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On Selberg's approximation to the twin prime problem

by

R. Balasubramanian and
Priyamvad Srivastav (Chennai and Mumbai)

1. Introduction. The *twin prime conjecture* asserts that there are infinitely many primes p such that $p+2$ is a prime. In spite of numerous efforts of various mathematicians, we seem to be nowhere close to settling this problem with the current methods. The best known approximation to this problem is due to J. R. Chen [Che], who proved that for infinitely many primes p, $p+2$ has at most two prime factors.

The last few years have seen major developments on a related problem, called the *bounded gaps problem*, which asks whether the quantity

$$H_m = \liminf_{n\to\infty}(p_{n+m}-p_n)$$

is finite. It was a remarkable breakthrough in 2013 when Zhang [Zha] proved $H_1 \le 7\cdot 10^7$. His proof was based on the GPY sieve, which was used in [GPY] and [MP]. This bound was subsequently lowered to about 4680 by the Polymath 8a [P8a] project. In the same year, James Maynard [May] introduced a refinement to the GPY sieve by using multidimensional sieve weights, under which he showed that $H_1 \le 600$ and $H_m \ll m^3e^{4m}$. This latter result was also proved independently by Tao. The Polymath 8b [P8b] project, which extends Maynard's methods, has successfully shown $H_1 \le 246$. Under the Generalized Elliott Halberstam conjecture, they have shown $H_1 \le 6$.

In this paper, we are interested in revisiting Selberg's classical approach to this problem. In an unpublished manuscript [Sel] of his, Selberg considers the weighted sum

$$\sum_{\substack{x<n\le 2x\\ n\equiv -1\,(6)}}\left(1-\frac{2^{\Omega(n)}+2^{\Omega(n+2)}}{\lambda}\right)w_n \tag{1.1}$$

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where w_n's are non-negative weights. Selberg succeeded in showing that (1.1) is positive for any $\lambda > 14$.

When this sum is positive, there is an $x < n \le 2x$ for which $2^{\Omega(n)} + 2^{\Omega(n+2)} \le 14$. This forces $n(n+2)$ to have at most five prime factors, with one factor having two and the other having at most three prime factors. So, there must be infinitely many n for which $n(n+2) \in P_5$, where P_r denotes the set of integers having at most r prime factors counted with multiplicity.

The choice of the weights w_n by Selberg mimics typical Selberg sieve weights. He makes the choice

$$w_n = \Big(\sum_{\substack{d \mid n(n+2)\\ d \le z}} \lambda_d\Big)^2$$

with $z = x^{1/3-\epsilon}$ and

$$\lambda_d = \mu(d)\left(1 - \frac{\log d}{\log z}\right)^3.$$

By adjusting the λ_d's suitably, Gerd Hofmeister (unpublished) was able to bring down λ to "about 13", though it is not clear from Selberg's manuscript what "about 13" exactly means.

One needs to bring λ below 12 in order to obtain $n(n+2) \in P_4$ (since $12 = 2^2 + 2^3$). To reach Chen's theorem, λ needs to be brought below 8 (since $8 = 2^2 + 2^2$), which, as predicted by Selberg, is out of scope of this method.

This result relies heavily on the level of distribution of the divisor-type function $2^{\Omega(n)}$ in arithmetic progressions. We say that an arithmetic function f has a *level of distribution* θ if for any $A > 0$, there is a $B > 0$ such that

$$\sum_{q \le x^{\theta}/\log^B x} \max_{(a,q)=1} \Big| \sum_{\substack{n \le x\\ n \equiv a\,(q)}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \le x\\ (n,q)=1}} f(n) \Big| \ll \frac{x}{\log^A x}.$$

Using Weil's bound, Selberg was able to obtain a level of distribution $\theta = 2/3 - \epsilon$ for the divisor function. He obtained the following result.

Theorem 1.1. *Let $m = Wm'$ be an even squarefree integer, where m' is odd and $(W, m') = 1$. Suppose $(a, m) = 1$. Then for any $\epsilon > 0$ and $m \le x^{2/3-\epsilon}$,*

$$\sum_{\substack{n \le x\\ n \equiv a\,(m)}} 2^{\Omega(n)} = x \frac{\varphi(W)}{W^2} \frac{c(W)}{\gamma(m')} \Big(\log x + c + 2 \sum_{\substack{p \mid Wm'\\ p > 2}} \frac{\log p}{p-2}\Big) + O_{\epsilon}\Big(\frac{x^{1/2+\epsilon}}{m^{1/4}}\Big).$$

It is conjectured that functions like $\Lambda(n), \tau(n)$ and many other nice arithmetic functions have a level of distribution $\theta = 1 - \epsilon$. We know that the von Mangoldt function $\Lambda(n)$ has a level of distribution $\theta = 1/2 - \epsilon$, thanks to the

Bombieri–Vinogradov Theorem. Having a high level of distribution permits us to choose a larger value for z while keeping the error term in check, and the larger the value of z, the better the quality of the result.

As a concluding remark, Selberg [Sel, p. 245] recommends using two-dimensional sieve weights

$$w_n = \Big(\sum_{\substack{d_1 \mid n \\ d_2 \mid n+2}} \lambda_{d_1,d_2} \Big)^2 \tag{1.2}$$

which would run separately through the divisors of n and $n+2$, thus providing some flexibility. The weights λ_{d_1,d_2} would be supported on the set

$$S(z) = \{(d_1,d_2) : \mu^2(d_1 d_2 W) = 1,\ \max\{d_1^{2/3} d_2, d_1 d_2^{2/3}\} \le z\}. \tag{1.3}$$

Note that here $z = x^{1/3-\epsilon}$ and a typical choice for the weights would be

$$\lambda_{d_1,d_2} = \mu(d_1)\mu(d_2) F\bigg(\frac{\log d_1}{\log z}, \frac{\log d_2}{\log z} \bigg)$$

for some smooth function F.

2. The modified sieve. In this paper, we adopt Selberg's idea of using multi-dimensional sieve weights and improve the value of λ. We consider the weighted sum

$$\sum_{\substack{n \sim x \\ n \equiv v_0\ (W)}} \bigg(1 - \frac{2^{\Omega(n)} + 2^{\Omega(n+h_0)}}{\lambda} \bigg) \Big(\sum_{\substack{d_1 \mid n \\ d_2 \mid n+h_0}} \lambda_{d_1,d_2} \Big)^2 = S_1 - S_2/\lambda. \tag{2.1}$$

Here h_0 is an even number and v_0 is chosen such that $(v_0(v_0+h_0), W) = 1$. Following [May], we define

$$D = \log\log\log x \quad \text{and} \quad W = \prod_{p \le D} p. \tag{2.2}$$

Note that

$$S_1 = \sum_{\substack{n \sim x \\ n \equiv v_0\ (W)}} \Big(\sum_{\substack{d_1 \mid n \\ d_2 \mid n+h_0}} \lambda_{d_1,d_2} \Big)^2, \tag{2.3}$$

$$S_2 = \sum_{\substack{n \sim x \\ n \equiv v_0\ (W)}} (2^{\Omega(n)} + 2^{\Omega(n+h_0)}) \Big(\sum_{\substack{d_1 \mid n \\ d_2 \mid n+h_0}} \lambda_{d_1,d_2} \Big)^2. \tag{2.4}$$

We will prove

THEOREM 2.1. (2.1) *is positive for any* $\lambda > 12.59$.

Let us assess the above result. Unfortunately, the above improved value of λ still does not improve qualitatively upon Selberg's result, since one needs to go below 12 in order to reach an improvement. Therefore, the result obtained via this approach is much weaker than Chen's theorem, which states that $n(n+2) \in P_3$ infinitely often. However, the objective of this paper is only to adopt the classical approach of Selberg to the twin prime problem and see how much one can push the value of λ through this approach. In a way, it also shows the strengths and advantages of using multi-dimensional sieve weights over the one-dimensional ones, as was apparent in [May].

3. Notation. Throughout this paper, h_0 remains a fixed even number. The notation $n \sim x$ means that $x < n \leq 2x$. The symbol p is reserved for prime numbers. The numbers ϵ, x and z will always be positive real with $z \leq x$. We shall often assume that x is sufficiently large and ϵ is sufficiently small. We write (a, b) and $[a, b]$ for the GCD and LCM of positive integers a, b respectively. In many places, particularly in Section 3, we write $f(a, b)$ (or $f[a, b]$) to denote $f((a, b))$ (or $f([a, b])$) to simplify notation. For arithmetic functions f and g, $f * g$ denotes their Dirichlet convolution. The O-constants often depend on ϵ or on the functions P, Q_1 or Q_2, which will all be bounded functions. For the sake of simplicity, we do not specify the dependence of these O-constants.

In this paper, id denotes the function defined by $\mathrm{id}(n) = n$, μ is the Möbius function, φ is the Euler phi function, τ is the divisor function, Ω is the function defined additively by $\Omega(p^k) = k$, γ is defined by $\gamma(p) = \frac{p(p-1)}{p-2}$, ρ is defined by $\rho(p) = p/2$. The functions ρ_1, γ_1 satisfy $\rho = \rho_1 * 1$ and $\gamma = \gamma_1 * 1$. The function h is given by $h(p) = 1 - 3/(p+2)$, h_1, h_2 satisfy $h_1(p) = 1 - 3/p + 2/p^2$ and $h_2(p) = 1 - 2/p + 2/p^2$.

All these arithmetic functions barring $\Omega(n)$ are multiplicative. In some cases, we have defined the functions only on primes as we are only concerned with their values at squarefree integers.

4. Preliminary results. We review some well known results about partial sums of arithmetic functions. We define a class of arithmetic functions and give asymptotic estimates for their partial sums. This will prove useful during the computations. We begin with a lemma.

Lemma 4.1. *Let F be a continuously differentiable function on $[0, \kappa]$ and let $x^{1/\kappa} \leq z \leq x$. Then*

$$\sum_{p \leq x} \frac{\log p}{p} F\left(\frac{\log p}{\log z}\right) = (\log z + O(1)) \int_0^{\frac{\log x}{\log z}} F(t)\, dt.$$

Proof. We apply partial summation to the function $\frac{\log n}{n} F\big(\frac{\log n}{\log z}\big) 1_{\mathbb{P}}(n)$. Since $\sum_{p\le x} \frac{\log p}{p} = \log x + O(1)$, we get

$$\sum_{p\le x} \frac{\log p}{p} F\left(\frac{\log p}{\log z}\right) = \int_1^x F\left(\frac{\log t}{\log z}\right) d(\log t + O(1))$$
$$= \int_1^x F\left(\frac{\log t}{\log z}\right) \frac{dt}{t} + O(1) + O\left(\int_1^x F'\left(\frac{\log t}{\log z}\right) d\left(\frac{\log t}{\log z}\right)\right).$$

Note that the main term above is clearly $(\log z)\int_0^{\frac{\log x}{\log z}} F(t)\,dt$ and the error term is $O(1)$. This completes the proof. ∎

We define a class of multiplicative arithmetic functions and prove an asymptotic formula for their partial sums.

DEFINITION 4.2. For an integer $k \ge 1$, we define Ω_k to be the set of all multiplicative arithmetic functions f which are supported on the squarefree integers and have Dirichlet series of the form $D_f(s) = \zeta^k(1+s)G(s)$, where $G(s)$ is given by an absolutely convergent series in $\sigma \ge -\delta$ for some $\delta > 0$.

REMARK 4.3. A typical function in Ω_k takes the value $k/p + O(1/p^{1+\delta})$, $\delta > 0$, at primes.

PROPOSITION 4.4. *Let $f \in \Omega_k$ be an arithmetic function with Dirichlet series $D_f(s) = \zeta^k(s+1)G(s)$. Then for any non-integer $x > 1$,*

$$\sum_{n\le x} f(n) = G(0)\frac{\log^k x}{k!} + O(\log^{k-1} x).$$

Note that here $G(s) = \prod_p (1 - 1/p^{1+s})^k (1 + f(p)/p^s)$.

Proof. From [Iv, (13.10), p. 353] we have, for any $k \ge 1$,

$$\sum_{n\le x} \tau_k(n) = xP_k(\log x) + O_\epsilon(x^{\theta_k+\epsilon}) \tag{4.1}$$

where $0 < \theta_k < 1$, $\tau_k = 1 * \overset{k}{\cdots} * 1$ and $P_k(\log x) = \operatorname{Res}_{s=1} \zeta^k(s)\,\frac{x^{s-1}}{s}$ is a polynomial of degree $k-1$ in $\log x$, with leading term $\log^{k-1} x/(k-1)!$. By partial summation, we obtain

$$\sum_{n\le x} \frac{\tau_k(n)}{n} = \left(1 + O\left(\frac{1}{\log x}\right)\right)\frac{\log^k x}{k!}. \tag{4.2}$$

As the Dirichlet series of τ_k/id is $\zeta^k(s+1)$, we have $f = \tau_k/\mathrm{id} * g$. Therefore, by the convolution method,

$$\sum_{n\le x} f(n) = \sum_{ab\le x} g(a)\frac{\tau_k(b)}{b} = \sum_{a\le qx} g(a)\sum_{b\le x/a}\frac{\tau_k(b)}{b} \tag{4.3}$$
$$= \sum_{a\le x} g(a)\biggl(1+O\biggl(\frac{1}{\log(x/a)}\biggr)\biggr)\frac{\log^k(x/a)}{k!}$$
$$= \frac{\log^k x}{k!}\sum_{a\le x} g(a)\biggl(1-\frac{\log a}{\log x}\biggr)^k + O(\log^{k-1} x).$$

Since $G(s)$ is well defined at $s=0$, it follows that $\sum_{a\le t} g(a) = G(0)+A^*(t)$, where $A^*(x)=\sum_{n>x} g(n)$. Since $G(s)=\sum_n g(n)/n^s$ converges absolutely for $\sigma\ge-\delta$, it follows that $A^*(t)\ll t^{-\delta}$. Letting $Q(x)=(1-x)^k$, we have

$$\sum_{n\le x} g(n)Q\biggl(\frac{\log n}{\log x}\biggr) = \int_1^x Q\biggl(\frac{\log t}{\log x}\biggr)\,d(G(0)-A^*(t))$$
$$= AQ(0)+O(x^{-\delta})+O\biggl(\int_1^x \frac{1}{t^{1+\delta}\log x}Q'\biggl(\frac{\log t}{\log x}\biggr)\,dt\biggr)$$
$$= G(0)Q(0)+O(x^{-\delta}).$$

Substituting the above expression into (4.3) yields the desired result. ∎

We now redefine the constant $G(0)$ occurring in the above proposition.

Definition 4.5. Let $f\in\Omega_k$ and let $\bar f = f*1$. Then for any positive integer m, we define

$$c(m,f)=\prod_{p\nmid m}\biggl(1-\frac{1}{p}\biggr)^k \bar f(p). \tag{4.4}$$

Proposition 4.4 leads to the following corollary, which gives us the asymptotic formula for partial sums of functions in Ω_k.

Proposition 4.6. *Let $f\in\Omega_k$ and let $\bar f = f*1$. Then for any positive integer m and $(d,m)=1$,*

$$\sum_{\substack{n\le z\\ n\equiv 0\,(d)\\ (n,m)=1}} f(n) = \frac{f(d)}{\bar f(d)}\biggl(\frac{\varphi(m)\log z}{m}\biggr)^k \frac{c(m,f)}{k!}\biggl(1-\frac{\log d}{\log z}\biggr)^k + O(\log^{k-1} z).$$

Proof. Firstly, we note that

$$\sum_{\substack{n\le z\\ n\equiv 0\,(d)\\ (n,m)=1}} f(n) = f(d)\sum_{\substack{n\le z/d\\ (n,dm)=1}} f(n).$$

We now apply Proposition 4.4 to the function $f(n)1_{(n,dm)=1}$ with $x = z/d$. Note that the Dirichlet series of this function is $\zeta^k(1+s)G_{dm}(s)$, where $G_{dm}(s) = G(s)\prod_{p|dm}(1+f(p)/p^s)^{-1}$. We therefore obtain

$$\sum_{\substack{n\le z\\ n\equiv 0\,(d)\\ (n,m)=1}} f(n) = f(d)\frac{\log^k(z/d)}{k!}G_{dm}(0) + O(\log^{k-1}(z/d)).$$

Writing

$$G_{dm}(0) = G(0)\prod_{p|dm}(1+f(p))^{-1} = \frac{1}{\bar f(d)}\,\frac{\varphi^k(m)}{m^k}\prod_{p\nmid m}(1-1/p)^k(1+f(p))$$

we obtain the desired result. ■

The next theorem gives an asymptotic formula for functions in Ω_k multiplied by a smoothing function.

THEOREM 4.7. *Let $f\in\Omega_k$ and let $\bar f = f*1$. Suppose $0 < w_1 < w_2 \ll z^\kappa$ for some $\kappa > 0$. Let $d \le w_2$ and let F be a continuously differentiable function on $[0,\kappa]$. Then for any positive integer m,*

$$\sum_{\substack{w_1<n\le w_2\\ n\equiv 0\,(d)\\ (n,m)=1}} f(n)F\Big(\frac{\log n}{\log z}\Big)$$

$$= \Big(\frac{\varphi(m)\log z}{m}\Big)^k \frac{f(d)}{\bar f(d)}\,\frac{c(m,f)}{(k-1)!}\int_{\max\{\frac{\log w_1}{\log z},\frac{\log d}{\log z}\}}^{\frac{\log w_2}{\log z}} F(t)\Big(t-\frac{\log d}{\log z}\Big)^{k-1}dt$$

$$+\ O(\log^{k-1} z).$$

Proof. Let $f_{m,d}$ be defined by

$$f_{m,d}(n) = \begin{cases} f(n), & n\equiv 0 \pmod d \text{ and } (n,m)=1,\\ 0, & \text{otherwise.}\end{cases}$$

The sum to be estimated then is

$$\sum_{w_1<n\le w_2} f_{m,d}(n)F\Big(\frac{\log n}{\log z}\Big).$$

We have seen in Proposition 4.6 that for any $x > w_1 > 0$,

$$\sum_{w_1<n\le x} f_{m,d}(n) = M(x) + E(x)$$

where $M(x)$ is the main term and $E(x)$ is the error term. We also know that $E(x) \ll \log^{k-1} x$ and $M(x)$ is a differentiable function on $\mathbb{R}_{>0}$. We apply partial summation to get

$$\sum_{w_1<n\le w_2} f_{m,d}(n)F\left(\frac{\log n}{\log z}\right)=\int_{w_1}^{w_2} F\left(\frac{\log t}{\log z}\right) d(M(t)+E(t))\,dt.$$

The main term above is

$$\int_{w_1}^{w_2} M'(t)F\left(\frac{\log t}{\log z}\right) dt$$

and the error term is

$$O\left(|E(w_2)|+|E(w_1)|+\int_{w_1}^{w_2}\frac{|E(t)|}{t\log z}\,dt\right).$$

Since $E(t)\ll \log^{k-1} t$, it follows that the error term above is $O(\log^{k-1} z)$. After substituting the value of the main term $M(t)$ from Proposition 4.6, the main term above is

$$\frac{1}{k!}\,\frac{f(d)}{\bar{f}(d)}\left(\frac{\varphi(m)\log z}{m}\right)^k c_m(f)\int_{\max\{w_1,d\}}^{w_2}\left(\frac{\log t}{\log z}-\frac{\log d}{\log z}\right)^{k-1} F\left(\frac{\log t}{\log z}\right)\frac{k\,dt}{t\log z}.$$

The change of variable $t\mapsto \frac{\log t}{\log z}$ yields the desired main term. ∎

Definition 4.8. For any real number s, we define

$$\eta(s)=\min\{1-2s/3,3(1-s)/2\}. \tag{4.5}$$

One easily sees that

$$\eta(s)=\begin{cases}1-2s/3 & \text{if } s\le 3/5,\\ 3(1-s)/2 & \text{if } s\ge 3/5.\end{cases}$$

We state the next lemma without proof.

Lemma 4.9. *Let d_1, d_2 be positive integers. Let $z\le x$, $S(z)$ be as in (1.3) and W be as in (2.2). Then*

$$\begin{aligned}(d_1,d_2)\in S(z) &\Leftrightarrow \mu^2(d_1d_2W)=1 \text{ and } \max\{d_1d_2^{2/3},d_1^{2/3}d_2\}\le z\\ &\Leftrightarrow \mu^2(d_1d_2W)=1 \text{ and } d_1\le z^{\eta\left(\frac{\log d_2}{\log z}\right)}.\end{aligned}$$

Definition 4.10. Let $s_1,s_2\in[0,1]$. We define

$$T_{s_1,s_2}=\{(x_1,x_2)\in\mathbb{R}^2 : s_i\le x_i,\ x_1+2x_2/3\le 1,\ 2x_1/3+x_2\le 1\}. \tag{4.6}$$

We shall denote the region $T_{0,0}$ by T.

The next corollary is the main result of this section and will be invoked quite frequently later on.

Corollary 4.11. *Let $f_1 \in \Omega_{k_1}$ and $f_2 \in \Omega_{k_2}$ for positive integers k_1 and k_2. Let $f = f_1 * f_2$ and let $F : T \to \mathbb{R}$ be differentiable in each variable. Let d_1, d_2 be positive integers and let*

$$s_i = \frac{\log d_i}{\log z}, \qquad B = \frac{\varphi(W)\log z}{W}.$$

Then with $S(z)$ as defined in (1.3), we have

$$\begin{aligned}\sum_{\substack{l_i \equiv 0\,(d_i)\\(l_1,l_2)\in S(z)}} & f_1(l_1)f_2(l_2)F\left(\frac{\log l_1}{\log z}, \frac{\log l_2}{\log z}\right)\\ &= B^{k_1+k_2}\frac{f_1(d_1)}{\bar{f}(d_1)}\,\frac{f_2(d_2)}{\bar{f}(d_2)}c(W,f)\\ &\quad\times\left(\iint_{T_{s_1,s_2}} F(t_1,t_2)\frac{(t_1-s_1)^{k_1-1}}{(k_1-1)!}\,\frac{(t_2-s_2)^{k_2-1}}{(k_2-1)!}\,dt_2\,dt_1\right)\\ &\quad+\ O(\log^{k_1+k_2-1} z).\end{aligned}$$

Proof. Let us rephrase the conditions $l_1 \equiv 0 \pmod{d_1}$, $l_2 \equiv 0 \pmod{d_2}$ and $(l_1, l_2) \in S(z)$. We know from Lemma 4.9 that

$$(l_1,l_2)\in S(z) \;\Leftrightarrow\; \mu^2(l_1l_2W) = 1 \text{ and } l_2 \le z^{\eta\left(\frac{\log l_1}{\log z}\right)}.$$

From now on, we let $t_i = \frac{\log l_i}{\log z}$ for $i = 1, 2$. We can write the given summation as

$$\sum_{\substack{l_1\equiv 0\,(d_1)\\ l_2\equiv 0\,(d_2)\\ (l_1,l_2)\in S(z)}} = \sum_{\substack{l_1\equiv 0\,(d_1)\\ (l_1,d_2)\in S(z)}}\ \sum_{\substack{l_2\equiv 0\,(d_2)\\ (l_2,l_1)\in S(z)}} = \sum_{\substack{l_1\equiv 0\,(d_1)\\ l_1\le z^{\eta(s_2)}\\ (l_1,d_2W)=1}}\ \sum_{\substack{l_2\equiv 0\,(d_2)\\ l_2\le z^{\eta(t_1)}\\ (l_2,l_1W)=1}}$$

The summation above is such that one can directly apply Theorem 4.7 to the inner sum. We therefore have

$$\begin{aligned}\sum_{\substack{l_i\equiv 0\,(d_i)\\(l_1,l_2)\in S(z)}} & f_1(l_1)f_2(l_2)F\left(\frac{\log l_1}{\log z}, \frac{\log l_2}{\log z}\right)\\ &= \sum_{\substack{l_1\equiv 0\,(d_1)\\ l_1\le z^{\eta(s_2)}\\ (l_1,d_2W)=1}} f_1(l_1) \sum_{\substack{l_2\equiv 0\,(d_2)\\ l_2\le z^{\eta(t_1)}\\ (l_2,l_1W)=1}} f_2(l_2)F\left(\frac{\log l_1}{\log z}, \frac{\log l_2}{\log z}\right)\\ &= \frac{f_2(d_2)}{\bar{f}_2(d_2)} \sum_{\substack{l_1\equiv 0\,(d_1)\\ l_1\le z^{\eta(s_2)}\\ (l_1,d_2W)=1}} f_1(l_1)\left(\frac{\varphi(Wl_1)\log z}{Wl_1}\right)^{k_2}\end{aligned}\tag{4.7}$$

$$\times\, c(Wl_1, f_2) \int_{s_2}^{\eta\left(\frac{\log l_1}{\log z}\right)} F\left(\frac{\log l_1}{\log z}, t_2\right) \frac{(t_2 - s_2)^{k_2-1}}{(k_2-1)!}\, dt_2$$
$$+\, O\Big((\log^{k_2-1} z) \sum_{l_1 \le z} f_1(l_1)\Big).$$

We can now write

$$c(Wl_1, f_2) = \prod_{p \nmid Wl_1} \left(1 - \frac{1}{p}\right)^{k_2} \bar{f}_2(p) = \left(\frac{l_1}{\varphi(l_1)}\right)^{k} \frac{1}{\bar{f}_2(l_1)} \prod_{p \nmid W} \left(1 - \frac{1}{p}\right)^{k_2} \bar{f}_2(p)$$
$$= \left(\frac{l_1}{\varphi(l_1)}\right)^{k} \frac{1}{\bar{f}_2(l_1)} c(W, f_2).$$

After substituting this expression for $c(Wl_1, f_2)$, the main term in (4.7) may be written as

$$\frac{f_2(d_2)}{\bar{f}_2(d_2)} \left(\frac{\varphi(W)\log z}{W}\right)^{k_2} c(W, f_2)$$
$$\times \sum_{\substack{l_1 \equiv 0\,(d_1) \\ l_1 \le z^{\eta(s_2)} \\ (l_1, d_2 W) = 1}} \frac{f_1(l_1)}{\bar{f}_2(l_1)} \int_{s_2}^{\eta\left(\frac{\log l_1}{\log z}\right)} F\left(\frac{\log l_1}{\log z}, t_2\right) \frac{(t_2 - s_2)^{k_2-1}}{(k_2-1)!}\, dt_2$$
$$= (f_1/\bar{f}_1)(d_1) \frac{(f_2/\bar{f}_2)(d_2)}{\overline{(f_2/\bar{f}_2)}(d_2)} \left(\frac{\varphi(W)\log z}{W}\right)^{k_1+k_2} c(W, f_2) c(W, f_1/\bar{f}_2)$$
$$\times \int_{s_1}^{\eta(s_2)} \int_{s_2}^{\eta(t_1)} F(t_1, t_2) \frac{(t_1 - s_1)^{k_1-1}}{(k_1-1)!} \frac{(t_2 - s_2)^{k_2-1}}{(k_2-1)!}\, dt_2\, dt_1.$$

Since

$$(f_1/\bar{f}_1)(d_1) \frac{(f_2/\bar{f}_2)(d_2)}{\overline{(f_2/\bar{f}_2)}(d_2)} = \frac{f_1(d_1)}{\bar{f}(d_1)} \frac{f_2(d_2)}{\bar{f}(d_2)}, \qquad c(W, f_2) c\left(W, \frac{f_1}{\bar{f}_2}\right) = c(W, f),$$

we obtain the desired main term. Moreover, since $f_1 \in \Omega_{k_1}$, it follows that the error term in (4.7) is $O(\log^{k_1+k_2-1} z)$. This completes the proof. ∎

The next corollary follows directly from Theorem 1.1 and gives the level of distribution of the divisor-type function $2^{\Omega(n)}$ in the interval $n \in (x, 2x)$.

Corollary 4.12. *Let $m = Wm'$ be an even squarefree integer, where m' is odd and $(W, m') = 1$. Suppose $(a, m) = 1$. Then for any $\epsilon > 0$ and $m \le x^{2/3-\epsilon}$,*

$$\sum_{\substack{n\sim x\\ n\equiv a\,(m)}} 2^{\Omega(n)} = x\frac{\varphi(W)}{W^2}\frac{c(W)}{\gamma(m')}\biggl(\log x + c' + 2\sum_{\substack{p\mid Wm'\\ p>2}}\frac{\log p}{p-2}\biggr) + O_\epsilon\biggl(\frac{x^{1/2+\epsilon}}{m^{1/4}}\biggr)$$

where $n \sim x$ means that $x < n \le 2x$.

5. Proof of Theorem 2.1. We first provide asymptotic expressions for S_1 and S_2 (given by (2.3) and (2.4)) in Proposition 5.2. Then we evaluate a suitable value of λ by using a Sage program to optimize the sieve weights. The proof of Proposition 5.2 will be given at the end of this section.

NOTATION 5.1. We set

$$B = \frac{\varphi(W)\log z}{W}. \tag{5.1}$$

We choose the sieve weights λ_{d_1,d_2} to satisfy

$$\frac{\lambda_{d_1,d_2}}{\rho(d_1)\rho(d_2)} = \mu(d_1)\mu(d_2)\sum_{\substack{l_i\equiv 0\,(d_i)\\ (l_1,l_2)\in S(z)}}\frac{\mu^2(l_1)\mu^2(l_2)}{\rho(l_1)\rho(l_2)}P\biggl(\frac{\log l_1}{\log z},\frac{\log l_2}{\log z}\biggr) \tag{5.2}$$

where $\rho = \mathrm{id}/\tau$ and $P : \mathbb{R}^2 \to \mathbb{R}$ is a symmetric differentiable function.

Then the following proposition gives us the asymptotic expressions for S_1 and S_2 in terms of the function P defined above.

PROPOSITION 5.2. *Let the sieve weights λ_{d_1,d_2} be given by* (5.2). *Let $S(z)$* (*as in* (1.3)) *be the support of the sieve weights, let T_{s_1,s_2} and T be the regions defined in* (4.6) *and let η be as in Definition* 4.5. *Then with $z = x^{1/3-\epsilon}$, one has*

$$S_1 = \frac{x}{W}(1+o(1))B^6R_1(P), \qquad S_2 = \frac{2x}{W}(1+o(1))B^6R_2(P),$$

where

$$\begin{aligned} R_1(P) &= \iint_T Q_1^2(s_1,s_2)\,ds_2\,ds_1,\\ R_2(P) &= \iint_T (s_1(3-s_1)Q_2^2(s_1,s_2) + 4s_1Q_1(s_1,s_2)Q_2(s_1,s_2))\,ds_2\,ds_1. \end{aligned} \tag{5.3}$$

Here

$$Q_1(s_1,s_2) = \iint_{T_{s_1,s_2}} P(t_1,t_2)\,dt_2\,dt_1, \qquad Q_2(s_1,s_2) = \int_{s_2}^{\eta(s_1)} P(s_1,t_2)\,dt_2. \tag{5.4}$$

Now that we have asymptotic expressions for S_1 and S_2 in terms of the symmetric and differentiable (in each variable) function P, we wish to find a suitable value of λ. Keeping in mind that P should be symmetric, we make the choice

$$P(x,y)=\sum_{i,j=0}^{7} a_{i,j}(x+y)^i(x^2+y^2)^j. \tag{5.5}$$

The choice of the coefficients is optimized by means of a Sage program. One finds that both $R_1(P)$ and $R_2(P)$ are quadratic forms in terms of the coefficients a_{ij}. The program performs the integration and gives quadratic form expressions for $R_1(P)$ and $R_2(P)$.

In order to show that (2.1) is positive, we must have $\lambda S_1 > S_2$. By Proposition 5.2, one must have

$$\lambda > \frac{2R_2(P)}{R_1(P)}. \tag{5.6}$$

So, we need to minimize $R_2(P)/R_1(P)$. This ratio is entirely dependent on the choice of P and since both $R_2(P)$ and $R_1(P)$ are quadratic forms in the coefficients a_{ij}, we are essentially minimizing the ratio of two quadratic forms. This is a well known problem with the following solution.

PROPOSITION 5.3. *Let $R_1 = \mathbf{a}^T M_1 \mathbf{a}$ and $R_2 = \mathbf{a}^T M_2 \mathbf{a}$ be two quadratic forms, where M_1 and M_2 are positive definite real symmetric matrices. Then the ratio R_2/R_1 is minimized when* $\mathbf{a}$ *is an eigenvector corresponding to the smallest eigenvalue of $M_1^{-1}M_2$. The value of the ratio at its minimum is this minimum eigenvalue.*

Proof. See [May, Lemma 7.3, p. 20]. ∎

Let A be the matrix corresponding to the form $R_1(P)$ and B the one corresponding to $R_2(P)$. The matrices A and B are clearly real symmetric. If one looks at the expressions (2.3) and (2.4) for S_1 and S_2 respectively, it is clear why both A and B are positive definite.

So by Theorem 5.3, the the minimum value of $R_2(P)/R_1(P)$ is the smallest eigenvalue of C, which is calculated by the Sage program and turns out to be

$$6.290731135292344.$$

Therefore, from (5.6), it follows that (2.1) is positive for any

$$\lambda > 12.5814622705847. \tag{5.7}$$

This completes the proof of Theorem 2.1

REMARK 5.4. It is possible to reduce the value of λ a little further, provided we take $P(x,y)$ to be of a higher degree. However, the improvement will be insignificant and not worth any program's effort as λ is unlikely to go below 12. Moreover, taking a higher degree polynomial can make the program increasingly time consuming.

It now remains to give a proof of Proposition 5.2.

6. Proof of Proposition 5.2

NOTATION 6.1. Throughout this section, $\{i, j\}$ will be a permutation of the set $\{1, 2\}$. So, if we write $s_{ij} \mid (d_i, l_j)$, it means that both $s_{12} \mid (d_1, l_2)$ and $s_{21} \mid (d_2, l_1)$. Moreover, if we write $d_i \equiv 0 \pmod{l_i}$, it will mean that both $d_1 \equiv 0 \pmod{l_1}$ and $d_2 \equiv 0 \pmod{l_2}$.

We define two new quantities which will arise when computing S_1 and S_2:

$$\begin{aligned} B_{r_1,r_2} &= \varphi(r_1)\varphi(r_2) \sum_{d_i \equiv 0\,(r_i)} \frac{\lambda_{d_1,d_2}}{d_1 d_2}, \\ C_{r_1,r_2} &= \rho_1(r_1)\gamma_1(r_2) \sum_{d_i \equiv 0\,(r_i)} \frac{\lambda_{d_1,d_2}}{\rho(d_1)\gamma(d_2)}, \end{aligned} \tag{6.1}$$

where $\mathrm{id}/\tau = \rho = \rho_1 * 1$ and γ, γ_1 are defined by

$$\gamma(p) = \frac{p(p-1)}{p-2} \quad \text{and} \quad \gamma_1 * 1 = \gamma. \tag{6.2}$$

LEMMA 6.2. *For any* $(r_1, r_2) \in S(z)$*:*

(a) $|\lambda_{r_1,r_2}| \ll \log^4 z$.

(b) $B_{r_1,r_2} = \kappa_1(W) B^2 \mu(r_1)\mu(r_2)h(r_1)h(r_2)\ Q_1\Big(\frac{\log r_1}{\log z}, \frac{\log r_2}{\log z}\Big) + O(\log z)$ *where* Q_1 *is defined in* (5.4). *Here*

$$\kappa_1(W) = \prod_{p \nmid W} \Big(1 - \frac{1}{p}\Big)^2 \Big(1 + \frac{2}{p}\Big) \quad \textit{and} \quad h(p) = 1 - 3/(p+2).$$

(c) $C_{r_1,r_2} = B\mu(r_1)\mu(r_2)h_1(r_1)h_2(r_2)\ Q_2\Big(\frac{\log r_1}{\log z}, \frac{\log r_2}{\log z}\Big) + O(1)$ *where* Q_2 *is defined in* (5.4). *Here*

$$h_1(p) = 1 - 3/p + 2/p^2 \quad \textit{and} \quad h_2(p) = 1 - 2/p + 2/p^2.$$

Proof. (a) From (5.2), we have

$$\lambda_{r_1,r_2} = \mu(r_1)\mu(r_2)\rho(r_1)\rho(r_2) \sum_{\substack{l_i \equiv 0\,(r_i) \\ (l_1,l_2) \in S(z)}} \frac{\mu^2(l_1)\mu^2(l_2)}{\rho(l_1)\rho(l_2)} P\Big(\frac{\log l_1}{\log z}, \frac{\log l_2}{\log z}\Big).$$

Applying Corollary 4.11 to the RHS and writing $s_i = \frac{\log r_i}{\log z}$, we obtain

$$\begin{aligned} \lambda_{r_1,r_2} &= c(W, \rho * \rho) B^4 \frac{\mu(r_1)\mu(r_2)}{\prod_{p \mid r_1 r_2}(1 + 4/p)} \iint_{T_{s_1,s_2}} P(t_1,t_2)(t_1 - s_1)(t_2 - s_2)\, dt_2\, dt_1 \\ &\qquad + O(\log^3 z) \\ &\ll \log^4 z \end{aligned}$$

Here $c(W, \rho * \rho)$ is as defined in (4.4) and tends to 1 as $x \to \infty$.

(b) We use (6.1) and (5.2) to get

$$\begin{aligned}\frac{B_{r_1,r_2}}{\varphi(r_1)\varphi(r_2)} &= \sum_{d_i\equiv 0\,(r_i)} \frac{\lambda_{d_1,d_2}}{d_1 d_2}\\ &= \sum_{d_i\equiv 0\,(r_i)} \frac{\mu(d_1)\mu(d_2)}{\tau(d_1)\tau(d_2)} \sum_{\substack{l_i\equiv 0\,(d_i)\\ (l_1,l_2)\in S(z)}} \frac{\mu^2(l_1)\mu^2(l_2)}{\rho(l_1)\rho(l_2)} P\left(\frac{\log l_1}{\log z}, \frac{\log l_2}{\log z}\right)\\ &= \sum_{\substack{l_i\equiv 0\,(r_i)\\ (l_1,l_2)\in S(z)}} \frac{\mu^2(l_1)\mu^2(l_2)}{\rho(l_1)\rho(l_2)} P\left(\frac{\log l_1}{\log z}, \frac{\log l_2}{\log z}\right) \sum_{r_i|d_i|l_i} \frac{\mu(d_1)\mu(d_2)}{\tau(d_1)\tau(d_2)}\\ &= \mu(r_1)\mu(r_2) \sum_{\substack{l_i\equiv 0\,(r_i)\\ (l_1,l_2)\in S(z)}} \frac{\mu^2(l_1)\mu^2(l_2)}{l_1 l_2} P\left(\frac{\log l_1}{\log z}, \frac{\log l_2}{\log z}\right).\end{aligned}$$

Applying Corollary 4.11 to the last equality above with the functions $f_1 = f_2 = \mu^2/\mathrm{id} \in \Omega_1$, we obtain the desired result.

(c) Again from (6.1) and (5.2), we have

$$\begin{aligned}\frac{C_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)} &= \sum_{d_i\equiv 0\,(r_i)} \frac{\lambda_{d_1,d_2}}{\rho(d_1)\gamma(d_2)}\\ &= \sum_{d_i\equiv 0\,(r_i)} \mu(d_1)\frac{\mu(d_2)\rho(d_2)}{\gamma(d_2)} \sum_{\substack{l_i\equiv 0\,(d_i)\\ (l_1,l_2)\in S(z)}} \frac{\mu^2(l_1)\mu^2(l_2)}{l_1 l_2} P\left(\frac{\log l_1}{\log z}, \frac{\log l_2}{\log z}\right)\\ &= \sum_{\substack{l_i\equiv 0\,(r_i)\\ (l_1,l_2)\in S(z)}} \frac{\mu^2(l_1)\mu^2(l_2)}{l_1 l_2} P\left(\frac{\log l_1}{\log z}, \frac{\log l_2}{\log z}\right) \sum_{r_i|d_i|l_i} \mu(d_1)\frac{\mu(d_2)\rho(d_2)}{\gamma(d_2)}\\ &= \frac{\mu(r_1)}{\rho(r_1)}\,\frac{\mu(r_2)\varphi(r_2)}{\gamma(r_2)} \sum_{\substack{l_2\equiv 0\,(r_2)\\ (r_1,l_2)\in S(z)}} \frac{\mu^2(l_2)}{\varphi(l_2)}.\end{aligned} \tag{6.3}$$

In the last equality in (6.3), we note from Lemma 4.9 that

$$(r_1, l_2) \in S(z) \Leftrightarrow \mu^2(r_1 l_2 W) = 1 \text{ and } l_2 \le z^{\eta\left(\frac{\log r_1}{\log z}\right)}.$$

By applying Theorem 4.7 to the sum in the last equality of (6.3) with the function $\mu^2/\varphi \in \Omega_1$, we obtain the desired result. ∎

We now begin with the evaluation of S_1.

6.1. Evaluation of S_1. From (2.3), we have

$$S_1 = \sum_{\substack{n\sim x\\ n\equiv v_0\,(W)}} \Big(\sum_{\substack{d_1|n\\ d_2|n+h_0}} \lambda_{d_1,d_2}\Big)^2 = \sum_{\substack{n\sim x\\ n\equiv v_0\,(W)}} \sum_{\substack{d_1,l_1|n\\ d_2,l_2|n+h_0}} \lambda_{d_1,d_2}\lambda_{l_1,l_2} \tag{6.4}$$
$$= \sum_{(d_i,l_j)=1} \lambda_{d_1,d_2}\lambda_{l_1,l_2} \sum_{\substack{n\sim x\\ n\equiv 0\,([d_1,l_1])\\ n\equiv -h_0\,([d_2,l_2])\\ n\equiv v_0\,(W)}} 1.$$

In the above sum, we have the conditions $d_1, l_1 \,|\, n$ and $d_2, l_2 \,|\, n + h_0$. We first choose x large enough so that $D > h_0$. This ensures that $\mathrm{rad}(h_0) \,|\, W$. It then follows $(d_i, d_j) = (d_i, l_j) = 1$ for $i \neq j$ and $(d_1d_2l_1l_2, W) = 1$. This is because if a prime p divides (d_i, d_j) (or (d_i, l_j)) for $i \neq j$, then p must divide both n and $n + h_0$, and therefore $p \,|\, h_0$. But since $\mathrm{rad}(h_0) \,|\, W$, it follows that $p \,|\, W$. This is a contradiction because the numbers d_i and l_j are all coprime to W. Moreover, the conditions $(d_1, d_2) = 1$ and $(l_1, l_2) = 1$ can be dropped because they are already included in the definition of $S(z)$, the support of the sieve weights λ_{d_1,d_2}. So, we are only left with the conditions $(d_1, l_2) = (d_2, l_1) = 1$. Since the numbers $[d_1, l_1]$, $[d_2, l_2]$ and W are pairwise coprime, the inner sum in the last equality of (6.4) becomes

$$\frac{x}{W[d_1, l_1][d_2, l_2]} + O(1).$$

Therefore,

$$S_1 = \sum_{(d_i,l_j)=1} \lambda_{d_1,d_2}\lambda_{l_1,l_2}\Big(\frac{x}{W[d_1, l_1][d_2, l_2]} + O(1)\Big) \tag{6.5}$$
$$= \frac{x}{W}\sum_{(d_i,l_j)=1} \frac{\lambda_{d_1,d_2}\lambda_{l_1,l_2}}{[d_1, l_1][d_2, l_2]} + O\Big(\sum_{d_i,l_i} |\lambda_{d_1,d_2}|\,|\lambda_{l_1,l_2}|\Big)$$
$$= M_1 + E_{11}.$$

The error term in (6.5) is

$$E_{11} \ll \Big(\sum_{d_1,d_2} |\lambda_{d_1,d_2}|\Big)^2 \ll \Big(\sum_{(d_1,d_2)\in S(z)} \log^4 z\Big)^2 \tag{6.6}$$
$$\ll |S(z)|^2 \log^8 z \ll z^{8/3}\log^8 z$$

where we have used the bound $|\lambda_{d_1,d_2}| \ll \log^4 z$ from Lemma 6.2 and the fact that $|S(z)| \ll z^{4/3}$.

In the main term M_1, we can write

$$\frac{1}{[d_i, l_i]} = \frac{1}{d_il_i}\sum_{r_i|(d_i,l_i)} \varphi(r_i)$$

This gives

$$M_1 = \frac{x}{W} \sum_{(d_i,l_j)=1} \frac{\lambda_{d_1,d_2}\lambda_{l_1,l_2}}{d_1 d_2\ l_1 l_2} \sum_{r_i \mid (d_i,l_i)} \varphi(r_1)\varphi(r_2)$$

To get rid of the conditions $(d_1, l_2) = (d_2, l_1) = 1$, we multiply by a factor of

$$\sum_{s_{ij} \mid (d_i,l_j)} \mu(s_{12})\mu(s_{21}).$$

This idea was used by Maynard [May].

Now, $r_i \mid (d_i, l_i)$ and $s_{ij} \mid (d_i, l_j)$, for $i \neq j$. This implies that $(r_i, s_{ij}) = (r_i, s_{ji}) = 1$ because if there were a prime p dividing r_i and s_{ij}, then $p \mid (l_i, l_j)$. This contradicts $(l_1, l_2) = 1$. Also note that both r_i and s_{ij} divide d_i. Since $(r_i, s_{ij}) = 1$, it follows that $r_i s_{ij} \mid d_i$. Similarly, we get $r_j s_{ji} \mid d_j$. Therefore both $(r_1 s_{12}, r_2 s_{21})$ and $(r_1 s_{21}, r_2 s_{12})$ are in $S(z)$. Summarizing, we have

$$\begin{aligned} (6.7)\qquad M_1 &= \frac{x}{W} \sum_{r_i} \varphi(r_1)\varphi(r_2) \sum_{d_i, l_i \equiv 0 (r_i)} \frac{\lambda_{d_1,d_2}}{d_1 d_2} \frac{\lambda_{l_1,l_2}}{l_1 l_2} \sum_{s_{ij} \mid (d_i,l_j)} \mu(s_{12})\mu(s_{21}) \\ &= \frac{x}{W} \sum_{r_i} \varphi(r_1)\varphi(r_2) \sum_{s_{ij}} \mu(s_{12})\mu(s_{21}) \Bigg(\sum_{d_i \equiv 0\ (r_i s_{ij})} \frac{\lambda_{d_1,d_2}}{d_1 d_2} \Bigg) \Bigg(\sum_{l_i \equiv 0\ (r_i s_{ji})} \frac{\lambda_{l_1,l_2}}{l_1 l_2} \Bigg). \end{aligned}$$

REMARK 6.3. We need not write the conditions $(r_1 s_{12}, r_2 s_{21}) \in S(z)$ and $(r_1 s_{21}, r_2 s_{12}) \in S(z)$ at every step because one of the two bracketed quantities in the last equality of (6.7) will vanish when any of these conditions does not hold (because the support of λ_{d_1,d_2} is $S(z)$).

Recalling the definition of B_{r_1,r_2} from (6.1), note that the two bracketed quantities in (6.7) can be replaced by appropriate B_{r_1,r_2}'s, i.e. we obtain

$$(6.8)\qquad M_1 = \frac{x}{W} \sum_{r_i} \frac{\mu^2(r_1)}{\varphi(r_1)} \frac{\mu^2(r_2)}{\varphi(r_2)} \sum_{s_{ij}} \frac{\mu(s_{12})}{\varphi^2(s_{12})} \frac{\mu(s_{21})}{\varphi^2(s_{21})} B_{r_1 s_{12}, r_2 s_{21}} B_{r_1 s_{21}, r_2 s_{12}}.$$

We would now like to get rid of the numbers s_{12} and s_{21} from the above summation. Since $(s_{12}, s_{21}) \in S(z)$, we have $(s_{12}, W) = (s_{21}, W) = 1$. Therefore, either $s_{ij} = 1$ or $s_{ij} > D$. The contribution to the main term M_1 of the above sum (6.8) when at least one of the s_{ij} is greater than D is

$$\begin{aligned} &\ll \frac{x}{W} \sum_{\substack{(s_{12},s_{21}) \in S(z) \\ s_{12} > D}} \frac{1}{\varphi^2(s_{12})\varphi^2(s_{21})} \sum_{(r_1,r_2) \in S(z)} \frac{\mu^2(r_1)}{\varphi(r_1)} \frac{\mu^2(r_1)}{\varphi(r_2)} \log^4 z \\ &\ll \frac{x \log^4 z}{W} \Bigg(\sum_{s_{12} > D} \frac{1}{\varphi^2(s_{12})} \Bigg) \Bigg(\sum_{s_{21}} \frac{1}{\varphi^2(s_{21})} \Bigg) \Bigg(\sum_{r_1, r_2 \leq z} \frac{1}{\varphi(r)\varphi(r_2)} \Bigg) \ll \frac{x \log^6 z}{WD} \end{aligned}$$

where we have used the estimate $B_{r_1,r_2} \ll \log^4 z$ from Lemma 6.2.

We may therefore assume that $s_{12} = s_{21} = 1$ with an error term of

$$E_{12} = O\left(\frac{x \log^6 z}{WD}\right). \tag{6.9}$$

Hence, one can now write

$$S_1 = \frac{x}{W} \sum_{r_i} \frac{\mu^2(r_1)}{\varphi(r_1)} \frac{\mu^2(r_2)}{\varphi(r_2)} B^2_{r_1,r_2} + E_{11} + E_{12}. \tag{6.10}$$

Substituting the value of B_{r_1,r_2} from Lemma 6.2, we obtain

$$\begin{aligned} M_1 = \frac{x}{W} \sum_{(r_1,r_2)\in S(z)} & \frac{\mu^2(r_1)}{\varphi(r_1)} \frac{\mu^2(r_2)}{\varphi(r_2)} \\ & \times \left(\kappa_1^2(W) B^4 h^2(r_1) h^2(r_2) Q_1^2\left(\frac{\log r_1}{\log z}, \frac{\log r_2}{\log z}\right) + O(\log^3 z)\right). \end{aligned} \tag{6.11}$$

The sum in (6.11) is precisely the type of sum estimated in Corollary 4.11. Applying Corollary 4.11 with $f_1 = f_2 = h^2/\varphi \in \Omega_1$, we obtain

$$M_1 = \frac{x}{W} B^6 \iint_T Q_1^2(s_1, s_2)\, ds_2\, ds_1 + O\left(\frac{x \log^5 z}{W}\right) \tag{6.12}$$

with Q_1 defined in (5.4).

When we choose $z = x^{1/3-\epsilon}$, we have

$$E_{11} \ll x^{8/9-8/3\epsilon} \quad \text{and} \quad E_{12} \ll \frac{x \log^6 z}{WD}$$

This gives the following final expression for S_1:

$$S_1 = \frac{x}{W}(1 + o(1)) B^6 R_1(P) \tag{6.13}$$

where

$$R_1(P) = \iint_T Q_1^2(s_1, s_2)\, ds_2\, ds_1 \tag{6.14}$$

6.2. Evaluation of S_2. We will use the same ideas here as we did while computing S_1. The conditions and observations given in the paragraph right after (6.4) will be implemented.

From (2.4), we have

$$\begin{aligned} S_2 &= \sum_{\substack{n\sim x \\ n\equiv v_0\,(W)}} (2^{\Omega(n)} + 2^{\Omega(n+h_0)}) \sum_{\substack{d_1,l_1|n \\ d_2,l_2|n+h_0}} \lambda_{d_1,d_2}\lambda_{l_1,l_2} \\ &= \sum_{(d_i,l_j)=1} \lambda_{d_1,d_2}\lambda_{l_1,l_2} \sum_{\substack{n\sim x \\ n\equiv 0\,([d_1,l_1]) \\ n\equiv -h_0\,([d_2,l_2]) \\ n\equiv v_0\,(W)}} (2^{\Omega(n)} + 2^{\Omega(n+h_0)}) \end{aligned} \tag{6.15}$$

$$= \sum_{(d_i,l_j)=1} \lambda_{d_1,d_2}\lambda_{l_1,l_2}\tau[d_1,l_1] \sum_{\substack{n\sim x/[d_1,l_1] \\ n\equiv\alpha'\,([d_2,l_2]) \\ n\equiv\beta'\,(W)}} 2^{\Omega(n)}$$

$$+ \sum_{(d_i,l_j)=1} \lambda_{d_1,d_2}\lambda_{l_1,l_2}\tau[d_2,l_2] \sum_{\substack{n\sim x/[d_2,l_2] \\ n\equiv\alpha''\,([d_1,l_1]) \\ n\equiv\beta''\,(W)}} 2^{\Omega(n)}$$

Here, we have taken out $[d_1, l_1]$ and $[d_2, l_2]$ respectively from the first and second term in the last equality of (6.15) and replaced the summation over $n + h_0$ with a summation over n. This is permitted because h_0 is very small compared to x. Note that therefore α' is coprime to $[d_2, l_2]$, α'' is coprime to $[d_1, l_1]$ and β', β'' are both coprime to W.

We assume that λ_{d_1,d_2}'s are *symmetric*, i.e. $\lambda_{d_1,d_2} = \lambda_{d_2,d_1}$ for any d_1, d_2. This will allow us to interchange the indices 1 and 2 in the second term in the last equality of (6.15) to get

$$S_2 = \sum_{(d_i,l_j)=1} \lambda_{d_1,d_2}\lambda_{l_1,l_2}\tau[d_1,l_1]\Bigg(\sum_{\substack{n\sim x/[d_1,l_1] \\ n\equiv\alpha'\,([d_2,l_2]) \\ n\equiv\beta'\,(W)}} 2^{\Omega(n)} + \sum_{\substack{n\sim x/[d_1,l_1] \\ n\equiv\alpha''\,([d_2,l_2]) \\ n\equiv\beta''\,(W)}} 2^{\Omega(n)}\Bigg).$$

We make use of Corollary 4.12 with $m = W[d_2, l_2]$ to evaluate the inner sums of the above expression. In order to apply the corollary, the following conditions must hold:

$$W[d_2,l_2] \le (x/[d_1,l_1])^{2/3-\epsilon}, \qquad W[d_1,l_1] \le (x/[d_2,l_2])^{2/3-\epsilon}.$$

The second condition holds because we have assumed λ_{d_1,d_2}'s to be symmetric. So, their support is also symmetric in the indices 1 and 2. Since W is very small compared to x, it can be absorbed into the x^ϵ term. So it is enough to have

$$\max\{d_1^{2/3}d_2, d_1 d_2^{2/3}\} \le z \tag{6.16}$$

where $z = x^{1/3-\epsilon}$. This explains why we have chosen the support of the sieve weights to be $S(z)$. Now, Corollary 4.12 gives

$$S_2 = 2x\frac{\varphi(W)}{W^2}c(W) \sum_{(d_i,l_j)=1} \lambda_{d_1,d_2}\lambda_{l_1,l_2} \frac{\tau[d_1,l_1]}{[d_1,l_1]}\,\frac{1}{\gamma[d_2,l_2]} \times \Bigg(\log x - \log{[d_1,l_1]} + c' + 2\sum_{\substack{p\mid W[d_2,l_2] \\ p>2}} \frac{\log p}{p-2}\Bigg) + O_\epsilon\Bigg(x^{1/2+\epsilon}\sum_{d_i,l_i} \frac{|\lambda_{d_1,d_2}|\,|\lambda_{l_1,l_2}|}{W^{1/4}[d_1,l_1]^{1/2}[d_2,l_2]^{1/4}}\Bigg) \tag{6.17}$$

where we have (with the notation of Corollary 4.12),

$$\gamma(p) = \frac{p(p-1)}{p-2} \quad \text{and} \quad c(W) = \prod_{p\nmid W} \frac{(p-1)^2}{p(p-2)}.$$

We can therefore write

$$S_2 = 2x\frac{\varphi(W)}{W^2}c(W)(M_2 - M_{21} + M_{22}) + E_2 \tag{6.18}$$

where

$$\begin{aligned} M_2 &= (\log x + c') \sum_{(d_i,l_j)=1} \frac{\lambda_{d_1,d_2}\lambda_{l_1,l_2}}{\rho[d_1,l_1]\gamma[d_2,l_2]}, \\ M_{21} &= \sum_{(d_i,l_j)=1} \frac{\lambda_{d_1,d_2}\lambda_{l_1,l_2}}{\rho[d_1,l_1]\gamma[d_2,l_2]} \log\,[d_1,l_1], \\ M_{22} &= 2\sum_{(d_i,l_j)=1} \frac{\lambda_{d_1,d_2}\lambda_{l_1,l_2}}{\rho[d_1,l_1]\gamma[d_2,l_2]} \sum_{\substack{p\mid W[d_2,l_2]\\ p>2}} \frac{\log p}{p-2}, \\ E_2 &\ll_\epsilon \frac{x^{1/2+\epsilon}}{W^{1/4}} \sum_{d_i,l_i} \frac{|\lambda_{d_1,d_2}|\,|\lambda_{l_1,l_2}|}{[d_1,l_1]^{1/2}[d_2,l_2]^{1/4}}. \end{aligned} \tag{6.19}$$

Of all these terms, only M_2 and M_{21} will be contributing to the main term. The terms M_{22} and E_2 will be error terms.

To compute these terms, we prove a few lemmas concerning additive functions. The purpose is to simplify the computation for M_{21} and the estimation of M_{22}. In both of these cases, Lemma 6.5 below is directly applicable. In simpler words, Lemma 6.5 helps us give an asymptotic formula for M_{21} and an upper bound for M_{22}.

LEMMA 6.4. *Let $L(n)$ be an additive function defined on squarefree integers by $L(n) = \sum_{p\mid n} L(p)$. Let C_{r_1,r_2} be as in* (6.1). *Then*

$$\sum_{d_i\equiv 0\,(r_i)} \frac{\lambda_{d_1,d_2}}{\rho(d_1)\gamma(d_2)}L(d_1) = \frac{C_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)}L(r_1) + 2\sum_p \frac{C_{pr_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)}\frac{L(p)}{p-2}$$

where $\rho_1 * 1 = \rho$ and $\gamma_1 * 1 = \gamma$.

Proof. We have

$$\begin{aligned} \sum_{d_i\equiv 0\,(r_i)} \frac{\lambda_{d_1,d_2}}{\rho(d_1)\gamma(d_2)}L(d_1) &= \sum_{d_i\equiv 0\,(r_i)} \frac{\lambda_{d_1,d_2}}{\rho(d_1)\gamma(d_2)} \sum_{p\mid d_1} L(p) \\ &= \sum_p L(p) \sum_{\substack{d_1\equiv 0\,([p,r_1])\\ d_2\equiv 0\,(r_2)}} \frac{\lambda_{d_1,d_2}}{\rho(d_1)\gamma(d_2)} = \sum_p L(p)\frac{C_{[p,r_1],r_2}}{\rho_1[p,r_1]\gamma_1(r_2)} \\ &= \frac{C_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)}L(r_1) + 2\sum_p \frac{C_{pr_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)}\frac{L(p)}{p-2}. \end{aligned}$$

In the last step above, we have split the sum into two cases, depending on whether $p \mid r_1$ or not. ∎

PROPOSITION 6.5. *Let $L(n)$ be as in Lemma 6.4 with the further restriction that $L(n) \ll \log n$. Then*

$$\sum_{(d_i,l_j)=1} \frac{\lambda_{d_1,d_2}\lambda_{l_1,l_2}}{\rho[d_1,l_1]\gamma[d_2,l_2]} L[d_1,l_1]$$
$$= \sum_{r_i} \frac{C_{r_1,r_2}^2}{\rho_1(r_1)\gamma_1(r_2)} L(r_1) + 4\sum_{r_i} \frac{C_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)} \sum_p \frac{L(p)}{p-2} C_{pr_1,r_2} + O\left(\frac{\log^6 z}{D}\right).$$

Proof. Now,

$$\begin{aligned}\mathrm{LHS} &= \sum_{(d_i,l_j)=1} \frac{\lambda_{d_1,d_2}\lambda_{l_1,l_2}}{\rho[d_1,l_1]\gamma[d_2,l_2]} L[d_1,l_1] \\ &= \sum_{(d_i,l_j)=1} \frac{\lambda_{d_1,d_2}\lambda_{l_1,l_2}}{\rho[d_1,l_1]\gamma[d_2,l_2]} (L(d_1)+L(l_1)-L(d_1,l_1)) \\ &= \sum\nolimits_1 + \sum\nolimits_2 - \sum\nolimits_3 .\end{aligned}$$

The fact that $L[d_1,l_1] = L(d_1)+L(l_1)-L(d_1,l_1)$ follows from the identity $[d_1,l_1] = d_1 l_1/(d_1,l_1)$ and the fact that L is additive.

As we have done when computing S_1, we want to get rid of the conditions $(d_1,l_2) = (d_2,l_1) = 1$. So we multiply by a factor

$$\sum_{\substack{s_{12}\mid(d_1,l_2)\\ s_{21}\mid(d_2,l_1)}} \mu(s_{12})\mu(s_{21}).$$

We then have the same conditions $(r_1 s_{12}, r_2 s_{21})$, $(r_1 s_{21}, r_2 s_{12}) \in S(z)$ as before.

Moreover, one can write

$$\frac{1}{\rho[d_1,l_1]\gamma[d_2,l_2]} = \frac{1}{\rho(d_1)\rho(l_1)\gamma(d_2)\gamma(l_2)} \sum_{r_i\mid(d_i,l_i)} \rho_1(r_1)\gamma_1(r_2).$$

Therefore,

$$\sum\nolimits_1 = \sum_{d_i,l_i} \frac{\lambda_{d_1,d_2}\lambda_{l_1,l_2}}{\rho(d_1)\gamma(d_2)\rho(l_1)\gamma(l_2)} L(d_1) \times \sum_{r_i\mid(d_i,l_i)} \rho_1(r_1)\gamma_1(r_2) \sum_{s_{ij}\mid(d_i,l_j)} \mu(s_{12})\mu(s_{21}) \tag{6.20}$$

$$= \sum_{r_i, s_{ij}} \mu(s_{12})\mu(s_{21})\rho_1(r_1)\gamma_1(r_2)$$

$$\times \Big(\sum_{l_i \equiv 0\,(r_i s_{ji})} \frac{\lambda_{l_1,l_2}}{\rho(l_1)\gamma(l_2)} \Big) \Big(\sum_{d_i \equiv 0\,(r_i s_{ij})} \frac{\lambda_{d_1,d_2}}{\rho(d_1)\gamma(d_2)} L(d_1) \Big).$$

The first bracketed quantity above is $C_{r_1 s_{21}, r_2 s_{12}}$ divided by $\rho_1(r_1 s_{21}) \times \gamma_1(r_2 s_{12})$. For the second bracketed quantity, we invoke Lemma 6.4. Thus

$$\sum\nolimits_1 = \sum_{r_i, s_{ij}} \frac{\mu(s_{12}s_{21})}{\rho_1(s_{12}s_{21})\gamma_1(s_{12}s_{21})} \frac{C_{r_1 s_{21}, r_2 s_{12}}}{\rho_1(r_1)\gamma_1(r_2)} \times \bigg(L(r_1 s_{12}) C_{r_1 s_{12}, r_2 s_{21}} + 2 \sum_p \frac{L(p)}{p-2} C_{p r_1 s_{12}, r_2 s_{21}} \bigg). \tag{6.21}$$

Above, we have used that fact that $\rho_1(p) = (p-2)/2$. Again, the contribution to (6.21) from all those (s_{12}, s_{21}) for which at least one s_{ij} is $> D$ is

$$\ll \sum_{\substack{(s_{12}, s_{21}) \in S(z) \\ s_{12} > D \\ (r_1, r_2) \in S(z)}} \frac{1}{\rho_1(s_{12}s_{21})\gamma_1(s_{12}s_{21})} \frac{\log^3 z}{\rho_1(r_1)\gamma_1(r_2)} \ll \frac{\log^6 z}{D}$$

where we have used the estimate $C_{r_1, r_2} \ll \log z$ from Proposition 6.2, the assumption that $L(n) \ll \log n$ and the fact that $\rho_1 \in \Omega_2$, $\gamma_1 \in \Omega_1$.

So, we can assume $s_{12} = s_{21} = 1$ at the cost of an error $O\big(\frac{\log^6 z}{D}\big)$. This gives

$$\sum\nolimits_1 = \sum_{r_i} \frac{C_{r_1, r_2}}{\rho_1(r_1)\gamma_1(r_2)} \bigg(L(r_1) C_{r_1, r_2} + 2 \sum_{p \le z} \frac{L(p)}{p-2} C_{p r_1, r_2} \bigg) + O\bigg(\frac{\log^6 z}{D} \bigg).$$

By the same argument, one obtains

$$\sum\nolimits_2 = \sum\nolimits_1 . \tag{6.22}$$

We do similar operations as above for $\sum_3$ to get

$$\sum\nolimits_3 = \sum_{d_i, l_i} \frac{\lambda_{d_1, d_2} \lambda_{l_1, l_2}}{\rho[d_1, l_1]\gamma[d_2, l_2]} \sum_{p | (d_1, l_1)} L(p) \sum_{s_{ij} | (d_i, l_j)} \mu(s_{12})\mu(s_{21})$$

$$= \sum_{r_i, s_{ij}} \mu(s_{12}s_{21})\rho_1(r_1)\gamma_1(r_2) \sum_p L(p)$$

$$\times \Big(\sum_{\substack{d_1 \equiv 0\,([p, r_1 s_{12}]) \\ d_2 \equiv 0 \pmod{r_2 s_{21}}}} \frac{\lambda_{d_1, d_2}}{\rho(d_1)\gamma(d_2)} \Big) \Big(\sum_{\substack{l_1 \equiv 0\,([p, r_1 s_{21}]) \\ l_2 \equiv 0 \pmod{r_2 s_{12}}}} \frac{\lambda_{l_1, l_2}}{\rho(l_1)\gamma(l_2)} \Big)$$

Replacing the two bracketed quantities with appropriate C_{r_1,r_2}'s, we obtain

$$\sum\nolimits_3 = \sum_{r_i,s_{ij}} \frac{\mu(s_{12}s_{21})}{\rho_1(s_{12}s_{21})\gamma_1(s_{12}s_{21})} \times \sum_p L(p) \frac{\rho_1(r_1)\gamma_1(r_2)C_{[p,r_1s_{12}],r_2s_{21}}C_{[p,r_1s_{21}],r_2s_{12}}}{\rho_1[p,r_1s_{12}]\rho_1[p,r_1s_{21}]\gamma_1(r_2s_{21})\gamma_1(r_2s_{12})} \tag{6.23}$$

We now split the sum (6.23) into four cases depending on whether $p \mid r_1$, $p \mid s_{12}$, $p \mid s_{21}$ or $p \nmid r_1s_{12}s_{21}$. These are the only cases that can occur because $\mu^2(r_1s_{12}s_{21}) = 1$. We have

$$\begin{aligned}\sum\nolimits_3 &= \sum_{r_i,s_{ij}} \frac{\mu(s_{12}s_{21})}{\rho_1(s_{12}s_{21})\gamma_1(s_{12}s_{21})} \frac{C_{r_1s_{12},r_2s_{21}}C_{r_1s_{21},r_2s_{12}}}{\rho_1(r_1)\gamma_1(r_2)} L(r_1) \\ &+ 2\sum_{r_i,s_{ij}} \frac{\mu(s_{12}s_{21})}{\rho_1(s_{12}s_{21})\gamma_1(s_{12}s_{21})} \sum_p \frac{L(p)}{p-2} \frac{C_{r_1s_{12},r_2s_{21}}C_{pr_1s_{21},r_2s_{12}}}{\rho_1(r_1)\gamma_1(r_2)} \\ &+ 2\sum_{r_i,s_{ij}} \frac{\mu(s_{12}s_{21})}{\rho_1(s_{12}s_{21})\gamma_1(s_{12}s_{21})} \sum_p \frac{L(p)}{p-2} \frac{C_{pr_1s_{12},r_2s_{21}}C_{r_1s_{21},r_2s_{12}}}{\rho_1(r_1)\gamma_1(r_2)} \\ &+ 4\sum_{r_i,s_{ij}} \frac{\mu(s_{12}s_{21})}{\rho_1(s_{12}s_{21})\gamma_1(s_{12}s_{21})} \sum_p \frac{L(p)}{(p-2)^2} \frac{C_{pr_1s_{12},r_2s_{21}}C_{pr_1s_{21},r_2s_{12}}}{\rho_1(r_1)\gamma_1(r_2)}.\end{aligned} \tag{6.24}$$

Of the four terms in (6.24) above, the fourth term is $O(\log^5 z)$. This follows by using the estimates $C_{r_1,r_2} \ll \log z$, $L(n) \ll \log n$ and the fact that $\rho_1 \in \Omega_2$ and $\gamma_1 \in \Omega_1$. The first, second and third terms will be contributing to the main term.

Again, the contribution to the first term, second term and third term of (6.24) when at least one s_{ij} is $> D$ is

$$\ll \sum_{\substack{(s_{12},s_{21})\in S(z)\\ s_{12}>D}} \frac{1}{\rho_1(s_{12}s_{21})\gamma_1(s_{12}s_{21})} \sum_{(r_1,r_2)\in S(z)} \frac{\log^3 z}{\rho_1(r_1)\gamma_1(r_2)} \ll \frac{\log^6 z}{D}.$$

So we may assume $s_{12} = s_{21} = 1$ with error $O\big(\frac{\log^6 z}{D}\big)$. Hence,

$$\sum\nolimits_3 = \sum_{r_i} \frac{C^2_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)} L(r_1) + 4\sum_{r_i} \frac{C_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)} \sum_p \frac{\log p}{p-2} C_{pr_1,r_2} + O\bigg(\frac{\log^6 z}{D}\bigg).$$

Putting together the expressions for $\sum_1$, $\sum_2$ and $\sum_3$, we get the desired result. ∎

Evaluation of M_2. We multiply by $\sum_{s_{ij}|(d_i,l_j)} \mu(s_{12})\mu(s_{21})$ to get rid of the conditions $(d_1,l_2) = (d_2,l_1) = 1$. Then as before $\mu^2(r_1r_2s_{12}s_{21}) = 1$. Hence

$$
\begin{aligned}
(6.25)\qquad \frac{M_2}{\log x + c'} &= \sum_{(d_i,l_j)=1} \frac{\lambda_{d_1,d_2}\lambda_{l_1,l_2}}{\rho[d_1,l_1]\gamma[d_2,l_2]} \\
&= \sum_{d_i,l_i} \frac{\lambda_{d_1,d_2}}{\rho(d_1)\gamma(d_2)} \frac{\lambda_{l_1,l_2}}{\rho(l_1)\gamma(l_2)} \sum_{r_i|(d_i,l_i)} \rho_1(r_1)\gamma_1(r_2) \sum_{s_{ij}|(d_i,l_j)} \mu(s_{12})\mu(s_{21}) \\
&= \sum_{s_{ij}} \frac{\mu(s_{12}s_{21})}{\rho_1(s_{12}s_{21})\gamma_1(s_{12}s_{21})} \sum_{r_i} \frac{C_{r_1s_{12},r_2}C_{r_1s_{21},r_2s_{12}}}{\rho_1(r_1)\gamma_1(r_2)}.
\end{aligned}
$$

As before, the contribution to the above sum when at least one s_{ij} is $> D$ is $O\big(\frac{\log^5 z}{D}\big)$. So we may assume $s_{12} = s_{21} = 1$ with error $O\big(\frac{\log^5 z}{D}\big)$. Therefore,

$$
\begin{aligned}
\frac{M_2}{\log x + c'} &= \sum_{r_i} \frac{C^2_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)} + O\bigg(\frac{\log^6 x}{D}\bigg) \\
&= \sum_{(r_1,r_2)\in S(z)} \frac{\mu^2(r_1)}{\rho_1(r_1)} \frac{\mu^2(r_2)}{\gamma_1(r_2)} \bigg(B^2h_1^2(r_1)h_2^2(r_2)Q_2^2\bigg(\frac{\log r_1}{\log z}, \frac{\log r_2}{\log z}\bigg) + O(\log z)\bigg) \\
&\quad + O\bigg(\frac{\log^6 x}{D}\bigg)
\end{aligned}
$$

where we have substituted the expression for C_{r_1,r_2} from Lemma 6.2. By applying Corollary 4.11 to the main term above, we obtain

$$
\begin{aligned}
(6.26)\qquad M_2 &= B^5 \log x \iint_T s_1 Q_2^2(s_1,s_2)\, ds_2\, ds_1 + O(\log^5 z) + O\bigg(\frac{\log^6 z}{D}\bigg) \\
&= (1+o(1))B^5 \log x \iint_T s_1 Q_2^2(s_1,s_2)\, ds_2\, ds_1
\end{aligned}
$$

where Q_2 is given in (5.4).

Evaluation of M_{21}. Applying Proposition 6.5 to the expression for M_{21} in (6.19) with $L(n) = \log n$, we get

$$
\begin{aligned}
M_{21} &= \sum_{r_i} \frac{C^2_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)} \log r_1 + 4\sum_{r_i} \frac{C_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)} \sum_p \frac{\log p}{p-2} C_{pr_1,r_2} \\
&\qquad + O\bigg(\frac{\log^6 z}{D}\bigg) \\
&= M_{21}^{(1)} + M_{21}^{(2)} + O\bigg(\frac{\log^6 z}{D}\bigg).
\end{aligned}
$$

Substituting the expression for C_{r_1,r_2}, we obtain

$$M_{21}^{(1)} = (\log z) \sum_{(r_1,r_2)\in S(z)} \frac{\mu^2(r_1)}{\rho_1(r_1)} \frac{\mu^2(r_2)}{\gamma_1(r_2)} \times \left(B^2 h_1^2(r_1) h_2^2(r_2) \frac{\log r_1}{\log z} Q_2^2\left(\frac{\log r_1}{\log z}, \frac{\log r_2}{\log z}\right) + O(\log z)\right).$$

By Corollary 4.11 applied to the above sum, we get

$$M_{21}^{(1)} = (1+o(1)) B^5 \log z \iint_T s_1^2 Q_2^2(s_1,s_2)\, ds_2\, ds_1. \tag{6.27}$$

Next, we look at $M_{21}^{(2)}$, substitute the value of C_{pr_1,r_2} and write $s_i = \frac{\log r_i}{\log z}$ to get

$$M_{21}^{(2)} = -4 \sum_{(r_1,r_2)\in S(z)} \frac{\mu(r_1)\mu(r_2)C_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)} \times \sum_{(pr_1,r_2)\in S(z)} \frac{(p-1)^2}{p(p-2)} \frac{\log p}{p} \left(BQ_2\left(s_1 + \frac{\log p}{\log z}, s_2\right) + O(1)\right) \tag{6.28}$$

$$= -4 \sum_{(r_1,r_2)\in S(z)} \frac{\mu(r_1)\mu(r_2)C_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)} \times \left[B \sum_{(pr_1,r_2)\in S(z)} \frac{(p-1)^2}{p(p-2)} \frac{\log p}{p} Q_2\left(s_1 + \frac{\log p}{\log z}, s_2\right) + O(\log z)\right].$$

First, we focus on the inner sum of (6.28). The condition $(pr_1, r_2) \in S(z)$ is equivalent to

$$p \le z^{\eta(s_2)-s_1} \quad \text{and} \quad (p, Wr_1r_2) = 1$$

with $\eta(s)$ as defined in (4.5). By observing that $\frac{(p-1)^2}{p(p-2)} = 1 + O(1/p^2)$, it follows that the inner sum of (6.28) is

$$B \sum_{2<p\le z^{\eta(s_2)-s_1}} \frac{\log p}{p} Q_2\left(s_1 + \frac{\log p}{\log z}, s_2\right) + O\left(B \sum_{p\mid Wr_1r_2} \frac{\log p}{p}\right) + O(\log z).$$

By Proposition 4.1, the main term above becomes

$$B(\log z + O(1)) \int_0^{\eta(s_2)-s_1} Q_2(s_1+t, s_2)\, dt = B(\log z + O(1)) \int_{s_1}^{\eta(s_2)} Q_2(t_1, s_2)\, dt$$

$$= B(\log z + O(1)) \int_{s_1}^{\eta(s_2)} \int_{s_2}^{\eta(s_1)} P(t_1,t_2)\, dt_2\, dt_1 = B(\log z + O(1)) Q_1(s_1, s_2).$$

Plugging the expression for the inner sum back into (6.28), we obtain

$$M_{21}^{(2)} = -4 \sum_{(r_1,r_2)\in S(z)} \frac{\mu^2(r_1)\mu^2(r_2)}{\rho_1(r_1)\gamma_1(r_2)}\biggl(B^2\log z\; h_1(r_1)h_2(r_2) \times Q_1\Bigl(\frac{\log r_1}{\log z},\frac{\log r_2}{\log z}\Bigr)Q_2\Bigl(\frac{\log r_1}{\log z},\frac{\log r_2}{\log z}\Bigr) + O(\log^2 z)\biggr) + O\biggl(\log^2 z \sum_{(r_1,r_2)\in S(z)} \frac{h_1(r_1)}{\rho_1(r_1)}\frac{h_2(r_2)}{\gamma_1(r_2)} \sum_{p\mid Wr_1r_2}\frac{\log p}{p}\biggr). \tag{6.29}$$

The second error term in (6.29) turns out to be $O(\log^5 z)$. This is seen by replacing

$$\sum_{(r_1,r_2)\in S(z)} \text{ by } \ll \sum_{\substack{r_1\le z\\ r_2\le z}} \quad\text{and}\quad \sum_{p\mid Wr_1r_2} \text{ by } \sum_{p\mid W}+\sum_{p\mid r_1}+\sum_{p\mid r_2}$$

and interchanging the order of summation. Coming back to the main term in (6.29), apply Corollary 4.11 to the sum to get

$$M_{21}^{(2)} = -4(1+o(1))B^5\log z \iint_T s_1 Q_1(s_1,s_2)Q_2(s_1,s_2)\,ds_2\,ds_1. \tag{6.30}$$

Estimation of M_{22}. To estimate M_{22}, we can apply Proposition 6.5 with $L(n)=\sum_{p\mid n,\,p>2}\frac{\log p}{p-2}$ to obtain

$$M_{22} = \sum_{r_i} \frac{C_{r_1,r_2}}{\rho_1(r_1)\gamma_1(r_2)}\biggl(C_{r_1,r_2}\sum_{\substack{p\mid r_1\\ p>2}}\frac{\log p}{p-2} + 4\sum_{2<p\le z}\frac{\log p}{(p-2)^2}C_{pr_1,r_2}\biggr) + O\Bigl(\frac{\log^6 z}{D}\Bigr). \tag{6.31}$$

Using the estimate $C_{r_1,r_2}\ll \log z$ in (6.31) we get

$$\begin{aligned} M_{22} &\ll \sum_{\substack{r_1\le z\\ r_2\le z}} \frac{\log^2 z}{\rho_1(r_1)\gamma_1(r_2)}\biggl(\sum_{\substack{p\mid r_1\\ p>2}}\frac{\log p}{p-2} + \sum_{2<p\le z}\frac{\log p}{(p-2)^2}\biggr) + O\Bigl(\frac{\log^z}{D}\Bigr)\\ &\ll (\log^2 z)\biggl(\sum_{r_2\le z}\frac{1}{\gamma_1(r_2)}\biggr)\biggl(\sum_{2<p\le z}\frac{\log p}{p(p-2)}\sum_{r_1\le z/p}\frac{1}{\rho_1(r_1)}\biggr) + \log^5 z + \frac{\log^6 z}{D}\\ &\ll \frac{\log^6 z}{D}. \end{aligned} \tag{6.32}$$

Estimation of E_2. From (6.19), we have

$$E_2 \ll_\epsilon \frac{x^{1/2+\epsilon}}{W^{1/4}} \sum_{d_i,l_i} \frac{|\lambda_{d_1,d_2}|\,|\lambda_{l_1,l_2}|}{[d_1,l_1]^{1/2}[d_2,l_2]^{1/4}}.$$

Using the estimate $\lambda_{d_1,d_2} \ll \log^4 z$ from Lemma 6.2 and writing $[d_i, l_i] = d_i l_i/(d_i, l_i)$, we get

$$E_2 \ll_\epsilon \frac{x^{1/2+\epsilon}\log^8 z}{W^{1/4}} \sum_{\substack{(d_1,d_2)\in S(z)\\(l_1,l_2)\in S(z)}} \frac{(d_1,l_1)^{1/2}(d_2,l_2)^{1/4}}{d_1^{1/2}d_2^{1/4}l_1^{1/2}l_2^{1/4}}.$$

Since $W^{1/4} \ll (\log\log x)^{1/4}$ and $\log^8 z$ are small, these terms can be absorbed into the term x^ϵ. Therefore,

$$E_2 \ll_\epsilon x^{1/2+\epsilon} \sum_{(g_1,g_2)\in S(z)} g_1^{1/2}g_2^{1/4} \sum_{\substack{(d_1,d_2)\in S(z)\\(l_1,l_2)\in S(z)\\(d_i,l_i)=g_i}} d_1^{-1/2}d_2^{-1/4}l_1^{-1/2}l_2^{-1/4} \tag{6.33}$$

$$\ll_\epsilon x^{1/2+\epsilon} \sum_{(g_1,g_2)\in S(z)} g_1^{-1/2}g_2^{-1/4}\Big(\sum_{\substack{d_1\le z/g_1\\ g_2d_2\le z/(g_1d_1)^{2/3}}} d_1^{-1/2}d_2^{-1/4}\Big)^2.$$

Now,

$$\sum_{\substack{d_1\le z/g_1\\ g_2d_2\le z/(g_1d_1)^{2/3}}} d_1^{-1/2}d_2^{-1/4}$$

$$= \sum_{d_1\le z/g_1} d_1^{-1/2} \sum_{g_2d_2\le z/(g_1d_1)^{2/3}} d_2^{-1/4} \ll \sum_{d_1\le z/g_1} d_1^{-1/2}\Big(\frac{z}{g_2(g_1d_1)^{2/3}}\Big)^{3/4}$$

$$\ll z^{3/4}g_1^{-1/2}g_2^{-3/4}\sum_{d_1\le z/g_1} d_1^{-1} \ll g_1^{-1/2}g_2^{-3/4}z^{3/4}\log z.$$

Substituting the above expression into (6.33), we obtain

$$E_2 \ll_\epsilon x^{1/2+\epsilon}z^{3/2}\log^2 z \sum_{\substack{g_1\le z\\ g_2\le z/g_1^{2/3}}} g_1^{-3/2}g_2^{-7/4} \ll_\epsilon x^{1/2+\epsilon}z^{3/2}\log^2 z.$$

Under the choice $z = x^{1/3-\epsilon}$, we get

$$E_2 \ll_\epsilon x^{1-\epsilon/2}. \tag{6.34}$$

Substituting $z = x^{1/3-\epsilon}$ and noting that $c(W) = 1 + o(1)$, we deduce from (6.26), (6.27), (6.30), (6.32), (6.34) and (6.18) that

$$S_2 = \frac{2x}{W}(1+o(1))B^6 R_2(P)$$

where

$$R_2(P) = \iint_T s_1(3+\epsilon-s_1)Q_2^2(s_1,s_2)\,ds_2\,ds_1 + 4\iint_T s_1Q_1(s_1,s_2)Q_2(s_1,s_2)\,ds_2\,ds_1.$$

This completes the proof of Proposition 5.2.

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References

[Che] J. R. Chen, *On the representation of a large even integer as the sum of a prime and a product of at most two primes*, Sci. Sinica 16 (1973), 157–176.

[GPY] D. A. Goldston, J. Pintz and C. Y. Yıldırım, *Primes in tuples I*, Ann. of Math. 170 (2009), 819–862.

[Iv] A. Ivić, *The Riemann Zeta-function: Theory and Applications*, Wiley, 1985.

[May] J. Maynard, *Small gaps between primes*, Ann. of Math. 181 (2015), 383–413.

[MP] Y. Motohashi and J. Pintz, *A smoothed GPY sieve*, Bull. London Math. Soc. 40 (2008), 298–310.

[P8a] D. H. J. Polymath, *New equidistribution estimates of Zhang type*, Algebra Number Theory 8 (2014), 2067–2199.

[P8b] D. H. J. Polymath, *Variants of the Selberg sieve, and bounded intervals containing many primes*, Res. Math. Sci. 1 (2014), art. 12, 83 pp..

[Sel] A. Selberg, *Collected Papers, Vol. II*, Springer, New York, 1991.

[Zha] Y. Zhang, *Bounded gaps between primes*, Ann. of Math. 179 (2014), 1121–1174.

R. Balasubramanian, Priyamvad Srivastav
Institute of Mathematical Sciences
Taramani, Chennai, India 600113
and
Homi Bhabha National Institute
Training School Complex
Anushakti Nagar, Mumbai, India 400094
E-mail: balu@imsc.res.in
priyamvads@imsc.res.in