ACTA ARITHMETICA
179.4 (2017)

Rational period functions and indefinite binary quadratic forms in higher level cases

by

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1. Introduction and statement of results. For $k \in \mathbb{Z}$ and any meromorphic function f on the complex upper half-plane $\mathfrak{H}$, we define the action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ by

$$(f|_k\gamma)(z) = (cz+d)^{-k} f(\gamma z).$$

For a positive integer p, we let

$$\Gamma_0(p) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{p} \right\}$$

be the Hecke subgroup of $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$, and $\Gamma_0^+(p)$ be the group generated by the Hecke group $\Gamma_0(p)$ and the Fricke involution $W_p := \left(\begin{smallmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{smallmatrix}\right)$. Let $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $U := TW_p = \left(\begin{smallmatrix} \sqrt{p} & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{smallmatrix}\right)$. For $p \in \{1,2,3\}$, we consider a rational function $q(z)$ satisfying

$$q|_{2k}W_p + q = 0, \tag{1.1}$$

$$q|_{2k}U^{n_p-1} + q|_{2k}U^{n_p-2} + \cdots + q|_{2k}U + q = 0, \tag{1.2}$$

where

$$n_p = \begin{cases} 3, & \text{if } p = 1, \\ 2p, & \text{if } p = 2,3. \end{cases}$$

We call such a function $q(z)$ a *rational period function of weight* $2k$ *for* $\Gamma_0^+(p)$, and denote by $\mathrm{RPF}_{2k}(\Gamma_0^+(p))$ the set of all such functions. In particular, if the rational period function is a polynomial, then we call it a *period polynomial*. Many examples of period polynomials come from Eichler integrals; by using weakly holomorphic modular forms, we can find the space of period polynomials, as described in [10, 5]. On the other hand, it is natural

2010 *Mathematics Subject Classification*: Primary 11F11; Secondary 11F67.
Key words and phrases: rational period functions, indefinite binary quadratic forms.
Received 23 June 2016; revised 12 December 2016.
Published online 1 August 2017.

to investigate rational period functions that are not period polynomials. The classical example is $1/z$ of weight 2 for $\Gamma_0(1)$. This rational period function comes from a modular integral E_2 that is an Eisenstein series of weight 2 for $\Gamma_0(1)$.

Knopp [13] introduced the notion of a rational period function for a modular integral of weight $2k \in 2\mathbb{Z}$ on Γ, where Γ is any Fuchsian group acting on $\mathfrak{H}$. An explicit characterization of the rational period functions on $\Gamma_0(1)$ was given in [14, 1, 6, 7, 8, 15]. Ash [1] used cohomological techniques to provide such a characterization, and later Schmidt [16] generalized Ash's work by giving an abstract characterization of rational period functions on any finitely generated Fuchsian group of the first kind with parabolic elements.

Let $n > 1$ be an integer coprime to p. Following [13, 4] we define a Hecke operator $\widehat{T}_{2k,n}$ on $\mathrm{RPF}_{2k}(\Gamma_0^+(p))$. More precisely, suppose that $q(z)$ is a rational function corresponding to a modular integral $F(z)$ of weight $2k$ that satisfies

$$F|_{2k}T = F \quad \text{and} \quad F|_{2k}W_p = F + q(z). \tag{1.3}$$

We define $\widehat{T}_{2k,n}$ by

$$\widehat{T}_{2k,n}(q(z)) = F_n|_{2k}(W_p - 1) \quad \text{where} \quad F_n = n^{k-1} \sum_{\substack{ad=n \\ b\ (d)}} F|_{2k}\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right). \tag{1.4}$$

Let $q(z) \in \mathrm{RPF}_{2k}(\Gamma_0^+(p))$. As for the location of poles of $q(z)$, it was shown in [13, 4] that if z_0 is a finite pole of $q(z)$, then $z_0 \in \mathbb{Q}(\sqrt{n})$ for some square-free positive integer n, and if z_0 is a finite rational pole of $q(z)$, then $z_0 = 0$. Moreover, in the case of $p = 1$, Gethner [11] proved that rational period functions with irrational poles are not Hecke eigenfunctions. In [4] we gave evidence for that phenomenon in higher level cases and suggested that one can construct rational period polynomials by using binary quadratic forms.

In this paper, by extending Gethner's [11] and Choie and Parson's [6] results, we solve these problems in higher level cases, in the following theorems.

THEOREM 1.1. *Let $p \in \{2,3\}$ and $q(z) \in \mathrm{RPF}_{2k}(\Gamma_0^+(p))$ with at least one quadratic irrational pole. Then $q(z)$ is not an eigenfunction of $\widehat{T}_{2k,n}$ for any integer $n > 1$ with* $\gcd(n,p) = 1$.

The simplest example of a rational period function is given by $q(z) = 1/z \in \mathrm{RPF}_2(\Gamma_0^+(p))$, which is shown in [13, 4] to be a Hecke eigenfunction. Let $p \in \{1,2,3\}$, and k be an odd integer. We set

$$z_0 = \frac{p + \sqrt{p^2+4p}}{2p}, \qquad z_0' = \frac{1}{pz_0} = \frac{-p + \sqrt{p^2+4p}}{2p}$$

and define

$$q_{2k}^{+}(z) = (z-z_0)^{-k}(z+z_0')^{-k} + (z+z_0)^{-k}(z-z_0')^{-k}$$
$$= \left(z^2 - z - \frac{1}{p}\right)^{-k} + \left(z^2 + z - \frac{1}{p}\right)^{-k}.$$

It then follows from [13, 4] that the function $q_{2k}^{+}(z)$ belongs to the space $\mathrm{RPF}_{2k}(\Gamma_0^+(p))$, and $q_{2k}^{+}(z)$ is not a Hecke eigenfunction of $\widehat{T}_{2k,n}$ for any integer $n > 1$ with $\gcd(n,p) = 1$, which supports Theorem 1.1.

Let D be a positive integer that is not a perfect square. We further assume that D is congruent to a square modulo $4p$, and let $\mathcal{Q}_{D,p}$ be the set of all integral binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2 = [a,b,c]$ with $p \mid a$ and $b^2 - 4ac = D$. For $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0^+(p)$, we set $X' = \alpha X + \beta Y$, $Y' = \gamma X + \delta Y$, and $Q(X,Y) \circ M = Q(X',Y')$. Two quadratic forms Q_1 and Q_2 are said to be *equivalent in the narrow sense*, written $Q_1 \sim Q_2$, if there exists $M \in \Gamma_0^+(p)$ such that $Q_1 \circ M = Q_2$. Also, a quadratic form $Q = [a,b,c]$ is called *reduced* if $a, c > 0$ and $b > a + c$.

We note that

$$[a,-b,c] = [a,b,c] \circ \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right),$$
$$[-a,-b,-c] = [a,b,c] \circ \left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right),$$
$$[-a,b,-c] = [a,b,c] \circ \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right).$$

Using these identities one can show that the maps on $\mathcal{Q}_{D,p}$ which send

$$\begin{aligned} &[a,b,c] \quad \text{to} \quad [a,-b,c], \\ &[a,b,c] \quad \text{to} \quad [-a,b,-c], \\ &[a,b,c] \quad \text{to} \quad [-a,-b,-c], \end{aligned}$$

induce maps on $\mathcal{Q}_{D,p}/\Gamma_0^+(p)$. Thus each narrow equivalence class $\mathcal{A}$ of quadratic forms in $\mathcal{Q}_{D,p}$ is associated with three other, not necessarily distinct, narrow equivalence classes of quadratic forms in $\mathcal{Q}_{D,p}$:

$$\mathcal{A}' := \{[a,-b,c] \mid [a,b,c] \in \mathcal{A}\},$$
$$\theta\mathcal{A} := \{[-a,b,-c] \mid [a,b,c] \in \mathcal{A}\},$$
$$\theta\mathcal{A}' := \{[-a,-b,-c] \mid [a,b,c] \in \mathcal{A}\}.$$

For each integer k and a narrow equivalence class $\mathcal{A}$ of quadratic forms in $\mathcal{Q}_{D,p}$, we set

$$Q_{k,D,\mathcal{A}} := \sum_{\substack{Q \in \mathcal{A} \\ Q \text{ reduced}}} Q(z,-1)^{-k} = \sum_{\substack{[a,b,c] \in \mathcal{A} \\ [a,b,c] \text{ reduced}}} \frac{1}{(az^2 - bz + c)^k}.$$

Since there are only finitely many reduced forms in $\mathcal{A}$ (see Lemma 3.1(i)), $Q_{k,D,\mathcal{A}}(z)$ is a rational function. As an extension of the case for $p = 1$ [6], the following assertion provides more examples of rational period functions with irrational poles when $p = 2$.

THEOREM 1.2. *Let D be a positive integer that is not a perfect square. Assume that D is congruent to a square modulo 8. Let $\mathcal{A}$ be a narrow equivalence class of quadratic forms in $\mathcal{Q}_{D,2}$. Set*

$$q_{\mathcal{A}}(z) := \sum_{\substack{a>0>c \\ [a,b,c]\in\mathcal{A}}} \frac{1}{(az^2-bz+c)^k} - \sum_{\substack{a<0<c \\ [a,b,c]\in\mathcal{A}}} \frac{1}{(az^2-bz+c)^k}$$

$$R_{\mathcal{A}}(z) := \sum_{\substack{[a,b,c]\in\theta\mathcal{A}' \\ [a,b,c]:\,\text{reduced} \\ a-2b+4c<0}} (-1)^k \frac{1}{(az^2-bz+c)^k} - \sum_{\substack{[a,b,c]\in\mathcal{A} \\ [a,b,c]:\,\text{reduced} \\ a-2b+4c<0}} \frac{1}{(az^2-bz+c)^k}.$$

Then, for any integer k:

(a) $q_{\mathcal{A}}(z) = (Q_{k,D,\mathcal{A}}(z) - (-1)^k Q_{k,D,\mathcal{A}'}(z))|_{2k}(-U+U^2) + R_{\mathcal{A}}(z)|_{2k}U^2$.

(b) $q_{\mathcal{A}} \in \mathrm{RPF}_{2k}(\Gamma_0^+(2))$ *and it is not an eigenfunction of* $\widehat{T}_{2k,n}$ *for any odd integer* $n > 1$.

EXAMPLE 1.3. Let $D = 12$. It follows from [3] that

$$\mathcal{Q}_{12}/\Gamma(1) = \{[2,2,-1],[-2,2,1]\}.$$

Moreover we know from [12] that there is a natural bijection between $\mathcal{Q}_{12,2,2}/\Gamma_0(2)$ and $\mathcal{Q}_{12}/\Gamma(1)$. Since $\mathcal{Q}_{12,2,2}/\Gamma_0(2) = \mathcal{Q}_{12,-2,2}/\Gamma_0(2)$, we see that W_2 acts on the set $\mathcal{Q}_{12,2,2}/\Gamma_0(2)$. Observing the identities

$$\begin{aligned} [2,2,-1]|\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) &= [2,6,3] \quad (=\text{ the unique reduced form}),\\ [2,6,3]|\left(\begin{smallmatrix}-1&-1\\2&1\end{smallmatrix}\right) &= [2,-2,-1],\\ [2,-2,-1]|W_2 &= [-2,2,1], \end{aligned}$$

we get

$$\mathcal{Q}_{12,2}/\Gamma_0^+(2) = \{[2,2,-1]\}.$$

Let $\mathcal{A}$ be the class of $[2,2,-1]$ in $\mathcal{Q}_{12,2}$. We then observe that

$$\mathcal{A} = \mathcal{A}' = \theta\mathcal{A} = \theta\mathcal{A}' = \mathcal{Q}_{12,2}.$$

For odd k, we compute that the simple forms in $\mathcal{Q}_{12,2}$ are $[2,2,-1]$, $[2,-2,-1]$, $[-2,-2,1]$, $[-2,2,1]$, so that

$$q_{\mathcal{A}}(z) = \frac{2}{(2z^2-2z-1)^k} + \frac{2}{(2z^2+2z-1)^k}.$$

We also compute that

$$\mathcal{Q}_{k,12,\mathcal{A}}(z) = \frac{1}{(2z^2-6z+3)^k} \quad \text{and} \quad R_{\mathcal{A}}(z) = 0.$$

Thus Theorem 1.2 gives

$$q_{\mathcal{A}}(z) = \frac{2}{(2z^2-6z+3)^k}|_{2k}(-U+U^2) \in \mathrm{RPF}_{2k}(\Gamma_0^+(2)).$$

2. Proof of Theorem 1.1

Definition 2.1. Suppose z_0 is a quadratic irrationality. If $P(z) = az^2 + bz + c$ is in $\mathbb{Z}[z]$ with $p \mid a$, $\gcd(a/p, b, c) = 1$, and $P(z_0) = 0$, then $P(z)$ is said to be a *minimal polynomial for z_0 in level p*.

Remark 2.2. (i) A quadratic irrationality z_0 always has a minimal polynomial in level p. Indeed, if $az_0^2 + bz_0 + c = 0$, then we see that

$$p\frac{a}{\gcd(a,pb,pc)}z^2 + \frac{pb}{\gcd(a,pb,pc)}z + \frac{pc}{\gcd(a,pb,pc)}$$

is a minimal polynomial in level p.

(ii) If $Q(z) = a'z^2 + b'z + c'$ is another minimal polynomial for z_0 in level p, then $a'/a = b'/b = c'/c = t$ for some t. Thus $\gcd(a'/p, b', c') = \gcd(at/p, bt, ct) = |t| = 1$, and therefore $Q(z) = \pm P(z)$.

Definition 2.3. Suppose z_0 is a quadratic irrationality with minimal polynomial $P(z) = az^2 + bz + c$ in level p. Then we define the *discriminant of z_0 in level p* by

$$\mathrm{disc}^{(p)}(z_0) := \mathrm{disc}(P(z)) = b^2 - 4ac.$$

Lemma 2.4. *Suppose z_0 is a quadratic irrationality with minimal polynomial $P(z) = az^2 + bz + c$ in level p. Then for $M \in \Gamma_0^+(p)$:*

(i) *A minimal polynomial for Mz_0 in level p is given by $(P|_{-2}M^{-1})(z)$.*
(ii) $\mathrm{disc}^{(p)}(Mz_0) = \mathrm{disc}^{(p)}(z_0)$.

Proof. Let $Q := [a, b, c]$. Then $P(z) = Q(z, 1)$, $\mathrm{disc}(P(z)) = \mathrm{disc}\, Q$, and

$$(P|_{-2}M^{-1})(z) = (Q \circ M^{-1})(z, 1). \tag{2.1}$$

Moreover,

$$(P|_{-2}M^{-1})(Mz_0) = 0. \tag{2.2}$$

Since $\mathcal{Q}_{d,p}$ and $\{[a,b,c] \in \mathcal{Q}_{d,p} \mid \gcd(a/p, b, c) = 1\}$ are invariant under the action of $\Gamma_0^+(p)$, by (2.1) and (2.2) we see that $(P|_{-2}M^{-1})(z)$ is a minimal polynomial for Mz_0 in level p and

$$\begin{aligned}\mathrm{disc}^{(p)}(Mz_0) &= \mathrm{disc}(P|_{-2}M^{-1}) = \mathrm{disc}(Q \circ M^{-1}) = \mathrm{disc}(Q)\\ &= \mathrm{disc}(P(z)) = \mathrm{disc}^{(p)}(z_0),\end{aligned}$$

as desired. ∎

Lemma 2.5. *Let $p \in \{2, 3\}$ and r be a prime with $r \nmid p$. Let $q(z)$ be a rational period function with at least one quadratic irrational pole z_0. Then* *there exists a quadratic irrational pole z_1 of $q(z)$ satisfying* $\mathrm{disc}^{(p)}(z_0) = \mathrm{disc}^{(p)}(z_1)$ *and with minimal polynomial $\alpha z^2 + \beta z + \gamma$ in level p such that* $\gcd(\alpha, r) = 1$.

Proof. Suppose $P(z) = az^2 + bz + c$ is a minimal polynomial for z_0 in level p. Without loss of generality, assume $\gcd(a, r) = r$. Since $q|_{2k}W_p + q = 0$, we note that $W_p z_0 = -1/(pz_0)$ is a pole of $q(z)$ and by Lemma 2.4, $pcz^2 - bz + a/p$ is a minimal polynomial for $-1/(pz_0)$ in level p. If $r \nmid c$, then we are done by taking $z_1 = -1/(pz_0)$. Otherwise, we have $r \mid c$, i.e. $\gcd(r, c) = r$. Since $q|_{2k}U^{2p-1} + q|_{2k}U^{2p-2} + \cdots + q_{2k}U + q = 0$, we note that one of $U^{2p-1}z_0, U^{2p-2}z_0, \ldots, Uz_0$ is a pole of $q(z)$.

CASE 1: *Uz_0 is a pole of $q(z)$.* In this case Lemma 2.4 shows that

$$\begin{aligned} P|_{-2}U^{-1} &= P|_{-2}\left(\begin{smallmatrix} 0 & 1/\sqrt{p} \\ -\sqrt{p} & \sqrt{p} \end{smallmatrix}\right) = cpz^2 - (b + 2cp)z + (cp + b + a/p) \\ &= [cp, -b - 2cp, cp + b + a/p](z, 1) \end{aligned}$$

is minimal for Uz_0 in level p and $\mathrm{disc}^{(p)}(Uz_0) = \mathrm{disc}^{(p)}(z_0)$. As $\gcd(c, -b-2cp, cp + b + a/p) = 1$, $r \mid c$, $r \mid a$, and $r \nmid p$, we must have $r \nmid b$. Since Uz_0 is assumed to be a pole, so is $z_1 := W_p(Uz_0)$, and z_1 has a minimal polynomial $P|_{-2}(W_pU)^{-1}$ in level p, which is equal to $(cp^2 + bp + a)z^2 + (2cp + b)z + c$. Since $r \nmid b$, $r \mid a$, $r \mid c$, and $r \nmid p$, we must have $r \nmid (cp^2 + bp + a)$.

CASE 2: *U^2z_0 is a pole of $q(z)$.* In this case we take $z_1 := U^2z_0$ and observe from Lemma 2.4 that

$$P|_{-2}U^{-2} = P|_{-2}\left(\begin{smallmatrix} -1 & 1 \\ -p & p-1 \end{smallmatrix}\right) = [cp^2 + bp + a, *, *](z, 1)$$

is minimal for z_1 in level p and $\mathrm{disc}^{(p)}(z_1) = \mathrm{disc}^{(p)}(z_0)$. Since $\gcd(a/p, b, c) = 1$, $r \mid a$, $r \mid c$, and $r \nmid p$, we must have $r \nmid (cp^2 + bp + a)$.

CASE 3: *U^3z_0 is a pole of $q(z)$.* In this case we let $z_1 := U^3z_0$ and deduce from Lemma 2.4 that

$$P|_{-2}U^{-3} = P|_{-2}\left(\begin{smallmatrix} -\sqrt{p} & (p-1)/\sqrt{p} \\ \sqrt{p}(1-p) & \sqrt{p}(p-2) \end{smallmatrix}\right) = [ap + bp(p-1) + cp(p-1)^2, *, *](z, 1)$$

is minimal for z_1 in level p, and $\mathrm{disc}^{(p)}(z_1) = \mathrm{disc}^{(p)}(z_0)$. Since $\gcd(a/p, b, c) = 1$, $r \mid a$, $r \mid c$, and $r \nmid p$, we should have $r \nmid (ap + bp(p-1) + cp(p-1)^2)$.

CASE 4: *$p = 3$ and U^4z_0 is a pole of $q(z)$.* In this case we take $z_1 := U^4z_0$ and deduce from Lemma 2.4 that

$$P|_{-2}U^{-4} = P|_{-2}\left(\begin{smallmatrix} -2 & 1 \\ -3 & 1 \end{smallmatrix}\right) = [4a + 6b + 9c, *, *](z, 1)$$

is minimal for z_1 in level 3 and $\mathrm{disc}^{(p)}(z_1) = \mathrm{disc}^{(p)}(z_0)$. Since $\gcd(a/3, b, c) = 1$, $r \mid a$, $r \mid c$, and $r \nmid 3$, we must have $r \nmid (4a + 6b + 9c)$ provided that $r \neq 2$. Now suppose $p = 3$ and $r = 2$. We note that $z_2 := W_3U^4z_0$ is a pole of $q(z)$, and

$$P|_{-2}U^{-4}W_3^{-1} = P|_{-2}\left(\begin{smallmatrix} -\sqrt{3} & -2/\sqrt{3} \\ -\sqrt{3} & -\sqrt{3} \end{smallmatrix}\right) = [3a + 3b + 3c, *, *](z, 1)$$

is minimal for z_2 in level 3 and $\mathrm{disc}^{(p)}(z_2) = \mathrm{disc}^{(p)}(z_0)$. Since $\gcd(a/3, b, c) = 1$, $2 \mid a$, $2 \mid c$, and $2 \nmid 3$, we have $2 \nmid (3a + 3b + 3c)$.

Case 5: *$p = 3$ and $U^5 z_0$ is a pole of $q(z)$.* In this case we set $z_1 := U^5 z_0$ and infer from Lemma 2.4 that

$$P|_{-2}U^{-5} = P|_{-2}\left(\begin{smallmatrix} -\sqrt{3} & 1/\sqrt{3} \\ -\sqrt{3} & 0 \end{smallmatrix}\right) = [3a + 3b + 3c, *, *](z, 1)$$

is minimal for z_1 in level p and $\mathrm{disc}^{(p)}(z_1) = \mathrm{disc}^{(p)}(z_0)$. Since $\gcd(a/3, b, c) = 1$, $r \mid a$, $r \mid c$, and $r \nmid 3$, we see that $r \nmid (3a + 3b + 3c)$. ∎

We remark that for $p \in \{2, 3\}$ and a given prime $r \nmid p$, a rational period function $q(z)$ has an irrational pole with minimal polynomial in level p whose leading coefficient is relatively prime to r. Let $n > 1$ be an integer coprime to p and set

$$S_n := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \;\middle|\; ad = n,\ 0 \le b < d \right\}.$$

Lemma 2.6. *Suppose $\left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right) \in S_{r^s}$, where r is a prime with $r \nmid p$ and $s \in \mathbb{N}$. Suppose z_0 is a quadratic irrationality with minimal polynomial $az^2 + bz + c$ in level p satisfying* $\mathrm{disc}^{(p)}(z_0) = D$. *Then* $\mathrm{disc}^{(p)}\left(\left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right) r^s z_0\right)$ *is at most $r^{4s} D$.*

Proof. Let

$$X := \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right) r^s z_0 = \frac{2ab' - a'r^s b \pm a' r^s \sqrt{D}}{2ad'}$$

so that

$$a(d')^2 X^2 + d'(-2ab' + a'r^s b)X + a(b')^2 - ba'b'r^s + (a')^2 r^{2s} c = 0.$$

Then X is a root of

$$\begin{aligned} P(z) &= [A, B, C](z, 1) \\ &:= a(d')^2 z^2 + d'(-2ab' + a'r^s b)z + a(b')^2 - ba'b'r^s + (a')^2 r^{2s} c \end{aligned}$$

and $\mathrm{disc}(P(z)) = r^{4s} D$. Thus $\mathrm{disc}^{(p)}(X)$ is at most $r^{4s} D$. ∎

Lemma 2.7. *Suppose the minimal polynomial for z_0 in level p is given by $az^2 + bz + c$ with $\gcd(a, r) = 1$ for some fixed prime r. Then* $\mathrm{disc}^{(p)}(r^s z_0) > \mathrm{disc}^{(p)}(z_0)$ *for each $s \in \mathbb{N}$.*

Proof. Since $0 = az_0^2 + bz_0 + c$, we observe that

$$0 = r^{2s}(az_0^2 + bz_0 + c) = a(r^s z_0)^2 + br^s(r^s z_0) + r^{2s} c = P(r^s z_0)$$

where $P(z) = az^2 + br^s z + r^{2s} c$ with $\gcd(a/p, br^s, r^{2s} c) = 1$. Hence

$$\mathrm{disc}^{(p)}(r^s z_0) = r^{2s}(b^2 - 4ac) = r^{2s}\mathrm{disc}^{(p)}(z_0) > \mathrm{disc}^{(p)}(z_0). \ \blacksquare$$

Now we are ready to prove Theorem 1.1. In order to show that $q(z)$ is not an eigenfunction of $\widehat{T}_{2k,n}$ we intend to prove that $\widehat{T}_{2k,n}(q(z))$ has a pole which cannot be a pole of $q(z)$.

Step 1: *$n = r^s$ for some odd prime r with $r \nmid p$ and $s \in \mathbb{N}$.* Let z_1 be a quadratic irrational pole of $q(z)$ such that $\mathrm{disc}^{(p)}(z_1)$ is maximal with respect to all quadratic irrational poles of $q(z)$. By Lemma 2.5 we can find a quadratic irrational pole z_0 of $q(z)$ with minimal polynomial $az^2 + bz + c$ in level p satisfying $\gcd(a, r) = 1$ and $\mathrm{disc}^{(p)}(z_1) = \mathrm{disc}^{(p)}(z_0) =: D$. We note from Lemma 2.7 and maximality of D that $r^s z_0$ cannot be a pole of $q(z)$. On the other hand, we claim that $r^s z_0$ is a pole of $\widehat{T}_{2k,r^s}(q(z))$. We recall that

$$\widehat{T}_{2k,r^s}(q(z)) = F_{r^s}|_{2k}(W_p - 1)$$

where

$$F_{r^s}(z) = (r^s)^{k-1} \sum_{\substack{d|r^s \\ 0\le \mathfrak{b}<d}} F|_{2k} M_{\mathfrak{b},d} \quad \text{with} \quad M_{\mathfrak{b},d} = \left(\begin{smallmatrix} r^s/d & \mathfrak{b} \\ 0 & d \end{smallmatrix}\right).$$

It follows from [4, Lemma 4.2] that for each $M_{\mathfrak{b},d} \in S_{r^s}$, there exists a unique $M_{b',d'} \in S_{r^s}$ such that

$$M_{\mathfrak{b},d} W_p = \gamma_{b',d'} M_{b',d'} \qquad \text{for some } \gamma_{b',d'} \in \Gamma_0(p) W_p. \tag{2.3}$$

Moreover the map f which sends $M_{\mathfrak{b},d} \in S_{r^s}$ to $M_{b',d'} \in S_{r^s}$ is injective. We then have

$$F_{r^s}(z)|_{2k} W_p = (r^s)^{k-1} \sum_{\mathfrak{b},d} F|_{2k} M_{\mathfrak{b},d} W_p = (r^s)^{k-1} \sum_{b',d'} F|_{2k} \gamma_{b',d'} M_{b',d'}. \tag{2.4}$$

Since the group $\Gamma_0^+(p)$ is generated by T and W_p, the element $\gamma_{b',d'}$ can be expressed as a word in T and W_p. This allows us to write

$$F|_{2k} \gamma_{b',d'} = F(z) + q_{b',d'}(z) \tag{2.5}$$

where $q_{b',d'}(z)$ is the sum of terms of the form $q|_{2k} M$ with $M \in \Gamma_0^+(p)$. Combining (2.4) and (2.5) we obtain

$$\begin{aligned} F_{r^s}(z)|_{2k} W_p &= (r^s)^{k-1} \sum_{b',d'} (F(z) + q_{b',d'}(z))|_{2k} M_{b',d'} \\ &= F_{r^s}(z) + (r^s)^{k-1} \sum_{b',d'} q_{b',d'}(z)|_{2k} M_{b',d'}. \end{aligned}$$

Thus

$$\widehat{T}_{2k,r^s}(q(z)) = F_{r^s}(z)|_{2k}(W_p - 1) = (r^s)^{k-1} \sum_{b',d'} q_{b',d'}(z)|_{2k} M_{b',d'}. \tag{2.6}$$

To prove the claim we will show that

$$q_{b',d'}(z)|_{2k} M_{b',d'} \text{ has a pole at } z = r^s z_0 \;\Leftrightarrow\; (b', d') = (0, r^s). \tag{2.7}$$

Suppose that $q_{b',d'}(z)|_{2k} M_{b',d'}$ has a pole at $z = r^s z_0$ for some $M_{b',d'} \in S_{r^s}$. Then $z_2 := M M_{b',d'} r^s z_0$ is a pole of $q(z)$ for some $M \in \Gamma_0^+(p)$. We compute that

$$\begin{aligned}\mathrm{disc}^{(p)}(z_2) &= \mathrm{disc}^{(p)}(MM_{b',d'}r^s z_0)\\ &= \mathrm{disc}^{(p)}(M_{b',d'}r^s z_0) \qquad \text{by Lemma 2.4}\\ &\le r^{4s}D \qquad \text{by Lemma 2.6.}\end{aligned}$$

Since D is maximal, we have

$$\mathrm{disc}^{(p)}(z_2) = \mathrm{disc}^{(p)}(M_{b',d'}r^s z_0) \le D. \tag{2.8}$$

Write $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ and $M_{b',d'} = \left(\begin{smallmatrix}a' & b'\\ 0 & d'\end{smallmatrix}\right)$. We then note from the proof of Lemma 2.6 that $X := \left(\begin{smallmatrix}a' & b'\\ 0 & d'\end{smallmatrix}\right)r^s z_0 = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)^{-1} z_2$ is a root of

$$P(z) = a(d')^2z^2 + d'(-2ab' + a'r^sb)z + a(b')^2 - ba'b'r^s + (a')^2r^{2s}c \tag{2.9}$$

and $\mathrm{disc}(P(z)) = r^{4s}D$. Let $Q(z)$ be a minimal polynomial of z_2 in level p. Then by Lemma 2.4(i), $Q(z)|_{-2}M$ is a minimal polynomial of $X = M^{-1}z_2$ in level p, and hence $P(z) = mQ(z)|_{-2}M$ for some $m \in \mathbb{N}$. It then follows from (2.8) that

$$\begin{aligned}r^{4s}D = \mathrm{disc}(P(z)) = m^2\mathrm{disc}(Q(z)|_{-2}M) = m^2\mathrm{disc}(Q) = m^2\mathrm{disc}^{(p)}(z_2)\\ \le m^2D,\end{aligned}$$

which gives

$$m \ge r^{2s}. \tag{2.10}$$

If we write

$$\begin{aligned}P(z) &=: pAz^2 + Bz + C\\ &= \gcd(A,B,C)\Big(p\frac{A}{\gcd(A,B,C)}z^2 + \frac{B}{\gcd(A,B,C)}z + \frac{C}{\gcd(A,B,C)}\Big),\end{aligned} \tag{2.11}$$

then $m = \gcd(A,B,C)$. Assume that l is a prime divisor of m and $l \ne r$. Comparing (2.9) and (2.11) we have

$$A = \frac{a}{p}(d')^2, \quad B = d'(-2ab' + a'r^sb), \quad C = a(b')^2 - ba'b'r^s + (a')^2r^{2s}c. \tag{2.12}$$

Combining (2.12) and the condition that l ($\ne r$) is a common divisor of A, B, and C we find that $l \mid \gcd(a/p, b, c) = 1$, which is impossible. Thus $m = r^{t_0}$ for some $t_0 \ge 1$, and the inequality (2.10) renders $2s \le t_0$, so that r^{2s} is a common divisor of A, B, and C. We then have $r^{2s} \mid A$ ($= (a/p)(d')^2$). Moreover since $r \nmid a/p$, we should have $r^{2s} \mid (d')^2$. Then we deduce from the inequality $d' \le r^s$ that $d' = r^s$, and hence $a' = 1$. So the condition $B/r^{2s} = -2ab'/r^s + ba' \in \mathbb{Z}$ implies that $r^s \mid b'$ since r is odd and $r \nmid a$. Finally, since $0 \le b' < r^s$, we have $b' = 0$. Thus one direction of (2.7) is verified.

Conversely, suppose $(b', d') = (0, r^s)$. Observing $M_{0,1}W_p = W_p M_{0,r^s}$, in (2.3) one has $\gamma_{0,r^s} = W_p$. Thus in (2.5) we find that $q_{0,r^s}(z) = q(z)$, so

$$q_{0,r^s}(z)|_{2k} M_{0,r^s} = q(z)|_{2k}\left(\begin{smallmatrix} 1 & 0 \\ 0 & r^s \end{smallmatrix}\right) = r^{-sk} q(z/r^s)$$

has a pole at $z = r^s z_0$, as desired. Thus the claim that $r^s z_0$ is a pole of $\widehat{T}_{2k,r^s}(q(z))$ is proved and Theorem 1.1 is established in the case $n = r^s$.

Step 2: *$n = 2^s$ and $p = 3$.* Following the same arguments of Step 1, we have $a' = 1$ and $d' = 2^s$. And we have only to show that $b' = 0$. Since $B/2^{2s} = -2ab'/2^s + ba' \in \mathbb{Z}$ and $2 \nmid a$, we have $2^{s-1} \mid b'$. Since $0 \le b' < 2^s$, we infer that $b' = 0$ or $b' = 2^{s-1}$. Suppose $b' = 2^{s-1}$. Since we know that

$$C = a(b')^2 - ba'b'2^s + (a')^2 2^{2s} c = a \cdot 2^{2s-2} - b \cdot 2^{2s-1} + 2^{2s} c$$

is divisible by 2^{2s}, we must have $2^{2s-1} \mid a \cdot 2^{2s-2}$, which contradicts $2 \nmid a$. Thus b' should be zero.

Step 3: *$n > 1$ and* $\gcd(n, p) = 1$. Let $n = r_t^{s_t} r_{t-1}^{s_{t-1}} \cdots r_1^{s_1}$ where $r_1, \dots, r_t$ are distinct primes different from p and $s_1, \dots, s_t$ are positive integers. By the multiplicative property of the operator $\widehat{T}_{2k,n}$, one has

$$\widehat{T}_{2k,n}(q(z)) = \widehat{T}_{2k,r_t^{s_t}} \circ \widehat{T}_{2k,r_{t-1}^{s_{t-1}}} \circ \cdots \circ \widehat{T}_{2k,r_1^{s_1}}(q(z)).$$

Let $h_0(z) := q(z)$ and $h_i(z)$ be the rational period function given by

$$h_i(z) := \widehat{T}_{2k,r_i^{s_i}} \circ \widehat{T}_{2k,r_{i-1}^{s_{i-1}}} \circ \cdots \circ \widehat{T}_{2k,r_1^{s_1}}(q(z))$$

for $i = 1, \dots, t$. For each i, $h_i(z)$ has a quadratic irrational pole x_i with the property that for any quadratic irrational pole z_{ρ_i} of $h_{i-1}(z)$,

$$\mathrm{disc}^{(p)}(x_i) > \mathrm{disc}^{(p)}(z_{\rho_i}).$$

In particular, $h_t(z) = \widehat{T}_{2k,n}(q(z))$ has a quadratic irrational pole which is not a quadratic irrational pole of $q(z)$. Thus $q(z)$ is not an eigenfunction for $\widehat{T}_{2k,n}$ for any n with $p \nmid n$.

3. Proof of Theorem 1.2. For $Q = [a, b, c] \in \mathcal{Q}_{D,2}$, we let

$$x_Q = \frac{b + \sqrt{D}}{2a} \quad \text{and} \quad x_Q^\sigma = \frac{b - \sqrt{D}}{2a}.$$

We note that x_Q and x_Q^σ are the two roots of the polynomial $Q(x, -1)$. We call $Q = [a, b, c] \in \mathcal{Q}_{D,2}$ *simple* if $a > 0 > c$.

Lemma 3.1. *Let $\mathcal{B}$ be a narrow equivalence class of quadratic forms in $\mathcal{Q}_{D,2}$. Let $\mathcal{B}^s$ denote the subset of $\mathcal{B}$ consisting of simple forms in $\mathcal{B}$. Define*

$$f_1([a, b, c]) := [-2(a - b + c), b - 2a, -a/2],$$
$$f_2([a, b, c]) := [a - 2b + 4c, 2a - 3b + 4c, a - b + c].$$

Then:

(i) *The map* f_1 *gives a bijection between the set of reduced forms in the class* $\theta\mathcal{B}'$ *and the set* $\{Q \in \mathcal{B}^s \mid x_Q > 1/2\}$.

(ii) *The map* f_2 *gives a bijection between*

$$\{Q = [a, b, c] \in \mathcal{B} \mid Q \textit{ is reduced and } a - 2b + 4c > 0\}$$

and $\{Q \in \mathcal{B}^s \mid x_Q < 1/2\}$.

Proof. (i) Let $Q = [a, b, c]$ be a reduced form in $\theta\mathcal{B}'$. We then have a simple form

$$Q_1 := Q|\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) = [a_1, b_1, c_1] \in \theta\mathcal{B}'$$

with $a_1 = a > 0, b_1 = b - 2a, c_1 = c - b + a < 0$. If we set $f(x) := Q_1(x, -1)$, then we see that $f(0) < 0$ and $f(-1) > 0$, from which it follows that $-1 < x^\sigma_{Q_1} < 0$ and $x_{Q_1} > 0$. Next let

$$Q_2 := Q_1|W_2 = [2c_1, -b_1, a_1/2] = [2(c - b + a), -(b - 2a), a/2] \in \theta\mathcal{B}',$$
$$Q_3 := -Q_2 = [-2(c - b + a), b - 2a, -a/2] \in \mathcal{B}.$$

We compute that $x_{Q_2} = -1/(2x_{Q_1}) < 0$, $x^\sigma_{Q_2} = -1/(2x^\sigma_{Q_1}) > 1/2$, $x_{Q_3} = x^\sigma_{Q_2} > 1/2$, and $x^\sigma_{Q_3} = x_{Q_2} < 0$. We then observe that $Q_3 = f_1(Q)$ belongs to the set $\{Q \in \mathcal{B}^s \mid x_Q > 1/2\}$.

Conversely, given a simple form $Q_s = [A, B, C]$ in $\mathcal{B}$ with $x_{Q_s} > 1/2$, we observe that x_{Q_s} and $x^\sigma_{Q_s}$ are the roots of the polynomial $f_s(x) := Q_s(x, -1)$. Since $f_s(1/2) < 0$ and $f_s(0) < 0$, we find that $A - 2B + 4C < 0$ and $C < 0$, respectively. We then see that

$$(-Q_s)|W_2\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = Q_s|\left(\begin{smallmatrix} 0 & -1/\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{smallmatrix}\right) = [-2C, B - 4C, -A/2 + B - 2C]$$

defines a reduced form in $\theta\mathcal{B}'$.

(ii) Let $Q = [a, b, c]$ be a reduced form in $\mathcal{B}$ satisfying $a - 2b + 4c > 0$. Define

$$Q_1 := Q|W_2 = [2c, -b, a/2],$$
$$Q_2 := Q_1|\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = [2c, 4c - b, 2c - b + a/2],$$
$$Q_3 := Q_2|W_2 = [4c - 2b + a, b - 4c, c],$$
$$Q_4 := Q_3|\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = [a - 2b + 4c, 2a - 3b + 4c, a - b + c].$$

Then we observe that $a - 2b + 4c > 0$ and $a - b + c < 0$. If we let $g(x) := Q_4(x, -1)$, then $g(0) < 0$ and $g(1/2) > 0$, from which we see that $x_{Q_4} < 1/2$. Thus $Q_4 = f_2(Q)$ belongs to the set $\{Q \in \mathcal{B}^s \mid x_Q < 1/2\}$.

Conversely, given a simple form $Q_s = [A, B, C]$ in $\mathcal{B}$ with $0 < x_{Q_s} < 1/2$, we note that x_{Q_s} and $x^\sigma_{Q_s}$ are the two roots of the polynomial $f_s(x) := Q_s(x, -1)$. Since $f_s(1/2) > 0$, $f_s(0) < 0$, and $f_s(1) > 0$, we find that $A-2B+4C>0$, $C<0$, $A-B+C>0$, respectively. We then observe that

$$Q_s|(W_2\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)W_2\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right))^{-1}=Q_s|\left(\begin{smallmatrix}1&1\\-2&-1\end{smallmatrix}\right)$$
$$=[A-2B+4C,2A-3B+4C,A-B+C]$$

defines a reduced form in $\mathcal{B}$ with $A-2B+4C>0$. ∎

LEMMA 3.2. *We have*

$$q_{\mathcal{A}}|_{2k}(W_2+1)=0, \tag{3.1}$$
$$R_{\mathcal{A}}(z)|_{2k}U^2=-R_{\mathcal{A}}(z). \tag{3.2}$$

Proof. Since

$$q_{\mathcal{A}}(z)=\sum_{\substack{a>0>c\\ [a,b,c]\in\mathcal{A}}}\frac{1}{(az^2-bz+c)^k}-\sum_{\substack{a<0<c\\ [a,b,c]\in\mathcal{A}}}\frac{1}{(az^2-bz+c)^k},$$

we have

$$\begin{aligned}q_{\mathcal{A}}|_{2k}W_2&=\sum_{\substack{a>0>c\\ [a,b,c]\in\mathcal{A}}}\frac{1}{(2cz^2+bz+a/2)^k}-\sum_{\substack{a<0<c\\ [a,b,c]\in\mathcal{A}}}\frac{1}{(2cz^2+bz+a/2)^k}\\&=\sum_{\substack{C>0>A\\ [A,B,C]\in\mathcal{A}}}\frac{1}{(Az^2-Bz+C)^k}-\sum_{\substack{C<0<A\\ [A,B,C]\in\mathcal{A}}}\frac{1}{(Az^2-Bz+C)^k}\\&\quad(\text{since } [a,b,c]\circ W_2=[2c,-b,a/2]=:[A,B,C]\in\mathcal{A}\text{ for }[a,b,c]\in\mathcal{A})\\&=-q_{\mathcal{A}}.\end{aligned}$$

This proves the identity (3.1). Next we observe that

$$\begin{aligned}R_{\mathcal{A}}(z)&=\sum_{\substack{[a,b,c]\in\theta\mathcal{A}'\\ [a,b,c]\text{ reduced}\\ a-2b+4c<0}}(-1)^k\frac{1}{(az^2-bz+c)^k}-\sum_{\substack{[a,b,c]\in\mathcal{A}\\ [a,b,c]\text{ reduced}\\ a-2b+4c<0}}\frac{1}{(az^2-bz+c)^k}\\&=\sum_{\substack{[a,b,c]\in\mathcal{A}\\ [-a,-b,-c]\text{ reduced}\\ a-2b+4c>0}}\frac{1}{(az^2-bz+c)^k}-\sum_{\substack{[a,b,c]\in\mathcal{A}\\ [a,b,c]\text{ reduced}\\ a-2b+4c<0}}\frac{1}{(az^2-bz+c)^k}\\&=\sum_{\substack{[a,b,c]\in\mathcal{A}\\ ac>0\\ \operatorname{sgn}(c)(a-b+c)<0\\ \operatorname{sgn}(c)(a-2b+4c)<0}}(-\operatorname{sgn}(c))\frac{1}{(az^2-bz+c)^k}\end{aligned}$$

and

$$R_{\mathcal{A}}(z)|_{2k}U^2 = R_{\mathcal{A}}(z)|_{2k}\left(\begin{smallmatrix}1 & -1\\ 2 & -1\end{smallmatrix}\right)$$

$$= - \sum_{\substack{[a,b,c]\in\mathcal{A}\\ ac>0\\ \mathrm{sgn}(c)(a-b+c)<0\\ \mathrm{sgn}(c)(a-2b+4c)<0}} \frac{\mathrm{sgn}(c)(2z-1)^{-2k}}{\left(a\left(\frac{z-1}{2z-1}\right)^2 - b\left(\frac{z-1}{2z-1}\right) + c\right)^k}$$

$$= - \sum_{\substack{[a,b,c]\in\mathcal{A}\\ ac>0\\ \mathrm{sgn}(c)(a-b+c)<0\\ \mathrm{sgn}(c)(a-2b+4c)<0}} \frac{\mathrm{sgn}(c)}{(z^2(a-2b+4c) + z(-2a+3b-4c) + (a-b+c))^k}.$$

We note that $\mathcal{A} = \circ\left(\begin{smallmatrix}1 & 1\\ -2 & -1\end{smallmatrix}\right)$. For each $[A,B,C]\in\mathcal{A}$, we observe that

$$Q' := [A,B,C] = [a,b,c]\circ\left(\begin{smallmatrix}1 & 1\\ -2 & -1\end{smallmatrix}\right) = [a-2b+4c, 2a-3b+4c, a-b+c]$$

for some $[a,b,c]\in\mathcal{A}$. Then $a = A+4C-2B, b = 2A-3B+4C$ and $c = A+C-B$. This means that

$$R_{\mathcal{A}}(z)|_{2k}U^2$$

$$= - \sum_{\substack{[a,b,c]\in\mathcal{A}\\ ac>0\\ \mathrm{sgn}(c)(a-b+c)<0\\ \mathrm{sgn}(c)(a-2b+4c)<0}} \frac{\mathrm{sgn}(c)}{(z^2(a-2b+4c) + z(-2a+3b-4c) + (a-b+c))^k}$$

$$= - \sum_{\substack{[A,B,C]\in\mathcal{A}\\ AC>0\\ \mathrm{sgn}(C)(A+C-B)<0\\ \mathrm{sgn}(C)(A-2B+4C)<0}} \frac{-\mathrm{sgn}(C)}{(Az^2-Bz+C)^k} = -R_{\mathcal{A}}(z). \blacksquare$$

We are now ready to prove Theorem 1.2. For any narrow equivalence class $\mathcal{B}$, we have

$$Q_{k,D,\mathcal{B}}|_{2k}U = Q_{k,D,\mathcal{B}}|_{2k}\left(\begin{smallmatrix}\sqrt{2} & -1/\sqrt{2}\\ \sqrt{2} & 0\end{smallmatrix}\right)$$

$$= \sum_{\substack{[a,b,c]\in\mathcal{B}\\ [a,b,c]\ \text{reduced}}} \frac{(-1)^k}{(z^2(-2a+2b-2c) + z(2a-b) - a/2)^k}$$

and

$$Q_{k,D,\mathcal{B}}|_{2k}U^2 = Q_{k,D,\mathcal{B}}|_{2k}\left(\begin{smallmatrix}1 & -1\\ 2 & -1\end{smallmatrix}\right)$$

$$= \sum_{\substack{[a,b,c]\in\mathcal{B}\\ [a,b,c]\ \text{reduced}}} \frac{1}{(z^2(a-2b+4c) + z(-2a+3b-4c) + (a-b+c))^k}.$$

Therefore

$$
\begin{aligned}
&(Q_{k,D,\mathcal{A}}(z) - (-1)^k Q_{k,D,\theta\mathcal{A}'}(z))|_{2k}(-U+U^2) \\
&\quad = \sum_{\substack{[a,b,c]\in\mathcal{A} \\ [a,b,c] \text{ reduced}}} \frac{-(-1)^k}{(z^2(-2a+2b-2c)+z(2a-b)-a/2)^k} \\
&\qquad + \sum_{\substack{[a,b,c]\in\theta\mathcal{A}' \\ [a,b,c] \text{ reduced}}} \frac{1}{(z^2(-2a+2b-2c)+z(2a-b)-a/2)^k} \\
&\qquad + \sum_{\substack{[a,b,c]\in\mathcal{A} \\ [a,b,c] \text{ reduced}}} \frac{1}{(z^2(a-2b+4c)+z(-2a+3b-4c)+(a-b+c))^k} \\
&\qquad + \sum_{\substack{[a,b,c]\in\theta\mathcal{A}' \\ [a,b,c] \text{ reduced}}} \frac{-(-1)^k}{(z^2(a-2b+4c)+z(-2a+3b-4c)+(a-b+c))^k}.
\end{aligned}
$$

By taking $\mathcal{A} = \theta\mathcal{B}'$ (equivalently $\mathcal{B} = \theta\mathcal{A}'$) this can be written as

$$
\begin{aligned}
&(Q_{k,D,\mathcal{A}}(z) - (-1)^k Q_{k,D,\theta\mathcal{A}'}(z))|_{2k}(-U+U^2) \\
&\quad = \sum_{\substack{[a,b,c]\in\theta\mathcal{B}' \\ [a,b,c] \text{ reduced}}} \frac{-(-1)^k}{(z^2(-2a+2b-2c)+z(2a-b)-a/2)^k} \\
&\qquad + \sum_{\substack{[a,b,c]\in\theta\mathcal{A}' \\ [a,b,c] \text{ reduced}}} \frac{1}{(z^2(-2a+2b-2c)+z(2a-b)-a/2)^k} \\
&\qquad + \sum_{\substack{[a,b,c]\in\mathcal{A} \\ [a,b,c] \text{ reduced} \\ a-2b+4c>0}} \frac{1}{(z^2(a-2b+4c)+z(-2a+3b-4c)+(a-b+c))^k} \\
&\qquad + \sum_{\substack{[a,b,c]\in\mathcal{A} \\ [a,b,c] \text{ reduced} \\ a-2b+4c<0}} \frac{1}{(z^2(a-2b+4c)+z(-2a+3b-4c)+(a-b+c))^k} \\
&\qquad + \sum_{\substack{[a,b,c]\in\mathcal{B} \\ [a,b,c] \text{ reduced} \\ a-2b+4c>0}} \frac{-(-1)^k}{(z^2(a-2b+4c)+z(-2a+3b-4c)+(a-b+c))^k} \\
&\qquad + \sum_{\substack{[a,b,c]\in\theta\mathcal{A}' \\ [a,b,c] \text{ reduced} \\ a-2b+4c<0}} \frac{-(-1)^k}{(z^2(a-2b+4c)+z(-2a+3b-4c)+(a-b+c))^k}.
\end{aligned}
$$

It then follows from Lemma 3.1 that

$$\begin{aligned}
&(Q_{k,D,\mathcal{A}}(z) - (-1)^k Q_{k,D,\theta\mathcal{A}'}(z))|_{2k}(-U+U^2) \\
&= \sum_{[a,b,c]\in\mathcal{B}^s} \frac{-(-1)^k}{(az^2-bz+c)^k} + \sum_{[a,b,c]\in\mathcal{A}^s} \frac{1}{(az^2-bz+c)^k} \\
&\quad + \sum_{\substack{[a,b,c]\in\mathcal{A} \\ [a,b,c]\text{ reduced} \\ a-2b+4c<0}} \frac{1}{(z^2(a-2b+4c)+z(-2a+3b-4c)+(a-b+c))^k} \\
&\quad + \sum_{\substack{[a,b,c]\in\theta\mathcal{A}' \\ [a,b,c]\text{ reduced} \\ a-2b+4c<0}} \frac{-(-1)^k}{(z^2(a-2b+4c)+z(-2a+3b-4c)+(a-b+c))^k} \\
&= \sum_{[a,b,c]\in\mathcal{B}^s} \frac{-(-1)^k}{(az^2-bz+c)^k} + \sum_{[a,b,c]\in\mathcal{A}^s} \frac{1}{(az^2-bz+c)^k} \\
&\quad + \sum_{\substack{[a,b,c]\in\mathcal{A} \\ [a,b,c]\text{ reduced} \\ a-2b+4c<0}} \frac{1}{(az^2-bz+c)^k}|_{2k}U^2 + \sum_{\substack{[a,b,c]\in\theta\mathcal{A}' \\ [a,b,c]\text{ reduced} \\ a-2b+4c<0}} \frac{-(-1)^k}{(az^2-bz+c)^k}|_{2k}U^2 \\
&= \sum_{[a,b,c]\in\mathcal{B}^s} \frac{-(-1)^k}{(az^2-bz+c)^k} + \sum_{[a,b,c]\in\mathcal{A}^s} \frac{1}{(az^2-bz+c)^k} - R_{\mathcal{A}}(z)|_{2k}U^2 \\
&= q_{\mathcal{A}}(z) - R_{\mathcal{A}}(z)|_{2k}U^2,
\end{aligned}$$

which proves Theorem 1.2(a). Letting $h(z) = Q_{k,D,\mathcal{A}}(z) - (-1)^k Q_{k,D,\theta\mathcal{A}'}(z)$ we obtain

$$(h|_{2k}(-U+U^2))|_{2k}(U^3+U^2+U+1) = 0.$$

Utilizing (3.2) we observe that

$$(R_{\mathcal{A}}(z)|_{2k}U^2)|_{2k}(U^3+U^2+U+1) = 0.$$

Consequently, $q_{\mathcal{A}}(z)|_{2k}(U^3+U^2+U+1) = 0$, and hence we infer from (3.1) that $q_{\mathcal{A}}(z)$ is a rational period function on $\Gamma_0^+(2)$. Moreover applying Theorem 1.1 we conclude that $q_{\mathcal{A}}(z)$ is not an eigenfunction of $\widehat{T}_{2k,n}$ for any odd integer $n > 1$. This completes the proof of Theorem 1.2(b).

Acknowledgements. We appreciate the referee's valuable comments. We would also like to thank KIAS (Korea Institute for Advanced Study) for its hospitality.

Choi was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of

Education (2014R1A1A2054915). Kim was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (NRF-2015R1D1A1A01057428 and 2016R1A5A1008055).

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