark 6.5(i), we find that

$$\begin{aligned} &\int_{\widehat{B}}|(t^{2m}L^*)^{M+1}e^{-t^{2m}L^*}(h)(x,t)|^2\,\frac{dx\,dt}{t} \\ &\qquad\lesssim \int_0^{r_B}\left\{2^{k\theta_1}\left[\Upsilon\left(\frac{2^k r_B}{t}\right)\right]^{\theta_2}|B|^{1/2}|2^kB|^{-1/q'}\right. \\ &\qquad\qquad\left.\times\exp\left[-\left(\frac{2^k r_B}{t}\right)^{2m/(2m-1)}\right]\right\}^2\frac{dt}{t} \\ &\qquad\lesssim 2^{2\theta_1 k}|B|\,|2^kB|^{-2/q'}\int_0^{r_B}\left[\frac{2^k r_B}{t}\right]^{-2(\epsilon-n+\theta_1)}\frac{dt}{t} \\ &\qquad\lesssim 2^{-2(\epsilon-n)k}|B|\,|2^kB|^{-2/q'}, \end{aligned}$$

which, together with (6.8), implies that, for any $k \in \mathbb{N}$ with $k \geq 5$,

$$|\langle \pi_{L,M}(A), h\rangle| \lesssim 2^{-\epsilon k}|2^kB|^{1/q}\|\chi_B\|_X^{-1}.$$

From this and the choice of h, we deduce that, for each $k \in \mathbb{N}$ with $k \geq 5$,

$$\|\alpha\|_{L^q(S_k(B))} = \|\pi_{L,M}(A)\|_{L^q(S_k(B))} \lesssim 2^{-\epsilon k}|2^k B|^{1/q}\|\chi_B\|_X^{-1}. \tag{6.9}$$

Moreover, let $j \in \{1, \dots, M\}$. When $k \in \{1, \dots, 4\}$, we dualize

$$\|(r_B^{-2m}L^{-1})^j\alpha\|_{L^q(S_k(B))};$$

take $h \in L^{q'}(\mathbb{R}^n)$ satisfying $\|h\|_{L^{q'}(\mathbb{R}^n)} \leq 1$ and $\mathrm{supp}(h) \subset S_k(B)$. A calculation shows that $\epsilon > n/\theta \geq n/q$. Since $0 < \theta \leq 1 \leq 2 \leq q$, and $S_{L^*,\ell}$ is bounded on $L^q(\mathbb{R}^n)$, from Hölder's inequality it follows that

$$\begin{aligned}
&|\langle (r_B^{-2m}L^{-1})^j\pi_{L,M}(A), h\rangle| \\
&\quad\lesssim \int_0^{r_B}\int_B \left(\frac{t}{r_B}\right)^{2jm} |A(x,t)|\,|(t^{2m}L^*)^{M+1-j}e^{-t^{2m}L^*}(h)(x)|\,\frac{dx\,dt}{t} \\
&\quad\lesssim \|\mathcal{A}(A)\|_{L^q(\mathbb{R}^n)}\|S_{L^*,\,M+1-j}(h)\|_{L^{q'}(\mathbb{R}^n)} \\
&\quad\lesssim \|A\|_{T_2^q(\mathbb{R}^{n+1}_+)} \lesssim |B|^{1/q}\|\chi_B\|_X^{-1} \lesssim 2^{-\epsilon k}|2^k B|^{1/q}\|\chi_B\|_X^{-1}.
\end{aligned}$$

From this, we conclude that, for each $j \in \{1, \dots, M\}$ and $k \in \{0, \dots, 4\}$,

$$\|(r_B^{-2m}L^{-1})^j\alpha\|_{L^q(S_k(B))} \lesssim 2^{-\epsilon k}|2^k B|^{1/q}\|\chi_B\|_X^{-1}. \tag{6.10}$$

When $k \in \mathbb{N}$ with $k \geq 5$, similarly to the proof of (6.9), we know that, for each j in $\{1, \dots, M\}$,

$$\|(r_B^{-2m}L^{-1})^j\alpha\|_{L^q(S_k(B))} \lesssim 2^{-\epsilon k}|2^k B|^{1/q}\|\chi_B\|_X^{-1},$$

which, together with (6.7), (6.9) and (6.10), implies that α is an $(X,\, q,\, M,\, \epsilon)_L$-molecule, up to a harmless constant multiple. This finishes the proof of Lemma 6.10. ∎

Now we prove Proposition 6.8 by using Lemmas 6.9 and 6.10.

Proof of Proposition 6.8. The proof of (i) is similar to that of [72, Proposition 4.1(i)], the details being omitted here.

Now we prove (ii). Let $f \in T_X^c(\mathbb{R}^{n+1}_+)$. Then, by Theorem 3.19, Lemma 6.9 and (i), we know that

$$\pi_{L,M}(f) = \sum_{k=1}^{\infty} \lambda_k \pi_{L,M}(A_k) =: \sum_{k=1}^{\infty} \lambda_k \alpha_k \quad \text{in } L^2(\mathbb{R}^n),$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{A_k\}_{k\in\mathbb{N}}$ satisfy (3.11) and

$$\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}) \lesssim \|f\|_{T_X(\mathbb{R}^{n+1}_+)}. \tag{6.11}$$

Recall that, for each $k \in \mathbb{N}$, $\mathrm{supp}(A_k) \subset \widehat{B}_k$ and B_k is a ball in $\mathbb{R}^n$. Moreover, from (6.3), we deduce that $S_L(\pi_{L,M}(f))(x) \leq \sum_{k=1}^{\infty} \lambda_k S_L(\alpha_k)(x)$ for almost every $x \in \mathbb{R}^n$, which, together with Definition 2.2(ii), implies that

$$\|S_L(\pi_{L,M}(f))\|_X \leq \left\|\sum_{k=1}^{\infty} \lambda_k S_L(\alpha_k)\right\|_X. \tag{6.12}$$

By Lemma 6.10, we know that, for some $\epsilon \in (n/\theta, \infty)$ and each $k \in \mathbb{N}$, $\alpha_k = \pi_{L,M}(A_k)$ is an $(X, M, \epsilon)_L$-molecule, up to a harmless constant multiple, associated with the ball B_k.

Moreover, let α be an (X,M,ϵ)-molecule associated with a ball $B:=B(x_B,r_B)$ for some $x_B\in\mathbb{R}^n$ and $r_B\in(0,\infty)$, where $\epsilon\in(n/\theta,\infty)$ and $q\in[2,p_L)$. Then, for any $x\in\mathbb{R}^n$, we have

$$\begin{aligned}S_L(\alpha)(x)&\le\sum_{j=0}^\infty\left\{\int_0^{r_B}\int_{B(x,t)}|t^{2m}Le^{-t^{2m}L}(\alpha\chi_{S_j(B)})(y)|^2\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\&\quad+\sum_{j=0}^\infty\left\{\int_{r_B}^\infty\int_{B(x,t)}|t^{2m}Le^{-t^{2m}L}(r_B^{2m}L)^M\right.\\&\qquad\left.\times(\chi_{S_j(B)}(r_B^{2m}L)^{-M}\alpha)(y)|^2\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\&=:\sum_{j=0}^\infty \mathrm{E}_j(x)+\sum_{j=0}^\infty \mathrm{F}_j(x).\end{aligned}\tag{6.13}$$

For any $i,j\in\mathbb{Z}_+$, let $B_j:=2^jB$ and

$$\widetilde{S}_i(B_j):=\{y\in\mathbb{R}^n:2^{i-3}2^jr_B\le|y-x_B|<2^{i+1}2^jr_B\}.$$

Now we estimate $\int_{S_i(B_j)}[\mathrm{E}_j(x)]^q\,dx$. When $i\in\{0,\dots,4\}$, by the boundedness of S_L on $L^q(\mathbb{R}^n)$ we conclude that, for any $j\in\mathbb{Z}_+$,

$$\begin{aligned}\int_{S_i(B_j)}[\mathrm{E}_j(x)]^q\,dx&\le\int_{S_i(B_j)}[S_L(\alpha\chi_{S_j(B)})(x)]^q\,dx\\&\lesssim\|\alpha\chi_{S_j(B)}\|_{L^q(\mathbb{R}^n)}^q\lesssim2^{-\epsilon jq}|2^jB|\,\|\chi_B\|_X^{-q}.\end{aligned}\tag{6.14}$$

It is easy to see that, when $i\ge5$, $d(S_j(B),S_i(B_j))\gtrsim2^{i+j}r_B$. As $M>\frac{n}{2m}\left[\frac{\theta_1}{n}+\frac{1}{\theta}\right]$, we know that

$$2mMq+\left(\theta_2+\frac12\right)q>\left(\theta_1+\theta_2+\frac{n}{\theta}+\frac12\right)q.$$

Let

$$\left[\theta_1+\theta_2+\frac{n}{\theta}+\frac12\right]q-1<v<2mMq+\left[\theta_2+\frac12\right]q-1.$$

Then, from Hölder's inequality, Fubini's theorem and Assumption 6.4, it follows that, for each $i\in\mathbb{N}$ with $i\ge5$ and $j\in\mathbb{Z}_+$,

$$\begin{aligned}\int_{S_i(B_j)}[\mathrm{E}_j(x)]^q\,dx&\le\int_{S_i(B_j)}\left\{\int_0^{r_B}\int_{B(x,t)}|t^{2m}Le^{-t^{2m}L}(\alpha\chi_{S_j(B)})(y)|^q\,\frac{dy\,dt}{t^nt^{q/2}}\right\}\\&\quad\times\left\{\int_0^{r_B}\int_{B(x,t)}\frac{dy\,dt}{t^n}\right\}^{(q-2)/2}dx\\&\lesssim r_B^{(q-2)/2}\int_0^{r_B}\int_{\widetilde{S}_i(B_j)}|t^{2m}Le^{-t^{2m}L}(\alpha\chi_{S_j(B)})(y)|^q\,\frac{dy\,dt}{t^{q/2}}\\&\lesssim r_B^{(q-2)/2}\int_0^{r_B}\left\{2^{i\theta_1}\left[\Upsilon\left(\frac{2^{i+j}r_B}{t}\right)\right]^{\theta_2}|2^{i+j}B|^{1/q}|2^jB|^{-1/q}\right.\\&\quad\left.\times e^{-(\frac{2^{i+j}r_B}{t})^{2m/(2m-1)}}\|\alpha\|_{L^q(S_j(B))}\right\}^q\frac{dt}{t^{q/2}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim r_B^{(q-2)/2} 2^{i\theta_1 q} 2^{-\epsilon q j} (2^{i+j} r_B)^{\theta_2 q} \|\chi_B\|_X^{-q} |2^{i+j} B| \\
&\quad \times \left\{ \int_0^{r_B} \left(\frac{t}{2^{i+j} r_B} \right)^v t^{-(\theta_2 q + q/2)}\, dt \right\} \\
&\sim 2^{-i[v-(\theta_1+\theta_2)q]} 2^{-j[v+(\epsilon-\theta_2)q]} |2^{i+j} B|\, \|\chi_B\|_X^{-q}. \qquad (6.15)
\end{aligned}
$$

Now we estimate $\int_{S_i(B_j)} [\mathrm{F}_j(x)]^q\, dx$. When $i \in \{0, \ldots, 4\}$, by the boundedness of $S_{L,M+1}$ on $L^q(\mathbb{R}^n)$ (see, for example, [13, Theorem 2.13]) we conclude that, for any $j \in \mathbb{Z}_+$,

$$
\begin{aligned}
\int_{S_i(B_j)} [\mathrm{F}_j(x)]^q\, dx &\le \int_{S_i(B_j)} [S_{L,M+1}(\chi_{S_j(B)} (r_B^{2m} L)^{-M} \alpha)(x)]^q\, dx \\
&\lesssim \|\chi_{S_j(B)} (r_B^{2m} L)^{-M} \alpha\|_{L^q(\mathbb{R}^n)}^q \\
&\lesssim 2^{-\epsilon j q} |2^j B|\, \|\chi_B\|_X^{-q}. \qquad (6.16)
\end{aligned}
$$

Notice that, for any $i, j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$
\begin{aligned}
\mathrm{F}_j(x) &\le \left\{ \int_{r_B}^{2^{i+j-3} r_B} \int_{B(x,t)} |t^{2m} L e^{-t^{2m} L} (r_B^{2m} L)^M \right. \\
&\qquad \left. \times (\chi_{S_j(B)} (r_B^{2m} L)^{-M} \alpha)(y)|^2 \frac{dy\, dt}{t^{n+1}} \right\}^{1/2} + \left\{ \int_{2^{i+j-3} r_B}^{\infty} \cdots \right\}^{1/2} \\
&=: \mathrm{F}_{j,1}(x) + \mathrm{F}_{j,2}(x). \qquad (6.17)
\end{aligned}
$$

For $\mathrm{F}_{j,1}$, similarly to (6.15), we know that, for any $i \in \mathbb{N}$ with $i \ge 5$, $j \in \mathbb{Z}_+$ and $v \in (0, \infty)$,

$$
\begin{aligned}
\int_{S_i(B_j)} [\mathrm{F}_{j,1}(x)]^q\, dx &\lesssim 2^{-i[v+1-(\theta_1+\theta_2+1/2)q]} \\
&\qquad \times 2^{-j[v+1+(\epsilon-\theta_2-1/2)q]} |2^{i+j} B|\, \|\chi_B\|_X^{-q}. \qquad (6.18)
\end{aligned}
$$

For $\mathrm{F}_{j,2}$, from the boundedness of $S_{L,M+1}$ on $L^q(\mathbb{R}^n)$, we deduce that

$$
\begin{aligned}
\int_{S_i(B_j)} [\mathrm{F}_{j,2}(x)]^q\, dx &\lesssim \frac{r_B^{2mMq}}{(2^{i+j} r_B)^{2mMq}} \int_{S_i(B_j)} \left[S_{L,M+1}\big(\chi_{S_j(B)} (r_B^{2m} L)^{-M} \alpha\big)(x) \right]^q dx \\
&\lesssim 2^{-2mMq(i+j)} \|(r_B^{2m} L)^{-M} \alpha\|_{L^q(S_j(B))}^q \\
&\lesssim 2^{-(2mMq+n)i} 2^{-(2mM+\epsilon)qj} |2^{i+j} B|\, \|\chi_B\|_X^{-q},
\end{aligned}
$$

which, together with (6.17), (6.18) and the facts that $2mMq+n > v+1+n-(\theta_1+\theta_2+1/2)q$ and $(2mM+\epsilon)q > v+1+(\epsilon-\theta_2-1/2)q$, implies that, for any $i \in \mathbb{N}$ with $i \ge 5$ and $j \in \mathbb{Z}_+$,

$$
\int_{S_i(B_j)} [\mathrm{F}_j(x)]^q\, dx \lesssim 2^{-i[v+1-(\theta_1+\theta_2+1/2)q]} 2^{-j[v+1+(\epsilon-\theta_2-1/2)q]} |2^{i+j} B|\, \|\chi_B\|_X^{-q}.
$$

From this and (6.14)–(6.16), we conclude that, for any $i, j \in \mathbb{Z}_+$,

$$
\|\chi_{B_j} \mathrm{E}_j\|_{L^q(\mathbb{R}^n)} \lesssim 2^{-j\epsilon} |2^j B|^{1/q} \|\chi_B\|_X^{-1}, \qquad (6.19)
$$

$$
\|\chi_{S_i(B_j)} \mathrm{E}_j\|_{L^q(\mathbb{R}^n)} \lesssim 2^{-i[v/q-(\theta_1+\theta_2)]} 2^{-j(v/q+\epsilon-\theta_2)} |2^{i+j} B|^{1/q} \|\chi_B\|_X^{-1}, \qquad (6.20)
$$

$$
\|\chi_{B_j} \mathrm{F}_j\|_{L^q(\mathbb{R}^n)} \lesssim 2^{-j\epsilon} |2^j B|^{1/q} \|\chi_B\|_X^{-1} \qquad (6.21)
$$

and

$$\|\chi_{S_i(B_j)}\mathrm{F}_j\|_{L^q(\mathbb{R}^n)} \lesssim 2^{-i[(v+1)/q-(\theta_1+\theta_2+1/2)]}2^{-j[(v+1)/q+\epsilon-\theta_2-1/2]}|2^{i+j}B|^{1/q}\|\chi_B\|_X^{-1}. \tag{6.22}$$

For each α_k with $k\in\mathbb{N}$, let $\{\mathrm{E}_{k,j}\}_{j\in\mathbb{Z}_+}$ and $\{\mathrm{F}_{k,j}\}_{j\in\mathbb{Z}_+}$ be as in (6.13) with α replaced by α_k. Then, for any $k\in\mathbb{N}$, $\{\mathrm{E}_{k,j}\}_{j\in\mathbb{Z}_+}$ and $\{\mathrm{F}_{k,j}\}_{j\in\mathbb{Z}_+}$ satisfy the estimates in (6.19) through (6.22) upon replacing B by B_k.

By the triangle inequality, we find that

$$\begin{aligned}\|S_L(\pi_{L,M}(f))\|_X &\le \Big\|\sum_{k=1}^{\infty}\lambda_k S_L(\alpha_k)\Big\|_X \le \Big\|\sum_{k=1}^{\infty}\lambda_k\sum_{j=0}^{\infty}(\mathrm{E}_{k,j}+\mathrm{F}_{k,j})\Big\|_X\\ &\lesssim \Big\|\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\lambda_k\chi_{S_i(2^jB_k)}\mathrm{E}_{k,j}\Big\|_X\\ &\quad+\Big\|\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\lambda_k\chi_{S_i(2^jB_k)}\mathrm{F}_{k,j}\Big\|_X.\end{aligned}$$

A calculation shows that $v>\big(\theta_1+\theta_2+\frac{n}{\theta}+\frac12\big)q-1$ implies $\frac{v-n}{q}-(\theta_1+\theta_2)>n\big[\frac1\theta-\frac1q\big]$. Thus, if we use (6.19)–(6.22) and Theorem 2.11, we then have

$$\|S_L(\pi_{L,M}(f))\|_X \lesssim \left\|\left\{\sum_{k,\,j=1}^{\infty}\left(\frac{\lambda_k 2^{-j\epsilon}}{\|\chi_{B_k}\|_X}\right)^s\chi_{2^jB_k}\right\}^{1/s}\right\|_X,$$

which, combined with the definition of the Hardy–Littlewood maximal operator, the assumption that $\epsilon>n/\theta$ and (2.8), further implies that

$$\|S_L(\pi_{L,M}(f))\|_X \lesssim \left\|\left\{\sum_{k,j=1}^{\infty}\left[\frac{\lambda_k 2^{-j(\epsilon-n/\theta)}}{\|\chi_{B_k}\|_X}M^{(\theta)}(\chi_{B_k})\right]^s\right\}^{1/s}\right\|_X \lesssim \Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}).$$

From this and (6.11), it follows that

$$\|S_L(\pi_{L,M}(f))\|_X \lesssim \|f\|_{T_X(\mathbb{R}^{n+1}_+)}. \tag{6.23}$$

Hence, for any $f\in T_X^c(\mathbb{R}^{n+1}_+)$,

$$\|\pi_{L,M}(f)\|_{H_{X,L}(\mathbb{R}^n)} \lesssim \|f\|_{T_X(\mathbb{R}^{n+1}_+)}. \tag{6.24}$$

Finally, assume that $f\in T_X(\mathbb{R}^{n+1}_+)$. Let $\{O_k\}_{k\in\mathbb{Z}}$, $\{\lambda_{k,j}\}_{k\in\mathbb{Z},\,j\in I_k}$ and $\{a_{k,j}\}_{k\in\mathbb{Z},\,j\in I_k}$ be as in the proof of Proposition 4.9. Then, from the definition of O_k, we deduce that $O_k\uparrow\operatorname{supp}(\mathcal{A}(f))$ as $k\to-\infty$, which, together with the absolute continuity of the quasi-norm of X, further implies that

$$\|\mathcal{A}(f)\chi_{\mathbb{R}^n\setminus O_k}\|_X\to 0 \tag{6.25}$$

as $k\to-\infty$. For any $N\in\mathbb{Z}$, let $f_N:=\sum_{k\in\mathbb{Z},\,k\ge N}\sum_{j\in I_k}\lambda_{k,j}a_{k,j}$. Then an argument similar to (4.49) yields, for any $N\in\mathbb{Z}$,

$$
\begin{aligned}
\|\mathcal{A}(f-f_N)\|_X &\le \Big\| \sum_{k\in\mathbb{Z},\,k<N} \sum_{j\in I_k} \lambda_{k,j}\mathcal{A}(a_{k,j}) \Big\|_X \\
&\le \Big\| \Big\{ \sum_{k\in\mathbb{Z},\,k<N} \sum_{j\in I_k} [\lambda_{k,j}\mathcal{A}(a_{k,j})]^s \Big\}^{1/s} \Big\|_X \\
&\lesssim \Big\| \Big\{ \sum_{k\in\mathbb{Z},\,k<N} \sum_{j\in I_k} \Big(\frac{\lambda_{k,j}}{\|\chi_{B_{k,j}}\|_X} \Big)^s \chi_{B_{k,j}} \Big\}^{1/s} \Big\|_X \\
&\sim \Big\| \Big\{ \sum_{k\in\mathbb{Z},\,k<N} \sum_{j\in I_k} 2^{ks}\chi_{B_{k,j}} \Big\}^{1/s} \Big\|_X. \qquad (6.26)
\end{aligned}
$$

Furthermore, similarly to the proofs of (4.46) and (4.48), we find that, for any $N \in \mathbb{Z}$,

$$
\begin{aligned}
\Big\| \Big\{ \sum_{k\in\mathbb{Z},\,k<N} \sum_{j\in I_k} 2^{ks}\chi_{B_{k,j}} \Big\}^{1/s} \Big\|_X &\lesssim \Big\| \Big\{ \sum_{k\in\mathbb{Z},\,k<N} 2^{ks}\chi_{O_k} \Big\}^{1/s} \Big\|_X \\
&\lesssim \Big\| \Big\{ \sum_{k\in\mathbb{Z},\,k<N} 2^{ks}\chi_{O_k\setminus O_{k+m_0}} \Big\}^{1/s} \Big\|_X \\
&\lesssim \|\mathcal{A}(f)\chi_{\mathbb{R}^n\setminus O_{N+m_0}}\|_X,
\end{aligned}
$$

where m_0 is as in (4.48), which, combined with (6.25) and (6.26), implies that

$$
\|f-f_N\|_{T_X(\mathbb{R}^{n+1}_+)} = \|\mathcal{A}(f-f_N)\|_X \to 0 \qquad (6.27)
$$

as $N \to -\infty$. Let $\{O_k^*\}_{k\in\mathbb{Z}}$ be as in the proof of Proposition 4.9. Then, from the definition of $\{a_{k,j}\}_{k\in\mathbb{Z},\,j\in I_k}$, it follows that, for any $N \in \mathbb{Z}$, $\mathrm{supp}(f_N) \subset \widehat{O_N^*}$, which implies that $f_N \in T_X^c(\mathbb{R}^{n+1}_+)$. By this and (6.27), we conclude that $T_X^c(\mathbb{R}^{n+1}_+)$ is dense in $T_X(\mathbb{R}^{n+1}_+)$, which, together with (6.24) and a density argument, implies that (6.24) holds true for any $f \in T_X(\mathbb{R}^{n+1}_+)$. This finishes the proof of (ii), and hence of Proposition 6.8. ■

PROPOSITION 6.11. *Assume that X is a ball quasi-Banach function space satisfying* (2.8) *for some $\theta, s \in (0,1]$ with $\theta < s$, and* (2.9) *for some $q \in [2, q_L)$. Let L satisfy Assumptions* 6.3 *and* 6.4, *and let $\epsilon \in (n/\theta, \infty)$ and $M \in \mathbb{N}$ with $M > \frac{n}{2m}[\frac{\theta_1}{n} + \frac{1}{\theta}]$. Assume further that X has an absolutely continuous quasi-norm. Then, for all $f \in H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, there exist $\{\lambda_j\}_{j\in\mathbb{N}} \subset [0,\infty)$ and a sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ of $(X, M, \epsilon)_L$-molecules, respectively, associated with balls $\{B_j\}_{j\in\mathbb{N}}$ such that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $L^2(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for any $f \in H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,*

$$
\Big\| \Big\{ \sum_{j=1}^\infty \Big(\frac{\lambda_j}{\|\chi_{B_j}\|_X} \Big)^s \chi_{B_j} \Big\}^{1/s} \Big\|_X \le C\|f\|_{H_{X,L}(\mathbb{R}^n)}.
$$

Proof. Let $f \in H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then, by the H_∞-functional calculus for L and the boundedness of S_L on $L^2(\mathbb{R}^n)$, we know that

$$
f = C_{(m,M)} \int_0^\infty (t^{2m}L)^{M+2} e^{-2t^{2m}L}(f)\,\frac{dt}{t} = \pi_{L,M}(t^{2m}Le^{-t^{2m}L}(f))
$$

in $L^2(\mathbb{R}^n)$, where $C_{(m,M)}$ is as in (6.6). Moreover, from $f \in H_{X,L}(\mathbb{R}^n)$ and the boundedness of S_L on $L^2(\mathbb{R}^n)$ again, we deduce that $t^{2m}Le^{-t^{2m}L}(f) \in T_X(\mathbb{R}^{n+1}_+) \cap T_2^2(\mathbb{R}^{n+1}_+)$.

Applying Theorem 3.19, Lemma 6.9 and Proposition 6.8 to $t^{2m}Le^{-t^{2m}L}(f)$, we conclude that

$$f=\pi_{L,M}(t^{2m}Le^{-t^{2m}L}(f))=\sum_{j=1}^{\infty}\lambda_j\pi_{L,M}(A_j)=:\sum_{j=1}^{\infty}\lambda_j\alpha_j$$

in $L^2(\mathbb{R}^n)$ and

$$\left\|\left\{\sum_{j=1}^{\infty}\left(\frac{\lambda_j}{\|\chi_{B_j}\|_X}\right)^s\chi_{B_j}\right\}^{1/s}\right\|_X\lesssim\|t^{2m}Le^{-t^{2m}L}(f)\|_{T_X(\mathbb{R}^{n+1}_+)}\sim\|f\|_{H_{X,L}(\mathbb{R}^n)},$$

where, for any $j\in\mathbb{N}$, α_j is an $(X,M,\epsilon)_L$-molecule associated with the ball B_j, up to a harmless constant multiple, which completes the proof of Proposition 6.11. ■

THEOREM 6.12. *Assume that X is a ball quasi-Banach function space satisfying* (2.8) *for some $\theta,s\in(0,1]$ with $\theta<s$, and* (2.9) *for $q\in[2,q_L)$, where q_L is as in Assumption* 6.4. *Let L satisfy Assumptions* 6.3 *and* 6.4, *$\epsilon\in(n/\theta,\infty)$ and $M\in\mathbb{N}$ with $M>\frac{n}{2m}\left[\frac{\theta_1}{n}+\frac{1}{\theta}\right]$. Assume further that X has an absolutely continuous quasi-norm. Then $H_{X,L}(\mathbb{R}^n)$ and $H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Proof. We first prove that

$$H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\subset H_{X,L}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n),$$

and the inclusion is continuous. Let $f\in H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Then there exist $\{\lambda_j\}_{j\in\mathbb{N}}\subset[0,\infty)$ and a sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ of $(X,q,M,\epsilon)_L$-molecules such that $f=\sum_{j=1}^{\infty}\lambda_j\alpha_j$ in $L^2(\mathbb{R}^n)$ and

$$\|f\|_{H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)}\sim\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}).\tag{6.28}$$

Furthermore, from the fact that $f=\sum_{j=1}^{\infty}\lambda_j\alpha_j$ in $L^2(\mathbb{R}^n)$ and the boundedness of S_L on $L^2(\mathbb{R}^n)$, it follows that $S_L(f)\le\sum_{j=1}^{\infty}\lambda_jS_L(\alpha_j)$ almost everywhere, which, together with the proof of (6.23), implies that

$$\|S_L(f)\|_X\le\left\|\sum_{j=1}^{\infty}\lambda_jS_L(\alpha_j)\right\|_X\lesssim\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}).$$

From this and (6.28), we conclude that $f\in H_{X,L}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and

$$\|f\|_{H_{X,L}(\mathbb{R}^n)}\lesssim\|f\|_{H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)}.\tag{6.29}$$

Now we show that

$$H_{X,L}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\subset H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n),$$

and the inclusion is continuous. Let $f\in H_{X,L}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Then, from Proposition 6.11, it follows that there exist $\{\lambda_j\}_{j\in\mathbb{N}}\subset[0,\infty)$ and a sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ of $(X,q,M,\epsilon)_L$-molecules, associated with the balls $\{B_j\}_{j\in\mathbb{N}}$, such that $f=\sum_{j=1}^{\infty}\lambda_j\alpha_j$ in $L^2(\mathbb{R}^n)$ and

$$\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}})\lesssim\|f\|_{H_{X,L}(\mathbb{R}^n)},$$

which implies that $f\in H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and

$$\|f\|_{H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)}\lesssim\|f\|_{H_{X,L}(\mathbb{R}^n)}.$$

From this and (6.29), we deduce that

$$H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) = H_{X,L}^{M,q,\epsilon}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

and, for all $f \in H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\|f\|_{H_{X,L}(\mathbb{R}^n)} \sim \|f\|_{H_{X,L}^{M,q,\epsilon}(\mathbb{R}^n)},$$

which, combined with the fact that $H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $H_{X,L}^{M,q,\epsilon}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ are dense in $H_{X,L}(\mathbb{R}^n)$ and $H_{X,L}^{M,q,\epsilon}(\mathbb{R}^n)$, respectively, implies that the spaces $H_{X,L}(\mathbb{R}^n)$ and $H_{X,L}^{M,q,\epsilon}(\mathbb{R}^n)$ coincide with equivalent quasi-norms. This finishes the proof of Theorem 6.12. ∎

6.3. Atomic characterization of $H_{X,L}(\mathbb{R}^n)$. In this subsection, we establish an atomic characterization of the Hardy type space $H_{X,L}(\mathbb{R}^n)$. To obtain the support condition of atoms via the finite propagation speed for the wave equation, we have to restrict ourselves to a special case of operators satisfying Assumptions 6.3 and 6.4. More precisely, we postulate the following assumptions on the operator L.

ASSUMPTION 6.13. *The operator L is a non-negative and self-adjoint linear operator in $L^2(\mathbb{R}^n)$.*

ASSUMPTION 6.14. *There exists a positive constant $p_L \in [1,2)$ such that the semigroup $\{e^{-tL}\}_{t>0}$ generated by L satisfies the reinforced $(p_L, p_L', 1)$ off-diagonal estimates on balls as in Assumption 6.4.*

REMARK 6.15.

(i) It is easy to see that if an operator L satisfying Assumptions 6.13 and 6.14 is one-to-one, then it also satisfies Assumptions 6.3 and 6.4, and hence all results obtained in Subsection 6.2 hold true in this case. If an operator T satisfies Assumptions 6.13 and 6.14, but is not one-to-one and hence does not satisfy Assumptions 6.3 and 6.4, thanks to the functional calculus via the spectral theorem, all the results obtained in Subsection 6.2 still hold true.

(ii) The following definition of the L^q off-diagonal estimates is from [5]: For all q in $(1,\infty)$, a family $\{T_t\}_{t>0}$ of linear operators is said to satisfy the L^q *off-diagonal estimates* if there exist positive constants C and c such that

$$\|T_t(f)\|_{L^q(F)} \leq Ce^{-[d(E,F)]^2/(ct)}\|f\|_{L^q(E)}$$

for any closed sets $E, F \subset \mathbb{R}^n$, $t \in (0,\infty)$ and $f \in L^q(E)$. From [8], we deduce that $\{T_t\}_{t>0} \in \mathcal{O}_1(L^q\text{-}L^q)$ if and only if $\{T_t\}_{t>0}$ satisfies the L^q off-diagonal estimates. Thus, Assumption 6.14 implies that $\{e^{-tL}\}_{t>0}$ satisfies the L^q off-diagonal estimates with $q \in (p_L, p_L')$.

To establish the atomic characterization of $H_{X,L}(\mathbb{R}^n)$, we first introduce the following notions of atoms and atomic Hardy type spaces.

DEFINITION 6.16. Let X be a ball quasi-Banach function space, let L satisfy Assumptions 6.13 and 6.14, and let p_L be as in Assumption 6.14. Assume that $q \in (p_L, p_L')$, $M \in \mathbb{N}$ and $B \subset \mathbb{R}^n$ is a ball.

(i) A function $\alpha \in L^q(\mathbb{R}^n)$ is called an $(X,q,M)_L$-*atom* associated with B if there exists a function $b \in \mathcal{D}(L^M)$, the domain of L^M, such that

(a) $\alpha = L^M b$;

(b) $\operatorname{supp}(L^k b) \subset B$ for any $k \in \{0,\dots,M\}$;

(c) $\|(r_B^2 L)^k b\|_{L^q(\mathbb{R}^n)} \le r_B^{2M} |B|^{1/q} \|\chi_B\|_X^{-1}$ for any $k \in \{0,\dots,M\}$, where r_B denotes the radius of B.

Moreover, if α is an $(X,q,M)_L$-atom associated with B for all $q \in (p_L, p_L')$, then α is called an $(X,\,M)_L$-*atom* associated with B.

(ii) For any $f \in L^2(\mathbb{R}^n)$, $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ is called an *atomic* $(X,q,M)_L$-*representation* of f if, for all $j \in \mathbb{N}$, α_j is an $(X,q,M)_L$-atom associated with the ball $B_j \subset \mathbb{R}^n$, the summation converges in $L^2(\mathbb{R}^n)$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset [0,\infty)$ satisfies $\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}) < \infty$, where $\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}})$ is as in (6.4). Let

$$\widetilde{H}^{M,q}_{X,L,\mathrm{at}}(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : f \text{ has an atomic } (X,q,M)_L\text{-representation}\}$$

equipped with the *quasi-norm* given by

$$\|f\|_{H^{M,q}_{X,L,\mathrm{at}}(\mathbb{R}^n)} := \inf\Big\{\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}) :\ f = \sum_{j=1}^\infty \lambda_j\alpha_j \text{ is an atomic } (X,q,M)_L\text{-representation}\Big\},$$

where the infimum is taken over all the atomic $(X,q,M)_L$-representations of f as above and $\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}})$ is as in (6.4).

The *atomic Hardy type space* $H^{M,q}_{X,L,\mathrm{at}}(\mathbb{R}^n)$ is then defined as the completion of the set $\widetilde{H}^{M,q}_{X,L,\mathrm{at}}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{M,q}_{X,L,\mathrm{at}}(\mathbb{R}^n)}$.

Then we have the following atomic characterization of the Hardy space $H_{X,L}(\mathbb{R}^n)$.

THEOREM 6.17. *Assume that $q \in [2, p_L')$ with p_L' as in Assumption* 6.14. *Let X be a ball quasi-Banach function space satisfying* (2.8) *for some $\theta, s \in (0,1]$ with $\theta < s$, and* (2.9) *for q as above, let L satisfy Assumptions* 6.13 *and* 6.14, *and let $M \in \mathbb{N}$ satisfy $M > \frac{n}{2}(\frac{1}{\theta} - \frac{1}{p_L'})$. Assume further that X has an absolutely continuous quasi-norm. Then the spaces $H_{X,L}(\mathbb{R}^n)$ and $H^{M,q}_{X,L,\mathrm{at}}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

To prove Theorem 6.17, we need to introduce some operators $\pi_{\Phi,L,k}$. To this end, we first recall the following notation: For any operator T, let K_T be its *integral kernel.* Let *cos*$(t\sqrt{L})$ with $t \in (0,\infty)$ be the *cosine function operator* generated by L. By [25, Theorem 3.4] (see also [61, Proposition 3.4]), we know that there exists a positive constant C_0 such that

$$\operatorname{supp}(K_{\cos(t\sqrt{L})}) \subset \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| \le C_0 t\}. \tag{6.30}$$

Moreover, let $\psi \in C_c^\infty(\mathbb{R})$ be even and $\operatorname{supp}\psi \subset (-C_0^{-1}, C_0^{-1})$, where C_0 is as in (6.30). Let Φ denote the *Fourier transform* of ψ. Then, for all $k \in \mathbb{N}$ and $t \in (0,\infty)$, the kernel of $(t^2L)^k\Phi(t\sqrt{L})$ satisfies

$$\operatorname{supp}(K_{(t^2L)^k\Phi(t\sqrt{L})}) \subset \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| \le t\} \tag{6.31}$$

(see, for example, [61, Lemma 3.5] for the details). Let $M\in\mathbb{N}$ with $M>\frac{n}{2}(\frac{1}{\theta}-\frac{1}{p_L'})$. Assume that Φ is as in (6.31). Then, for all $k\in\mathbb{N}$, $f\in L^2_c(\mathbb{R}^{n+1}_+)$ and $x\in\mathbb{R}^n$, the operator $\pi_{\Phi,L,k}$ is defined by setting

$$\pi_{\Phi,L,k}(f)(x):=C_{(\Phi,\,k)}\int_0^\infty (t^2L)^{k+1}\Phi(t\sqrt{L})(f(\cdot,\,t))(x)\,\frac{dt}{t},$$

where $C_{(\Phi,k)}$ is a positive constant such that

$$C_{(\Phi,k)}\int_0^\infty t^{2(k+1)}\Phi(t)t^2e^{-t^2}\,\frac{dt}{t}=1. \tag{6.32}$$

Using Minkowski's integral inequality and quadratic estimates (see [61, (3.14)]), we easily find that $\pi_{\Phi,L,k}$ can be continuously extended from $T^2_2(\mathbb{R}^{n+1}_+)$ to $L^2(\mathbb{R}^n)$. Moreover, we have the following boundedness of $\pi_{\Phi,L,M}$, which can be viewed as an extension of [132, Proposition 4.6].

PROPOSITION 6.18. *Assume that X is a ball quasi-Banach function space satisfying* (2.8) *for some $\theta,s\in(0,1]$ with $\theta<s$, and* (2.9) *for some $q\in[2,p_L')$, where p_L is as in Assumption* 6.14. *Let L satisfy Assumptions* 6.13 *and* 6.14, *and let $M\in\mathbb{N}$ satisfy $M>\frac{n}{2}(\frac{1}{\theta}-\frac{1}{p_L'})$. Assume further that X has an absolutely continuous quasi-norm. Then the operator $\pi_{\Phi,L,M}$, initially defined on $T^c_X(\mathbb{R}^{n+1}_+)$, extends to a bounded linear operator from $T_X(\mathbb{R}^{n+1}_+)$ to $H_{X,L}(\mathbb{R}^n)$.*

Proof. Similarly to the proof of Proposition 6.8, we only need to show Proposition 6.18 for the space $T^c_X(\mathbb{R}^{n+1}_+)$. Let $f\in T^c_X(\mathbb{R}^{n+1}_+)$. From Theorem 3.19, Lemma 6.9 and the fact that $\pi_{\Phi,L,M}$ is bounded from $T^2_2(\mathbb{R}^{n+1}_+)$ to $L^2(\mathbb{R}^n)$, we deduce that there exists a sequence $\{A_j\}_{j\in\mathbb{N}}$ of (T_X,∞)-atoms associated with the balls $\{B_j\}_{j\in\mathbb{N}}$, respectively, and $\{\lambda_j\}_{j\in\mathbb{N}}\subset[0,\infty)$ such that

$$\pi_{\Phi,L,M}(f)=\sum_{j=1}^\infty\lambda_j\pi_{\Phi,L,M}(A_j)=:\sum_{j=1}^\infty\lambda_j\alpha_j\quad\text{ in } L^2(\mathbb{R}^n)$$

and

$$\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}})\lesssim\|f\|_{T_X(\mathbb{R}^{n+1}_+)}. \tag{6.33}$$

Moreover, by the boundedness of S_L on $L^2(\mathbb{R}^n)$, for almost every $x\in\mathbb{R}^n$,

$$S_L(\pi_{\Phi,L,M}(f))(x)\le\sum_{j=1}^\infty\lambda_jS_L(\alpha_j)(x). \tag{6.34}$$

Now we prove that, for each $j\in\mathbb{N}$, α_j is an $(X,M)_L$-atom associated with $B_j:=B(x_{B_j},r_{B_j})$ for some $x_{B_j}\in\mathbb{R}^n$ and $r_{B_j}\in(0,\infty)$, up to a harmless constant multiple. Indeed, let

$$b_j:=C_{(\Phi,\,M)}\int_0^\infty t^{2(M+1)}L\Phi(t\sqrt{L})(A_j(\cdot,\,t))\,\frac{dt}{t}, \tag{6.35}$$

where $C_{(\Phi,\,M)}$ is as in (6.32) with k replaced by M. From (6.31), it follows that $\operatorname{supp}(L^kb_j)\subset B_j$ for all $k\in\{0,\dots,M\}$.

Let $q\in(p_L,p_L')$. For any $h\in L^{q'}(B_j)\cap L^2(B_j)$, by (6.34), Assumption 6.13, Fubini's theorem, the fact that $\operatorname{supp}(A_j)\subset\widehat{B}_j$ and Hölder's inequality, we conclude that, for all

$k \in \{0, \ldots, M\}$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} (r_{B_j}^2 L)^k b_j(x) h(x)\,dx\right| \\
&\qquad \sim \left|\int_0^\infty \int_{\mathbb{R}^n} A_j(x,t)(r_{B_j}^2 L)^k t^{2(M+1)} L\Phi(t\sqrt{L})(h)(x)\,\frac{dx\,dt}{t}\right| \\
&\qquad \lesssim r_{B_j}^{2M} \int_0^\infty \int_{\mathbb{R}^n} |A_j(x,t)(t^2L)^{k+1}\Phi(t\sqrt{L})(h)(x)|\,\frac{dx\,dt}{t} \\
&\qquad \lesssim r_{B_j}^{2M} \|\mathcal{A}(A_j)\|_{L^q(\mathbb{R}^n)} \left\|\left[\int_{\Gamma(\cdot)} |(t^2L)^{k+1}\Phi(t\sqrt{L})(h)(x)|^2\,\frac{dx\,dt}{t^{n+1}}\right]^{1/2}\right\|_{L^{q'}(\mathbb{R}^n)}.
\end{aligned}$$

Moreover, it follows from [13, Lemma 5.3] that, for all $q' \in (p_L, p_L')$,

$$\left\|\left[\int_{\Gamma(\cdot)} |(t^2L)^{k+1}\Phi(t\sqrt{L})(h)(x)|^2\,\frac{dx\,dt}{t^{n+1}}\right]^{1/2}\right\|_{L^{q'}(\mathbb{R}^n)} \lesssim \|h\|_{L^{q'}(\mathbb{R}^n)},$$

which, together with the arbitrariness of h and the fact that A_j is a (T_X, ∞)-atom associated with B_j, implies that, for any $k \in \{0, \ldots, M\}$ and $j \in \mathbb{N}$,

$$\|(r_{B_j}^2 L)^k b_j\|_{L^q(\mathbb{R}^n)} \lesssim r_{B_j}^{2M} |B_j|^{1/q} \|\chi_{B_j}\|_X^{-1}.$$

Thus, for each $j \in \mathbb{N}$, α_j is an $(X, M)_L$-atom associated with the ball B_j up to a harmless constant multiple.

To prove that $\pi_{\Phi,L,M}(f) \in H_{X,L}(\mathbb{R}^n)$, via (6.33), it suffices to show that

$$\|S_L(\pi_{\Phi,L,M}(f))\|_X \lesssim \Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}). \tag{6.36}$$

Let α be an (X, M)-atom associated with $B := B(x_B, r_B)$ for some $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, as in Definition 6.16. From $M > \frac{n}{2}\left(\frac{1}{\theta} - \frac{1}{p_L'}\right)$, we deduce that there exists $q \in [2, p_L')$ such that $M > \frac{n}{2}\left(\frac{1}{\theta} - \frac{1}{q}\right)$. To show (6.36), via (6.34) and Theorem 2.11, we only need to prove that, for any $k \in \mathbb{Z}_+$,

$$\|\chi_{S_k(B)} S_L(\alpha)\|_{L^q(\mathbb{R}^n)} \lesssim 2^{-2Mk} |B|^{1/q} \|\chi_B\|_X^{-1}. \tag{6.37}$$

Now we show (6.37). By the boundedness of S_L on $L^q(\mathbb{R}^n)$ with $q \in (p_L, p_L')$, we know that, for any $k \in \{0, \ldots, 4\}$,

$$\|\chi_{S_k(B)} S_L(\alpha)\|_{L^q(\mathbb{R}^n)} \lesssim \|\alpha\|_{L^q(\mathbb{R}^n)} \lesssim |B|^{1/q} \|\chi_B\|_X^{-1}. \tag{6.38}$$

For any $k \in \mathbb{N}$ with $k \geq 5$, let $\mathrm{D}_k := \|S_L(\alpha)\|_{L^q(S_k(B))}^q$. To estimate D_k, we write

$$\begin{aligned}
\mathrm{D}_k &\lesssim \int_{S_k(B)} \left[\int_0^{|x-x_B|/4} \int_{B(x,t)} |t^2Le^{-t^2L}(\alpha)(y)|^2\,\frac{dy\,dt}{t^{n+1}}\right]^{q/2} dx \\
&\quad + \int_{S_k(B)} \left[\int_{|x-x_B|/4}^\infty \int_{B(x,t)} \cdots \frac{dy\,dt}{t^{n+1}}\right]^{q/2} dx =: \mathrm{H}_k + \mathrm{I}_k.
\end{aligned} \tag{6.39}$$

We first estimate H_k. For $k \geq 5$, let

$$G_k(B) := \{y \in \mathbb{R}^n : \text{there exists } x \in S_k(B) \text{ such that } |y - x| < |x - x_B|/4\}.$$

By Hölder's inequality, we have

$$\mathrm{H}_k \lesssim (2^k r_B)^{q/2-1} \int_{S_k(B)} \left[\int_0^{|x-x_B|/4} \int_{B(x,t)} |t^2 L e^{-t^2 L}(\alpha)(y)|^q \, \frac{dy\,dt}{t^{n+q/2}} \right] dx.$$

A geometric observation shows that, if $y \in G_k(B)$, then $2^{k-2} r_B \le |y - x_B| < 2^{k+1} r_B$. Thus, $G_k(B) \subset \bigcup_{i=k-1}^{k+1} S_i(B) =: \widetilde{\widetilde{S}}_k(B)$. If we use Fubini's theorem and $\alpha = L^M b$ (see Definition 6.16), we then obtain

$$\mathrm{H}_k \lesssim (2^k r_B)^{q/2-1} \int_0^{2^{k+1} r_B} \int_{\widetilde{\widetilde{S}}_k(B)} |(t^2 L)^{M+1} e^{-t^2 L}(b)(y)|^q \, \frac{dy\,dt}{t^{q(1/2+2M)}}.$$

From Assumption 6.14 and the fact that α is an $(X, M)_L$-atom, we deduce that

$$\begin{aligned} \mathrm{H}_k &\lesssim (2^k r_B)^{q/2-1} \|b\|_{L^q(B)}^q \int_0^{2^{k+1} r_B} \exp\left\{ -C \frac{[2^k r_B]^2}{t^2} \right\} \frac{dt}{t^{q(1/2+2M)}} \\ &\sim 2^{-2kqM} |B| \, \|\chi_B\|_X^{-q}, \end{aligned} \tag{6.40}$$

where C is a positive constant. The estimation of I_k is similar: replacing Assumption 6.14 by the boundedness of the family $\{(t^2 L)^M e^{-t^2 L}\}_{t>0}$ of operators on $L^q(\mathbb{R}^n)$ (see, for example, [8]), we know that

$$\begin{aligned} \mathrm{I}_k &\lesssim (2^k r_B)^{-4M(q/2-1)} \int_{S_k(B)} \left[\int_{|x-x_B|/4}^{\infty} \int_{B(x,t)} |t^2 L e^{-t^2 L}(b)(y)|^q \, \frac{dy\,dt}{t^{n+4M+1}} \right] dx \\ &\lesssim (2^k r_B)^{-4M(q/2-1)} \|b\|_{L^q(\mathbb{R}^n)}^q \int_{2^{k-3} r_B}^{\infty} \frac{dt}{t^{4M+1}} \sim 2^{-2kqM} |B| \, \|\chi_B\|_X^{-q}, \end{aligned}$$

which, together with (6.39) and (6.40), implies that, for any $k \in \mathbb{N}$ with $k \ge 5$,

$$\mathrm{D}_k \lesssim 2^{-2kqM} |B| \, \|\chi_B\|_X^{-q}.$$

From this and (6.38), it follows that (6.37) holds true, which completes the proof of Proposition 6.18. ■

Now we prove Theorem 6.17 using Proposition 6.18.

Proof of Theorem 6.17. Let $q \in [2, p_L')$ and $M > \frac{n}{2}\left(\frac{1}{\theta} - \frac{1}{p_L'}\right)$. To show Theorem 6.17, it suffices to prove that

$$L^2(\mathbb{R}^n) \cap H_{X,L}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap H_{X,L,\mathrm{at}}^{M,q}(\mathbb{R}^n)$$

with equivalent quasi-norms.

We first show the inclusion

$$L^2(\mathbb{R}^n) \cap H_{X,L}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap H_{X,L,\mathrm{at}}^{M,q}(\mathbb{R}^n). \tag{6.41}$$

For any $f \in L^2(\mathbb{R}^n) \cap H_{X,L}(\mathbb{R}^n)$, by the functional calculus in $L^2(\mathbb{R}^n)$, we know that

$$f = C_{(\Phi,\,M)} \int_0^\infty (t^2 L)^{M+1} \Phi(t\sqrt{L}) t^2 L e^{-t^2 L}(f) \, \frac{dt}{t} = \pi_{\Phi,L,M}(t^2 L e^{-t^2 L}(f))$$

in $L^2(\mathbb{R}^n)$, where $C_{(\Phi,\,M)}$ is as in (6.32) with k replaced by M. Moreover, from the fact that $t^2 L e^{-t^2 L}(f) \in T_X(\mathbb{R}_+^{n+1}) \cap T_2^2(\mathbb{R}_+^{n+1})$, Lemma 6.9(i) and Theorem 3.19, it follows that there exist $\{\lambda_j\}_{j\in\mathbb{N}} \subset [0, \infty)$ and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of (T_X, ∞)-atoms associated with balls $\{B_j\}_{j\in\mathbb{N}}$, respectively, such that

$$t^2Le^{-t^2L}(f)=\sum_{j=1}^{\infty}\lambda_jA_j \quad \text{in } T_2^2(\mathbb{R}_+^{n+1})$$

and

$$\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}})\lesssim\|t^2Le^{-t^2L}(f)\|_{T_X(\mathbb{R}_+^{n+1})}\sim\|f\|_{H_{X,L}(\mathbb{R}^n)}, \tag{6.42}$$

which, together with Proposition 6.18, implies that

$$f=\pi_{\Phi,L,M}(t^2Le^{-t^2L}(f))=\sum_{j=1}^{\infty}\lambda_j\pi_{\Phi,L,M}(A_j) \quad \text{in } L^2(\mathbb{R}^n). \tag{6.43}$$

By the proof of Proposition 6.18, for any $j\in\mathbb{N}$, $\pi_{\Phi,L,M}(A_j)$ is an $(X,M)_L$-atom associated with B_j, which, combined with (6.42) and (6.43), implies that $f\in L^2(\mathbb{R}^n)\cap H^{M,q}_{X,L,\mathrm{at}}(\mathbb{R}^n)$. Thus, (6.41) holds true.

We now prove the inclusion

$$L^2(\mathbb{R}^n)\cap H^{M,q}_{X,L,\mathrm{at}}(\mathbb{R}^n)\subset L^2(\mathbb{R}^n)\cap H_{X,L}(\mathbb{R}^n). \tag{6.44}$$

Let $f\in L^2(\mathbb{R}^n)\cap H^{M,q}_{X,L,\mathrm{at}}(\mathbb{R}^n)$. Then there exist $\{\lambda_j\}_{j\in\mathbb{N}}\in[0,\infty)$ and a sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ of $(X,q,M)_L$-atoms such that $f=\sum_{j=1}^{\infty}\lambda_j\alpha_j$ in $L^2(\mathbb{R}^n)$ and

$$\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}})\lesssim\|f\|_{H^{M,q}_{X,L,\mathrm{at}}(\mathbb{R}^n)}. \tag{6.45}$$

From (6.37) and Theorem 2.11, we deduce that

$$\|S_L(f)\|_X\le\Big\|\sum_{j=1}^{\infty}\lambda_jS_L(\alpha_j)\Big\|_X\lesssim\Lambda(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}),$$

which, together with (6.45), further implies that $\|S_L(f)\|_X\lesssim\|f\|_{H^{M,q}_{X,L,\mathrm{at}}(\mathbb{R}^n)}$ and thus $f\in L^2(\mathbb{R}^n)\cap H_{X,L}(\mathbb{R}^n)$. This finishes the proof of (6.44) and hence of Theorem 6.17. ■

6.4. $H_{X,L}(\mathbb{R}^n)$ with L satisfying Poisson estimates. In this subsection, we study the Hardy type space $H_{X,L}(\mathbb{R}^n)$ associated with the operator L satisfying Poisson estimates. More precisely, we obtain a molecular characterization of $H_{X,L}(\mathbb{R}^n)$.

We begin with the following assumptions on L.

Assumption 6.19. *The linear operator L is one-to-one, has dense range in $L^2(\mathbb{R}^n)$ and a bounded H_∞-calculus in $L^2(\mathbb{R}^n)$.*

Assumption 6.20. *For any $t\in(0,\infty)$, the distribution kernel p_t of e^{-tL} belongs to $L^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ and satisfies the Poisson estimate $|p_t(x,y)|\le h_t(x,y)$ for all $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$, where h_t is given by*

$$h_t(x,y):=t^{-n/m}g\left(\frac{|x-y|}{t^{1/m}}\right) \quad \text{for all } (x,y)\in\mathbb{R}^n\times\mathbb{R}^n, \tag{6.46}$$

where $m\in\mathbb{N}$ and g is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty}r^{n+\epsilon}g(r)=0 \tag{6.47}$$

for some $\epsilon\in(0,\infty)$.

In this subsection, we do not assume that L satisfies Assumptions 6.13 and 6.14.

From Assumptions 6.19 and 6.20, we deduce some useful estimates related to L.

If $\{e^{-tL}\}_{t>0}$ is a bounded analytic semigroup in $L^2(\mathbb{R}^n)$ whose kernels $\{p_t\}_{t>0}$ satisfy (6.46) and (6.47), then, for any $k \in \mathbb{N}$, there exists a positive constant $C_{(k)}$, depending on k, such that, for any $t \in (0, \infty)$ and almost every $x, y \in \mathbb{R}^n$,

$$\left| t^k \frac{\partial^k p_t(x, y)}{\partial t^k} \right| \le \frac{C_{(k)}}{t^{n/m}} g\left(\frac{|x - y|}{t^{1/m}} \right). \tag{6.48}$$

It is worth pointing out that, for any $k \in \mathbb{N}$, the function g may depend on k but it always satisfies (6.47) (see, for example, [108, Theorem 6.17]).

Let $\nu \in [0, \pi/2)$. Then L has a bounded H_∞-calculus in $L^2(\mathbb{R}^n)$ if and only if, for any $\gamma \in (\nu, \pi]$ and each non-zero function $\psi \in \Psi(S^0_\gamma)$, L satisfies the square function estimate and its reverse, namely, there exists a positive constant C such that, for any $f \in L^2(\mathbb{R}^n)$,

$$C^{-1} \|f\|_{L^2(\mathbb{R}^n)} \le \left\{ \int_0^\infty \|\psi_t(L) f\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t} \right\}^{1/2} \le C \|f\|_{L^2(\mathbb{R}^n)}, \tag{6.49}$$

where, for any $t \in (0, \infty)$ and $x \in \mathbb{R}^n$, $\psi_t(x) := \psi(tx)$. Notice that different choices of $\gamma > \nu$ and $\psi \in \Psi(S^0_\gamma)$ lead to equivalent quadratic norms of f (see [98] for details).

For any operator L, let

$$\theta(L) := \sup\{\epsilon \in (0, \infty) : (6.47) \text{ holds true}\}. \tag{6.50}$$

Furthermore, for any $s \in \mathbb{Z}_+$, $f \in L^2(\mathbb{R}^n)$ and $(x, t) \in \mathbb{R}^{n+1}_+$, let

$$P_t(f)(x) := e^{-tL}(f)(x) \quad \text{and} \quad Q_{s,t}(f)(x) := t^{s+1} L^{s+1} e^{-tL}(f)(x). \tag{6.51}$$

In particular, when $s = 0$, we denote $Q_{s,t}$ simply by Q_t. Moreover, by (6.48), we find that the kernels p_t and $q_{s,t}$ of P_t and $Q_{s,t}$, respectively, satisfy, for any $t \in (0, \infty)$ and almost every $x, y \in \mathbb{R}^n$,

$$|p_{t^m}(x, y)| + |q_{s,t^m}(x, y)| \lesssim t^{-n} g(|x - y|/t), \tag{6.52}$$

where m is as in (6.46), g satisfies (6.47) and the implicit positive constant is independent of t, x and y.

Some examples of operators satisfying Assumptions 6.19 and 6.20 are given in the following remark.

Remark 6.21.

(i) If $L := -\Delta$, then $\theta(L) = \infty$ and L satisfies Assumptions 6.19 and 6.20.

(ii) If $L := \sqrt{-\Delta}$, then $\theta(L) = 1$ and L satisfies Assumptions 6.19 and 6.20.

(iii) Let $L := -\mathrm{div}(A\nabla)$ be a second-order divergence form elliptic operator, where $A := \{a_{i,j}\}_{1 \le i,j \le n}$ is an $n \times n$ matrix with real entries $a_{i,j} \in L^\infty(\mathbb{R}^n, \mathbb{R})$ satisfying $\sum_{i,\,j=1}^n a_{i,j}(x) \xi_i \xi_j \ge \lambda |\xi|^2$ for all $x \in \mathbb{R}^n$, $\xi := (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and a certain $\lambda \in (0, \infty)$. In this case, $\theta(L) = \infty$ and L satisfies Assumptions 6.19 and 6.20 (see, for example, [60]).

(iv) Let $L := -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$ with $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where $n \ge 3$. Then $\theta(L) = \infty$ and L satisfies Assumptions 6.19 and 6.20 (see, for example, [61]).

For any $f \in L^2(\mathbb{R}^n)$, the *Lusin-area function* $S_L(f)$ associated with L is defined by setting, for any $x \in \mathbb{R}^n$,

$$S_L(f)(x) := \left\{ \int_{\Gamma(x)} |Q_{t^m}(f)(y)|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}, \tag{6.53}$$

where Q_{t^m} is as in (6.51) and $\Gamma(x)$ as in Definition 3.16. From Assumption 6.19 and (6.49), it follows that S_L is bounded on $L^2(\mathbb{R}^n)$. Moreover, Auscher et al. [6] proved that, for any $p \in (1,\infty)$, there exists a positive constant $C_{(p)}$ such that, for any $f \in L^p(\mathbb{R}^n)$,

$$C_{(p)}^{-1}\|f\|_{L^p(\mathbb{R}^n)} \le \|S_L(f)\|_{L^p(\mathbb{R}^n)} \le C_{(p)}\|f\|_{L^p(\mathbb{R}^n)}. \tag{6.54}$$

Now we introduce the Hardy type space $H_{X,L}(\mathbb{R}^n)$ associated with L in a similar way to Definition 6.6.

Definition 6.22. Let X be a ball quasi-Banach space and suppose the operator L satisfies Assumptions 6.19 and 6.20. Then the *Hardy type space* $H_{X,L}(\mathbb{R}^n)$ is defined by replacing the operator L, satisfying Assumptions 6.3 and 6.4, and the Lusin-area function $S_L(f)$ in (6.3) of Definition 6.6, respectively, by the operator L, satisfying Assumptions 6.19 and 6.20, and $S_L(f)$ in (6.53).

Definition 6.23. Let X be a ball quasi-Banach function space and let L satisfy Assumptions 6.19 and 6.20. Assume that $q \in (1,\infty)$, $M \in \mathbb{N}$ and $\eta \in (0,\infty)$. Denote by $\mathcal{R}(L^M)$ the range of L^M. A function $\alpha \in L^q(\mathbb{R}^n)$ is called an $(X,q,M,\eta)_L$-*molecule* associated with the ball $B := B(x_B,r_B) \subset \mathbb{R}^n$ for some $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$ if $\alpha \in \mathcal{R}(L^M)$ and, for each $k \in \{0,\dots,M\}$ and $j \in \mathbb{Z}_+$,

$$\|(r_B^{-m}L^{-1})^k\alpha\|_{L^q(S_j(B))} \le 2^{-j\eta}|2^jB|^{1/q}\|\chi_B\|_X^{-1}.$$

Moreover, if α is an $(X,q,M,\eta)_L$-molecule associated with a ball B for all $q \in (1,\infty)$, then α is called an $(X, M, \eta)_L$-*molecule* associated with B.

Then the *molecular Hardy space* $H_{X,L}^{M,q,\eta}(\mathbb{R}^n)$ is defined via replacing $(X,q,M,\epsilon)_L$-molecules of Definition 6.7(ii) by $(X,q,M,\eta)_L$-molecules here.

Let $M \in \mathbb{N}$ and $M > n/(m\theta)$, where m and θ are as in (6.46) and (2.8), respectively. For any $f \in L_c^2(\mathbb{R}_+^{n+1})$ and $x \in \mathbb{R}^n$, define

$$\widetilde{\pi}_{L,M}(f)(x) := \widetilde{C}_{(m,M)} \int_0^\infty (t^mL)^{M+1}e^{-t^mL}(f(\cdot,t))(x)\,\frac{dt}{t}, \tag{6.55}$$

where $\widetilde{C}_{(m,M)}$ is a positive constant such that

$$\widetilde{C}_{(m,M)} \int_0^\infty t^{m(M+2)}e^{-2t^m}\,\frac{dt}{t} = 1.$$

For the operator $\widetilde{\pi}_{L,M}$, similarly to Proposition 6.8, we have the following boundedness.

Proposition 6.24. *Let L satisfy Assumptions* 6.19 *and* 6.20, *and $\widetilde{\pi}_{L,M}$ be as in* (6.55). *Assume that X is a ball quasi-Banach function space satisfying* (2.8) *for some $\theta, s \in (0,1]$ with $\theta < s$, and* (2.9) *for some $q \in [2,\infty)$. Let $M \in \mathbb{N}$ be such that $M > n/(m\theta)$ and $\theta(L) > n(1/\theta - 1) + 1/2$, where $\theta(L)$ is as in* (6.50). *Assume further that X has an absolutely continuous quasi-norm. Then*

(i) *the operator* $\widetilde{\pi}_{L,M}$*, initially defined on* $T_2^{p,c}(\mathbb{R}^{n+1}_+)$ *with* $p \in (1,\infty)$*, extends to a bounded linear operator from* $T_2^p(\mathbb{R}^{n+1}_+)$ *to* $L^p(\mathbb{R}^n)$*;*
(ii) *the operator* $\widetilde{\pi}_{L,M}$*, initially defined on* $T_X^c(\mathbb{R}^{n+1}_+)$*, extends to a bounded linear operator from* $T_X(\mathbb{R}^{n+1}_+)$ *to* $H_{X,L}(\mathbb{R}^n)$*.*

To show Proposition 6.24(ii), similarly to Lemma 6.10, we need the following technical lemma.

LEMMA 6.25. *Let* L *satisfy Assumptions* 6.19 *and* 6.20*,* $\widetilde{\pi}_{L,M}$ *be as in* (6.55) *and* X *as in Proposition* 6.24*. Assume that* $M \in \mathbb{N}$ *satisfies* $M > n/(m\theta)$ *and* $\theta(L) > n(1/\theta - 1) + 1/2$*, where* $\theta(L)$ *is as in* (6.50)*. Then, for any* (T_X,∞)*-atom* A *and some*

$$\eta \in (n[1/\theta - 1] + 1/2, \infty),$$

$\alpha := \pi_{L,M}(A)$ *is an* $(X,\, M,\, \eta + n)_L$*-molecule, up to a harmless constant multiple, associated with the ball* B*, where* $\widehat{B}$ *appears in the support of* A*.*

Proof. Assume that $\eta \in (n[1/\theta - 1] + 1/2, \infty)$, A is a (T_X,∞)-atom associated with the ball $B := B(x_B, r_B)$ for some $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, and $q \in [2,\infty)$. To prove Lemma 6.25, similarly to the proof of Lemma 6.10, it suffices to show that $\alpha := \widetilde{\pi}_{L,M}(A)$ is an $(X, q, M, n+\eta)_L$-molecule, up to a harmless constant multiple, associated with B.

For any $k \in \mathbb{N}$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$S_{L^*,k}(f)(x) := \left\{ \int_{\Gamma(x)} |(t^m L^*)^{k+1} e^{-t^m L^*}(f)(y)|^2 \, \frac{dy\, dt}{t^{n+1}} \right\}^{1/2}.$$

Since $\{e^{-tL}\}_{t>0}$ is an analytic semigroup in $L^2(\mathbb{R}^n)$, so is $\{e^{-tL^*}\}_{t>0}$. Let $p \in (1,\infty)$. Similarly to the proof of (6.54), we can show that, for any $f \in L^p(\mathbb{R}^n)$,

$$\|S_{L^*,k}(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{6.56}$$

From the assumption $\theta(L) > n(1/\theta - 1) + 1/2$, we deduce that there exists a number $\eta \in (n[1/\theta - 1] + 1/2, \min\{\theta(L), n/\theta\})$. When $j \in \{0,\dots,4\}$, we deal with the term $\|(r_B^{-m} L^{-1})^k \alpha\|_{L^q(S_j(B))}$ via duality; take $h \in L^{q'}(\mathbb{R}^n)$ satisfying $\mathrm{supp}(h) \subset S_j(B)$ and $\|h\|_{L^{q'}(\mathbb{R}^n)} \le 1$. Then, by (6.56) and Hölder's inequality, we conclude that, for any k in $\{0,\dots,M\}$,

$$\begin{aligned}
|\langle (r_B^{-m} L^{-1})^k \alpha, h\rangle| &= \left| \int_0^\infty \int_{\mathbb{R}^n} r_B^{-mk} t^{m(M+1)} L^{M+1-k} e^{-t^m L}(A(\cdot,t))(x) h(x) \, \frac{dx\, dt}{t} \right| \\
&= \left| \int_0^\infty \int_{\mathbb{R}^n} r_B^{-mk} A(x,t) t^{m(M+1)} (L^*)^{M+1-k} e^{-t^m L^*}(h)(x) \, \frac{dx\, dt}{t} \right| \\
&\le \int_0^{r_B} \int_{\mathbb{R}^n} \int_{B(x,t)} t^{mk} r_B^{-mk} |A(x,t)| \\
&\qquad \times |(t^m L^*)^{M+1-k} e^{-t^m L^*}(h)(x)| \, \frac{dy\, dx\, dt}{t^{n+1}} \\
&\le \int_{\mathbb{R}^n} \int_{\Gamma(y)} |A(x,t)| |(t^m L^*)^{M+1-k} e^{-t^m L^*}(h)(x)| \, \frac{dx\, dt\, dy}{t^{n+1}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}^n} \mathcal{A}(A)(y) S_{L^*,M+1-k}(h)(y)\,dy \\
&\leq \|\mathcal{A}(A)\|_{L^q(\mathbb{R}^n)} \|S_{L^*,M+1-k}(h)\|_{L^{q'}(\mathbb{R}^n)} \lesssim \|A\|_{T_2^q(\mathbb{R}^{n+1}_+)}.
\end{aligned}
$$

Since $\eta + n > n/\theta > n/q$, it follows that

$$
|\langle (r_B^{-m} L^{-1})^k \alpha, h\rangle| \lesssim 2^{-j(n+\eta)} |2^j B|^{1/q} \|\chi_B\|_X^{-1},
$$

which, by the arbitrariness of $h \in L^{q'}(\mathbb{R}^n)$, implies that, for any $j \in \{0,\ldots,4\}$ and $k \in \{0,\ldots,M\}$,

$$
\|(r_B^{-m} L^{-1})^k \alpha\|_{L^q(S_j(B))} \lesssim 2^{-j(n+\eta)} |2^j B|^{1/q} \|\chi_B\|_X^{-1}. \tag{6.57}
$$

Moreover, for any $k \in \{0,\ldots,M\}$, $t \in (0,\infty)$, $h \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$
\Psi_t^k(L,M)(h)(x) := t^{m(M+1-k)} \left[\left. \frac{d^{M+1-k} P_r}{dr^{M+1-k}} \right|_{r=t^m} h \right](x),
$$

where P_r is as in (6.51) with t replaced by r. From (6.47) and (6.52), together with $\eta \in (n[1/\theta - 1] + 1/2, \theta(L))$, it follows that the kernel of $\Psi_t^k(L,M)$, $\psi_t^k(L,M)$, satisfies, for any $t \in (0,\infty)$ and almost every $x, y \in \mathbb{R}^n$,

$$
|\psi_t^k(L,M)(x,y)| \leq C_{(k,n)} \frac{t^\eta}{(t + |x-y|)^{n+\eta}}, \tag{6.58}
$$

where $C_{(k,n)}$ is a positive constant depending only on k and n. From this pointwise estimate, we deduce that, for any $k \in \{0,\ldots,M\}$ and $x \in \mathbb{R}^n$,

$$
\begin{aligned}
|(r_B^{-m} L^{-1})^k \alpha(x)| &\sim \int_0^\infty |(r_B^{-m} L^{-1})^k t^{m(M+1)} L^{M+1} e^{-t^m L}(A(\cdot,t))(x)| \,\frac{dt}{t} \\
&\lesssim \int_0^\infty |r_B^{-mk} t^{mk} \Psi_t^k(L,M)(A(\cdot,t))(x)| \,\frac{dt}{t} \\
&\lesssim r_B^{-mk} \int_0^{r_B} \int_B t^{mk} \frac{t^\eta}{(t+|x-y|)^{n+\eta}} |A(y,t)| \,\frac{dy\,dt}{t} \\
&\lesssim r_B^{-mk} \int_0^{r_B} \int_B \int_{B(y,t)} \frac{t^{mk+\eta}}{(t+|x-y|)^{n+\eta}} |A(y,t)| \,\frac{dz\,dy\,dt}{t^{n+1}}.
\end{aligned}
$$

By Fubini's theorem, Hölder's inequality and $\operatorname{supp}(A) \subset \widehat{B}$, for any $x \in \mathbb{R}^n$, we have

$$
\begin{aligned}
|(r_B^{-m} L^{-1})^k \alpha(x)| &\lesssim r_B^{-mk} \int_{\mathbb{R}^n} \int_{\Gamma(z)} \frac{t^{mk+\eta} \chi_{\widehat{B}}(y,t)}{(t+|x-y|)^{n+\eta}} |A(y,t)| \,\frac{dy\,dt\,dz}{t^{n+1}} \\
&\lesssim r_B^{-mk} \int_{\mathbb{R}^n} \mathcal{A}(A)(z) \left[\int_{\Gamma(z)} \left| \frac{t^{mk+\eta} \chi_{\widehat{B}}(y,t)}{(t+|x-y|)^{n+\eta}} \right|^2 \frac{dy\,dt}{t^{n+1}} \right]^{1/2} dz \\
&\lesssim r_B^{-mk} \|\mathcal{A}(A)\|_{L^q(\mathbb{R}^n)} \\
&\quad \times \left\{ \int_{\mathbb{R}^n} \left[\int_{\Gamma(z)} \left| \frac{t^{mk+\eta} \chi_{\widehat{B}}(y,t)}{(t+|x-y|)^{n+\eta}} \right|^2 \frac{dy\,dt}{t^{n+1}} \right]^{q'/2} dz \right\}^{1/q'}.
\end{aligned} \tag{6.59}
$$

When $j \in \mathbb{N}$ and $j \geq 5$, we know that, for any $x \in S_j(B)$ and $y \in B$, $|x-y| \sim 2^j r_B$.

From this and (6.59), we deduce that, for any $k \in \{0, \ldots, M\}$,

$$\begin{aligned}\|(r_B^{-m}L^{-1})^k\alpha\|_{L^q(S_j(B))} &\lesssim r_B^{-mk}|2^jB|^{1/q}\|A\|_{T_2^q(\mathbb{R}^n)}\\ &\quad\times\left\{\int_{\mathbb{R}^n}\left[\int_{\Gamma(z)}\left|\frac{t^{mk+\eta}\chi_{\widehat{B}}(y,t)}{(2^jr_B)^{n+\eta}}\right|^2\frac{dy\,dt}{t^{n+1}}\right]^{q'/2}dz\right\}^{1/q'}\\ &\lesssim 2^{-j(n+\eta)}|2^jB|^{1/q}\|\chi_B\|_X^{-1}. \qquad (6.60)\end{aligned}$$

Thus, by (6.57) and (6.60), α is an $(X, q, M, n+\eta)$-molecule, up to a harmless constant multiple, associated with B, which completes the proof of Lemma 6.25. ∎

Now we prove Proposition 6.24 using Lemma 6.25.

Proof of Proposition 6.24. The proof of (i) is similar to that of Proposition 6.8(i), the details being omitted.

Moreover, the proof of (ii) is similar to that of Proposition 6.8(ii). Here we content ourselves with indicating the necessary changes.

Let $f \in T_X^c(\mathbb{R}_+^{n+1})$. Then, by Theorem 3.19, Lemma 6.9 and (i),

$$\widetilde{\pi}_{L,M}(f) = \sum_{k=1}^{\infty}\lambda_k\widetilde{\pi}_{L,M}(A_k) =: \sum_{k=1}^{\infty}\lambda_k\alpha_k \quad \text{in } L^2(\mathbb{R}^n),$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{A_k\}_{k\in\mathbb{N}}$ satisfy (3.11) and (6.11), respectively. Recall that, for each $k \in \mathbb{N}$, $\mathrm{supp}(A_k) \subset \widehat{B}_k$ and B_k is a ball in $\mathbb{R}^n$. Moreover, from the boundedness of S_L on $L^2(\mathbb{R}^n)$, we deduce that, for almost every $x \in \mathbb{R}^n$,

$$S_L(\widetilde{\pi}_{L,M}(f))(x) \le \sum_{k=1}^{\infty}\lambda_kS_L(\alpha_k)(x),$$

which, together with Definition 2.2(ii), implies that (6.12) holds true in this case. Furthermore, by Lemma 6.25, we know that, for some $\eta \in (n[1/\theta - 1] + 1/2, \infty)$ and each $k \in \mathbb{N}$, $\alpha_k = \widetilde{\pi}_{L,M}(A_k)$ is an $(X,\ M,\ n+\eta)_L$-molecule, up to a harmless constant multiple, associated with the ball B_k.

Let α be an $(X, M, n+\eta)$-molecule with $\eta \in (n[1/\theta - 1] + 1/2, \infty)$, and $q \in [2, \infty)$. Then, for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned}S_L(\alpha)(x) &\le \sum_{j=0}^{\infty}\left\{\int_0^{r_B}\int_{B(x,t)}|Q_{t^m}(\alpha\chi_{S_j(B)})(y)|^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\ &\quad+\sum_{j=0}^{\infty}\left\{\int_{r_B}^{\infty}\int_{B(x,t)}|Q_{t^m}(r_B^mL)^M\right.\\ &\qquad\left.\times(\chi_{S_j(B)}(r_B^mL)^{-M}\alpha)(y)|^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\ &=: \sum_{j=0}^{\infty}\mathrm{J}_j(x) + \sum_{j=0}^{\infty}\mathrm{K}_j(x). \qquad (6.61)\end{aligned}$$

For any $i, j \in \mathbb{Z}_+$, let $B_j := 2^jB$ and

$$\widetilde{S}_i(B_j) := \{y \in \mathbb{R}^n : 2^{i-3}2^jr_B \le |y - x_B| < 2^{i+1}2^jr_B\}.$$

Take $q\in[2,\infty)$ such that X satisfies (2.9) for q. Now we estimate $\int_{S_i(B_j)}[\mathrm{J}_j(x)]^q\,dx$. When $i\in\{0,\dots,4\}$, by the boundedness of S_L on $L^q(\mathbb{R}^n)$, we have, for any $j\in\mathbb{Z}_+$,

$$\begin{aligned}\int_{S_i(B_j)}[\mathrm{J}_j(x)]^q\,dx&\le\int_{S_i(B_j)}[S_L(\alpha\chi_{S_j(B)})(x)]^q\,dx\\&\lesssim\|\alpha\chi_{S_j(B)}\|^q_{L^q(\mathbb{R}^n)}\lesssim 2^{-(n+\eta)jq}|2^jB|\,\|\chi_B\|_X^{-q}.\end{aligned}\tag{6.62}$$

From Hölder's inequality, Fubini's theorem, (6.58) and the fact that, for any $i\in\mathbb{N}$ with $i\ge5$, $d(S_j(B),S_i(B_j))\gtrsim 2^{i+j}r_B$, it follows that, for each $i\in\mathbb{N}$ with $i\ge5$ and $j\in\mathbb{Z}_+$,

$$\begin{aligned}\int_{S_i(B_j)}[\mathrm{J}_j(x)]^q\,dx&\le\int_{S_i(B_j)}\left\{\int_0^{r_B}\int_{B(x,t)}|Q_{t^m}(\alpha\chi_{S_j(B)})(y)|^q\,\frac{dy\,dt}{t^nt^{q/2}}\right\}\\&\quad\times\left\{\int_0^{r_B}\int_{B(x,t)}\frac{dy\,dt}{t^n}\right\}^{(q-2)/2}dx\\&\lesssim r_B^{(q-2)/2}\int_0^{r_B}\int_{\widetilde{S}_i(B_j)}|Q_{t^m}(\alpha\chi_{S_j(B)})(y)|^q\,\frac{dy\,dt}{t^{q/2}}\\&\lesssim r_B^{(q-2)/2}\int_0^{r_B}\int_{\widetilde{S}_i(B_j)}\left[\int_{S_j(B)}\frac{t^\eta|\alpha(z)|}{(t+|y-z|)^{n+\eta}}\,dz\right]^q\frac{dy\,dt}{t^{q/2}}\\&\lesssim r_B^{(q-2)/2}[2^{i+j}r_B]^{n-(n+\eta)q}\|\alpha\|^q_{L^1(S_j(B))}r_B^{(\eta-1/2)q+1}\\&\lesssim 2^{-i(n+\eta)q}2^{-j(n+2\eta)q}|2^{i+j}B|\,\|\chi_B\|_X^{-q}.\end{aligned}\tag{6.63}$$

Now we estimate $\int_{S_i(B_j)}[\mathrm{K}_j(x)]^q\,dx$. When $i\in\{0,\dots,4\}$, by the boundedness of $S_{L,M+1}$ on $L^q(\mathbb{R}^n)$, similarly to the proof of (6.16), we know that, for any $j\in\mathbb{Z}_+$,

$$\int_{S_i(B_j)}[\mathrm{K}_j(x)]^q\,dx\lesssim 2^{-(n+\eta)qj}|2^jB|\,\|\chi_B\|_X^{-q}.\tag{6.64}$$

Notice that, for any $i,j\in\mathbb{Z}_+$ and $x\in\mathbb{R}^n$,

$$\begin{aligned}\mathrm{K}_j(x)&\le\left\{\int_{r_B}^{2^{i+j-3}r_B}\int_{B(x,t)}|Q_{t^m}(r_B^mL)^M\big(\chi_{S_j(B)}(r_B^mL)^{-M}\alpha\big)(y)|^2\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\&\quad+\left\{\int_{2^{i+j-3}r_B}^\infty\cdots\right\}^{1/2}\\&=:\mathrm{K}_{j,1}(x)+\mathrm{K}_{j,2}(x).\end{aligned}\tag{6.65}$$

For $\mathrm{K}_{j,1}$, similarly to (6.63), we know that, for any $i\in\mathbb{N}$ with $i\ge5$ and $j\in\mathbb{Z}_+$,

$$\int_{S_i(B_j)}[\mathrm{K}_{j,1}(x)]^q\,dx\lesssim 2^{-i[(n+\eta-1/2)q+1]}2^{-j[(n+2\eta-1/2)q+1]}|2^{i+j}B|\,\|\chi_B\|_X^{-q}.\tag{6.66}$$

For $\mathrm{K}_{j,2}$, from the boundedness of $S_{L,M+1}$ on $L^q(\mathbb{R}^n)$, we deduce that

$$\begin{aligned}\int_{S_i(B_j)}[\mathrm{K}_{j,2}(x)]^q\,dx&\lesssim\frac{r_B^{mMq}}{(2^{i+j}r_B)^{mMq}}\int_{S_i(B_j)}[S_{L,M+1}(\chi_{S_j(B)}(r_B^mL)^{-M}\alpha)(x)]^q\,dx\\&\lesssim 2^{-mMq(i+j)}\|(r_B^mL)^{-M}\alpha\|^q_{L^q(S_j(B))}\\&\lesssim 2^{-i(mMq+n)}2^{-j(mM+n+\eta)q}|2^{i+j}B|\,\|\chi_B\|_X^{-q},\end{aligned}$$

which, together with (6.65), (6.66), $M > n/(m\theta)$ and $\eta < n/\theta$, implies that, for any $i \in \mathbb{N}$ with $i \geq 5$ and $j \in \mathbb{Z}_+$,

$$\int_{S_i(B_j)} [\mathrm{K}_j(x)]^q\,dx \lesssim 2^{-i[(n+\eta-1/2)q+1]} 2^{-j[(n+2\eta-1/2)q+1]} |2^{i+j}B|\,\|\chi_B\|_X^{-q}.$$

From this and (6.62)–(6.64), we conclude that, for any $i, j \in \mathbb{Z}_+$,

$$\|\chi_{B_j}\mathrm{J}_j\|_{L^q(\mathbb{R}^n)} \lesssim 2^{-j(n+\eta)}|2^jB|^{1/q}\|\chi_B\|_X^{-1}, \tag{6.67}$$

$$\|\chi_{S_i(B_j)}\mathrm{J}_j\|_{L^q(\mathbb{R}^n)} \lesssim 2^{-i(n+\eta-n/q)}2^{-j(n+2\eta)}|2^jB|^{1/q}\|\chi_B\|_X^{-1}, \tag{6.68}$$

$$\|\chi_{B_j}\mathrm{K}_j\|_{L^q(\mathbb{R}^n)} \lesssim 2^{-j(n+\eta)}|2^jB|^{1/q}\|\chi_B\|_X^{-1} \tag{6.69}$$

and

$$\|\chi_{S_i(B_j)}\mathrm{K}_j\|_{L^q(\mathbb{R}^n)} \lesssim 2^{-i(n+\eta+1/q-1/2-n/q)}2^{-j(n+2\eta+1/q-1/2)}|2^jB|^{1/q}\|\chi_B\|_X^{-1}. \tag{6.70}$$

For each α_k with $k \in \mathbb{N}$, let $\{\mathrm{J}_{k,j}\}_{j\in\mathbb{Z}_+}$ and $\{\mathrm{K}_{k,j}\}_{j\in\mathbb{Z}_+}$ be as in (6.61) with α replaced by α_k. Then, for any $k \in \mathbb{N}$, $\{\mathrm{J}_{k,j}\}_{j\in\mathbb{Z}_+}$ and $\{\mathrm{K}_{k,j}\}_{j\in\mathbb{Z}_+}$ satisfy the estimates in (6.67)–(6.70) with B replaced by B_k. Moreover, from $\eta > n(1/\theta - 1) + 1/2$ and $q \geq 2$, it follows that

$$n + 2\eta > n + 2\eta + 1/q - 1/2 > n + \eta > n/\theta + 1/2$$

and

$$n + \eta - n/q \geq n + \eta + 1/q - 1/2 - n/q > n(1/\theta - 1/q).$$

From this, (6.55), (6.11), (6.12), (6.61), (6.67)–(6.70), (2.8) and Theorem 2.11, similarly to the proof of (6.23), we conclude that

$$\|S_L(\widetilde{\pi}_{L,M}(f))\|_X \lesssim \|f\|_{T_X(\mathbb{R}^{n+1}_+)},$$

which further implies that

$$\|\widetilde{\pi}_{L,M}(f)\|_{H_{X,L}(\mathbb{R}^n)} \lesssim \|f\|_{T_X(\mathbb{R}^{n+1}_+)}.$$

This finishes the proof of (ii) and hence of Proposition 6.24. ■

Repeating the proof of Theorem 6.12 with the application of Proposition 6.8 replaced by Proposition 6.24, we obtain the following molecular characterization of the Hardy type space $H_{X,L}(\mathbb{R}^n)$, the details being omitted here.

THEOREM 6.26. *Let L satisfy Assumptions* 6.19 *and* 6.20*, $q \in [2,\infty)$ and X be a ball quasi-Banach function space satisfying* (2.8) *for some $\theta, s \in (0,1]$ with $\theta < s$, and* (2.9) *for q. Assume that $\theta(L) > n(1/\theta - 1) + 1/2$, $\eta \in (n[1/\theta - 1] + 1/2, \min\{\theta(L), n/\theta\})$ and $M \in \mathbb{N}$ satisfies $M > n/(m\theta)$, where $\theta(L)$ is as in* (6.50)*. Assume further that X has an absolutely continuous quasi-norm. Then $H_{X,L}(\mathbb{R}^n)$ and $H^{M,q,n+\eta}_{X,L}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Finally, let us look back on the results on Hardy spaces associated with operators. The strong thrust for the study of Hardy spaces associated with operators is the Kato conjecture solved affirmatively in [7, 60]. Let L be a linear operator in $L^2(\mathbb{R}^n)$ satisfying Assumptions 6.19 and 6.20. Auscher et al. [6] initially studied the Hardy space

$H^1_L(\mathbb{R}^n)$ associated with L. Moreover, Duong and Yan introduced the BMO type space $\mathrm{BMO}_L(\mathbb{R}^n)$ associated with L, and established the dual relation between $H^1_L(\mathbb{R}^n)$ and $\mathrm{BMO}_{L^*}(\mathbb{R}^n)$, where L^* denotes the adjoint operator of L in $L^2(\mathbb{R}^n)$ (see [34, 35]). Furthermore, Yan [127] further generalized these results to the Hardy spaces $H^p_L(\mathbb{R}^n)$ with p slightly less than 1. The Orlicz–Hardy space $H_{\Phi,\,L}(\mathbb{R}^n)$ and its (pre-)dual space were studied in [74, 76, 89]. Very recently, the variable Hardy space associated with L was studied in [138, 139, 144].

Let L be the second-order divergence form elliptic operator on $\mathbb{R}^n$ with complex bounded measurable coefficients. Hofmann et al. [62, 63] studied the Hardy space $H^p_L(\mathbb{R}^n)$ associated with L, where $p \in (0, 1]$, and its dual space. Moreover, Jiang and Yang [72, 73] studied the Orlicz–Hardy space $H_{\Phi,\,L}(\mathbb{R}^n)$ associated with L and its (pre-)dual space. Recently, the Musielak–Orlicz–Hardy space $H_{\varphi,\,L}(\mathbb{R}^n)$ was studied in [14]. As was explained in [62], investigating $H^1_L(\mathbb{R}^n)$ is not a mere quest to generalization. Indeed, using a counterexample showing that the Riesz transform $\nabla L^{-1/2}$ is not bounded on $L^p(\mathbb{R}^n)$ for some $p \in (1, 2)$, Hofmann and Mayboroda showed that the Riesz transform $\nabla L^{-1/2}$ need not be bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, but it is bounded from $H^1_L(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ (see [62, Theorem 3.4]). We refer the reader to [3, 15, 16, 63, 64, 72, 130, 131, 132] for more recent progress in this direction.

Let L be a non-negative self-adjoint operator in $L^2(\mathcal{X})$ satisfying the Davies–Gaffney estimate, where $\mathcal{X}$ is a metric measure space with doubling measure. Hofmann et al. [61] studied the Hardy space $H^1_L(\mathcal{X})$ associated with L and its dual space $\mathrm{BMO}_L(\mathcal{X})$, which were extended to the Orlicz–Hardy space in [75] and the Musielak–Orlicz–Hardy space in [132], respectively. It is worth pointing out that the Schrödinger operator $L := -\Delta + V$ with $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and the magnetic Schrödinger operator $L_A := -(\nabla - iA)^2 + V$ with $A \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ are examples of this kind of operators.

Let L be a sectorial operator having a bounded H_∞-functional calculus on $L^2(\mathcal{X})$ and satisfying the reinforced (p_L, q_L, m) off-diagonal estimates on balls, where $\mathcal{X}$ is a metric measure space with doubling measure, $p_L \in [1, 2)$, $q_L \in (2, \infty]$ and $m \in \mathbb{N}$. The Musielak–Orlicz–Hardy space $H_{\varphi,\,L}(\mathcal{X})$ was studied in [14]. In particular, when $m := 1$, Duong and Li [33] studied the Hardy space $H^p_L(\mathcal{X})$ with $p \in (0, 1]$ and its dual space. Recently, when $\mathcal{X} := \mathbb{R}^n$ and $m := 1$, the variable Hardy spaces associated with L were studied in [138].

7. Examples of function spaces

In this section, we present several concrete examples of ball quasi-Banach function spaces X and associated Hardy type spaces $H_X(\mathbb{R}^n)$.

7.1. Weighted Lebesgue spaces. If X is a weighted Lebesgue space, then $H_X(\mathbb{R}^n)$ is just a *weighted Hardy space*. The weighted Hardy space $H^p_w(\mathbb{R}^n)$ defined via weighted Lebesgue spaces, with Muckenhoupt weights, was studied in [12, 39, 123].

Now we recall the definition of Muckenhoupt weights.

Definition 7.1. Let $\mathbb{B}$ be as in (2.2). A locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is called an $A_p(\mathbb{R}^n)$-*weight*, with $p \in (1, \infty)$, if

$$A_p(\omega) := \sup_{B\in\mathbb{B}} \left\{\frac{1}{|B|}\int_B \omega(x)\,dx\right\}\left\{\frac{1}{|B|}\int_B [\omega(x)]^{-p'/p}\,dx\right\}^{p/p'} < \infty.$$

Moreover, a locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is called an $A_1(\mathbb{R}^n)$-*weight* if

$$A_1(\omega) := \frac{1}{|B|}\int_B \omega(x)\,dx \left\{\operatorname*{ess\,sup}_{x\in B}\,[\omega(x)]^{-1}\right\} < \infty.$$

Then define $A_\infty(\mathbb{R}^n) := \bigcup_{p\in[1,\infty)} A_p(\mathbb{R}^n)$.

It is worth pointing out that a weighted Lebesgue space with an $A_\infty(\mathbb{R}^n)$-weight may not be a Banach function space. For example, let $X := L^2_{w_0}(\mathbb{R})$, $w_0(x) := 1 + |x|$ for any $x \in \mathbb{R}$, and $E = \bigcup_{l=1}^\infty [2^l, 2^l + 2^{-l}]$. Then, by a simple calculation, we find that $|E| < \infty$ but $\|\chi_E\|_X = \infty$. Thus, X is not a Banach function space.

On the other hand, let $q \in (0, \infty)$ and $w \in A_\infty(\mathbb{R}^n)$. From the definition of $A_\infty(\mathbb{R}^n)$, it follows that, for any $B \in \mathbb{B}$ with $\mathbb{B}$ as in (2.2), $\chi_B \in L^q_w(\mathbb{R}^n)$ (see (2.20) for the definition of $L^q_w(\mathbb{R}^n)$). Therefore, $L^q_w(\mathbb{R}^n)$ is a ball quasi-Banach function space. Furthermore, by [2, Theorem 3.1(b)], we know that, for any $p \in (0, \infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$, the space $L^p_w(\mathbb{R}^n)$ satisfies the inequality (2.8) for any $\theta \in (0, \min\{1, p\})$ satisfying $\omega \in A_{p/\theta}(\mathbb{R}^n)$ and any $s \in (\theta, 1]$.

We refer the reader to [123] for more details on weighted Hardy spaces.

7.2. Herz spaces. Recall that, for the cube $Q(\vec{0}_n, 1)$ and $j \in \mathbb{N}$,

$$S_j(Q(\vec{0}_n, 1)) := Q(\vec{0}_n, 2^{j+1}) \setminus Q(\vec{0}_n, 2^j).$$

Then, for any $\alpha \in \mathbb{R}$ and $p, q \in (0, \infty]$, the *Herz space* $K^\alpha_{p,q}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{K^\alpha_{p,q}(\mathbb{R}^n)} := \|\chi_{Q(\vec{0}_n,2)} f\|_{L^p(\mathbb{R}^n)} + \left\{\sum_{j=1}^\infty [2^{j\alpha}\|\chi_{S_j(Q(\vec{0}_n,1))} f\|_{L^p(\mathbb{R}^n)}]^q\right\}^{1/q} < \infty.$$

In 1989, Chen and Lau [20] and García-Cuerva [40] introduced Herz–Hardy spaces. Later, García-Cuerva and Herrero [41] and Lu and Yang [95] further developed the theory of these spaces. Moreover, Lu and Yang [96] considered weighted Herz–Hardy spaces.

Let $\alpha \in \mathbb{R}$ and $p, q \in (0, \infty]$. It is easy to see that, for any $B \in \mathbb{B}$ with $\mathbb{B}$ as in (2.2), $\chi_B \in K^\alpha_{p,q}(\mathbb{R}^n)$. Therefore, $K^\alpha_{p,q}(\mathbb{R}^n)$ is a ball quasi-Banach function space. Furthermore, according to [97, Corollary 1.2.1], when $p, q \in (1, \infty)$,

$$(K^\alpha_{p,q}(\mathbb{R}^n))^* = K^{-\alpha}_{p',q'}(\mathbb{R}^n) \quad \text{with equivalent norms,}$$

which implies that, for any $f \in K^\alpha_{p,q}(\mathbb{R}^n)$ and $B \in \mathbb{B}$ with $\mathbb{B}$ as in (2.2),

$$\left|\int_{\mathbb{R}^n} \chi_B(x) f(x)\,dx\right| \le \|\chi_B\|_{K^{-\alpha}_{p',q'}(\mathbb{R}^n)} \|f\|_{K^\alpha_{p,q}(\mathbb{R}^n)}.$$

Thus, when $p, q \in (1, \infty)$, $K^\alpha_{p,q}(\mathbb{R}^n)$ is a ball Banach function space.

Moreover, for any $p, q \in (0, \infty)$ and $\alpha \in (-n/p, \infty)$, the space $K^{\alpha}_{p,q}(\mathbb{R}^n)$ satisfies the inequality (2.8) for any $\theta \in (0, \min\{1, p, [\alpha/n + 1/p]^{-1}\})$ and $s \in (\theta, 1]$ (see, for example, [67, 96]).

7.3. Lorentz spaces. Recall that the *Lorentz space* $L^{p,q}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that, when $p, q \in (0, \infty)$,

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} := \left\{ \int_0^\infty [t^{1/p} f^*(t)]^q \, \frac{dt}{t} \right\}^{1/q} < \infty,$$

and, when $p \in (0, \infty)$ and $q = \infty$,

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} := \sup_{t \in (0,\infty)} t^{1/p} f^*(t) < \infty,$$

where f^* denotes the decreasing rearrangement of f, which is defined by setting, for any $t \in [0, \infty)$,

$$f^*(t) := \inf\{s \in (0, \infty) : \mu_f(s) \leq t\}$$

with $\mu_f(s) := |\{x \in \mathbb{R}^n : |f(x)| > s\}|$.

The space $L^{p,\infty}(\mathbb{R}^n)$ with $p \in (0, \infty)$ is also called a *weak Lebesgue space*. Obviously, when $p, q \in (1, \infty)$ or $p \in (1, \infty)$ and $q = \infty$, the Lorentz space $L^{p,q}(\mathbb{R}^n)$ is a Banach function space and hence a ball Banach function space; when $p, q \in (0, \infty)$ or $p \in (0, \infty)$ and $q = \infty$, $L^{p,q}(\mathbb{R}^n)$ is a quasi-Banach function space and hence a ball quasi-Banach function space (see, for example, [43, Theorem 1.4.11]). Furthermore, for any $p \in (0, \infty)$ and $q \in (0, \infty]$, the space $L^{p,q}(\mathbb{R}^n)$ satisfies the inequality (2.8) for any $\theta \in (0, \min\{1, p\})$ and $s \in (\theta, 1]$ (see, for example, [29, Theorem 2.3(iii)]).

If $X := L^{p,q}(\mathbb{R}^n)$ with $p, q \in (0, \infty)$ or $p \in (0, \infty)$ and $q = \infty$, then the Hardy type space $H_X(\mathbb{R}^n)$ is just the well-known *Hardy–Lorentz space* (when $p, q \in (0, \infty)$) or *weak Hardy space* (when $p \in (0, \infty)$ and $q = \infty$). The Hardy–Lorentz space was originally introduced and studied by Abu-Shammala and Torchinsky [1, Section 2]. More precisely, the atomic characterization and interpolation of the Hardy–Lorentz space were obtained in [1, Theorems 2.1] and [1, Theorem 2.5], respectively. Ho [57] obtained the boundedness of fractional integral operators on the Hardy–Lorentz space. Furthermore, the weak Hardy space $WH^p(\mathbb{R}^n)$ was studied by Fefferman and Soria [38] and Liu [92]. Recently, the anisotropic Hardy–Lorentz spaces were fully studied in [93, 94].

7.4. Morrey spaces. Let $0 < q \leq p \leq \infty$. Recall that the *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is defined to be the set of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} := \sup_{B \in \mathbb{B}} |B|^{1/p-1/q} \left\{ \int_B |f(y)|^q \, dy \right\}^{1/q} < \infty, \tag{7.1}$$

where the supremum is taken over all balls $B \in \mathbb{B}$ with $\mathbb{B}$ as in (2.2).

The space $\mathcal{M}^p_q(\mathbb{R}^n)$ was introduced by Morrey [100] in 1938. It is well known that $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 \leq q < p < \infty$ is not a Banach function space because it fails condition (2.1) (see [117]). Clearly, $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 < q \leq p < \infty$ is a ball Banach function space. Furthermore, for any $0 < q \leq p < \infty$, $\mathcal{M}^p_q(\mathbb{R}^n)$ satisfies (2.8) for any $\theta \in (0, \min\{1, q\})$ and $s \in (\theta, 1]$ (see, for example, [22, 55, 58]).

Moreover, Jia and Wang [71] and Sawano [113] introduced Hardy–Morrey spaces. Theorem 3.1 also generalizes the maximal function characterizations of Hardy–Morrey spaces. As shown in [113], Hardy–Morrey spaces can be described by means of the Littlewood–Paley decomposition. Once such a decomposition is available, we can combine the theory of Hardy–Morrey spaces and the theory of Triebel–Lizorkin–Morrey spaces; see [54, 116, 112, 114, 118, 124, 133, 134, 135, 136, 137, 141, 142, 143, 103] for a more detailed study of the latter.

7.5. Weighted Morrey spaces. We begin with the definition of Morrey weight functions.

DEFINITION 7.2. Let $p \in (0,\infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$. A Lebesgue measurable function $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ is called a *Morrey weight function* for ω if there exist $\lambda \in [0,1/p)$ and positive constants C_1 and C_2 such that, for any $x, y \in \mathbb{R}^n$, $u(x,r) > C_1$ when $r \in [1,\infty)$,

$$\frac{u(x,2r)}{u(x,r)} \le \left[\frac{\omega(B(x,2r))}{\omega(B(x,r))}\right]^\lambda \quad \text{when } r \in (0,\infty),$$

and

$$C_2^{-1} u(y,r) \le u(x,t) \le C_2 u(y,r)$$

when $0 < r \le t \le 2r$ and $|x-y| \le t$, where $\omega(B(x,r)) := \int_{B(x,r)} \omega(y)\,dy$. Denote by $\mathcal{W}_{\omega,p}(\mathbb{R}^n)$ the class of Morrey weight functions for ω.

Let $p \in (0,\infty)$, $\omega \in A_\infty(\mathbb{R}^n)$ and $u \in \mathcal{W}_{\omega,p}(\mathbb{R}^n)$. The *weighted Morrey space* $\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)$ is defined to be the set of all Lebesgue measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)} := \sup_{(z,r)\in\mathbb{R}^{n+1}_+} \frac{1}{u(z,r)} \|\chi_{B(z,r)} f\|_{L^p_\omega(\mathbb{R}^n)} < \infty.$$

Weighted Hardy–Morrey spaces were studied by Ho [56]. In view of [56, Lemma 3.2], for any $B \in \mathbb{B}$ with $\mathbb{B}$ as in (2.2), we have $\chi_B \in \mathcal{M}^p_{\omega,u}(\mathbb{R}^n)$. Therefore, when $p \in (0,1)$, $\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)$ is a ball quasi-Banach function space. Moreover, if $p \in [1,\infty)$, $\omega \in A_p(\mathbb{R}^n)$ and $u \in \mathcal{W}_{\omega,p}(\mathbb{R}^n)$, then the space $\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)$ satisfies the inequality (2.8) (see [56, Theorem 2.1 and Lemma 3.1] and [55] for the details). Thus, the atomic decompositions for weighted Hardy–Morrey spaces given in [56] can be recovered from our general results in Subsection 3.4.

7.6. Orlicz spaces. In this subsection, we recall Orlicz spaces. Recall that there are many operators that are not bounded on Lebesgue spaces $L^p(\mathbb{R}^n)$ with $p \in [1,\infty]$, especially on $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$. Orlicz spaces are used to cover the failure of the boundedness of some integral operators. For example, the Hardy–Littlewood maximal operator M fails to be bounded on $L^1(\mathbb{R}^n)$. Thus, one is fascinated with the Orlicz space $L^1(\log L)^{1+\varepsilon}(\mathbb{R}^n)$ with $\varepsilon \in (0,\infty)$, since these function spaces have finer and subtler structures. These examples show that Orlicz spaces are useful when we study the boundedness properties of operators appearing in PDE and potential theory.

We first recall Young functions. A function $\Phi : [0,\infty) \to [0,\infty)$ is called a *Young function* if:

(i) Φ is continuous;
(ii) Φ is convex: for any $t_1, t_2 \in (0,\infty)$ and $\theta \in (0,1)$,

$$\Phi((1-\theta)t_1 + \theta t_2) \le (1-\theta)\Phi(t_1) + \theta\Phi(t_2).$$

Let f be a measurable function on $\mathbb{R}^n$. Set

$$\|f\|_{L^\Phi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \Phi(|f(x)|/\lambda)\,dx \le 1\right\}.$$

The *Orlicz space* $L^\Phi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that $\|f\|_{L^\Phi(\mathbb{R}^n)} < \infty$. For any Young function Φ, $L^\Phi(\mathbb{R}^n)$ is a Banach function space (see, for example, [10, p. 269, Theorem 8.9]).

As shown in [79], the class of Young functions can be widely extended. Let

$$\Xi := \left\{\Phi : [0,\infty) \to [0,\infty) \text{ measurable} : \lim_{t\to 0} \Phi(t) = \Phi(0) = 0,\ \lim_{t\to\infty} \Phi(t) = \infty\right\}.$$

If $\Phi \in \Xi$ and $\Phi(t) > 0$ for all $t \in (0,\infty)$, then Φ is called an *Orlicz function.*

To discuss the boundedness of the Hardy–Littlewood maximal operator on $L^\Phi(\mathbb{R}^n)$, we recall the following notion of convexity and concavity.

DEFINITION 7.3. Let $\Phi, \Psi \in \Xi$ and $\ell \in (0,\infty)$.

(i) If there exists a constant $\widetilde{C} \in [1,\infty)$ such that, for any $r \in [0,\infty)$,

$$\Phi(r/\widetilde{C}) \le \Psi(r) \le \Phi(\widetilde{C}r),$$

then we write $\Phi \approx \Psi$ and say that Φ is *equivalent* to Ψ.
(ii) Φ is *ℓ-convex* if $\Phi((\cdot)^{1/\ell})$ is convex. Moreover, Φ is *quasi-ℓ-convex* if $\Phi \approx \Psi$ for some ℓ-convex $\Psi \in \Xi$. In particular, if $\ell = 1$, one abbreviates quasi-ℓ-convex to quasi-convex.
(iii) Φ is *ℓ-concave* if $\Phi((\cdot)^{1/\ell})$ is concave. Moreover, Φ is *quasi-ℓ-concave* if $\Phi \approx \Psi$ for some ℓ-concave function $\Psi \in \Xi$. In particular, if $\ell = 1$, one abbreviates quasi-ℓ-concave to quasi-concave.

It is easy to see that, for any $\Phi \in \Xi$, $L^\Phi(\mathbb{R}^n)$ is a ball quasi-Banach function space. Moreover, by [79, Theorem 1.3.3], if $\Phi \in \Xi$ is such that $\Phi(2t) \lesssim \Phi(t)$ for any $t \in (0,\infty)$ with the implicit positive constant independent of t, and that there exists $\beta \in (0,1)$ such that Φ^β is quasi-convex, then the space $L^\Phi(\mathbb{R}^n)$ satisfies (2.8) for some $\theta, s \in (0,1]$ (see also Theorem 7.14(i) below).

According to [102, Theorem 2.6], we have the following boundedness result.

PROPOSITION 7.4. *Let $1 < \ell_- \le \ell_+ < \infty$. If Φ is a convex function that is ℓ_--convex and ℓ_+-concave, then $\ell_- \le l_{L^\Phi(\mathbb{R}^n)} \le u_{L^\Phi(\mathbb{R}^n)} \le \ell_+$, where $l_{L^\Phi(\mathbb{R}^n)}$ and $u_{L^\Phi(\mathbb{R}^n)}$ are, respectively, as in* (2.27) *and* (2.28)*.*

As a corollary of this proposition and Theorem 2.9, we have the following decomposition results.

THEOREM 7.5 (Reconstruction). *Let $1 < \ell_- \le \ell_+ < r < \infty$. Assume that Φ is a convex function that is ℓ_--convex and ℓ_+-concave. Let $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, $\{a_j\}_{j=1}^\infty \subset L^r(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ be such that, for any $j \in \mathbb{N}$,*

$$\|a_j\|_{L^r(\mathbb{R}^n)} \le \frac{|Q_j|^{1/r}}{\|\chi_{Q_j}\|_{L^\Phi(\mathbb{R}^n)}}, \quad \operatorname{supp}(a_j) \subset Q_j$$

and

$$\left\| \sum_{j=1}^\infty \frac{\lambda_j}{\|\chi_{Q_j}\|_{L^\Phi(\mathbb{R}^n)}} \chi_{Q_j} \right\|_{L^\Phi(\mathbb{R}^n)} < \infty.$$

Then $f := \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and satisfies

$$\|f\|_{L^\Phi(\mathbb{R}^n)} \lesssim_{r,\Phi} \left\| \sum_{j=1}^\infty \frac{\lambda_j}{\|\chi_{Q_j}\|_{L^\Phi(\mathbb{R}^n)}} \chi_{Q_j} \right\|_{L^\Phi(\mathbb{R}^n)},$$

where the implicit positive constant is independent of f, but depends on r and Φ.

REMARK 7.6. In [102, Theorem 4.14], the requirement on the r of Theorem 7.5 is that r is large enough. Thus, Theorem 7.5 improves [102, Theorem 4.14].

We point out that Hardy–Orlicz spaces date back to [66, 119], where they were used to investigate the properties of the Jacobian. Hardy–Orlicz spaces for more general Φ are defined by letting X be the Orlicz space $L^\Phi(\mathbb{R}^n)$ (see, for example, [102, 126]). In 1987, Viviani [126] defined Hardy–Orlicz spaces in the context of preduals of generalized BMO spaces. Indeed, Viviani [126, Theorem 2.2] showed that Hardy–Orlicz spaces naturally arise as such preduals. Compared with [126], Nakai and Sawano [102, 126] removed the condition on ρ and redefined Hardy–Orlicz spaces. Moreover, D. Yang and S. Yang [130, 131] generalized Hardy–Orlicz spaces on $\mathbb{R}^n$ to those on Lipschitz domains of $\mathbb{R}^n$. Theorem 3.1 generalizes the maximal function characterizations of Hardy–Orlicz spaces.

7.7. Musielak–Orlicz spaces. Recently, more and more attention has been paid to (weak) Musielak–Orlicz–Hardy spaces (see, for example, [16, 17, 64, 81, 87, 88, 128, 132]).

In this subsection, we recall the setting of [81]. Let us start with the notions of the upper type and lower type of Orlicz functions.

DEFINITION 7.7. Let Φ be an Orlicz function and $p \in (0,\infty)$. The function Φ is said to be of *upper* (resp. *lower*) *type* p if there exists a positive constant C such that $\Phi(st) \le Ct^p\Phi(s)$ for all $t \in [1,\infty)$ (resp. $t \in (0,1]$) and $s \in (0,\infty)$.

PROPOSITION 7.8. *Let $p \in (1,\infty)$, Φ be a Young function and Ψ its conjugate.*

(i) *If Φ is of upper type p, then Ψ is of lower type p'.*
(ii) *If Φ is of lower type p, then Ψ is of upper type p'.*

Proof. We first prove (i). Recall that, for any $t \in (0,\infty)$,

$$\Psi(t) := \sup_{s\in(0,\infty)} \{st - \Phi(s)\}.$$

Let $u \in (0,1]$ and $t \in (0,\infty)$. Then, from the definition of upper type, it follows that there exists a constant $C \in [1,\infty)$ such that

$$\begin{aligned}\Psi(u^{-1}t) &= \sup_{s\in(0,\infty)}\{u^{-1}st - \Phi(s)\} = \sup_{s\in(0,\infty)}\{st - \Phi(us)\}\\ &\le \sup_{s\in(0,\infty)}\{st - C^{-1}u^p\Phi(s)\} \le C^{-1}u^p \sup_{s\in(0,\infty)}\{Cstu^{-p} - \Phi(s)\}\\ &= C^{-1}u^p\Psi(Cu^{-p}t).\end{aligned}$$

By replacing t with u^pt, we conclude that $\Psi(u^{p-1}t) \le C^{-1}u^p\Psi(Ct)$, which implies that, $\Psi(vt) \le C^{-1}v^{p'}\Psi(Ct)$ for any $v \in (0,1]$ and $t \in (0,\infty)$. Via replacing Ct and v by t and Cv, respectively, we find that, for any $v \in (0, C^{-1}]$ and $t \in (0,\infty)$,

$$\Psi(vt) \le C^{p'-1}v^{p'}\Psi(t). \tag{7.2}$$

Moreover, from the fact that Φ is increasing and the definition of Ψ, it follows that Ψ is also increasing, which further implies that, for any $v \in [C^{-1},1]$ and $t \in (0,\infty)$,

$$\Psi(vt) \le \Psi(t) \le C^{p'}v^{p'}\Psi(t).$$

From this and (7.2), we conclude that $\Psi(vt) \le C^{p'}v^{p'}\Psi(t)$ for any $v \in (0,1]$ and $t \in (0,\infty)$. Thus, Ψ is of lower type p'.

The proof of (ii) is similar, the details being omitted. This finishes the proof of Proposition 7.8. ∎

Now we recall the definition of the Musielak–Orlicz function.

Definition 7.9. Let $p \in [0,\infty)$ and $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be such that $\varphi(x,\cdot)$ is an Orlicz function for any $x \in \mathbb{R}^n$. The function φ is said to be of *uniformly upper* (resp. *lower*) *type* $p \in [0,\infty)$ if there exists a positive constant C such that $\varphi(x,st) \le Cs^p\varphi(x,t)$ for all $x \in \mathbb{R}^n$, $t \in [0,\infty)$ and $s \in [1,\infty)$ (resp. $s \in [0,1]$).

In addition to the conditions on the second variable of φ, we need another condition on the first variable.

Definition 7.10. Let $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be such that $\varphi(\cdot,t)$ is measurable for any $t \in [0,\infty)$.

(i) The function φ is said to satisfy the *uniform Muckenhoupt condition for* $q \in [1,\infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, if

$$\mathbb{A}_q(\varphi) := \sup_{t\in(0,\infty)} A_q(\varphi(\cdot,t)) < \infty,$$

where the supremum is taken over all $t \in (0,\infty)$ and, for any given $t \in (0,\infty)$, $A_q(\varphi(\cdot,t))$ is as in Definition 7.1.

(ii) The class $\mathbb{A}_\infty(\mathbb{R}^n)$ is defined by setting

$$\mathbb{A}_\infty(\mathbb{R}^n) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathbb{R}^n).$$

(iii) For any $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$, the *critical weight index* $q(\varphi)$ is defined by setting

$$q(\varphi) := \inf\{q \in [1,\infty) : \varphi \in \mathbb{A}_q(\mathbb{R}^n)\}.$$

Now we recall the notion of growth functions from Ky [81].

DEFINITION 7.11. A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *growth function* if:

(i) φ is a *Musielak–Orlicz function*, namely,

  (a) $\varphi(x,\cdot)$ is an Orlicz function for all $x \in \mathbb{R}^n$;
  (b) $\varphi(\cdot,t)$ is a measurable function for all $t \in [0,\infty)$.

(ii) $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$.

(iii) The function φ is of uniformly lower type p for some $p \in (0,1]$ and of uniformly upper type 1.

For a Musielak–Orlicz function φ as in Definition 7.11, a measurable function f on $\mathbb{R}^n$ is said to be in the *Musielak–Orlicz space* $L^\varphi(\mathbb{R}^n)$ if $\int_{\mathbb{R}^n} \varphi(x,|f(x)|)\,dx < \infty$. Moreover, for any $f \in L^\varphi(\mathbb{R}^n)$, the *quasi-norm* of f is defined by setting

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \varphi(x,|f(x)|/\lambda)\,dx \le 1\right\}.$$

In view of the definition of $\mathbb{A}_\infty(\mathbb{R}^n)$, we find that $\chi_B \in L^\varphi(\mathbb{R}^n)$ for any $B \in \mathbb{B}$ with $\mathbb{B}$ as in (2.2). Therefore, the Musielak–Orlicz space $L^\varphi(\mathbb{R}^n)$ is a ball quasi-Banach function space.

Moreover, we have the following Fefferman–Stein vector-valued maximal inequalities for $L^\varphi(\mathbb{R}^n)$.

THEOREM 7.12. *Let $r \in (1,\infty]$ and $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ satisfy Definition 7.11(i). Assume in addition that φ is of uniformly lower type p^- and uniformly upper type p^+. If $q(\varphi) < p^- \le p^+ < \infty$, then, for any measurable functions $\{f_j\}_{j=1}^\infty \subset \mathcal{M}$,*

$$\left\|\left\{\sum_{j=1}^\infty (Mf_j)^r\right\}^{1/r}\right\|_{L^\varphi(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{j=1}^\infty |f_j|^r\right\}^{1/r}\right\|_{L^\varphi(\mathbb{R}^n)}, \tag{7.3}$$

where the implicit positive constant is independent of $\{f_j\}_{j=1}^\infty$.

Proof. Let $r \in (1,\infty]$. In view of [85, Theorem 2.10], we have the following modular vector-valued maximal inequality: for any measurable functions $\{f_j\}_{j=1}^\infty \subset \mathcal{M}$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \left\{\sum_{j=1}^\infty [Mf_j(x)]^r\right\}^{1/r}\right) dx \lesssim \int_{\mathbb{R}^n} \varphi\left(x, \left\{\sum_{j=1}^\infty |f_j(x)|^r\right\}^{1/r}\right) dx. \tag{7.4}$$

Therefore, (7.3) follows from the definition of $L^\varphi(\mathbb{R}^n)$ and (7.4), which completes the proof of Theorem 7.12. ■

Now we recall the definition of the Musielak–Orlicz–Hardy space introduced in [81].

DEFINITION 7.13. Let φ be a growth function as in Definition 7.11. The *Musielak–Orlicz–Hardy space* $H_\varphi(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{H_\varphi(\mathbb{R}^n)} := \|\mathcal{M}_N(f)\|_{L^\varphi(\mathbb{R}^n)} < \infty$, where $\mathcal{M}_N$ is as in (2.13) and $N \in [\lfloor b+2\rfloor,\infty) \cap \mathbb{N}$ with b as in (2.14).

We point out that Theorem 7.12 ensures that $L^\varphi(\mathbb{R}^n)$ satisfies (2.8) (see Theorem 7.14(i) below). Thus, our general results in Section 3 can be applied to the space $H_\varphi(\mathbb{R}^n)$. We also point out that the Musielak–Orlicz–Hardy space unifies the weighted Hardy space and the Orlicz–Hardy space, while the Musielak–Orlicz–Hardy space is a member of the family of Hardy type spaces introduced in Definition 2.22.

Using Theorem 7.12, we obtain the following results.

THEOREM 7.14. *Let $0 < p^- \le p^+ < \infty$ and $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ satisfy Definition 7.11(i) with uniformly lower type p^- and uniformly upper type p^+.*

(i) *Assume that $0 < \theta < s < \infty$ and $\theta q(\varphi) < p^-$, where $q(\varphi)$ is as in Definition 7.10(iii). Then, for any $\{f_j\}_{j=1}^\infty \subset \mathcal{M}$,*

$$\left\|\left\{\sum_{j=1}^\infty [M^{(\theta)}(f_j)]^s\right\}^{1/s}\right\|_{L^\varphi(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{j=1}^\infty |f_j|^s\right\}^{1/s}\right\|_{L^\varphi(\mathbb{R}^n)},$$

where the implicit positive constant is independent of $\{f_j\}_{j=1}^\infty$.

(ii) *Assume that $\mathbb{R}^n \times [0,\infty) \ni (x,t) \mapsto \varphi(x, \sqrt[s]{t})$ is a convex function with respect to t for some $s \in (0,p^-)$. Let $L^\varphi(\mathbb{R}^n)$ be strictly s-convex satisfying (2.8) for some $\theta \in (0,1]$ and $d_{L^\varphi(\mathbb{R}^n)}$ as in (2.29) with X replaced by $L^\varphi(\mathbb{R}^n)$. Then there exists $q \in (1,\infty)$ having the following property: Suppose that $\{a_j\}_{j=1}^\infty$ is a sequence of $(L^\varphi(\mathbb{R}^n), q, d_{L^\varphi(\mathbb{R}^n)})$-atoms supported, respectively, on cubes $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ satisfies*

$$\left\|\left\{\sum_{j=1}^\infty \left[\frac{\lambda_j}{\|\chi_{Q_j}\|_{L^\varphi(\mathbb{R}^n)}}\right]^s \chi_{Q_j}\right\}^{1/s}\right\|_{L^\varphi(\mathbb{R}^n)} < \infty.$$

Then $f := \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$, $f \in H_\varphi(\mathbb{R}^n)$ and

$$\|f\|_{H_\varphi(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{j=1}^\infty \left[\frac{\lambda_j}{\|\chi_{Q_j}\|_{L^\varphi(\mathbb{R}^n)}}\right]^s \chi_{Q_j}\right\}^{1/s}\right\|_{L^\varphi(\mathbb{R}^n)},$$

where the implicit positive constant is independent of f.

(iii) *Let φ be as in (ii), $d \in [d_{L^\varphi(\mathbb{R}^n)},\infty) \cap \mathbb{Z}_+$ and $f \in H_\varphi(\mathbb{R}^n)$. Then there exist a sequence $\{a_j\}_{j=1}^\infty$ of $(L^\varphi(\mathbb{R}^n), \infty, d)$-atoms supported, respectively, on the cubes $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, and a sequence $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and*

$$\left\|\left\{\sum_{j=1}^\infty \left[\frac{\lambda_j}{\|\chi_{Q_j}\|_{L^\varphi(\mathbb{R}^n)}}\right]^s \chi_{Q_j}\right\}^{1/s}\right\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|f\|_{H_\varphi(\mathbb{R}^n)},$$

where the implicit positive constant is independent of f.

Proof. We first show (i). For any $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, let $\varphi_\theta(x,t) := \varphi(x, \sqrt[\theta]{t})$. Then φ_θ is of uniformly lower type p^-/θ and of uniformly upper type p^+/θ, which, together with $q(\varphi) < p^-/\theta$ and Theorem 7.12, implies (i).

We now prove (ii). For any $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, let $\varphi_s(x,t) := \varphi(x, \sqrt[s]{t})$. Then φ_s is of uniformly lower type p^-/s and of uniformly upper type p^+/s, and $p^-/s > 1$. For any $x \in \mathbb{R}^n$, let $\psi_s(x,\cdot)$ be the conjugate function of $\varphi_s(x,\cdot)$. Then, for any $x \in \mathbb{R}^n$, $\psi_s(x,\cdot)$ is of uniformly upper type $(p^-/s)'$ and of uniformly lower type $(p^+/s)'$. Therefore, we can apply [85, Theorem 2.7] to conclude that, for any $f \in (L^{\varphi_s}(\mathbb{R}^n))'$,

$$\|Mf\|_{(L^{\varphi_s}(\mathbb{R}^n))'} \lesssim \|f\|_{(L^{\varphi_s}(\mathbb{R}^n))'}.$$

Thus, it follows from Lemma 2.15(ii) that, for some $\zeta \in (1,\infty)$,

$$\|M^{(\zeta)}(f)\|_{(L^{\varphi_s}(\mathbb{R}^n))'} \lesssim \|f\|_{(L^{\varphi_s}(\mathbb{R}^n))'}.$$

By letting $q = \zeta'$ and applying Theorem 3.6, we obtain (ii).

(iii) is just a conclusion of Theorem 3.7. This finishes the proof of Theorem 7.14. ∎

REMARK 7.15. The growth function φ is said to satisfy the *uniformly locally dominated convergence condition* if the following holds true: For every compact $K \subset \mathbb{R}^n$ and every sequence $\{f_m\}_{m\in\mathbb{N}}$ of measurable functions on $\mathbb{R}^n$, if $f_m \to f$ as $m \to \infty$ almost everywhere and $|f_m| \le g$ almost everywhere for all $m \in \mathbb{N}$ and some non-negative measurable function g satisfying

$$\sup_{t\in(0,\infty)} \int_K g(x) \frac{\varphi(x,t)}{\int_K \varphi(y,t)\,dy}\,dx < \infty,$$

then

$$\lim_{m\to\infty} \sup_{t\in(0,\infty)} \int_K |f(x) - f_m(x)| \frac{\varphi(x,t)}{\int_K \varphi(y,t)\,dy}\,dx = 0.$$

In [87], φ is required to satisfy the uniformly locally dominated convergence condition. Notice that, in Theorem 7.14, φ does not always satisfy this condition. Thus, the uniformly locally dominated convergence condition is superfluous; see also [64, 88]. Examples of functions satisfying this condition can be found in [77, 87, 129].

7.8. Variable Lebesgue spaces. In this subsection, we recall variable Lebesgue spaces. The variable Lebesgue spaces were introduced in [104, 105, 107]. The "modern" theory of variable Lebesgue spaces began from the article [80].

Let $p(\cdot) : \mathbb{R}^n \to [0,\infty)$ be a measurable function. Then the *variable Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} [|f(x)|/\lambda]^{p(x)} dx \le 1\right\}.$$

If $p(x) \ge 1$ for all $x \in \mathbb{R}^n$, then $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ is a complete norm and hence $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space (see, for example, [31, Theorem 3.2.13]). The reader is referred to [30, 31] for the details on variable Lebesgue spaces.

For any measurable function $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, let

$$p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x), \qquad p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \qquad \tilde{p} := p/p_-.$$

Then $\tilde{p}(x) \ge 1$ for all $x \in \mathbb{R}^n$. Therefore, for any $B \in \mathbb{B}$ with $\mathbb{B}$ as in (2.2),

$$\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|\chi_B\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^n)}^{1/p_-} < \infty$$

(see, for example, [31, Lemma 3.2.6]). Thus, whenever $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, $L^{p(\cdot)}(\mathbb{R}^n)$ is a ball quasi-Banach function space. Furthermore, for any $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, the space $L^{p(\cdot)}(\mathbb{R}^n)$ satisfies (2.8) for any $\theta \in (0,p_-)$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)/\theta}(\mathbb{R}^n)$, and for any $s \in (\theta,1]$ (see, for example, [26, 28]).

If $X := L^{p(\cdot)}(\mathbb{R}^n)$, then $H_X(\mathbb{R}^n)$ is the variable Hardy space studied in [28, 101, 140, 145]. Theorem 3.1 generalizes the maximal function characterizations of variable Hardy spaces.

7.9. Variable Morrey spaces. In this subsection, we recall variable Morrey spaces. Let $p(\cdot):\mathbb{R}^n\to(0,\infty)$ and

$$u:\mathbb{R}^n\times(0,\infty)\to(0,\infty).$$

Then the *variable Morrey space* $\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)}:=\sup_{z\in\mathbb{R}^n,\,r\in(0,\infty)}\frac{1}{u(z,r)}\|\chi_{B(z,r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}<\infty.$$

Let $\mathcal{B}$ be the set of all measurable functions $p(\cdot):\mathbb{R}^n\to(1,\infty)$ such that $1<p_-\le p_+<\infty$ and the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Denote by $\widetilde{\mathcal{B}}$ the set of all measurable functions $p(\cdot):\mathbb{R}^n\to(0,\infty)$ satisfying $0<p_-\le p_+<\infty$ and $p(\cdot)/\beta\in\mathcal{B}$ for some $\beta\in(0,\infty)$. Moreover, a function $p(\cdot):\mathbb{R}^n\to(0,\infty)$ is said to be *globally log-Hölder continuous* if, for any $x,y\in\mathbb{R}^n$,

$$|p(x)-p(y)|\lesssim\frac{1}{\log(1/|x-y|)}\quad\text{when }|x-y|\le1/2,$$
$$|p(x)-p(y)|\lesssim\frac{1}{\log(e+|x|)}\quad\text{when }|y|\ge|x|,$$

where the implicit positive constants are independent of x and y.

We point out that if $p(\cdot)\in\widetilde{\mathcal{B}}$ is globally log-Hölder continuous and $u\in\mathcal{W}_{\omega,p_+}(\mathbb{R}^n)$ with $\omega\equiv1$, where $\mathcal{W}_{\omega,p_+}(\mathbb{R}^n)$ is as in Definition 7.2, then the space $\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)$ fulfills the requirements of Definition 2.2(iv). Thus, $\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)$ is a ball quasi-Banach function space in this case (see [58, Proposition 3.4] for more details). Additionally, the reader is also referred to [59] for some further studies on variable Morrey spaces.

Whenever $p_-\in[1,\infty]$, by Hölder's inequality for $L^{p(\cdot)}(\mathbb{R}^n)$ (see, for example, [31, Lemma 3.2.20]), we find that, for any $B:=B(z,r)\in\mathbb{B}$ with $\mathbb{B}$ as in (2.2),

$$\left|\int_{\mathbb{R}^n}\chi_B(x)f(x)\,dx\right|\le2\|\chi_Bf\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $p'(\cdot)$ denotes the conjugate function of $p(\cdot)$, namely, for any $x\in\mathbb{R}^n$,

$$\frac{1}{p(x)}+\frac{1}{p'(x)}=1.$$

Consequently, for any $B:=B(z,r)\in\mathbb{B}$ with $\mathbb{B}$ as in (2.2),

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\chi_B(x)f(x)\,dx\right|&\le2\|\chi_Bf\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\&\le2u(z,r)\|f\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)},\end{aligned}$$

which implies that $\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)$ is a ball Banach function space. Indeed, the above result is a consequence of a more general result on the duality between the variable block space and the variable Morrey space. We refer the reader to [21, Theorem 2.2] for the details. Furthermore, if $p(\cdot)\in\mathcal{B}$ and u has the property that, for any $x\in\mathbb{R}^n$ and $r\in(0,\infty)$,

$$\sum_{j=0}^{\infty}\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}u(x,2^{j+1}r)\lesssim u(x,r),$$

then the space $\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)$ satisfies (2.8) (see [58, Theorem 3.1] for the details).

The study of variable Hardy spaces was extended to variable Hardy–Morrey spaces in [58]. Theorem 3.1 generalizes the maximal function characterizations of variable Hardy–Morrey spaces. Furthermore, the atomic decompositions in Subsection 3.4 are extensions of the atomic decompositions for variable Hardy–Morrey spaces in [58, Section 5].

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