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The spherical dual transform is an isometry for spherical Wulff shapes

by

Huhe Han (Yangling) and Takashi Nishimura (Yokohama)

Abstract. A spherical Wulff shape is the spherical counterpart of a Wulff shape which is the well-known geometric model of a crystal at equilibrium introduced by G. Wulff in 1901. Just as a Wulff shape, each spherical Wulff shape has its unique dual. The spherical dual transform for spherical Wulff shapes is the mapping which maps a spherical Wulff shape to its spherical dual Wulff shape. In this paper, it is shown that the spherical dual transform for spherical Wulff shapes is an isometry with respect to the Pompeiu–Hausdorff metric.

1. Introduction. Throughout this paper, let n (resp., S^n) be a positive integer (resp., the unit sphere in $\mathbb{R}^{n+1}$). For any point $P \in S^n$, let $H(P)$ be the closed hemisphere centered at P, that is, the set of $Q \in S^n$ satisfying $P \cdot Q \geq 0$, where the dot stands for the scalar product of vectors in $\mathbb{R}^{n+1}$. For any non-empty subset $W \subset S^n$, the *spherical polar set* of W, denoted by W°, is defined as follows:

$$W^\circ = \bigcap_{P \in W} H(P).$$

In [NS], the spherical polar set plays an essential role for investigating a Wulff shape, which is the geometric model of a crystal at equilibrium introduced by G. Wulff [W].

Let $\mathcal{H}(S^n)$ be the set of non-empty closed subsets of S^n. It is well-known that $\mathcal{H}(S^n)$ is a complete metric space with respect to the Pompeiu–Hausdorff metric. Let $\mathcal{H}^\circ(S^n)$ be the subspace of $\mathcal{H}(S^n)$ consisting of non-empty closed subsets W of S^n such that $W^\circ \neq \emptyset$. The *spherical polar transform* $\bigcirc : \mathcal{H}^\circ(S^n) \to \mathcal{H}^\circ(S^n)$ is defined by $\bigcirc(W) = W^\circ$. Since $W \subset W^{\circ\circ}$

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for any $W \in \mathcal{H}^\circ(S^n)$ by [NS, Lemma 2.2], it follows that $W^\circ \in \mathcal{H}^\circ(S^n)$ for any $W \in \mathcal{H}^\circ(S^n)$. Thus, the spherical polar transform $\bigcirc$ is well-defined (1).

In [KN], crystal growth is investigated by introducing a geometric model of a certain growing crystal in $\mathbb{R}^2$. One of the powerful tools in [KN] is the spherical polar transform $\bigcirc : \mathcal{H}^\circ(S^2) \to \mathcal{H}^\circ(S^2)$. Especially, for studying the dissolving process of the geometric model introduced in [KN], the spherical polar transform is indispensable since it enables one to analyze in detail the image of a dissolving one-parameter family of spherical Wulff shapes. Hence, it is important to establish the properties of the spherical polar transform.

In this paper, motivated by the above considerations, we investigate natural restrictions of the spherical polar transform. The most natural subspace for the restriction of spherical polar transform is $\mathcal{H}_{\mathrm{Wulff}}(S^n, P)$ defined as follows.

Definition 1.

(1) Let W be a subset of S^n. Suppose that there exists a point $P \in S^n$ such that $W \cap H(P) = \emptyset$. Then W is said to be *hemispherical.*

(2) Let $W \subset S^n$ be a hemispherical subset. Let $P, Q \in W$. Then we denote by PQ
$$PQ = \left\{ \frac{(1-t)P + tQ}{\|(1-t)P + tQ\|} \in S^n \;\middle|\; 0 \le t \le 1 \right\}.$$

(3) Let $W \subset S^n$ be a hemispherical subset. Suppose that $PQ \subset W$ for any $P, Q \in W$. Then W is said to be *spherical convex.*

(4) Let $W \subset S^n$ be a hemispherical subset. Suppose that W is closed, spherical convex and has an interior point. Then W is said to be a *spherical convex body.*

(5) For any $P \in S^n$, let
$$\mathcal{H}_{\mathrm{Wulff}}(S^n, P) = \{W \in \mathcal{H}(S^n) \mid W \cap H(-P) = \emptyset,\ P \in \mathrm{int}(W),\ W \text{ is a spherical convex body}\},$$
where int stands for interior. The closure of $\mathcal{H}_{\mathrm{Wulff}}(S^n, P)$ in $\mathcal{H}(S^n)$ is denoted by $\overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$.

(6) For any $P \in S^n$, an element of $\mathcal{H}_{\mathrm{Wulff}}(S^n, P)$ is called a *spherical Wulff shape.*

It is known that a Wulff shape in $\mathbb{R}^n$ can be characterized as a convex body in $\mathbb{R}^n$ with the origin in its interior [T]. Hence, the definition of spherical Wulff shape is reasonable. The restriction of $\bigcirc$ to $\mathcal{H}_{\mathrm{Wulff}}(S^n, P)$ (resp., $\overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$) is called the *spherical dual transform relative to* P (resp., the *extended spherical dual transform relative to* P) and is denoted by $\bigcirc_{\mathrm{Wulff},P}$ (resp., $\overline{\bigcirc_{\mathrm{Wulff},P}}$). The set $\bigcirc(W) = W^\circ$ is called the *spherical*

(1) Since $(S^n)^\circ = \emptyset$ for any $n \in \mathbb{N}$, the spherical polar transform defined in [KN] should be understood as $\bigcirc : \mathcal{H}^\circ(S^n) \to \mathcal{H}^\circ(S^n)$.

dual Wulff shape of W if W is a spherical Wulff shape. Thus, it is reasonable to call $\bigcirc_{\mathrm{Wulff},P}$ the spherical dual transform. It is not difficult to prove the following (Proposition 5.2):

$$\bigcirc_{\mathrm{Wulff},P} : \mathcal{H}_{\mathrm{Wulff}}(S^n, P) \to \mathcal{H}_{\mathrm{Wulff}}(S^n, P) \text{ is well-defined and bijective,}$$
$$\overline{\bigcirc_{\mathrm{Wulff},P}} : \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)} \to \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)} \text{ is well-defined and bijective.}$$

The main purpose of this paper is to show the following:

THEOREM 1. *Let $P \in S^n$. Then, with respect to the Pompeiu–Hausdorff metric, the following hold:*

(1) *The spherical dual transform relative to P,*

$$\bigcirc_{\mathrm{Wulff},P} : \mathcal{H}_{\mathrm{Wulff}}(S^n, P) \to \mathcal{H}_{\mathrm{Wulff}}(S^n, P),$$

is an isometry.

(2) *The extended spherical dual transform relative to P,*

$$\overline{\bigcirc_{\mathrm{Wulff},P}} : \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)} \to \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)},$$

is an isometry.

For any $r > 0$, let D_r be the set of $x \in \mathbb{R}^n$ satisfying $\|x\| \leq r$. Then D_r is a Wulff shape for any $r \in \mathbb{R}$ $(r > 0)$, and it is well-known that the dual Wulff shape of D_r is $D_{1/r}$. Moreover, it is easily seen that $h(D_{r_1}, D_{r_2}) = |r_1 - r_2|$ for any $r_1, r_2 > 0$, where h is the Pompeiu–Hausdorff metric. Thus, one cannot expect the Euclidean counterpart of assertion (1) of Theorem 1. This is an advantage of studying the spherical version of Wulff shapes. Moreover, the Euclidean counterpart of the extended of spherical dual transform relative to P is not well-defined. This, too, is an advantage of studying the spherical version of Wulff shapes.

Next, we investigate the restriction of $\bigcirc$ to $\overline{\mathcal{H}_{\text{s-conv}}(S^n)}$, the closure of the set of spherical convex closed subsets. The restriction of $\bigcirc$ to $\overline{\mathcal{H}_{\text{s-conv}}(S^n)}$ is denoted by $\bigcirc_{\text{s-conv}}$. It is not hard to see that (Proposition 5.2)

$$\bigcirc_{\text{s-conv}} : \overline{\mathcal{H}_{\text{s-conv}}(S^n)} \to \overline{\mathcal{H}_{\text{s-conv}}(S^n)} \text{ is well-defined and bijective.}$$

THEOREM 2. *With respect to the Pompeiu–Hausdorff metric, the restriction of the spherical polar transform*

$$\bigcirc_{\text{s-conv}} : \overline{\mathcal{H}_{\text{s-conv}}(S^n)} \to \overline{\mathcal{H}_{\text{s-conv}}(S^n)}$$

is bi-Lipschitz but never an isometry.

This paper is organized as follows. In Section 2, preliminaries for the proofs of Theorems 1 and 2 are given. Theorems 1 and 2 are proved in Sections 3 and 4 respectively. Section 5 is an appendix where, for the readers' convenience, it is proved that $\bigcirc_{\mathrm{Wulff},P}$, $\overline{\bigcirc_{\mathrm{Wulff},P}}$ and $\bigcirc_{\text{s-conv}}$ are all well-defined bijective mappings; and moreover it is explained why the restriction of $\bigcirc$ to $\overline{\mathcal{H}_{\text{s-conv}}(S^n)}$ is important.

2. Preliminaries

2.1. Convex geometry in S^n

Definition 2 ([NS]). Let W be a hemispherical subset of S^n. The *spherical convex hull of W* is

$$\text{s-conv}(W) = \left\{ \frac{\sum_{i=1}^{k} t_i P_i}{\| \sum_{i=1}^{k} t_i P_i \|} \;\middle|\; P_i \in W,\ \sum_{i=1}^{k} t_i = 1,\ t_i \geq 0,\ k \in \mathbb{N} \right\}.$$

Lemma 2.1 ([NS]). *Let W_1, W_2 be non-empty subsets of S^n with $W_1 \subset W_2$. Then $W_2^\circ \subset W_1^\circ$.*

Lemma 2.2 ([NS]). *For any non-empty closed hemispherical subset $X \subset S^n$,*

$$\text{s-conv}(X) = (\text{s-conv}(X))^{\circ\circ}.$$

The following proposition may be regarded as a spherical version of the separation theorem, which may be easily obtained from the separation theorem in Euclidean space (for the latter, see for instance [M]).

Proposition 2.1. *Let $P \in S^n$ and let W_1, W_2 be closed, disjoint spherical convex sets such that $W_i \cap H(P) = \emptyset$ $(i = 1, 2)$. Then there exists $Q \in S^n$ satisfying*

$$W_1 \subset H(Q) \quad \text{and} \quad W_2 \cap H(Q) = \emptyset.$$

2.2. Metric geometry in S^n. For any $P, Q \in S^n$, the length of the arc PQ is denoted by $|PQ|$.

Lemma 2.3. *For any $P, Q \in S^n$ such that $|PQ| \leq \pi/2$,*

$$h(H(P), H(Q)) = |PQ|.$$

Proof. By Figure 1, it is clear that $h(H(P), H(Q)) + r = |PQ| + r = \pi/2$, so $h(H(P), H(Q)) = |PQ|$. ∎

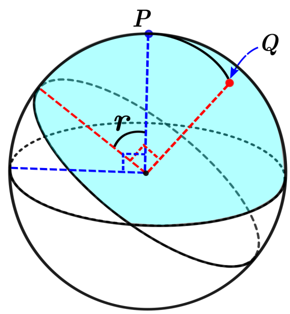

Fig. 1. $h(H(P), H(Q)) = |PQ|$

Definition 3.

(1) For any $P \in S^n$ and any $0 < r < \pi$, define
$$\overline{B(P,r)} = \{Q \in S^n \mid |PQ| \le r\}, \qquad \partial\overline{B(P,r)} = \{Q \in S^n \mid |PQ| = r\}.$$
(2) For any non-empty $W \subset S^n$ and any $0 < r < \pi$, define
$$\overline{B(W,r)} = \bigcup_{P \in W} \overline{B(P,r)}.$$

Lemma 2.4. *For any $W \subset S^n$ such that W° is a spherical convex set and any $0 < r < \pi/2$,*
$$\overline{B\Big(\bigcap_{P \in W} H(P), r\Big)} = \bigcap_{P \in W} \overline{B(H(P),r)}.$$

Proof. "$\subset$" Let $Q \in \overline{B(\bigcap_{P\in W} H(P), r)}$. Then $\overline{B(Q,r)} \cap \bigcap_{P \in W} H(P) \ne \emptyset$. Let $Q_1 \in \overline{B(Q,r)}$ be such that $Q_1 \in H(P)$ for any $P \in W$. Then $Q \in \bigcap_{P\in W} \overline{B(H(P),r)}$.

"$\supset$" Suppose there exists $Q \in \bigcap_{P\in W} \overline{B(H(P),r)} \setminus \overline{B(\bigcap_{P\in W} H(P),r)}$. Since $Q \notin \overline{B(\bigcap_{P\in W} H(P),r)} = \overline{B(W^\circ,r)}$, it follows that $\overline{B(Q,r)} \cap W^\circ = \emptyset$. Since W° and $\overline{B(Q,r)}$ are closed spherical convex sets, by Proposition 2.1 there exists $P \in S^n$ such that $W^\circ \subset H(P)$ and $\overline{B(Q,r)} \cap H(P) = \emptyset$. By Lemmas 2.1 and 2.2, it follows that $P \in W^{\circ\circ} = W$. Therefore, there exists $P \in W$ such that $Q \notin \overline{B(H(P),r)}$, contrary to assumption. ■

2.3. Lipschitz mappings

Proposition 2.2. *For any $n \in \mathbb{N}$, the spherical polar transform $\bigcirc : \mathcal{H}^\circ(S^n) \to \mathcal{H}^\circ(S^n)$ is Lipschitz with respect to the Pompeiu–Hausdorff metric.*

Proof. We first show that $\bigcirc$ is Lipschitz when restricted to sets W such that W° is a spherical convex set. Suppose it is not Lipschitz. Then for any $K > 0$ there exist $W_1, W_2 \in \mathcal{H}^\circ(S^n)$ with W_1°, W_2° spherical convex such that $Kh(W_1,W_2) < h(W_1^\circ, W_2^\circ)$. In particular, for $K = 2$ there exist $W_1, W_2 \in \mathcal{H}^\circ(S^n)$ with W_1°, W_2° spherical convex such that $2h(W_1,W_2) < h(W_1^\circ, W_2^\circ)$. Since $h(X,Y) \le \pi$ for any $X, Y \in \mathcal{H}^\circ(S^n)$, it follows that $h(W_1,W_2) < \pi/2$. Set $r = h(W_1,W_2)$. Then, since $2r < h(W_1^\circ, W_2^\circ)$, from the definition of the Pompeiu–Hausdorff metric, at least one of the following holds:

(1) There exists $P \in W_1^\circ$ such that $d(P,Q) > 2r$ for any $Q \in W_2^\circ$.
(2) There exists $Q \in W_2^\circ$ such that $d(Q,P) > 2r$ for any $P \in W_1^\circ$.

We will show that (1) implies a contradiction. Suppose that there exists $\widetilde{P} \in W_1^\circ$ such that $\widetilde{P} \notin \overline{B(W_2^\circ, 2r)}$. In particular, $\widetilde{P} \notin \overline{B(W_2^\circ, r)}$. Since W_2°

is a spherical convex set and $r < \pi/2$, by Lemma 2.4 we have

$$\widetilde{P} \notin \overline{B(W_2^\circ, r)} = \overline{B\Big(\bigcap_{Q\in W_2} H(Q), r\Big)} = \bigcap_{Q\in W_2} \overline{B(H(Q), r)}.$$

Hence, there exists $Q \in W_2$ such that $\widetilde{P} \notin \overline{B(H(Q), r)}$.

On the other hand, since $h(W_1, W_2) = r$, there exists $P_Q \in W_1$ such that $d(P_Q, Q) \le r$. Thus, by Lemma 2.3, $\widetilde{P} \in H(P_Q) \subset \overline{B(H(Q), r)}$, a contradiction.

In the same way, we can show that (2) implies a contradiction.

Next we show that for any $W, \widetilde{W} \in \mathcal{H}^\circ(S^n)$, we have $h(W^\circ, \widetilde{W}^\circ) \le 2h(W, \widetilde{W})$. Since $W, \widetilde{W} \in \mathcal{H}^\circ(S^n)$, there exist $P, \widetilde{P} \in S^n$ such that $W \subset H(P)$ and $\widetilde{W} \subset H(\widetilde{P})$. Set

$$W_i = \overline{B(W, 1/i)} \cap \overline{B(H(P), \pi/2 - 1/i)},$$
$$\widetilde{W}_i = \overline{B(\widetilde{W}, 1/i)} \cap \overline{B(H(\widetilde{P}), \pi/2 - 1/i)}$$

for any $i \in \mathbb{N}$. Since both $W_i^\circ, \widetilde{W}_i^\circ$ are black spherical convex, by the proof given above we have $h(W_i^\circ, \widetilde{W}_i^\circ) \le 2h(W_i, \widetilde{W}_i)$ for any $i \in \mathbb{N}$. Now, $W = \lim_{i\to\infty} W_i$ and $\widetilde{W} = \lim_{i\to\infty} \widetilde{W}_i$, so for any $i \in \mathbb{N}$,

$$\begin{aligned} h(W^\circ, \widetilde{W}^\circ) &\le h(W^\circ, W_i^\circ) + h(W_i^\circ, \widetilde{W}_i^\circ) + h(\widetilde{W}_i^\circ, \widetilde{W}^\circ) \\ &\le h(W^\circ, W_i^\circ) + 2h(W_i, \widetilde{W}_i) + h(\widetilde{W}_i^\circ, \widetilde{W}^\circ). \end{aligned}$$

In [KN], it has been shown that $\bigcirc : \mathcal{H}^\circ(S^2) \to \mathcal{H}^\circ(S^2)$ is continuous. It is easily seen that the proof given in [KN] works for all $n \in \mathbb{N}$. Thus, $\lim_{i\to\infty} h(W^\circ, W_i^\circ) = 0 = \lim_{i\to\infty} h(\widetilde{W}^\circ, \widetilde{W}_i^\circ)$. Therefore,

$$h(W^\circ, \widetilde{W}^\circ) \le 2 \lim_{i\to\infty} h(W_i, \widetilde{W}_i) = 2h(W, \widetilde{W}). \ \blacksquare$$

Claim 2.1. *The following example shows that* 2 *is the least Lipschitz constant of* $\bigcirc$.

Example. For any real $r \in (1, 2)$, there exist a real r_1 and $P_1, P_2 \in S^n$ such that $r\pi/2 < r_1 < \pi$ and $d(P_1, P_2) = r_1$. Since $H(P_i) \subset S^n = \overline{B(H(P_j), \pi/2)}$ for $\{i, j\} = \{1, 2\}$, we have $h(H(P_1), H(P_2)) \le \pi/2$. Set $W_1 = H(P_1)$ and $W_2 = H(P_2)$. Then

$$rh(W_1, W_2) \le r\pi/2 < r_1 = d(P_1, P_2) = h(\{P_1\}, \{P_2\}) = h(W_1^\circ, W_2^\circ).$$

It follows (see Figure 2) that

$$rh(W_1, W_2) < h(W_1^\circ, W_2^\circ).$$

By Proposition 2.2, we can extend Lemma 2.2 as follows.

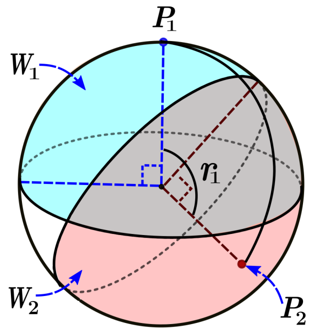

Fig. 2. $rh(W_1, W_2) < h(\{P_1\}, \{P_2\})$ $(1 < r < 2)$

LEMMA 2.5. *For* $X = \lim_{i\to\infty} X_i$, *where* $X_i \in \mathcal{H}_{\text{s-conv}}(S^n)$ $(i = 1, 2, \dots)$, *we have*

$$X = X^{\circ\circ}.$$

Proof. By Proposition 2.2, $\bigcirc \circ \bigcirc$ is Lipschitz. Thus, if $X = \lim_{i\to\infty} X_i$ then $X^{\circ\circ} = \lim_{i\to\infty} X_i^{\circ\circ}$. By Lemma 2.2, $X = \lim_{i\to\infty} X_i = \lim_{i\to\infty} X_i^{\circ\circ} = X^{\circ\circ}$. ∎

3. Proof of Theorem 1. We first show that for any $W_1, W_2 \in \mathcal{H}_{\text{Wulff}}(S^n, P)$,

$$(*) \qquad h(W_1^\circ, W_2^\circ) \le h(W_1, W_2).$$

Suppose that there exist $W_1, W_2 \in \mathcal{H}_{\text{Wulff}}(S^n, P)$ such that $h(W_1, W_2) < h(W_1^\circ, W_2^\circ)$. First, since $W_1, W_2 \in \mathcal{H}_{\text{Wulff}}(S^n, P)$, it follows that $h(W_1, W_2) < \pi/2$. Set $r = h(W_1, W_2)$. As $r < h(W_1^\circ, W_2^\circ)$, it follows that $d(W_1^\circ, W_2^\circ) > r$ or $d(W_2^\circ, W_1^\circ) > r$, where $d(\cdot,\cdot)$ is the distance defined in Section 2. Therefore, at least one of the following holds:

(a) There exists $Q_1 \in W_1^\circ$ such that $d(Q_1, R_2) > r$ for any $R_2 \in W_2^\circ$.
(b) There exists $Q_2 \in W_2^\circ$ such that $d(Q_2, R_1) > r$ for any $R_1 \in W_1^\circ$.

Suppose that (a) holds, so there exists $Q_1 \in W_1^\circ \setminus \overline{B(W_2^\circ, r)}$. Then, by Lemma 2.4,

$$Q_1 \notin \overline{B(W_2^\circ, r)} = \overline{B\Big(\bigcap_{\widetilde{Q}\in W_2} H(\widetilde{Q}), r\Big)} = \bigcap_{\widetilde{Q}\in W_2} \overline{B(H(\widetilde{Q}), r)}.$$

Hence, there exists $R \in W_2$ such that $Q_1 \notin \overline{B(H(R), r)}$.

On the other hand, since $h(W_1, W_2) = r$, there exists $\widetilde{P}_R \in W_1$ such that $d(\widetilde{P}_R, R) \le r$. By Lemma 2.3, $Q_1 \in H(\widetilde{P}_R) \subset \overline{B(H(R), r)}$, a contradiction. In the same way, we can show that (b) implies a contradiction. Thus, $(*)$ is proved.

By Lemma 2.2 and $(*)$, for any $W_1, W_2 \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$ we have

$$h(W_1, W_2) \le h(W_1^\circ, W_2^\circ) \le h(W_1, W_2).$$

Therefore, $h(W_1, W_2) = h(W_1^\circ, W_2^\circ)$ for any $W_1, W_2 \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$.

Next, we show assertion (2) of Theorem 1. Let $W_1 = \lim_{i\to\infty} W_{1_i}$ and $W_2 = \lim_{i\to\infty} W_{2_i}$, where $W_{1_i}, W_{2_i} \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$ for any $i \in \mathbb{N}$. By assertion (1), $h(W_{1_i}, W_{2_i}) = h(W_{1_i}^\circ, W_{2_i}^\circ)$. By Proposition 2.2,

$$\begin{aligned} h(W_1, W_2) &= h\Big(\lim_{i\to\infty} W_{1_i}, \lim_{i\to\infty} W_{2_i}\Big) \\ &= \lim_{i\to\infty} h(W_{1_i}, W_{2_i}) = \lim_{i\to\infty} h(W_{1_i}^\circ, W_{2_i}^\circ) \\ &= h\Big(\lim_{i\to\infty} W_{1_i}^\circ, \lim_{i\to\infty} W_{2_i}^\circ\Big) = h(W_1^\circ, W_2^\circ). \ \blacksquare \end{aligned}$$

4. Proof of Theorem 2. By the proof of Proposition 2.2, we have

$$h(W_1^\circ, W_2^\circ) \le 2h(W_1, W_2), \qquad h(W_1^{\circ\circ}, W_2^{\circ\circ}) \le 2h(W_1^\circ, W_2^\circ)$$

for any $W_1, W_2 \in \mathcal{H}_{\text{s-conv}}(S^n)$. By Lemma 2.5, for any $W_1, W_2 \in \overline{\mathcal{H}_{\text{s-conv}}(S^n)}$,

$$W_1^{\circ\circ} = W_1, \qquad W_2^{\circ\circ} = W_2,$$

so

$$\tfrac{1}{2} h(W_1, W_2) \le h(W_1^\circ, W_2^\circ) \le 2h(W_1, W_2).$$

Hence, $\bigcirc_{\text{s-conv}} : \overline{\mathcal{H}_{\text{s-conv}}(S^n)} \to \overline{\mathcal{H}_{\text{s-conv}}(S^n)}$ is bi-Lipschitz.

By Claim 2.1, it is clear that $\bigcirc_{\text{s-conv}} : \overline{\mathcal{H}_{\text{s-conv}}(S^\circ)} \to \overline{\mathcal{H}_{\text{s-conv}}(S^\circ)}$ is never isometric. $\blacksquare$

5. Appendix

5.1. Mappings in the theorems are well-defined bijections

PROPOSITION 5.1.

(1) *For any* $P \in S^n$, $\overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)} \subset \mathcal{H}^\circ(S^n)$.
(2) *For any* $P \in S^n$, $\bigcirc(\mathcal{H}_{\mathrm{Wulff}}(S^n, P)) = \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$.
(3) *For any* $P \in S^n$, $\bigcirc(\overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}) = \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$.
(4) *For any* $P \in S^n$, $\overline{\bigcirc_{\mathrm{Wulff},P}}$ *is injective.*
(5) $\overline{\mathcal{H}_{\text{s-conv}}(S^n)} \subset \mathcal{H}^\circ(S^n)$.
(6) $\bigcirc(\mathcal{H}_{\text{s-conv}}(S^n)) \neq \mathcal{H}_{\text{s-conv}}(S^n)$.
(7) $\bigcirc(\overline{\mathcal{H}_{\text{s-conv}}(S^n)}) = \overline{\mathcal{H}_{\text{s-conv}}(S^n)}$.
(8) *The mapping* $\bigcirc_{\text{s-conv}}$ *is injective.*

Proof. (1) It is clear that for any $W \in \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$, we have $W \subset H(P)$. Thus, by Lemma 2.1, $P \in W^\circ$, so $W^\circ \neq \emptyset$.

(2) For any $W \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$, there exist $0 < r_1 < r_2 < \pi/2$ such that $\overline{B(P, r_1)} \subset W \subset \overline{B(P, r_2)}$. By Lemma 2.1,

$$\big(\overline{B(P, r_2)}\big)^\circ \subset W^\circ \subset \big(\overline{B(P, r_1)}\big)^\circ.$$

It follows that $W^\circ \cap H(-P) = \emptyset$ and $P \in \mathrm{int}(W^\circ)$. Let $Q_1, Q_2 \in W^\circ = \bigcap_{Q \in W} H(Q)$. Since $W^\circ \cap H(-P) = \emptyset$, it follows that $(1-t)Q_1 + tQ_2$ is not the zero vector for any $t \in [0,1]$. Thus, for any $t \in [0,1]$,

$$\frac{(1-t)Q_1 + tQ_2}{\|(1-t)Q_1 + tQ_2\|} \in \bigcap_{Q \in W} H(Q) = W^\circ.$$

Hence W° is spherical convex, so $\bigcirc(W) \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$.

Conversely, for any $W \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$, set $\widetilde{W} = W^\circ$. We have already proved that $\widetilde{W} \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$. By Lemma 2.2, $\bigcirc(\widetilde{W}) = W$.

(3) For any $W \in \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$, there exists a sequence $W_i \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$ such that $W = \lim_{i\to\infty} W_i$. Since $\bigcirc : \mathcal{H}^\circ(S^n) \to \mathcal{H}^\circ(S^n)$ is continuous, $W^\circ = \lim_{i\to\infty} W_i^\circ$. By (2), $\bigcirc(W) \in \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$.

Conversely, for any $W \in \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$, set $\widetilde{W} = W^\circ$. We have proved that $\widetilde{W} \in \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$. By Lemma 2.5, $\bigcirc(\widetilde{W}) = W$.

(4) Suppose that $W_1 = \lim_{i\to\infty} W_{1_i}, W_2 = \lim_{i\to\infty} W_{2_i} \in \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$ and $W_1^\circ = W_2^\circ$, where $W_{1_i}, W_{2_i} \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$ for any $i \in \mathbb{N}$. Since W_{1_i}, W_{2_i} are spherical convex, by Lemma 2.5 we have

$$W_1 = W_1^{\circ\circ} = W_2^{\circ\circ} = W_2.$$

(5) Let $W \in \overline{\mathcal{H}_{\text{s-conv}}(S^n)}$. Then, by Proposition 5.3 below, there exists $P \in S^n$ such that $W \in \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$. By (1), $\bigcirc(W) \in \mathcal{H}^\circ(S^n)$.

(6) For any $P \in S^n$, $\bigcirc(\{P\}) = H(P)$ is not hemispherical. Therefore, $\bigcirc(\mathcal{H}_{\text{s-conv}}(S^n)) \neq \mathcal{H}_{\text{s-conv}}(S^n)$.

(7) For any $W \in \overline{\mathcal{H}_{\text{s-conv}}(S^n)}$, by Proposition 5.3, $W \in \bigcup_{P\in S^n} \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$. It follows that there exists a sequence $\widetilde{W}_i \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P_i)$ such that $W = \lim_{i\to\infty} \widetilde{W}_i$. Since $\bigcirc$ is continuous and $\widetilde{W}_i^\circ \in \overline{\mathcal{H}_{\text{s-conv}}(S^n)}$ $(i = 1, 2, \dots)$, we have $\lim_{i\to\infty} \widetilde{W}_i^\circ = W^\circ \in \overline{\mathcal{H}_{\text{s-conv}}(S^n)}$.

Conversely, for any $W \in \overline{\mathcal{H}_{\text{s-conv}}(S^n)}$, set $\widetilde{W} = W^\circ$. By Lemma 2.5, $W = \bigcirc(W^\circ)$.

(8) Suppose that $W_1 = \lim_{i\to\infty} W_{1_i}, W_2 = \lim_{i\to\infty} W_{2_i} \in \overline{\mathcal{H}_{\text{s-conv}}(S^n)}$ and $W_1^\circ = W_2^\circ$, where $W_{1_i}, W_{2_i} \in \mathcal{H}_{\text{s-conv}}(S^n)$ for any $i \in \mathbb{N}$. Since W_{1_i}, W_{2_i} are spherical convex, by Lemma 2.5 we have

$$W_1 = W_1^{\circ\circ} = W_2^{\circ\circ} = W_2. \ \blacksquare$$

By Proposition 5.1, we have the following:

PROPOSITION 5.2. *Each of the following is a well-defined bijective mapping:*

$$\bigcirc_{\mathrm{Wulff},P} : \mathcal{H}_{\mathrm{Wulff}}(S^n, P) \to \mathcal{H}_{\mathrm{Wulff}}(S^n, P),$$
$$\overline{\bigcirc_{\mathrm{Wulff},P}} : \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)} \to \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)},$$
$$\bigcirc_{\text{s-conv}} : \overline{\mathcal{H}_{\text{s-conv}}(S^n)} \to \overline{\mathcal{H}_{\text{s-conv}}(S^n)}.$$

Notice that $\bigcirc(\mathcal{H}_{\text{s-conv}}(S^n)) \not\subset \mathcal{H}_{\text{s-conv}}(S^n)$. Thus, the restriction to $\mathcal{H}_{\text{s-conv}}(S^n)$ is not investigated in this paper.

5.2. Why the restriction of $\bigcirc$ to $\overline{\mathcal{H}_{\text{s-conv}}(S^n)}$ is important? It is natural to expect that the isometric property still holds even for the restriction of $\bigcirc$ to $\bigcup_{P\in S^n} \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$. Since this subspace of $\mathcal{H}^\circ(S^n)$ seems to be complicated, the following proposition is useful.

PROPOSITION 5.3.
$$\bigcup_{P\in S^n} \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)} = \overline{\mathcal{H}_{\text{s-conv}}(S^n)}.$$

Proof. By Definition 1, the inclusion $\subset$ is clear. Thus, it is sufficient to show that
$$\overline{\mathcal{H}_{\text{s-conv}}(S^n)} \subset \bigcup_{P\in S^n} \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}.$$

Assume first that $W \in \overline{\mathcal{H}_{\text{s-conv}}(S^n)}$ is a hemispherical closed subset of S^n. Suppose that W has an interior point. Then, it is easily seen that there exists $P \in \mathrm{int}(W)$ such that $W \subset H(P)$. Since $H(P) \in \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$, it follows that $W \in \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}$. Next, suppose that W has no interior points. Since W is hemispherical and closed, there exist $P \in W$ and $N \in \mathbb{N}$ such that for any $i > N$ we have $\partial\overline{B(W, 2/i)} \cap H(-P) = \emptyset$. For any $i > N$, there exists a sequence
$$\{W_i\}_{i=1}^{\infty} \subset \mathcal{H}_{\text{s-conv}}(S^n)$$
such that $h(W_i, W) < 1/i$. Thus,
$$P \in W \subset \overline{B(W_i, 1/i)} \subset \overline{B(\overline{B(W, 1/i)}, 1/i)} = \overline{B(W, 2/i)} \subset H(P).$$
Therefore, as $\overline{B(W_i, 1/i)} \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$, we have
$$W = \lim_{i\to\infty} \overline{B(W_i, 1/i)} \in \bigcup_{P\in S^n} \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)}.$$

Finally, let W be any element of $\overline{\mathcal{H}_{\text{s-conv}}(S^n)}$. There exists $P \in S^n$ such that $W \subset H(P)$. For any positive integer i, define
$$W_i = \overline{B(W, 1/i)} \cap \overline{B(P, \pi/2 - 1/i)}.$$
Then it is easily seen that $W_i \in \mathcal{H}_{\mathrm{Wulff}}(S^n, P)$ for any $i \in \mathbb{N}$ and $W = \lim_{i\to\infty} W_i$. Therefore, $W \in \bigcup_{P\in S^n} \overline{\mathcal{H}_{\mathrm{Wulff}}(S^n, P)})$. ∎

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Huhe Han
College of Sciences
Northwest Agriculture and Forestry University
712100 Yangling, China
E-mail: han-huhe@nwafu.edu.cn

Takashi Nishimura
Research Institute of Environment
and Information Sciences
Yokohama National University
240-8501 Yokohama, Japan
E-mail: nishimura-takashi-yx@ynu.jp