

*ON SOME DIOPHANTINE EQUATIONS INVOLVING
GENERALIZED FIBONACCI AND LUCAS NUMBERS*

BY

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Abstract. We solve completely, in integers, some quadratic Diophantine equations involving generalized Fibonacci and Lucas numbers.

1. Introduction. Let $(u_n)_n$ and $(v_n)_n$ be the classical generalized Fibonacci and Lucas sequences respectively. For an integer p , the sequence $(u_n)_n$ is defined by the recurrence relation $u_n = pu_{n-1} - u_{n-2}$ for $n \geq 2$, with the initial conditions $u_0 = 0$ and $u_1 = 1$. The sequence $(v_n)_n$ is defined by the same recurrence relation $v_n = pv_{n-1} - v_{n-2}$ for $n \geq 2$, but with the initial conditions $v_0 = 2$ and $v_1 = p$. These generalized Fibonacci and Lucas numbers can be extended to negative indices by setting, for all $n \in \mathbb{N}$, $u_{-n} = -u_n$ and $v_{-n} = v_n$. For more detailed information about these sequences, see [3, 6, 7, 8].

Many authors have studied various Diophantine equations involving linear recurrence sequences, e.g. [1, 3, 6, 13]. In [3, Theorem 15], the authors claim that all integer solutions of $x^2 - (p^2 - 4)u_nxy - (p^2 - 4)y^2 = 1$, with $p \geq 3$, are given by $(x, y) = \pm(v_{(2k+1)n}/v_n, u_{2kn}/v_n)$ with $k \in \mathbb{Z}$. For $n = 1$ and $p = 3$, the equation becomes $x^2 - 5xy - 5y^2 = 1$. We note that $(x, y) = (-6, 7)$ is also a solution which is not of the form $\pm(v_{2k+1}/3, u_{2k}/3)$. It turns out that Theorems 8, 11, 13, 14 and 17 of [3] also miss some of the solutions.

In this paper, our purpose is to determine all integer solutions (x, y) of the following Diophantine equations:

$$(1.1) \quad x^2 - (p^2 - 4)u_nxy - (p^2 - 4)y^2 = v_n^2,$$

$$(1.2) \quad x^2 - pxy + y^2 = 1,$$

2010 *Mathematics Subject Classification*: Primary 11D09; Secondary 11D45, 11B37, 11B39.

Key words and phrases: generalized Fibonacci and Lucas sequences, Diophantine equations, Pell equation.

Received 9 February 2016; revised 21 November 2016.

Published online 11 August 2017.

$$\begin{aligned}
(1.3) \quad & x^2 - (p^2 - 4)y^2 = 4, \\
(1.4) \quad & x^2 - (p^2 - 4)y^2 = 4^k, \\
(1.5) \quad & x^2 - (p^2 - 4)y^2 = 1, \\
(1.6) \quad & x^2 - v_n xy + y^2 = u_n^2, \\
(1.7) \quad & x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = 1, \\
(1.8) \quad & x^2 - v_n xy + y^2 = 1, \\
(1.9) \quad & x^2 - v_{2n} xy + y^2 = u_n^2, \\
(1.10) \quad & x^2 - v_{2n} xy + y^2 = v_n^2, \\
(1.11) \quad & x^2 - pxy + y^2 = 4^k.
\end{aligned}$$

This paper completes and adjusts [3] in the sense that we solve some equations that do not occur in [3] and we give, for an integer p such that $|p| > 2$, a complete set of integer solutions of equations (1.1)–(1.3) and (1.6)–(1.10) which are given in [3]. The cases $|p| \leq 2$ are obvious and left to the reader.

We recall the following classical divisibility properties involving the sequences $(u_n)_n$ and $(v_n)_n$:

$$\begin{aligned}
(1.12) \quad & u_n \mid u_m \Leftrightarrow m = kn \text{ for some } k \in \mathbb{Z}, \\
(1.13) \quad & v_n \mid u_m \Leftrightarrow m = 2kn \text{ for some } k \in \mathbb{Z}, \\
(1.14) \quad & v_n \mid v_m \Leftrightarrow m = (2k + 1)n \text{ for some } k \in \mathbb{Z}.
\end{aligned}$$

These properties (proved in [4, 11, 12]) will be used to solve equations (1.7)–(1.10).

Throughout this paper, the signs in a formula of the type $(\pm a, \pm b)$ are independent.

2. Some identities for $(u_n)_n$ and $(v_n)_n$. The generalized Fibonacci and Lucas sequences $(u_n)_n$ and $(v_n)_n$ have many interesting properties. We list here those needed in our paper; they can be found in [5, 8, 9, 10]. For any integers n and m ,

$$\begin{aligned}
(2.1) \quad & u_{n+1} - u_{n-1} = v_n, \\
(2.2) \quad & v_{n+1} - v_{n-1} = (p^2 - 4)u_n, \\
(2.3) \quad & u_{m+1}v_n - v_{n-1}u_m = v_{n+m}, \\
(2.4) \quad & u_n^2 - pu_n u_{n-1} + u_{n-1}^2 = 1, \\
(2.5) \quad & v_n^2 - pv_n v_{n-1} + v_{n-1}^2 = -(p^2 - 4), \\
(2.6) \quad & v_m u_n - u_m v_n = 2u_{n-m}, \\
(2.7) \quad & v_m v_n - (p^2 - 4)u_m u_n = 2v_{n-m},
\end{aligned}$$

$$(2.8) \quad v_m u_n + u_m v_n = 2u_{n+m},$$

$$(2.9) \quad v_n^2 - (p^2 - 4)u_n^2 = 4.$$

THEOREM 2.1. *Let $k, m, n \in \mathbb{Z}$. Then*

$$(2.10) \quad v_{n+m}^2 - (p^2 - 4)u_{n-k}v_{n+m}u_{m+k} - (p^2 - 4)u_{m+k}^2 = v_{n-k}^2,$$

$$(2.11) \quad u_{n+m}^2 - v_{n-k}u_{n+m}u_{m+k} + u_{m+k}^2 = u_{n-k}^2.$$

Proof. Identities (2.10) and (2.11) are given in [3, Theorems 5 and 7] where the authors use matrix methods to obtain them and where (2.11) is given for $k \neq n$. Here we give another proof of (2.11), valid also if $k = n$.

From (2.8) we have

$$(2.12) \quad \begin{cases} u_n v_m + v_n u_m = 2u_{n+m}, \\ u_k v_m + v_k u_m = 2u_{m+k}. \end{cases}$$

Multiplying the first identity by u_k and the second by u_n , and subtracting the results, we obtain

$$u_m(u_k v_n - u_n v_k) = 2(u_k u_{n+m} - u_n u_{m+k}).$$

Using (2.6) we find

$$u_m u_{n-k} = u_n u_{m+k} - u_k u_{n+m}.$$

Multiplying the first identity of (2.12) by v_k and the second by v_n , and subtracting the results, we obtain

$$v_m(v_k u_n - v_n u_k) = 2(v_k u_{n+m} - v_n u_{m+k}).$$

Using (2.6) we find

$$v_m u_{n-k} = v_k u_{n+m} - v_n u_{m+k}.$$

From (2.9) we see that $(v_m u_{n-k})^2 - (p^2 - 4)(u_m u_{n-k})^2 = 4u_{n-k}^2$, i.e.,

$$(v_k u_{n+m} - v_n u_{m+k})^2 - (p^2 - 4)(u_n u_{m+k} - u_k u_{n+m})^2 = 4u_{n-k}^2.$$

Finally, using (2.7) and (2.9) we get (2.11). ■

3. Solutions of some Diophantine equations. The identities (2.10) and (2.11) suggest exploring the solutions of the Diophantine equations (1.1) and (1.6).

THEOREM 3.1. *All integer solutions (x, y) of*

$$x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = v_n^2$$

for $|p| > 2$ and $n \in \mathbb{Z}$ are $\pm(v_{n+m}, u_m)$ and $\pm(-v_{n-m}, u_m)$, where $m \in \mathbb{Z}$.

To prove Theorem 3.1, we need the following proposition which is given in [2, Proposition 6.3.16, p. 355].

PROPOSITION 3.2 (The Structure Theorem). *If $D > 0$ is not a square and is congruent to 0 or 1 modulo 4, the Pell equation $x^2 - Dy^2 = \pm 4$ has an infinity of solutions given in the following way. If (x_0, y_0) is a solution with the least strictly positive y_0 (and $x_0 > 0$, say), the **general solution** is given by*

$$\frac{x + y\sqrt{D}}{2} = \pm \left(\frac{x_0 + y_0\sqrt{D}}{2} \right)^k \quad \text{for any } k \in \mathbb{Z}.$$

REMARK 3.3. For our purpose, we need only integer solutions of $x^2 - Dy^2 = 4$ given by Proposition 3.2. In this case, if $k \geq 0$ then x and y are of the same sign. If $k < 0$, then it is easy to see that

$$\frac{x + y\sqrt{D}}{2} = \pm \left(\frac{x_0 - y_0\sqrt{D}}{2} \right)^{-k},$$

which gives x and y of different signs.

Proof of Theorem 3.1. Assume that $x^2 - (p^2 - 4)u_nxy - (p^2 - 4)y^2 = v_n^2$. Then

$$(2x - (p^2 - 4)u_ny)^2 - (u_n^2(p^2 - 4) + 4)(p^2 - 4)y^2 = 4v_n^2.$$

Using (2.9) we obtain

$$(3.1) \quad (2x - (p^2 - 4)u_ny)^2 - v_n^2(p^2 - 4)y^2 = 4v_n^2;$$

we deduce that $v_n \mid (2x - (p^2 - 4)u_ny)$, and so

$$\begin{aligned} (3.1) &\Leftrightarrow \left(\frac{2x - (p^2 - 4)u_ny}{v_n} \right)^2 - (p^2 - 4)y^2 = 4 \\ &\Leftrightarrow \left(\frac{2x - (p^2 - 4)u_ny}{v_n} \right)^2 - (py)^2 = 4(1 - y^2) \\ &\Leftrightarrow \left(\frac{2x - (p^2 - 4)u_ny}{v_n} - py \right) \left(\frac{2x - (p^2 - 4)u_ny}{v_n} + py \right) = 4s, \end{aligned}$$

where $s = 1 - y^2$. We deduce that $(2x - (p^2 - 4)u_ny)/v_n + py$ is even. Thus, taking

$$a = \frac{1}{2} \left(\frac{2x - (p^2 - 4)u_ny}{v_n} + py \right) = \frac{x + v_{n-1}y}{v_n} \quad \text{and} \quad b = y,$$

and using identities (2.2) and (2.5), we get

$$a^2 - pab + b^2 = \frac{x^2 - (p^2 - 4)u_nxy - (p^2 - 4)y^2}{v_n^2} = 1.$$

We have

$$a^2 - pab + b^2 = 1 \Leftrightarrow [2a - pb]^2 - (p^2 - 4)b^2 = 4.$$

Let $c = 2a - pb$. Then our equation becomes

$$(3.2) \quad c^2 - (p^2 - 4)b^2 = 4.$$

By assumption $p^2 - 4$ is positive. Obviously, it is congruent to 0 or 1 modulo 4. To see that it is not a square, assume that $p^2 - 4 = \delta^2$ for a positive integer δ ; this is equivalent to $(p - \delta)(p + \delta) = 4$, which has no solutions in positive integers.

The minimal solution of (3.2) is $(c_0, b_0) = (|p|, 1)$. Thus, according to the Structure Theorem, the solutions of (3.2) satisfy

$$\frac{c + b\sqrt{p^2 - 4}}{2} = \pm \left(\frac{|p| + \sqrt{p^2 - 4}}{2} \right)^m \quad (m \in \mathbb{Z}).$$

Thus, for the “+” sign we get

$$\begin{cases} \frac{c + b\sqrt{p^2 - 4}}{2} = \left(\frac{|p| + \sqrt{p^2 - 4}}{2} \right)^m & (m \in \mathbb{Z}), \\ \frac{c - b\sqrt{p^2 - 4}}{2} = \left(\frac{|p| - \sqrt{p^2 - 4}}{2} \right)^m & (m \in \mathbb{Z}). \end{cases}$$

By subtracting, we find, for $m \in \mathbb{Z}$,

$$b = \frac{1}{\sqrt{p^2 - 4}} \left[\left(\frac{|p| + \sqrt{p^2 - 4}}{2} \right)^m - \left(\frac{|p| - \sqrt{p^2 - 4}}{2} \right)^m \right].$$

• If $p > 0$, then $y = b = u_m$. Let us find the values of x . From (1.1) and (2.10) we obtain

$$\begin{cases} x^2 - (p^2 - 4)u_n x u_m - (p^2 - 4)u_m^2 = v_n^2, \\ v_{n+m}^2 - (p^2 - 4)u_n v_{n+m} u_m - (p^2 - 4)u_m^2 = v_n^2. \end{cases}$$

By subtracting we find

$$(x - v_{n+m})(x + v_{n+m} - (p^2 - 4)u_n u_m) = 0.$$

Thus,

$$x = v_{n+m} \text{ or } x = -v_{n+m} + (p^2 - 4)u_n u_m = -v_{n+m} + (v_{n+m} - v_{n-m}) = -v_{n-m}.$$

Hence, $(x, y) = (v_{n+m}, u_m)$ or $(-v_{n-m}, u_m)$.

• If $p < 0$, then $y = b = (-1)^{m+1}u_m$. From (1.1) and (2.10) we obtain

$$\begin{cases} x^2 - (-1)^{m+1}(p^2 - 4)u_n x u_m - (p^2 - 4)u_m^2 = v_n^2, \\ v_{n+m}^2 - (p^2 - 4)u_n v_{n+m} u_m - (p^2 - 4)u_m^2 = v_n^2. \end{cases}$$

By subtracting we find

$$x^2 - v_{n+m}^2 - (p^2 - 4)u_n u_m ((-1)^{m+1}x - v_{n+m}) = 0.$$

* If m is even, we obtain

$$(x + v_{n+m})(x - v_{n+m} + (p^2 - 4)u_n u_m) = 0.$$

Thus,

$$x = -v_{n+m} \text{ or } x = v_{n+m} - (p^2 - 4)u_n u_m = v_{n+m} - (v_{n+m} - v_{n-m}) = v_{n-m}.$$

Hence, $(x, y) = -(v_{n+m}, u_m)$ or $(-v_{n-m}, u_m)$.

* If m is odd, we obtain

$$(x - v_{n+m})(x + v_{n+m} - (p^2 - 4)u_n u_m) = 0.$$

Thus,

$$x = v_{n+m} \text{ or } x = -v_{n+m} + (p^2 - 4)u_n u_m = -v_{n+m} + (v_{n+m} - v_{n-m}) = -v_{n-m}.$$

Hence, $(x, y) = (v_{n+m}, u_m)$ or $(-v_{n-m}, u_m)$. Proceeding in the same way for the “-” sign, we find that all solutions are $\pm(v_{n+m}, u_m)$ and $\pm(-v_{n-m}, u_m)$.

Conversely, replacing in (2.10) n by $n + k$ and m by $m - k$ for $k \in \mathbb{Z}$, we obtain

$$v_{n+m}^2 - (p^2 - 4)u_n v_{n+m} u_m - (p^2 - 4)u_m^2 = v_n^2,$$

which means that $\pm(v_{n+m}, u_m)$ are solutions of (1.1). Also, replacing in (2.10) n by $n + k$ and m by $-m + k$ for $k \in \mathbb{Z}$, we obtain

$$v_{n-m}^2 - (p^2 - 4)u_n v_{n-m} u_{-m} - (p^2 - 4)u_m^2 = v_n^2,$$

which means that $\pm(-v_{n-m}, u_m)$ are solutions of (1.1). ■

LEMMA 3.4. *All integer solutions (a, b) of $a^2 - pab + b^2 = 1$ for $|p| > 2$ are $\pm(u_{m \pm 1}, u_m)$ with $m \in \mathbb{Z}$.*

Proof. Follows from the proof of Theorem 3.1 with the help of (2.3), using the fact that $a = (x + v_{n-1}y)/v_n$ and $b = y$. The converse is obvious by (2.4). ■

LEMMA 3.5. *All integer solutions (c, b) of $c^2 - (p^2 - 4)b^2 = 4$ for $|p| > 2$ are $(\pm v_n, \pm u_n)$ with $n \in \mathbb{Z}$.*

Proof. Follows from Theorem 3.1 with the help of (2.1) and (2.3), using the fact that $a = (x + v_{n-1}y)/v_n$, $b = y$ and $c = 2a - pb$. The converse is obvious by (2.9). ■

LEMMA 3.6. *Assume that $c^2 - (p^2 - 4)b^2 = 4^k$ with $k \geq 2$. If p is odd, then c and b are even numbers.*

Proof. Assume that p is odd. Then it is easy to see that $p^2 \equiv 1 \pmod{8}$. Since $k \geq 2$, we have $c^2 - (p^2 - 4)b^2 \equiv 0 \pmod{8}$. Thus, $c^2 - (p^2 - 4)b^2 \equiv c^2 + 3b^2 \pmod{8}$. But $c^2 + 3b^2 \equiv 0 \pmod{8}$ if and only if either $c^2, b^2 \equiv 0 \pmod{8}$ or $c^2, b^2 \equiv 4 \pmod{8}$. We conclude that either $c, b \equiv 0, 4 \pmod{8}$ or $c, b \equiv 2, 6 \pmod{8}$, i.e., c and b are even. ■

REMARK 3.7. If p is even, the conclusion of Lemma 3.6 is not true. Indeed, for $k = 4$ and $p = 18$, the equation $c^2 - (p^2 - 4)b^2 = 4^k$ becomes $c^2 - 2^6 \cdot 5b^2 = 2^8$. We deduce that $2^3 | c$. Let $c = 2^3 c_1$. Then the equation

$c^2 - 2^6 \cdot 5b^2 = 2^8$ becomes $c_1^2 - 5b^2 = 4$. It is clear that $(c_1, b) = (3, 1)$ is a solution and b is odd.

COROLLARY 3.8. *Let $k \geq 1$ be an integer. Then all integer solutions (c, b) of the equation $c^2 - (p^2 - 4)b^2 = 4^k$ for $|p| > 2$ with p odd are $(\pm 2^{k-1}v_n, \pm 2^{k-1}u_n)$ with $n \in \mathbb{Z}$.*

Proof. By induction using Lemmas 3.5 and 3.6. ■

THEOREM 3.9. *All integer solutions (c, d) of $c^2 - (p^2 - 4)b^2 = 1$ for $|p| > 2$ are $(\pm v_{2m}/2, \pm u_{2m}/2)$ if p is even, and $(\pm v_{3m}/2, \pm u_{3m}/2)$ if p is odd, where $m \in \mathbb{Z}$.*

Proof. Assume that $c^2 - (p^2 - 4)b^2 = 1$. Then $(2c)^2 - (p^2 - 4)(2b)^2 = 4$. Thus, we deduce from Lemma 3.5 that $(2c, 2b) = (\pm v_n, \pm u_n)$ with $n \in \mathbb{Z}$. It is easy to see that

- If p is even: v_n is always even and $2 | u_n \Leftrightarrow 2 | n$, so $n = 2m$ for some $m \in \mathbb{Z}$.
- If p is odd: $2 | u_n \Leftrightarrow 3 | n \Leftrightarrow 2 | v_n$, and so $n = 3m$ for $m \in \mathbb{Z}$.

Hence, $(c, b) = (\pm v_{2m}/2, \pm u_{2m}/2)$ if p is even, and $(c, b) = (\pm v_{3m}/2, \pm u_{3m}/2)$ if p is odd.

The converse is obvious by (2.9). ■

THEOREM 3.10. *If $n \neq 0$ is an integer and $|p| > 2$, then all integer solutions (x, y) of the equation $x^2 - v_nxy + y^2 = u_n^2$ are $\pm(u_{n+m}, u_m)$ and $\pm(u_{m-n}, u_m)$, where $m \in \mathbb{Z}$. If $n = 0$, the solutions are (x, x) with $x \in \mathbb{Z}$.*

Proof. Assume that $x^2 - v_nxy + y^2 = u_n^2$ with $n \neq 0$. Then

$$(2x - v_ny)^2 - (v_n^2 - 4)y^2 = 4u_n^2.$$

Using (2.9) we obtain

$$(3.3) \quad (2x - v_ny)^2 - (p^2 - 4)u_n^2y^2 = 4u_n^2;$$

we deduce that $u_n | (2x - v_ny)$; consequently,

$$\begin{aligned} (3.3) &\Leftrightarrow \left(\frac{2x - v_ny}{u_n}\right)^2 - (p^2 - 4)y^2 = 4 \\ &\Leftrightarrow \left(\frac{2x - v_ny}{u_n}\right)^2 - (py)^2 = 4(1 - y^2) \\ &\Leftrightarrow \left(\frac{2x - v_ny}{u_n} - py\right)\left(\frac{2x - v_ny}{u_n} + py\right) = 4(1 - y^2). \end{aligned}$$

We see that $(2x - v_ny)/u_n + py$ is even. Thus, taking

$$a = \frac{1}{2}\left(\frac{2x - v_ny}{u_n} + py\right) = \frac{x + u_{n-1}y}{u_n} \quad \text{and} \quad b = y,$$

and using (2.3) and (2.4), we find that

$$a^2 - pab + b^2 = \frac{x^2 - v_nxy + y^2}{u_n^2} = 1.$$

As in the proof of Theorem 3.1, for the “ + ” sign we obtain

$$b = \frac{1}{\sqrt{p^2 - 4}} \left[\left(\frac{|p| + \sqrt{p^2 - 4}}{2} \right)^m - \left(\frac{|p| - \sqrt{p^2 - 4}}{2} \right)^m \right].$$

• If $p > 0$, then $y = b = u_m$. Let us find the values of x . From (1.6) and (2.11) we get

$$\begin{cases} x^2 - v_nxu_m + u_m^2 = u_n^2, \\ u_{n+m}^2 - v_nu_{n+m}u_m + u_m^2 = u_n^2. \end{cases}$$

By subtracting we find

$$(x - u_{n+m})(x + u_{n+m} - v_nu_m) = 0.$$

Thus,

$$x = u_{n+m} \quad \text{or} \quad x = -u_{n+m} + v_nu_m = -u_{n+m} + (u_{n+m} - u_{n-m}) = -u_{n-m}.$$

Hence, $(x, y) = (u_{n+m}, u_m)$ or (u_{m-n}, u_m) .

• If $p < 0$, then $y = b = (-1)^{m+1}u_m$. From (1.6) and (2.11) we get

$$\begin{cases} x^2 - (-1)^{m+1}v_nxu_m + u_m^2 = u_n^2, \\ u_{n+m}^2 - v_nu_{n+m}u_m + u_m^2 = u_n^2. \end{cases}$$

By subtracting we find

$$x^2 - u_{n+m}^2 - v_nu_m((-1)^{m+1}x - u_{n+m}) = 0.$$

* If m is even, we obtain

$$(x + u_{n+m})(x - u_{n+m} + v_nu_m) = 0.$$

Thus,

$$x = -u_{n+m} \quad \text{or} \quad x = u_{n+m} - v_nu_m = u_{n+m} - (u_{n+m} - u_{n-m}) = u_{n-m}.$$

Hence, $(x, y) = -(u_{n+m}, u_m)$ or $-(u_{m-n}, u_m)$.

* If m is odd, we obtain

$$(x - u_{n+m})(x + u_{n+m} - v_nu_m) = 0.$$

Thus,

$$x = u_{n+m} \quad \text{or} \quad x = -u_{n+m} + v_nu_m = -u_{n+m} + (u_{n+m} - u_{n-m}) = -u_{n-m}.$$

Hence, $(x, y) = (u_{n+m}, u_m)$ or (u_{m-n}, u_m) . Proceeding in the same way for the “ - ” sign, we find that all solutions are $\pm(u_{n+m}, u_m)$ and $\pm(u_{m-n}, u_m)$.

Conversely, replacing in (2.11) n by $n + k$ and m by $m - k$ for $k \in \mathbb{Z}$, we obtain

$$u_{n+m}^2 - v_nu_{n+m}u_m + u_m^2 = u_n^2,$$

which means that $\pm(u_{n+m}, u_m)$ are solutions of (1.6). Also, replacing in (2.11) n by $-n + k$ and m by $m - k$ for $k \in \mathbb{Z}$, we obtain

$$u_{m-n}^2 - v_n u_{m-n} u_m + u_m^2 = u_n^2,$$

which means that $\pm(u_{m-n}, u_m)$ are solutions of (1.1).

If $n = 0$, the equation becomes $x^2 - 2xy + y^2 = 0$ and the solutions are (x, x) with $x \in \mathbb{Z}$. ■

THEOREM 3.11. *If $n \neq 0$ is an integer and $|p| > 2$, then all integer solutions (x, y) of the equation $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = 1$ are $\pm(v_{(2k+1)n}/v_n, u_{2kn}/v_n)$ and $\pm(-v_{(2k-1)n}/v_n, u_{2kn}/v_n)$ with $k \in \mathbb{Z}$. If $n = 0$, the solutions are $(\pm v_{2m}/2, \pm u_{2m}/2)$ if p is even, and $(\pm v_{3m}/2, \pm u_{3m}/2)$ if p is odd, with $m \in \mathbb{Z}$.*

Proof. Assume that $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = 1$ with $n \neq 0$. Then, multiplying both sides of the equation by v_n^2 , we find

$$(v_n x)^2 - (p^2 - 4)u_n (v_n x)(v_n y) - (p^2 - 4)(v_n y)^2 = v_n^2.$$

It follows from Theorem 3.1 that

$$(v_n x, v_n y) = \pm(v_{n+m}, u_m) \text{ or } \pm(-v_{n-m}, u_m).$$

Relation (1.13) tells us that $v_n \mid u_m$ if and only if $m = 2kn$ for some $k \in \mathbb{Z}$. Thus, we obtain

$$(x, y) = \pm(v_{(2k+1)n}/v_n, u_{2kn}/v_n) \text{ or } \pm(-v_{(2k-1)n}/v_n, u_{2kn}/v_n).$$

Conversely, replacing in (2.10) n by $n + k$ and m by $2nk - k$ for $k \in \mathbb{Z}$, we see that $\pm(v_{(2k+1)n}/v_n, u_{2kn}/v_n)$ are solutions of (1.7). Also, replacing in (2.10) n by $n + k$ and m by $-2nk - k$ for $k \in \mathbb{Z}$, we see that $\pm(-v_{(2k-1)n}/v_n, u_{2kn}/v_n)$ are solutions of (1.7).

If $n = 0$, the equation becomes $x^2 - (p^2 - 4)y^2 = 1$ and, from Theorem 3.9, the solutions are $(\pm v_{2m}/2, \pm u_{2m}/2)$ if p is even and $(\pm v_{3m}/2, \pm u_{3m}/2)$ if p is odd. ■

THEOREM 3.12. *If $n \neq 0$ is an integer and $|p| > 2$, then all integer solutions (x, y) of the equation $x^2 - v_n xy + y^2 = 1$ are $\pm(u_{(k+1)n}/u_n, u_{kn}/u_n)$ and $\pm(u_{(k-1)n}/u_n, u_{kn}/u_n)$ with $k \in \mathbb{Z}$. If $n = 0$, the solutions are $(r \pm 1, r)$ where $r \in \mathbb{Z}$.*

Proof. Assume that $x^2 - v_n xy + y^2 = 1$ with $n \neq 0$. Then, multiplying both sides of the equation by u_n^2 , we find

$$(u_n x)^2 - v_n (u_n x)(u_n y) + (u_n y)^2 = u_n^2.$$

It follows from Theorem 3.10 that

$$(u_n x, u_n y) = \pm(u_{n+m}, u_m) \text{ or } \pm(u_{m-n}, u_m).$$

Relation (1.12) tells us that $u_n \mid u_m$ if and only if $m = kn$ for some $k \in \mathbb{Z}$. Thus, we obtain

$$(x, y) = \pm(u_{(k+1)n}/u_n, u_{kn}/u_n) \text{ or } \pm(u_{(k-1)n}/u_n, u_{kn}/u_n).$$

Conversely, replacing in (2.11) n by $n + k$ and m by $kn - k$ for $k \in \mathbb{Z}$, we see that $\pm(u_{(k+1)n}/u_n, u_{kn}/u_n)$ are solutions of (1.8). Also, replacing in (2.11) n by $n + k$ and m by $-kn - k$ for $k \in \mathbb{Z}$, we see that $\pm(u_{(k-1)n}/u_n, u_{kn}/u_n)$ are solutions of (1.8).

If $n = 0$, we know that $v_0 = 2$. Thus, the equation becomes $(x - y)^2 = 1$ and the solutions are $(r \pm 1, r)$ where $r \in \mathbb{Z}$. ■

THEOREM 3.13. *If $n \neq 0$ is an integer and $|p| > 2$, then all integer solutions (x, y) of the equation $x^2 - v_{2n}xy + y^2 = u_n^2$ are $\pm(u_{2(k+1)n}/v_n, u_{2kn}/v_n)$ and $\pm(u_{2(k-1)n}/v_n, u_{2kn}/v_n)$ with $k \in \mathbb{Z}$. If $n = 0$, the solutions are (r, r) where $r \in \mathbb{Z}$.*

Proof. For $\alpha = (p + \sqrt{p^2 - 4})/2$ and $\beta = (p - \sqrt{p^2 - 4})/2$, we have the well known Binet formulas $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and $v_n = \alpha^n + \beta^n$. Thus, we see that

$$v_n u_n = (\alpha^n + \beta^n)(\alpha^n - \beta^n)/(\alpha - \beta) = u_{2n}.$$

Assume that $x^2 - v_{2n}xy + y^2 = u_n^2$ with $n \neq 0$. Then, multiplying both sides of the equation by v_n^2 , we find

$$(v_n x)^2 - v_{2n}(v_n x)(v_n y) + (v_n y)^2 = u_{2n}^2.$$

It follows from Theorem 3.10 that

$$(v_n x, v_n y) = \pm(u_{2n+m}, u_m) \text{ or } \pm(u_{m-2n}, u_m).$$

Relation (1.13) tells us that $v_n \mid u_m$ if and only if $m = 2kn$ for some $k \in \mathbb{Z}$. Thus, we obtain

$$(x, y) = \pm(u_{2(k+1)n}/v_n, u_{2kn}/v_n) \text{ or } \pm(u_{2(k-1)n}/v_n, u_{2kn}/v_n).$$

Conversely, replacing in (2.11) n by $2n + k$ and m by $2kn - k$ for $k \in \mathbb{Z}$, we see that $\pm(u_{2(k+1)n}/v_n, u_{2kn}/v_n)$ are solutions of (1.9). Also, replacing in (2.11) n by $2n + k$ and m by $-2kn - k$ for $k \in \mathbb{Z}$, we see that $\pm(u_{2(k-1)n}/v_n, u_{2kn}/v_n)$ are solutions of (1.9).

If $n = 0$, we know that $u_0 = 0$ and $v_0 = 2$. Thus, the equation becomes $(x - y)^2 = 0$ and the solutions are (r, r) where $r \in \mathbb{Z}$. ■

THEOREM 3.14. *If $n \neq 0$ is an integer and $|p| > 2$, then all integer solutions (x, y) of the equation $x^2 - v_{2n}xy + y^2 = v_n^2$ are $\pm(u_{(k+2)n}/u_n, u_{kn}/u_n)$ and $\pm(u_{(k-2)n}/u_n, u_{kn}/u_n)$ with $k \in \mathbb{Z}$. If $n = 0$, the solutions are $(r \pm 2, r)$ where $r \in \mathbb{Z}$.*

Proof. Assume that $x^2 - v_{2n}xy + y^2 = v_n^2$ with $n \neq 0$. Multiplying both sides of the equation by u_n^2 and using the fact that $v_n u_n = u_{2n}$, we find

$$(u_n x)^2 - v_{2n}(u_n x)(u_n y) + (u_n y)^2 = u_{2n}^2.$$

It follows from Theorem 3.10 that

$$(u_n x, u_n y) = \pm(u_{2n+m}, u_m) \text{ or } \pm(u_{m-2n}, u_m).$$

Relation (1.12) tells us that $u_n \mid u_m$ if and only if $m = kn$ for some $k \in \mathbb{Z}$. Thus, we obtain

$$(x, y) = \pm(u_{(k+2)n}/u_n, u_{kn}/u_n) \text{ or } \pm(u_{(k-2)n}/u_n, u_{kn}/u_n).$$

Conversely, replacing in (2.11) n by $2n+k$ and m by $kn-k$ for $k \in \mathbb{Z}$, we see that $\pm(u_{(k+2)n}/u_n, u_{kn}/u_n)$ are solutions of (1.10). Also, replacing in (2.11) n by $2n+k$ and m by $-kn-k$ for $k \in \mathbb{Z}$, we see that $\pm(u_{(k-2)n}/u_n, u_{kn}/u_n)$ are solutions of (1.10).

If $n = 0$, we know that $v_0 = 2$. Thus, the equation becomes $(x - y)^2 = 4$ and the solutions are $(r \pm 2, r)$ where $r \in \mathbb{Z}$. ■

THEOREM 3.15. *Let $k \geq 0$ be an integer. Then all integer solutions (x, y) of the equation $x^2 - pxy + y^2 = 4^k$ for $|p| > 2$ and p odd are $\pm(2^k u_{m \pm 1}, 2^k u_m)$ with $m \in \mathbb{Z}$.*

Proof. From the proof of Theorem 3.1 we get

$$4(x^2 - pxy + y^2) = c^2 - (p^2 - 4)b^2,$$

where $b = y$ and $c = 2x - py$. Thus

$$x^2 - pxy + y^2 = 4^k \Leftrightarrow c^2 - (p^2 - 4)b^2 = 4^{k+1}.$$

It follows from Corollary 3.8 that $(c, b) = (\pm 2^k v_n, \pm 2^k u_n)$ with $n \in \mathbb{Z}$. Using the identities $2u_{n+1} = pu_n + v_n$ and $2u_{n-1} = pu_n - v_n$, which come from (2.1), it is seen that $(x, y) = \pm(2^k u_{m \pm 1}, 2^k u_m)$. The converse is obvious by (2.4). ■

Acknowledgements. The authors are grateful to the referee for comments and suggestions which improved the quality of this paper.

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