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INTEGRAL FORMULAE FOR FOLIATIONS WITH SINGULARITIES

BY

PAWEŁ WALCZAK (Łódź)

Dedicated to the memory of Witold Roter

Abstract. We consider an oriented closed Riemannian manifold M equipped with a codimension-one foliation $\mathcal{F}$ defined outside a finite union Σ of pairwise disjoint closed submanifolds of sufficiently large codimension. Using a technical lemma we show that several integral formulae known for foliations of closed manifolds hold also in this case under some conditions (integrability of some functions). In particular, the results of this article generalize some observations of Andrzejewski et al. (2014), Lużyńczyk and Walczak (2015) and Rovenski and Walczak (2012).

Introduction. Integral formulae in Riemannian geometry of foliated manifolds provide obstructions to the existence of foliations with given geometric properties. Roughly speaking, they arise from Green's or Stokes's theorem applied to geometrically defined vector fields or differential forms. The oldest one,

$$\int_M \sigma_1 \, d\mathrm{vol} = 0, \tag{1}$$

follows from Green's theorem applied to the unit vector field N orthogonal to the leaves of a transversely oriented codimension-one foliation $\mathcal{F}$ of a closed Riemannian manifold M and the (first observed by Reeb [Re]) equality $\operatorname{div} N = -\sigma_1$, σ_1 being the mean curvature of (the leaves of) $\mathcal{F}$. Other formulae of this sort were obtained e.g. in [BLR], [RW1], [AW], in fact by applying Green's theorem to the traces of the Newton transformations of the Weingarten operator A of $\mathcal{F}$ (compare Section 1.1) on manifolds of constant sectional curvature, locally symmetric spaces and arbitrary Riemannian manifolds, respectively (see also [ARW] and [RW2] for more information).

In [BW], an energy estimate for unit vector fields with isolated singularities (that is, defined on a closed manifold M outside a finite set) has been established with the use of one of such integral formulae, namely (8)

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below, while in [BGV] a similar result has been obtained when the set of singularities consists of pairwise disjoint closed submanifolds of codimension ≥ 2 (including, of course, single points, that is, submanifolds of codimension equal to the dimension of the manifold).

Using the ideas of [BW] and [BGV], we show here that several integral formulae which appear in the papers mentioned in the first paragraph of this Introduction hold for foliations defined on $M \setminus \Sigma$, M being a closed oriented Riemannian manifold, and Σ a finite union of pairwise disjoint closed submanifolds of codimension large enough, under the additional assumption of convergence of the integrals $\int_{M\setminus\Sigma} f\, d\mathrm{vol}$ for suitable functions f.

1. Preliminaries

1.1. Extrinsic geometry. Let (M, g) be an $(n+1)$-dimensional Riemannian manifold, $\mathcal{F}$ a transversely orientable codimension-one foliation on M, and N the positive oriented unit vector field orthogonal to (the leaves) of $\mathcal{F}$. Denote by ∇ the Levi-Civita connection of $g = \langle\cdot,\cdot\rangle$ and by R its curvature tensor. Then the Weingarten operator A and the second fundamental form b of $\mathcal{F}$ are defined by

$$AX = -\nabla_X N, \quad b(X, Y) = \langle AX, Y\rangle, \quad X, Y \in T\mathcal{F}. \tag{2}$$

Thus, A is a self-adjoint endomorphism of the tangent bundle $T\mathcal{F}$, while b is a symmetric 2-form on $T\mathcal{F}$ adjoint to A.

The elementary symmetric functions σ_r, $0 \leq r \leq n$, of the *principal curvatures* (that is, eigenvalues of A) $k_1, \ldots, k_n$, given by

$$\sigma_r = \sum_{i_1<\ldots<i_r} k_{i_1} \cdot \ldots \cdot k_{i_r},$$

are called the *mean curvatures* of $\mathcal{F}$. The formula

$$T_r = T_r(A) = \sum_{j=0}^{r} (-1)^j \sigma_{r-j} A^j, \qquad r = 0, 1, \ldots, n, \tag{3}$$

defines the *Newton transformations* of A; these can also be defined inductively by

$$T_0 = \mathrm{id}, \quad T_r = \sigma_r \cdot \mathrm{id} - AT_{r-1}. \tag{4}$$

By the Cayley–Hamilton Theorem, $T_n = 0$.

Given a vector field Z on M, define an endomorphism $R(Z)$ of $T\mathcal{F}$ by

$$R(Z)(X) = R(X, Z)N. \tag{5}$$

In particular, one may set in (5) $Z = \nabla_N N$, the curvature vector of the curves orthogonal to $\mathcal{F}$.

1.2. Integrals. In [AW], the following formula for foliations of closed manifolds has been established:

$$\int_M \big(\langle D_r, \nabla_N N\rangle - (r+2)\sigma_{r+2} - \operatorname{Tr}(R(N)\circ T_r)\big)\, d\mathrm{vol} = 0, \tag{6}$$

where D_r is a vector field tangent to $\mathcal{F}$ given by

$$D_0 = 0 \quad \text{and} \quad \langle D_r, X\rangle = \sum_{j=1}^{r} (-1)^{j-1} \operatorname{Tr}(R(A^{j-1}(X))\circ T_{r-j}). \tag{7}$$

for $r \geq 1$. Note that

- for $r = 0$, (6) reduces to the equation (known since the early 1980's [No])

$$\int_M (2\sigma_2 - \operatorname{Ric}(N, N)) = 0, \tag{8}$$

- for manifolds of constant sectional curvature, (6) yields the formulae of [BLR], and
- for Riemannian foliations, the orthogonal curves are geodesics, therefore $\nabla_N N = 0$ and (6) reduces to

$$(r+2)\int_M \sigma_{r+2} = \int_M \operatorname{Tr}(R(N)T_r). \tag{9}$$

Formula (6) follows from Green's theorem and the equality [AW, Thm. 3.5]

$$\operatorname{div}(T_r(\nabla_N N) + \sigma_{r+1}N) = \langle D_r, \nabla_N N\rangle - (r+2)\sigma_{r+2} - \operatorname{Tr}(R(N)T_r). \tag{10}$$

2. Lemmas. First, let f be a nonnegative real function defined on $M \setminus S$, M being a closed oriented Riemannian manifold and S a closed submanifold of M of codimension k. Given a positive and sufficiently small r, denote by $N_S(r)$ the tube of radius r around S and by $\partial N_S(r)$ its boundary, the tubular surface at distance r. Also, let X be a vector field defined on $M \setminus S$.

The following two lemmas generalize Lemmas 1 and 2 of [LuW], respectively.

Lemma 1. *If $p > 1$, $(k-1)(p-1) \geq 1$ and*

$$\liminf_{r\to 0^+} \int_{\partial N_S(r)} f > 0,$$

then $\int_M f^p = \infty$.

Proof. Since M is compact, its geometry is bounded and there exist positive constants r_0, c and ε such that $\operatorname{vol}(\partial N_S(r)) \leq c r^{k-1}$ and $\int_{\partial N_S(r)} f \geq \varepsilon$ for all $r \leq r_0$. Hölder's inequality implies that

$$\int_{\partial N_S(r)} f \leq \Big(\int_{\partial N_S(r)} f^p\Big)^{1/p} \operatorname{vol}(\partial N_S(r))^{1/q},$$

where $1/p + 1/q = 1$. Consequently,

$$\int_{\partial N_S(r)} f^p \geq \frac{\varepsilon^p}{c^{p/q}} r^{p(1-k)/q}$$

for all $r \leq r_0$. Finally, if $r \leq r_0$, then

$$\int_M f^p \geq \int_{N_S(r)} f^p = \int_0^r \Big(\int_{\partial N_S(t)} f^p \Big) dt \geq \frac{\varepsilon^p}{c^{p/q}} \int_0^r t^{(1-k)p/q} \, dt = \infty$$

when $(1-k)p/q \leq -1$, equivalently when $(k-1)(p-1) \geq 1$. ∎

LEMMA 2. *If $(k-1)(p-1) \geq 1$ and there exists a vector field Z on $M \setminus S$ such that $\int_M \|Z\|^p \, d\mathrm{vol} < \infty$ and $\int_M \|X + Z\|^p \, d\mathrm{vol} < \infty$, then*

$$\int_M \operatorname{div} X \, d\mathrm{vol} = 0.$$

Proof. Let ν_r be the suitably oriented unit vector field orthogonal to the tubular surface $\partial N_S(r)$. By Stokes's theorem and Lemma 1 applied to $f = \|Z\|$, we obtain

$$\Big| \int_{M \setminus N_S(r)} \operatorname{div} Z \Big| = \Big| \int_{\partial N_S(r)} \langle Z, \nu_r \rangle \Big| \leq \int_{\partial N_S(r)} \|Z\| \to 0 \quad \text{as } r \to 0.$$

The same argument shows that

$$\lim_{r \to 0} \int_{M \setminus N_S(r)} \operatorname{div}(X + Z) = 0.$$

From the above, the statement follows. ∎

Certainly, Lemmas 1 and 2 remain valid when we replace S by a finite union of pairwise disjoint closed submanifolds S_i of variable codimensions $k_i \geq k$. Also, Lemma 2 stays valid when Z is defined globally on M and satisfies $\int_M \|X + Z\|^p \, d\mathrm{vol} < \infty$.

3. Results. The following is a consequence of Lemma 2 of the previous section.

THEOREM 3. *Assume that*

(i) *a codimension one foliation $\mathcal{F}$ is defined on $M \setminus \Sigma$, Σ being a union of finitely many, pairwise disjoint, closed submanifolds S_i of codimensions $k_i \geq k \geq 2$,*
(ii) *the norm $\|A\|$ of the Weingarten operator A of $\mathcal{F}$ is $L^{(r+1)p}$-integrable on M for some $r \in \mathbb{N}$ and $p > 1$ such that $(k-1)(p-1) \geq 1$, and*
(iii) *the curvature $\kappa = \|\nabla_N N\|$ of the 1-dimensional foliation by integral curves of the unit normal N of $\mathcal{F}$ is $L^{(r+1)p}$-integrable.*

Then equality (6) *holds. In particular, equality* (8) *holds when* κ *and* $\|A\|$ *are* L^p*-integrable on* M *for some* p *satisfying* $(p-1)(k-1) \geq 1$.

Proof. Since M is closed, it has finite volume and

$$\int_M \|A\|^{jp} < \infty \tag{11}$$

for all $j \leq r+1$. Let $\tau_j = \mathrm{Tr}(A^j)$ be the sum of the kth powers of the principal curvatures of the leaves of $\mathcal{F}$. From (11) and the inequality $|\mathrm{Tr}\, B| \leq c(n)\|B\|$ which holds for some constant $c(n)$ and all square $n \times n$ matrices B, it follows that $\tau_j \in L^{p(r+1)/j}(M)$ for all $j \leq r+1$. Using the classical Newton–Girard formulae (see, for example [Me] or [RW2, p. 6])

$$\tau_j - \tau_{j-1}\sigma_1 + \cdots + (-1)^{j-1}\tau_1\sigma_{j-1} + (-1)^j j\sigma_j = 0, \quad j = 1, \ldots, n,$$

one can show inductively that $\sigma_j \in L^{p(r+1)/j}(M)$ for all $j \leq r+1$. In fact, $\sigma_1 = \tau_1 \in L^{p(r+1)}(M)$ and if $\sigma_i \in L^{p(r+1)/i}(M)$ for all $i < j$, then

$$\int_M |\tau_{j-i}\sigma_i|^{\frac{p(r+1)}{j}} \leq \Big(\int_M |\tau_{j-i}|^{\frac{p(r+1)}{j-i}}\Big)^{\frac{j-i}{j}} \Big(\int_M |\sigma_i|^{\frac{p(r+1)}{i}}\Big)^{i/j} < \infty.$$

In the same way, formulae (3) and (11) imply that $\|T_r\| \in L^{p(r+1)/r}(M)$ and $\|T_r(\nabla_N N)\| \in L^p(M)$. Since σ_{r+1} is also L^p-integrable on M, applying Lemma 2 to the vector field $X = T_r(\nabla_N N) + \sigma_{r+1}N$ (and $Z = 0$) ends the proof. ∎

In [LaW], equation (8) was applied to show that codimension-one foliations of closed manifolds of negative Ricci curvature are far from being umbilical, in the following sense.

Let

$$U(\mathcal{F}) = \int_M \sum_{i<j} |k_i - k_j|^{n+1}\, d\mathrm{vol}, \tag{12}$$

where $k_1, \ldots, k_n$ are the principal curvatures of the leaves of $\mathcal{F}$. Note that (i) $U(\mathcal{F}) = 0$ if and only if $k_1 = \cdots = k_n$, that is, when the leaves of $\mathcal{F}$ are umbilical and (ii) $U(\mathcal{F})$ is a conformal invariant, that is, it stays the same when we change the Riemannian metric on M conformally. In [LaW], it has been shown that

$$U(\mathcal{F}) \geq \binom{n}{2}^{(1-n)/2} (-2c)^{(n+1)/2}\mathrm{vol}(M) \tag{13}$$

when M is a closed $(n+1)$-dimensional Riemannian manifold of negative Ricci curvature $\leq c < 0$. Inequality (13) is an almost direct consequence of (8). Therefore, Theorem 3 implies the following.

THEOREM 4. *If* $\mathcal{F}$ *is a foliation on* $M \setminus \Sigma$, M *being a closed oriented Riemannian manifold of negative Ricci curvature* $\leq c < 0$ *and* Σ *a finite* *union of pairwise disjoint closed submanifolds S_i of codimension $k_i \geq k$, $p > 1$, $(k-1)(p-1) \geq 1$ and both the norm $\|A\|$ of the Weingarten operator of $\mathcal{F}$ and the curvature $\kappa = \|\nabla_N N\|$ of the normal flow are L^p-integrable, then* (13) *holds.*

Finally, let $\mathcal{F}$ be of constant mean curvature. This means that the mean curvature vector H of $\mathcal{F}$, $H = hN$, h being the scalar mean curvature of $\mathcal{F}$, is covariantly constant, that is, $(\nabla_X H)^\perp = 0$ for all $X \in TM$, and consequently $h = \text{const}$ all over M. Assume also that the curvature vector $\mathcal{K} = \nabla_N N$ of the normal flow is constant, that is, $(\nabla_X \mathcal{K})^\top = 0$ for all $X \in TM$, and consequently the curvature $\kappa = \|\nabla_N N\|$ is constant on M. In [LuW], the divergence of the vector field

$$X = hA(\nabla_N N) + (\nabla_H \mathcal{K})^\perp \tag{14}$$

has been calculated (in fact, the situation described above is a particular case of that involving two complementary orthogonal plane fields, both of constant mean curvature, considered in [LuW]):

$$\operatorname{div} X = h\operatorname{Ric}(\nabla_N N, N) + h^2\kappa^2 - \langle A(\mathcal{K}), \mathcal{K}\rangle. \tag{15}$$

Applying Lemma 2 again we arrive at

THEOREM 5. *If, as before, $\mathcal{F}$ is a foliation on $M \setminus \Sigma$, M being a closed oriented Riemannian manifold and Σ a finite union of pairwise disjoint closed submanifolds S_i of codimension $k_i \geq k \geq 2$, $p > 1$, $(k-1)(p-1) \geq 1$, and both the norm $\|A\|$ of the Weingarten operator of $\mathcal{F}$ and the curvature κ of the normal flow are L^{4p}-integrable, then*

$$\int_M (h\operatorname{Ric}(\mathcal{K}, N) + h^2\kappa^2 - \langle A(\mathcal{K}), \mathcal{K}\rangle)\, d\text{vol} = 0. \tag{16}$$

Proof. We have only to verify L^p-integrability of the norm $\|X\|$. Certainly, by (14) and

$$|\langle \nabla_H \mathcal{K}, N\rangle| = |\langle \mathcal{K}, \nabla_H N\rangle| \leq \|\mathcal{K}\| \cdot |h| \cdot \|\nabla_N N\| \leq c(n)\kappa^2\|A\|$$

we obtain

$$\|X\| \leq |h| \cdot |\kappa| \cdot \|A\| + c(n)|h|\kappa^2 \leq c(n)(|\kappa| \cdot \|A\|^2 + \|A\|\kappa^2).$$

Therefore, Hölder's inequality applied several times yields

$$\int_M \|X\|^p \leq \operatorname{vol}(M)^{1/4}\Big(\int_M \kappa^{4p}\Big)^{1/4}\Big(\int_M \|A\|^{4p}\Big)^{1/4} < \infty. \ \blacksquare$$

The following particular case of the above seems to be interesting.

COROLLARY 6. *Under the assumptions of Theorem* 5, *if in addition M is an Einstein manifold and the foliation $\mathcal{F}$ is umbilical, then $h \cdot \kappa \equiv 0$ on $M \setminus \Sigma$.*

Proof. If M is Einstein, then $\mathrm{Ric}(\mathcal{K}, H) = 0$ since $H \perp \mathcal{K}$. If moreover $\mathcal{F}$ is umbilical, then $A = (1/n)h \cdot \mathrm{id}_{T\mathcal{F}}$, $\langle A(\mathcal{K}), \mathcal{K})\rangle = (1/n)hk^2$, and (16) reduces to

$$\int_M h^2 \kappa^2 \, d\mathrm{vol} = 0. \ \blacksquare$$

4. Remarks. 1. Observe that (1) holds for all codimension-one foliations defined on $M \setminus \Sigma$, M and Σ being as above with $k \geq 2$. Indeed, $\sigma_1 = -\operatorname{div} N$ and $\|N\| = 1$ is L^p-integrable for all $p > 1$.

2. A nice and simple example of a foliation with singularities of codimension 2 is provided by the family $\mathcal{T}$ of tori $T_{a,b} = \{(z, w) : |z|^2 = a^2, |w|^2 = b^2\}$, $a^2 + b^2 = 1$, which fill all the unit sphere $S^3 \in \mathbb{R}^4 = \mathbb{C}^2$ except for the union Σ of two geodesic circles given by intersecting S^3 with the planes $z = 0$ and $w = 0$. In this case, the reflection $(w, z) \mapsto (z, w)$ provides an isometry of S^3 which preserves $\mathcal{T}$ and maps a torus of (constant) mean curvature H onto a torus of mean curvature $-H$. Therefore, equality (1) holds for $\mathcal{T}$ in the obvious way. This simple example can be transferred to any odd-dimensional sphere.

3. We would be happy to find a reasonable application of Lemma 2 with a geometrically interesting $Z \neq 0$.

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Paweł Walczak
Katedra Geometrii
Wydział Matematyki i Informatyki
Uniwersytet Łódzki
Banacha 22
90-238 Łódź, Poland
E-mail: pawelwal@math.uni.lodz.pl