COLLOQUIUM MATHEMATICUM

VOL. 150 2017 NO. 2

A NOTE ON SOME COMPLEMENTED SPACES OF OPERATORS

BY

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Abstract. Let $W(X,Y)$, $Pwc(X,Y)$, $CC(X,Y)$, and $UC(X,Y)$ denote respectively the sets of all weakly compact, pseudo weakly compact, completely continuous, and unconditionally converging operators from X to Y. We use classical results of Kalton to study the complementability of the space $W(X,Y)$ in the spaces $Pwc(X,Y)$, $UC(X,Y)$, and $CC(X,Y)$.

1. Introduction. A bounded subset A of a Banach space X is called a *Dunford–Pettis* (DP) subset of X if every weakly null sequence (x_n^*) in X^* tends to 0 uniformly on A, i.e.,

$$\lim_n (\sup\{|x_n^*(x)| : x \in A\}) = 0.$$

A sequence (x_n) is DP if the set $\{x_n : n \in \mathbb{N}\}$ is DP.

A subset S of X is said to be *weakly precompact* provided that every sequence from S has a weakly Cauchy subsequence. Every DP set is weakly precompact (see e.g. [R, p. 377]).

An operator $T : X \to Y$ is called *weakly precompact* (or *almost weakly compact*) if $T(B_X)$ is weakly precompact, and *completely continuous* (or *Dunford–Pettis*) if T maps weakly convergent sequences to norm convergent sequences.

In [PV$^+$] the authors introduced the *Right topology* on a Banach space X. It is the restriction of the Mackey topology $\tau(X^{**}, X)$ to X and it is also the topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$ compact subsets of X^*. Furthermore, $\tau(X^{**}, X)$ can also be viewed as the topology of uniform convergence on relatively $\sigma(X^*, X^{**})$ compact subsets of X^* [K].

A sequence (x_n) in a Banach space X is Right null if and only if it is weakly null and DP [G1, Proposition 2].

2010 *Mathematics Subject Classification*: Primary 46B20, 46B25, 46B28; Secondary 47L20.
Key words and phrases: complemented spaces, sequentially Right spaces, pseudo weakly compact operators, completely continuous operators.
Received 3 May 2016.
Published online 24 July 2017.

An operator $T : X \to Y$ is called *pseudo weakly compact* (*pwc*) (or *Dunford–Pettis completely continuous* (*DPcc*)) if it takes Right null sequences to norm null sequences ([PV$^+$], [WC]). Every completely continuous operator $T : X \to Y$ is pseudo weakly compact. If $T : X \to Y$ is an operator with weakly precompact adjoint, then T is a pseudo weakly compact operator [G2, Corollary 5].

In [BBG] the authors studied the complementability of the spaces $W(X, \ell_\infty)$ and $CC(X, \ell_\infty)$ in $L(X, \ell_\infty)$. It was shown that if X is not reflexive, then $W(X, \ell_\infty)$ is not complemented in $L(X, \ell_\infty)$ [BBG, Theorem 3]. Further, if X does not have the Schur property, then $CC(X, \ell_\infty)$ is not complemented in $L(X, \ell_\infty)$ [BBG, Theorem 30].

In this paper we use classical results of Kalton and separably determined operator ideals to investigate the complementability of $W(X, Y)$ in $Pwc(X, Y)$, $UC(X, Y)$, and $CC(X, Y)$. Further, we study the complementability of $K(X, Y)$ in $Pwc(X, Y)$ and $CC(X, Y)$, and of $CC(X, Y)$ in $Pwc(X, Y)$.

2. Definitions and notation. Throughout this paper, X and Y will denote Banach spaces. The unit ball of X will be denoted by B_X, and X^* will denote the continuous linear dual of X. An *operator* $T : X \to Y$ will be a continuous and linear function. The set of all operators, weakly compact operators, and compact operators from X to Y will be denoted by $L(X, Y)$, $W(X, Y)$, and $K(X, Y)$. We will denote the canonical unit vector basis of c_0 by (e_n) and the canonical unit vector basis of ℓ_1 by (e_n^*).

A bounded subset A of X^* is called an *L-subset* of X^* if each weakly null sequence (x_n) in X tends to 0 uniformly on A, i.e.,

$$\lim_n (\sup\{|x^*(x_n)| : x^* \in A\}) = 0.$$

A Banach space X has *the Dunford–Pettis property* (*DPP*) if every weakly compact operator $T : X \to Y$ is completely continuous, for any Banach space Y. Schur spaces, $C(K)$ spaces, and $L_1(\mu)$ spaces have the DPP. The reader can consult [D1], [D2], and [DU] for a guide to the extensive classical literature dealing with the DPP.

A Banach space X has the *Dunford–Pettis relatively compact property* (*DPrcP*) if every Dunford–Pettis subset of X is relatively compact [E2]. Schur spaces have the DPrcP. The space X does not contain a copy of ℓ_1 if and only if X^* has the DPrcP if and only if every L-subset of X^* is relatively compact ([B, Corollary 7], [E2, Theorem 1], [E1, Theorem 2]).

A series $\sum x_n$ in X is said to be *weakly unconditionally convergent* (*wuc*) if for every $x^* \in X^*$, the series $\sum |x^*(x_n)|$ is convergent. An operator $T : X \to Y$ is called *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent ones.

A bounded subset A of X^* is called a *V-subset* of X^* provided that

$$\lim_n(\sup\{|x^*(x_n)| : x^* \in A\}) = 0$$

for each wuc series $\sum x_n$ in X.

A Banach space X has *property* (V) if every V-subset of X^* is relatively weakly compact [P]. A Banach space X has property (V) if every unconditionally converging operator T from X to any Banach space Y is weakly compact [P, Proposition 1]. $C(K)$ spaces and reflexive spaces have property (V) [P, Theorem 1, Proposition 7].

A Banach space X has the *reciprocal Dunford–Pettis property* (*RDPP*) if every completely continuous operator T from X to any Banach space Y is weakly compact. The space X has the RDPP if and only if every L-subset of X^* is relatively weakly compact [L]. Banach spaces with property (V) have the RDPP [P].

A subset K of X^* is called a *Right set* (*R-set*) if each Right null sequence (x_n) in X tends to 0 uniformly on K [K], i.e.,

$$\lim_n(\sup\{|x^*(x_n)| : x^* \in K\}) = 0.$$

A Banach space X is said to be *sequentially Right* (*SR*) (or to have *property* (*SR*)) if every pseudo weakly compact operator $T : X \to Y$ is weakly compact, for any Banach space Y [PV$^+$]. Banach spaces with property (V) are sequentially Right [PV$^+$, Corollary 15]. A Banach space X is sequentially Right if and only if every Right subset of X^* is relatively weakly compact [K, Theorem 3.25].

3. Complemented subspaces of operators. The sets of all pseudo weakly compact, completely continuous, and unconditionally converging operators from X to Y will be respecyively denoted by $Pwc(X,Y)$, $CC(X,Y)$, and $UC(X,Y)$.

We begin by investigating the complementability of the space $W(X,\ell_\infty)$ in $Pwc(X,\ell_\infty)$, $UC(X,\ell_\infty)$, and $CC(X,\ell_\infty)$.

Lemma 3.1 ([Ka, Proposition 5]). *Let X be a separable Banach space, and $\phi : \ell_\infty \to L(X,\ell_\infty)$ be a bounded linear operator such that $\phi(e_n) = 0$ for all n. Then there is an infinite subset M of $\mathbb{N}$ such that $\phi(b) = 0$ for each $b \in \ell_\infty(M)$, where $\ell_\infty(M)$ is the set of all $b = (b_n) \in \ell_\infty$ with $b_n = 0$ for each $n \notin M$.*

Observation. If $T : Y \to X^*$ be an operator such that $T^*|_X$ is [weakly] compact, then T is [weakly] compact. To see this, let $T : Y \to X^*$ be an operator such that $T^*|_X$ is [weakly] compact. Let $S = T^*|_X$. Suppose $x^{**} \in B_{X^{**}}$ and choose a net (x_α) in B_X which is w^*-convergent to x^{**}. Then $[T^*(x_\alpha)] \xrightarrow{w^*} T^*(x^{**})$. Now, $(T^*(x_\alpha)) \subseteq S(B_X)$, which is a relatively [weakly] compact set. Then $(T^*(x_\alpha)) \to T^*(x^{**})$ (resp. $T^*(x_\alpha) \xrightarrow{w} T^*(x^{**})$). Hence $T^*(B_{X^{**}}) \subseteq \overline{S(B_X)}$, which is relatively [weakly] compact. Therefore $T^*(B_{X^{**}})$ is relatively [weakly] compact, and thus T is [weakly] compact.

If T is a weakly compact operator, then T takes Dunford–Pettis sets to relatively compact sets ([A, Theorem 1], [G2, Corollary 5]), hence T is pseudo weakly compact.

THEOREM 3.2. *If X is not sequentially Right, then $W(X,\ell_\infty)$ is complemented neither in $Pwc(X,\ell_\infty)$ nor in $UC(X,\ell_\infty)$.*

Proof. Let A be a Right subset of X^* which is not relatively weakly compact [K, Theorem 3.25]. Let (x_n^*) be a sequence in A with no weakly convergent subsequence. Define $S : X \to \ell_\infty$ by $S(x) = (x_n^*(x))_n$, $x \in X$. Since $S^*(e_n^*) = x_n^*$, S^*, and thus S, is not weakly compact. Let (y_n) be a sequence in B_X such that $(S(y_n))$ has no weakly convergent subsequence. Let $X_0 = [y_n]$ be the closed linear span of $\{y_n : n \in \mathbb{N}\}$. Note that X_0 is a separable subspace of X and $L = S|_{X_0}$ is not weakly compact. If $y_n^* = x_n^*|_{X_0}$, then $(y_n^*) \subseteq X_0^*$ is bounded and has no weakly convergent subsequence. (If (y_n^*) is weakly convergent, then $L^*|_{\ell_1}$ is weakly compact, since $L^*(e_n^*) = y_n^*$. By the Observation above, L is weakly compact. This is a contradiction.)

Define $T : \ell_\infty \to L(X,\ell_\infty)$ by $T(b)(x) = (b_n\, x_n^*(x))_n$ for $b = (b_n) \in \ell_\infty$ and $x \in X$. Note that the operator T is well-defined and $T(e_n) = x_n^* \otimes e_n$ for each $n \in \mathbb{N}$. Let $b \in \ell_\infty$ and suppose that (x_m) is a weakly null DP sequence in X. Since (x_n^*) is a Right set,

$$\lim_m \|T(b)(x_m)\| = \lim_m \sup_n |b_n x_n^*(x_m)| = 0,$$

and thus $T(b)$ is pseudo weakly compact.

Suppose that $W(X,\ell_\infty)$ is complemented in $Pwc(X,\ell_\infty)$ and let $P : Pwc(X,\ell_\infty) \to W(X,\ell_\infty)$ be a projection. Let $R : L(X,\ell_\infty) \to L(X_0,\ell_\infty)$ be the natural restriction map. Define $\phi : \ell_\infty \to Pwc(X_0,\ell_\infty)$ by $\phi(b) = RT(b)$ and $\psi : \ell_\infty \to W(X_0,\ell_\infty)$ by $\psi(b) = RPT(b)$. Since $T(e_n)$ is a rank one operator, it is compact, hence weakly compact. Thus

$$\psi(e_n) = RPT(e_n) = RT(e_n) = \phi(e_n)$$

for each $n \in \mathbb{N}$. By Lemma 3.1, there is an infinite subset M of $\mathbb{N}$ such that $\psi(\chi_M) = \phi(\chi_M)$. Hence $\phi(\chi_M)$ is weakly compact. However, for $n \in M$ and $x \in X_0$,

$$\phi(\chi_M)^*(e_n^*)(x) = y_n^*(x).$$

Therefore $\phi(\chi_M)^*(e_n^*) = y_n^*$ for all $n \in M$. This contradiction proves that $W(X,\ell_\infty)$ is not complemented in $Pwc(X,\ell_\infty)$.

Since every pseudo weakly compact operator is unconditionally converging [PV$^+$, Proposition 14], $W(X,\ell_\infty)$ is not complemented in $UC(X,\ell_\infty)$

(otherwise $W(X, \ell_\infty)$ would be complemented in $Pwc(X, \ell_\infty)$, a contradiction). ∎

THEOREM 3.3. *If X^* does not have the Schur property, then $K(X, \ell_\infty)$ is complemented neither in $Pwc(X, \ell_\infty)$ nor in $UC(X, \ell_\infty)$.*

Proof. Since X^* does not have the Schur property, there is a Right subset of X^* which is not relatively compact [G1, Corollary 9]. The proof is similar to that of Theorem 3.2. ∎

If T is a weakly compact operator, then T is unconditionally converging, by the Orlicz–Pettis theorem.

THEOREM 3.4. *If X does not have property (V), then $W(X, \ell_\infty)$ is not complemented in $UC(X, \ell_\infty)$.*

Proof. The proof is similar to that of Theorem 3.2. ∎

THEOREM 3.5. *If X has the DPP and does not have the RDPP, then $W(X, \ell_\infty)$ is complemented neither in $CC(X, \ell_\infty)$ nor in $UC(X, \ell_\infty)$.*

Proof. Since X has the DPP, every weakly compact operator $T : X \to \ell_\infty$ is completely continuous. Let A be an L-subset of X^* which is not relatively weakly compact [L]. Let (x_n^*) be a sequence in A with no weakly convergent subsequence. Define $S : X \to \ell_\infty$ by $S(x) = (x_n^*(x))_n$ for $x \in X$. As in the proof of Theorem 3.2, S is not weakly compact. Let X_0 be a separable subspace of X such that $S|_{X_0}$ is not weakly compact. If $y_n^* = x_n^*|_{X_0}$, then $(y_n^*) \subseteq X_0^*$ has no weakly convergent subsequence.

Define $T : \ell_\infty \to L(X, \ell_\infty)$ as in Theorem 3.2. Since (x_n^*) is an L-set, $T(b)$ is completely continuous.

Suppose that $W(X, \ell_\infty)$ is complemented in $CC(X, \ell_\infty)$ and let $P : CC(X, \ell_\infty) \to W(X, \ell_\infty)$ be a projection. Let $R : L(X, \ell_\infty) \to L(X_0, \ell_\infty)$ be the natural restriction map. Define $\phi : \ell_\infty \to CC(X_0, \ell_\infty)$ and $\psi : \ell_\infty \to W(X_0, \ell_\infty)$ as in Theorem 3.2 and note that $\psi(e_n) = \phi(e_n)$ for each $n \in \mathbb{N}$.

By Lemma 3.1, there is an infinite subset M of $\mathbb{N}$ such that $\psi(\chi_M) = \phi(\chi_M)$. Hence $\phi(\chi_M)$ is weakly compact. However, $\phi(\chi_M)^*(e_n^*) = y_n^*$ for all $n \in M$. This contradiction concludes the proof. ∎

THEOREM 3.6. *The Banach space X contains a copy of ℓ_1 if and only if $K(X, \ell_\infty)$ is not complemented in $CC(X, \ell_\infty)$.*

Proof. Suppose X contains a copy of ℓ_1. Then X^* contains an L-set which is not relatively compact [E1, Theorem 2]. The proof is similar to the proof of Theorem 3.5.

If X does not contain a copy of ℓ_1, then $CC(X, \ell_\infty) = K(X, \ell_\infty)$ by a result of Odell [R, p. 377]. ∎

Corollary 3.7. *Suppose X and Y are Banach spacees, and $\ell_\infty \hookrightarrow Y$.*

(i) *If X is not sequentially Right, then $W(X,Y)$ is not complemented in $Pwc(X,Y)$.*
(ii) *If X does not have property (V), then $W(X,Y)$ is not complemented in $UC(X,Y)$.*
(iii) *If X has the DPP and does not have the RDPP, then $W(X,Y)$ is not complemented in $CC(X,Y)$.*
(iv) *If X^* does not have the Schur property, then $K(X,Y)$ is not complemented in $Pwc(X,Y)$.*
(v) *If X contains a copy of ℓ_1, then $K(X,Y)$ is not complemented in $CC(X,Y)$.*

Proof. We only prove (i). The other proofs are similar. Suppose that $W(X,Y)$ is complemented in $Pwc(X,Y)$. Since $\ell_\infty \hookrightarrow Y$ we have $\ell_\infty \overset{c}{\hookrightarrow} Y$, since ℓ_∞ is injective [D1, p. 71]. Then $W(X,\ell_\infty)$ is complemented in $W(X,Y)$, and thus in $Pwc(X,Y)$. Since $W(X,\ell_\infty) \subseteq Pwc(X,\ell_\infty) \subseteq Pwc(X,Y)$, it follows that $W(X,\ell_\infty)$ is complemented in $Pwc(X,\ell_\infty)$, contrary to Theorem 3.2. ∎

Corollary 3.8. *Suppose X and Y are Banach spaces, and $\ell_\infty \hookrightarrow Y$. Then the following are equivalent:*

(1) (i) *X is sequentially Right.*
(ii) *$Pwc(X,Y) = W(X,Y)$.*
(iii) *$W(X,Y)$ is complemented in $Pwc(X,Y)$.*

(2) (i) *X has property (V).*
(ii) *$UC(X,Y) = W(X,Y)$.*
(iii) *$W(X,Y)$ is complemented in $UC(X,Y)$.*

(3) (i) *X^* has the Schur property.*
(ii) *$K(X,Y) = Pwc(X,Y)$.*
(iii) *$K(X,Y)$ is complemented in $Pwc(X,Y)$.*

(4) (i) *X does not contain a copy of ℓ_1.*
(ii) *$CC(X,Y) = K(X,Y)$.*
(iii) *$K(X,Y)$ is complemented in $CC(X,Y)$.*

Proof. (i)⇒(ii). (1) Since X is sequentially Right, $Pwc(X,Y) \subseteq W(X,Y)$. Since $W(X,Y) \subseteq Pwc(X,Y)$ ([A, Theorem 1], [G2, Corollary 5]), it follows that $Pwc(X,Y) = W(X,Y)$. (2) By [P, Proposition 1], $UC(X,Y) \subseteq W(X,Y)$. Since $W(X,Y) \subseteq UC(X,Y)$ (by the Orlicz–Pettis theorem), it follows that $UC(X,Y) = W(X,Y)$. (3) By [G1, Corollary 9], every Right subset of X^* is relatively compact. Then every pseudo weakly compact operator $T : X \to Y$ is compact by [G1, Theorem 2]. Hence $K(X,Y) = Pwc(X,Y)$. (4) Apply a result of Odell [R, p. 377].

(iii)⇒(i) follows by Corollary 3.7. ∎

Corollary 3.9. *Suppose X has the DPP and $\ell_\infty \hookrightarrow Y$. Then the following are equivalent:*

(i) *X has the RDPP.*
(ii) *$CC(X,Y) = W(X,Y)$.*
(iii) *$W(X,Y)$ is complemented in $CC(X,Y)$.*

Proof. (i)⇒(ii) since X has the DPP and the RDPP; and (iii)⇒(i) by Corollary 3.7. ∎

We conclude this paper with another complementation theorem. We use the following notation. Let $A : X \to \ell_\infty$ be an operator and M be a nonempty subset of $\mathbb{N}$. We define $A_M : X \to \ell_\infty$ by

$$A_M(x) = \sum_{n\in M} e_n^*(A(x))e_n, \qquad x \in X.$$

Definition 3.10. A closed operator ideal $\mathcal{O}$ *has property* $(*)$ if whenever X is a Banach space and $A \notin \mathcal{O}(X,\ell_\infty)$, then there is an infinite subset M_0 of $\mathbb{N}$ such that $A_M \notin \mathcal{O}(X,\ell_\infty)$ for all infinite subsets M of M_0 [BBG].

Theorem 3.11. *If X^* contains a Right set which is not an L-set, then $CC(X,\ell_\infty)$ is complemented neither in $Pwc(X,\ell_\infty)$ nor in $UC(X,\ell_\infty)$.*

Proof. Let A be a Right subset of X^* which is not an L-set. Let (x_n^*) be a sequence in A and (x_n) be a weakly null sequence in X such that $|x_n^*(x_n)| \nrightarrow 0$. Without loss of generality assume that for some $\epsilon > 0$, $|x_n^*(x_n)| > \epsilon$ for all n. Define $S : X \to \ell_\infty$ by $S(x) = (x_n^*(x))_n$ for $x \in X$. Since $\|S(x_n)\| > \epsilon$, S is not completely continuous. Let $X_0 = [x_n]$ be the closed linear span of $\{x_n : n \in \mathbb{N}\}$. Note that X_0 is a separable subspace of X and $S|_{X_0}$ is not completely continuous. By [BBG, Theorem 25], the ideal of completely continuous operators has property $(*)$. Let M_0 be an infinite subset of $\mathbb{N}$ such that $S_M \notin CC(X_0,\ell_\infty)$ for all infinite subsets M of M_0.

Define $T : \ell_\infty \to L(X,\ell_\infty)$ by $T(b)(x) = (b_n\, x_n^*(x))_n$ for $b = (b_n) \in \ell_\infty$ and $x \in X$. Note that the operator T is well-defined and $T(e_n) = x_n^* \otimes e_n$ for each $n \in \mathbb{N}$. Since (x_n^*) is a Right set, $T(b)$ is pseudo weakly compact.

Suppose that $CC(X,\ell_\infty)$ is complemented in $Pwc(X,\ell_\infty)$ with projection $P : Pwc(X,\ell_\infty) \to CC(X,\ell_\infty)$, and let $R : L(X,\ell_\infty) \to L(X_0,\ell_\infty)$ be the natural restriction map. Define $\phi : \ell_\infty \to Pwc(X_0,\ell_\infty)$ by $\phi(b) = RT(b)$ and $\psi : \ell_\infty \to CC(X_0,\ell_\infty)$ by $\psi(b) = RPT(b)$. Since $T(e_n)$ is a rank one operator,

$$\psi(e_n) = RPT(e_n) = RT(e_n) = \phi(e_n)$$

for each $n \in \mathbb{N}$. By Lemma 3.1, there is an infinite subset M of $\mathbb{N}$ such that $\psi(\chi_M) = \phi(\chi_M)$. Hence $\phi(\chi_M)$ is completely continuous. However,

$\phi(\chi_M) = T(\chi_M)|_{X_0} = S_M$. This contradiction proves that $CC(X, \ell_\infty)$ is not complemented in $Pwc(X, \ell_\infty)$.

Since every pseudo weakly compact operator is unconditionally converging [PV$^+$, Proposition 14], $CC(X, \ell_\infty)$ is not complemented in $UC(X, \ell_\infty)$. ∎

COROLLARY 3.12. *If X does not have the DPP, then $CC(X, \ell_\infty)$ is complemented neither in $Pwc(X, \ell_\infty)$ nor in $UC(X, \ell_\infty)$.*

Proof. Since X does not have the DPP, there exist weakly null sequences (x_n) in X and (x_n^*) in X^* such that $|x_n^*(x_n)| \nrightarrow 0$ [D2, Theorem 1]. Since (x_n^*) is weakly null, it is a Right set [K, Corollary 3.26]. Thus (x_n^*) is a Right set which is not an L-set. Apply Theorem 3.11. ∎

COROLLARY 3.13. *If X does not have the DPP and $\ell_\infty \hookrightarrow Y$, then $CC(X, Y)$ is complemented neither in $Pwc(X, Y)$ nor in $UC(X, Y)$.*

COROLLARY 3.14. *Let X and Y be Banach spaces, and $\ell_\infty \hookrightarrow Y$. Then the following are equivalent:*

(i) *X has the DPP.*
(ii) *$CC(X, Y) = Pwc(X, Y)$.*
(iii) *$CC(X, Y)$ is complemented in $Pwc(X, Y)$.*

Proof. (i)⇒(ii) by [K, Proposition 3.17] and [G2, Theorem 10]; and (iii)⇒(i) by Corollary 3.13. ∎

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