

*SIMPLE CONFORMALLY RECURRENT SPACE-TIMES
ARE CONFORMALLY RECURRENT PP-WAVES*

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Dedicated to the memory of Witold Roter

Abstract. We show that in dimension $n \geq 4$ the class of simple conformally recurrent space-times coincides with the class of conformally recurrent pp-waves.

1. Introduction. Riemannian manifolds with recurrent Weyl tensor (conformally recurrent manifolds) were introduced by Adati and Miyazawa [1] and subsequently extended to pseudo-Riemannian manifolds by Derdziński [7], Roter [31, 32, 33, 34], and Suh and Kwon [38], and to Lorentzian manifolds (space-times) by Mc Lenaghan et al. [24, 25], Hall [15, 16], and De and Mantica [5]. In a recent study [21, 22] of conformally recurrent pseudo-Riemannian and Lorentzian manifolds in dimension $n \geq 5$, the authors showed that the Ricci tensor has at most two different eigenvalues, with restrictions on the metric structure, and gave an explicit representation of the Weyl tensor as a Kulkarni–Nomizu product.

A pseudo-Riemannian manifold of dimension $n \geq 4$ is *conformally recurrent* if the Weyl curvature tensor $(^1)$ is everywhere non-zero and satisfies the condition

$$(1.1) \quad \nabla_i C_{jklm} = \alpha_i C_{jklm}$$

where α_i is a non-zero 1-form named recurrence vector.

The natural question arises whether the recurrent vector may be locally cancelled by a conformal transformation. This “simple” case was investigated by Roter [31, 34].

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⁽¹⁾ The local components of the Weyl tensor are [29]

$$C_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-2}(\delta_j^m R_{kl} - \delta_k^m R_{jl} + g_{kl}R_j^m - g_{jl}R_k^m) - R \frac{\delta_j^m g_{kl} - \delta_k^m g_{jl}}{(n-1)(n-2)}$$

with Ricci tensor $R_{ij} = -R_{kij}{}^k$ and scalar curvature $R = R^k{}_k$.

DEFINITION 1.1 (Roter). A conformally recurrent manifold (\mathcal{M}, g) is *simple* if the metric is locally conformal to a conformally symmetric metric, i.e. at each point $x \in \mathcal{M}$ there exist a neighbourhood U_x and a scalar function σ on U_x such that $g'_{ij} = g_{ij}e^{2\sigma}$ and $\nabla'_i C'_{jklm} = 0$ on U_x .

The main result of this work is the statement that such spaces, with a Lorentzian metric, coincide with (conformally recurrent) pp-wave Lorentzian space-times.

pp-waves are important in general relativity (see for example [19] and references therein). They were studied in $n > 4$ by Schimming [36], and appear in Kaluza–Klein theories and in string theory. In the literature (see for example [27]), pp-wave space-times are identified with Brinkmann waves, i.e. space-times equipped with a null covariantly constant vector field, $X^i X_i = 0$, $\nabla_j X_k = 0$ [4]. Here we conform to the more restrictive definition used in [36, 19, 17, 18]:

DEFINITION 1.2. A *pp-wave* is a Brinkmann wave whose Riemann tensor satisfies the condition $R_{jk}{}^{pq}R_{pqlm} = 0$.

The organization of the paper is the following. In Section 2 we present preliminary material about simple conformally recurrent pseudo-Riemannian manifolds and pp-wave space-times. In Section 3 we prove that in dimension $n \geq 4$ a conformally recurrent pp-wave space-time is a simple conformally recurrent Lorentzian manifold, and establish the converse implication. We obtain new curvature properties of conformally recurrent pp-waves, such as $\nabla^m \nabla_m R_{ij} = 0$. In Section 4 we show that conformally recurrent pp-wave space-times are solutions to the extended theory of gravity with pure radiation source.

Throughout the paper, the manifolds are assumed to be connected, Hausdorff, with non-degenerate metric of arbitrary signature, i.e. n -dimensional pseudo-Riemannian manifolds. Some results are restricted to Lorentzian manifolds (space-times), i.e. to metrics of signature $n - 2$.

2. Preliminary results. In this section we present preliminary material about conformally recurrent pseudo-Riemannian manifolds, simple conformally recurrent manifolds, and pp-wave space-times.

A vector field belongs to the *Olszak distribution* [26] if

$$(2.1) \quad X_i C_{jklm} + X_j C_{kilm} + X_k C_{ijlm} = 0.$$

This condition was extensively studied in the geometric literature on pseudo-Riemannian manifolds (see for example [6, 9, 10, 11]).

Contraction of (2.1) with g^{im} gives $X^m C_{jklm} = 0$. Then, contraction with X^i gives $X^i X_i C_{jklm} = 0$. If the Weyl tensor is non-zero, the Olszak distribution consists of null vectors, $X^i X_i = 0$.

LEMMA 2.1. *On an n -dimensional Lorentzian manifold with everywhere non-zero Weyl curvature, the Olszak distribution is one-dimensional.*

Proof. Suppose that, besides X_i , there exists another covector Y_i satisfying (2.1). Then also $X_i + Y_i$ belongs to the Olszak distribution, $X^i X_i = 0$, $Y^i Y_i = 0$ and $X_i Y^i = 0$. On a Lorentzian manifold, two null orthogonal vectors must be collinear, $Y_i = \mu X_i$ [35]. ■

The following general identity holds for the Weyl tensor on a pseudo-Riemannian manifold [1, eq. 3.8]:

$$(2.2) \quad \nabla_i C_{jkl}{}^m + \nabla_j C_{kil}{}^m + \nabla_k C_{ijl}{}^m = \frac{1}{n-3} \nabla_p [\delta_j^m C_{kil}{}^p + \delta_k^m C_{ijl}{}^p + \delta_i^m C_{jkl}{}^p + g_{kl} C_{ji}{}^{mp} + g_{il} C_{kj}{}^{mp} + g_{jl} C_{ik}{}^{mp}].$$

PROPOSITION 2.2 (Adati and Myazawa [1]). *Let (\mathcal{M}, g) be a conformally recurrent pseudo-Riemannian manifold of dimension $n \geq 4$. Then:*

- (1) $\nabla_m C_{jkl}{}^m = 0$ if and only if α_i belongs to the Olszak distribution;
- (2) if $\nabla_m C_{jkl}{}^m = 0$, then the recurrent covector is null, $\alpha^i \alpha_i = 0$.

Proof. Obviously $\nabla_m C_{jkl}{}^m = 0$ if and only if $\alpha_m C_{jkl}{}^m = 0$. The vanishing of the right hand side of (2.2) and recurrence imply that α_i belongs to the Olszak distribution. Conversely, we have $\alpha^i \alpha_i C_{jklm} = 0$, i.e. either $\alpha^i \alpha_i = 0$ or $C_{jklm} = 0$. ■

THEOREM 2.3 (Roter [31, 34]). *A pseudo-Riemannian manifold \mathcal{M} of dimension $n \geq 4$ is simple conformally recurrent if and only if:*

- (1) $C_{jklm} \neq 0$ everywhere on \mathcal{M} ,
- (2) $\nabla_i C_{jklm} = \alpha_i C_{jklm}$, with recurrence 1-form α_i that is locally a gradient,
- (3) the Ricci tensor is a Codazzi tensor, $\nabla_k R_{jl} = \nabla_j R_{kl}$.

Contraction of the Codazzi property with the metric tensor gives $\nabla_k R = 0$, where R is the scalar curvature. Moreover, we have:

PROPOSITION 2.4 (Roter [31]). *On a non-locally symmetric simple conformally recurrent manifold, $R = 0$.*

PROPOSITION 2.5 (Roter [34, Lemma 12]). *If the Ricci tensor of a non-locally symmetric simple conformally recurrent manifold has the form $R_{ij} = \pm d_i d_j$, then*

$$(2.3) \quad d_i C_{jklm} + d_j C_{kilm} + d_k C_{ijlm} = 0,$$

$$(2.4) \quad d_i R_{jklm} + d_j R_{kilm} + d_k R_{ijlm} = 0.$$

PROPOSITION 2.6 (Roter [34, Theorem 1]). *Every simple conformally recurrent Lorentzian manifold is Ricci-recurrent, i.e. there exists a non-zero covector field ω_i such that $\nabla_k R_{jl} = \omega_k R_{jl}$.*

Brinkmann and pp-wave space-times arise in the presence of a null covariantly constant vector field. They are special cases of Lorentzian manifolds with a recurrent null vector field, which have been studied for a long time (see for example [39, 4, 36, 17, 18]). A characterization of the metric through a set of canonical coordinates was obtained by Walker [39]. We recall Proposition 1 of [18]:

PROPOSITION 2.7. *Let (\mathcal{M}, g) be a Lorentzian manifold of dimension $n = d + 2 > 2$ with a recurrent null vector field, $X^k X_k = 0$, $\nabla_k X_j = p_k X_j$.*

- (1) *There exist coordinates (u, x^1, \dots, x^d, v) in which the metric has the local expression*

$$ds^2 = 2dudv + H(u, \vec{x}, v)du^2 + \sum_{\rho=1}^d a_\rho(u, \vec{x})dudx^\rho + \sum_{\mu, \nu=1}^d g_{\mu\nu}^*(u, \vec{x})dx^\mu dx^\nu,$$

where $g_{\mu\nu}^*$ and a_ρ are independent of the coordinate v , and H is a smooth function. We refer to such coordinates as Walker coordinates.

- (2) $\nabla_k X_j = 0$ if and only if H does not depend on v . We refer to such coordinates as Brinkmann coordinates, and to the manifold as a Brinkmann wave (space-time) [4].

A pp-wave is a Brinkmann wave with some restrictions. Schimming [36] gave the following coordinate characterization (see also [19, 17, 18]):

PROPOSITION 2.8 (Schimming [36]). *A Lorentzian manifold of dimension $d + 2 > 2$ is a pp-wave if and only if there exist coordinates (u, \vec{x}, v) in which the metric has the following local expression:*

$$(2.5) \quad ds^2 = 2du dv + H(\vec{x}, u)du^2 + \sum_{\rho=1}^d dx^{\rho 2}$$

where $H(\vec{x}, u)$ is a smooth function independent of v , usually called the potential function of the pp-wave.

The expression of the metric yields the Ricci tensor of a pp-wave:

$$(2.6) \quad R_{kl} = \psi(\vec{x}, u)X_k X_l, \quad \psi(\vec{x}, u) = -\frac{1}{2} \sum_{\rho=1}^d \frac{\partial^2 H}{\partial x^{\rho 2}},$$

where $X_k = \nabla_k u$ is a covariantly constant null vector ($X^k X_k = 0$, $\nabla_j X_k = 0$) [28]. It follows that the scalar curvature is zero, $R = 0$.

REMARK 2.9. A metric such that $R_{kl} = \psi X_k X_l$ with a null recurrent covector is called a *pure radiation metric with parallel rays*, or *aligned pure radiation metric* [20].

PROPOSITION 2.10 (Schimming [36], see also [19, 17, 18]). *A Lorentzian manifold of dimension $d + 2 > 2$ with covariantly constant null vector field*

(i.e. a Brinkmann wave) is a pp-wave if and only if one of the following conditions is satisfied:

$$(2.7) \quad X_i R_{jklm} + X_j R_{kilm} + X_k R_{ijlm} = 0;$$

$$(2.8) \quad R_{jklm} = X_j X_m D_{kl} - X_j X_l D_{mk} - X_k X_m D_{jl} + X_k X_l D_{jm};$$

$$(2.9) \quad R^p{}_{jk}{}^q R_{plmq} = \chi X_j X_k X_l X_m.$$

Here D_{ij} is a symmetric tensor and χ is a suitable scalar function.

A contraction recovers the form of the Ricci tensor of a pp-wave, $R_{kl} = \psi X_k X_l$. Leistner and Nurowski [19] showed that in dimension $d+2 = 4$ the conditions are equivalent to $R_{jk}{}^{pq} R_{pqlm} = 0$.

REMARK 2.11. It is worth noting that for a Ricci tensor of the form (2.6) we have

$$(2.10) \quad X_i C_{jklm} + X_j C_{kilm} + X_k C_{ijlm} = X_i R_{jklm} + X_j R_{kilm} + X_k R_{ijlm}.$$

Contraction with g^{im} and the equality $R_{ij} = \psi X_i X_j$ imply $X^m C_{jklm} = X^m R_{jklm}$.

For a pp-wave, by (2.7), we have $X_i C_{jklm} + X_j C_{kilm} + X_k C_{ijlm} = 0$, and so $X^m C_{jklm} = X^m R_{jklm} = 0$.

3. Simple conformally recurrent Lorentzian manifolds and pp-waves. In this section we obtain two results. First we show, in several steps, that a conformally recurrent pp-wave space-time is a simple conformally recurrent space-time. Next, we show that a simple conformally recurrent space-time is a pp-wave space-time. Finally, some curvature properties of conformally recurrent pp-wave space-times are presented, as well as some technical results.

LEMMA 3.1. *On a pp-wave space-time of dimension $n \geq 4$, the Ricci tensor:*

- (1) *is recurrent with a closed recurrence 1-form,*
- (2) *satisfies $[\nabla_i, \nabla_j] R_{kl} = 0$,*
- (3) *satisfies*

$$(3.1) \quad \nabla^j \nabla^m C_{jklm} = -\frac{n-3}{n-2} \nabla^2 R_{kl}$$

where $\nabla^2 = \nabla_k \nabla^k$. In particular, $\nabla^2 R_{kl} = 0$ if and only if $\nabla^j \nabla^m C_{jklm} = 0$.

Proof. In a coordinate system with (2.5) one has $\nabla_j R_{kl} = (\nabla_j \psi) X_k X_l = (\nabla_j \log |\psi|) R_{kl}$. From this we infer $[\nabla_i, \nabla_j] R_{kl} = 0$, i.e. a pp-wave space-time is Ricci semi-symmetric [30].

The divergence of the Weyl tensor has the general expression

$$(3.2) \quad \begin{aligned} \nabla_m C_{jkl}{}^m &= \frac{n-3}{n-2} (\nabla_k R_{jl} - \nabla_j R_{kl}) + \frac{n-3}{2(n-1)(n-2)} (g_{kl} \nabla_j R - g_{jl} \nabla_k R). \end{aligned}$$

A further covariant divergence gives

$$\nabla^j \nabla^m C_{jklm} = -\frac{n-3}{n-2} \left[\nabla^2 R_{kl} - \frac{g_{kl} \nabla^2 R}{2(n-1)} - [\nabla_j, \nabla_k] R_l{}^j \right] + \frac{n-3}{2(n-1)} \nabla_k \nabla_l R.$$

Ricci semi-symmetry and the property $R = 0$ give (3.1). ■

LEMMA 3.2 (see [21] and [23, Theorem 6.1]). *A pp-wave space-time of dimension $d+2 \geq 4$ with the metric (2.5) satisfies $\nabla_m C_{jkl}{}^m = 0$ if and only if $\nabla_k \psi = \lambda X_k$.*

Proof. For a pp-wave space-time we have $\nabla_j R_{kl} = (\nabla_j \psi) X_k X_l$ and $R = 0$. (3.2) becomes

$$(3.3) \quad \nabla_m C_{jkl}{}^m = \frac{n-3}{n-2} [(\nabla_k \psi) X_j - (\nabla_j \psi) X_k] X_l.$$

It follows that $\nabla_m C_{jkl}{}^m = 0$ if and only if $(\nabla_k \psi) X_j = (\nabla_j \psi) X_k$, which is equivalent to the condition $\nabla_k \psi = \lambda X_k$ for some scalar function λ . ■

REMARK 3.3. For a pp-wave the condition $\nabla_m C_{jkl}{}^m = 0$ is equivalent to $\nabla_k R_{jl} = \nabla_j R_{kl}$, i.e. to the Ricci tensor being a Codazzi tensor. This follows from (3.2) and $R = 0$.

PROPOSITION 3.4. *Let (\mathcal{M}, g) be a pp-wave space-time of dimension $d+2 \geq 4$ with the metric (2.5). If $\psi = -\frac{1}{2} \sum_{\rho=1}^d \partial^2 H / \partial x_\rho^2$ only depends on u , then $\nabla^2 R_{kl} = 0$.*

Proof. If $\psi = \psi(u)$ we have $\nabla_k \psi = \partial_k \psi(u) = \psi'(u) X_k$ and $\nabla_m C_{jkl}{}^m = 0$ by the previous lemma. Therefore $\nabla_k R_{ij} = \psi' X_k X_i X_j$ and $\nabla^2 R_{ij} = \psi''(u) X^k X_k X_i X_j = 0$. ■

Galaev [12] classified the indecomposable conformally recurrent Lorentzian manifolds. He showed that either the manifold is conformally flat, or $\nabla_i R_{jklm} = 0$, or it is a pp-wave. Then he found all pp-wave potential functions $H(\vec{x}, u)$ such that the Weyl tensor in the metric (2.5) is recurrent, $\nabla_i C_{jklm} = \alpha_i C_{jklm}$. The potential solves the following system of $d+1$ equations:

$$(3.4) \quad (\alpha_\rho - \partial_\rho) \Omega_{\mu\nu} = 0 \quad (\rho = 1, \dots, d),$$

$$(3.5) \quad (\alpha_u - \partial_u) \Omega_{\mu\nu} = 0,$$

where $\Omega_{\mu\nu} = \partial_\mu \partial_\nu H - \frac{1}{n} \delta_{\mu\nu} \sum_\rho \partial_\rho^2 H$ ($\mu, \nu = 1, \dots, d$).

Let $\Omega^2 = \sum_{\mu,\nu}(\Omega_{\mu\nu})^2$. In an open set $\mathcal{O} \subset \mathcal{M}$ where $\Omega^2 \neq 0$ the equations give $\alpha_\mu = \frac{1}{2}\partial_\mu \log \Omega^2$ and $\alpha_u = \frac{1}{2}\partial_u \log \Omega^2$. The potential for which the metric (2.5) is conformally recurrent turns out to be

$$(3.6) \quad H(\vec{x}, u) = \sum_{\rho=1}^d x_\rho^2 [a(u) + F(u)\lambda_\rho^2]$$

with functions $a(u)$, $F(u)$ and real numbers $\lambda_1 + \dots + \lambda_d = 0$. Correspondingly, $\psi(\vec{x}, u) = -na(u)$, and since it only depends on u , we conclude that for conformally recurrent pp-waves the divergence of the conformal tensor vanishes:

PROPOSITION 3.5. *On a conformally recurrent pp-wave space-time with the metric (2.5), we have $\nabla_m C_{jkl}{}^m = 0$; moreover, there exists a domain \mathcal{O} where the recurrence covector is a gradient.*

The recurrence 1-form α_i being closed, it follows that $[\nabla_p, \nabla_q]C_{jklm} = 0$, i.e. the manifold is Weyl semi-symmetric [8]. Moreover, as $[\nabla_i, \nabla_j]R_{kl} = 0$, the manifold is also semi-symmetric, $[\nabla_p, \nabla_q]R_{jklm} = 0$.

Now, from Proposition 3.5, Remark 3.3 and Theorem 2.3, we have:

THEOREM 3.6. *In $n \geq 4$ any conformally recurrent pp-wave space-time is a simple conformally recurrent Lorentzian manifold.*

Some further properties of conformally recurrent pp-wave space-times are collected here:

PROPOSITION 3.7. *For a conformally recurrent pp-wave space-time:*

- (1) *the recurrence covector α_i is null and collinear with the covariantly constant covector $X_i = \nabla_i u$;*
- (2) *the recurrence covector is recurrent with closed recurrence 1-form.*

Proof. From Propositions 3.5 and 2.2 we get $\alpha_i C_{jklm} + \alpha_j C_{kilm} + \alpha_k C_{ijlm} = 0$, i.e. the recurrence covector of a conformally recurrent pp-wave space-time belongs to the Olszak distribution. In view of Remark 2.11, for a pp-wave space-time, X_i also belongs to the Olszak distribution; then, by Lemma 2.1, $\alpha_i = \mu X_i$. Taking the covariant derivative, we see that $\nabla_j \alpha_i = (\nabla_j \mu)X_i = (\nabla_j \log |\mu|)\alpha_i \equiv q_j \alpha_i$ (i.e. the recurrence vector is itself recurrent, with closed recurrence 1-form). ■

PROPOSITION 3.8. *Let (\mathcal{M}, g) be a pp-wave space-time of dimension $d+2 \geq 4$ with the metric (2.5). Then there exists a coordinate domain \mathcal{O} in \mathcal{M} where $\nabla^2 C_{jklm} = 0$ and $\nabla^2 R_{jklm} = 0$.*

Proof. From the closedness condition $\nabla_j \alpha_i = \nabla_i \alpha_j$ we infer $q_i \alpha_j = q_j \alpha_i$, so that $q_j = \rho \alpha_i$ and the recurrence 1-form α_i satisfies $\nabla_j \alpha_i = \rho \alpha_i \alpha_j$ and

$\nabla^i \alpha_i = 0$. Thus, since $\alpha^i \alpha_i = 0$, we have $\alpha^i \nabla_i C_{jklm} = 0$, and consequently

$$\nabla^i \nabla_i C_{jklm} = \nabla^i (\alpha_i C_{jklm}) = (\nabla^i \alpha_i) C_{jklm} + \alpha^i (\nabla_i C_{jklm}) = 0.$$

Moreover, $\nabla^i \nabla_i R_{kl} = 0$ so that $\nabla^i \nabla_i R_{jklm} = 0$. ■

Now we consider an n -dimensional Lorentzian simple conformally recurrent manifold and show that it is a conformally recurrent pp-wave space-time.

THEOREM 3.9. *Let (\mathcal{M}, g) be a Lorentzian simple conformally recurrent manifold of dimension $n \geq 4$. Then there exists a coordinate domain \mathcal{O} in \mathcal{M} where the manifold is a pp-wave space-time.*

Proof. From Theorem 2.3 and Proposition 2.4 we have $\nabla_m C_{jkl}{}^m = 0$ so that $\alpha^m C_{jklm} = 0$ and $\alpha_i C_{jklm} + \alpha_j C_{kilm} + \alpha_k C_{ijlm} = 0$. Taking the covariant derivative, we get $(\nabla_p \alpha_i) C_{jklm} + (\nabla_p \alpha_j) C_{kilm} + (\nabla_p \alpha_k) C_{ijlm} = 0$. Since in a Lorentzian manifold the Olszak distribution is one-dimensional (Lemma 2.1), this implies proportionality, $\nabla_i \alpha_j = p_i \alpha_j$ for some 1-form p_j .

Proposition 2.6 states that a simple conformally recurrent Lorentzian manifold is Ricci recurrent, i.e. $\nabla_k R_{jl} = \omega_k R_{jl}$ for some 1-form ω_k . Furthermore, by Theorem 2.3, the Ricci tensor is Codazzi, $\nabla_k R_{jl} = \nabla_j R_{kl}$. Then

$$(3.7) \quad \omega_j R_{kl} = \omega_k R_{jl}.$$

Contraction with g^{jl} gives $\omega^l R_{kl} = 0$ because $R = 0$ (Theorem 2.3). Thus on multiplying (3.7) by ω_k it follows that $(\omega^k \omega_k) R_{jl} = 0$, i.e. ω_k is a null vector.

Let θ^k be a vector such that $\theta^k \omega_k = 1$. Equation (3.7) gives $R_{jl} = \omega_j \theta^k R_{kl}$, and by symmetry $\omega_j \theta^k R_{kl} = \omega_l \theta^k R_{kj}$. Thus $\theta^j R_{jl} = \omega_l (\theta^k \theta^j R_{kj})$, from which we finally get

$$(3.8) \quad R_{ij} = \psi \omega_i \omega_j$$

where $\psi = \theta^m \theta^j R_{mj}$ is a scalar function.

Let $d = \psi/|\psi|$ and $d_j = \omega_j \sqrt{|\psi|}$. Then $R_{ij} = dd_i d_j$ with $d^k d_k = 0$. By Proposition 2.5, equations (2.3) and (2.4) are recovered. In this way the vector d_j belongs to the one-dimensional Olszak distribution; thus $d_i = \epsilon \alpha_i$ and the Ricci tensor takes the form $R_{jl} = \phi \alpha_j \alpha_l$. Furthermore, from

$$(3.9) \quad \alpha_i R_{jklm} + \alpha_j R_{kilm} + \alpha_k R_{ijlm} = 0,$$

Remark 2.11 gives $\alpha^m R_{jklm} = 0$. Taking the covariant derivative of $\nabla_k \alpha_l = p_k \alpha_l$ and using skew symmetrization, we obtain $\alpha^m R_{jklm} = (\nabla_j p_k - \nabla_k p_j) \alpha_l = 0$. Thus the covector p_j is locally a gradient, i.e. there exists a coordinate

domain \mathcal{O} of (\mathcal{M}, g) where $p_j = \nabla_j \eta$. The covector $\bar{\alpha}_j = \alpha_j e^{-\eta}$ is covariantly constant ($\nabla_k \bar{\alpha}_j = 0$). Equation (3.9) with $\bar{\alpha}_j$ is Schimming's condition (2.7) for the manifold to be locally a pp-wave space-time. ■

4. Some consequences for extended theories of gravitation. We derive some consequences for extended theories of gravitation in dimensions $n \geq 4$. Since for a pp-wave the scalar curvature is zero, a pp-wave solves Einstein's field equations (in natural units)

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi T_{ij}$$

with a pure radiation source, $T_{ij} = \Phi^2 k_i k_j$, where $k_i k^i = 0$ [37, eq. 5.8].

Generic theories of gravitation modify Einstein's theory at short distances by expressing the action integral with zero source as a power series

$$(4.1) \quad I = \int d^n x \sqrt{-g} \left[R - 2\Lambda_0 + \alpha R^2 + \beta R_{ij} R^{ij} + \gamma (R_{jklm} R^{jklm} - 4R_{jk} R^{jk} + R^2) + \sum_{p>2} C_p (\text{Riem}, \text{Ric}, \nabla \text{Riem}, \dots) \right]$$

where $\alpha, \beta, \gamma, C_p$ are parameters provided by some microscopic theory such as string theory. The quadratic part represents "quadratic gravity" (Weyl–Eddington terms), and the terms of higher order are contracted products of the Riemann tensor and its derivatives. Such corrections scale as powers of ℓ_P/L , where ℓ_P is Planck's length and L is a typical length for the variation of the metric [40], and could be significant in the early evolution of the universe.

The field equations of the full theory are complicated; nevertheless Gürses et al. [14] proved that for pp-wave metrics (2.5) the field equations with cosmological constant $\Lambda_0 = 0$ may be written as

$$(4.2) \quad [a_0 + a_1 \nabla^2 + a_2 (\nabla^2)^2 + \dots] R_{kl} = 0$$

where a_i are constants depending on $\alpha, \beta, \gamma, C_p$. If $\nabla^2 R_{kl} = 0$ the equation reduces to $a_0 R_{kl} = 0$, with the vacuum solution. In the same paper the authors noted that for pp-waves with $\nabla^2 R_{kl} = 0$ the field equations may include a pure radiation source:

$$(4.3) \quad a_0 R_{kl} = T_{kl}.$$

It is thus of interest to find non-vacuum pp-wave solutions. According to Proposition 3.4, if $\psi = -\frac{1}{2} \sum_{\rho=1}^d \partial_\rho^2 H$ depends only on u , then $\nabla^2 R_{kl} = 0$.

PROPOSITION 4.1. *Let (\mathcal{M}, g) be a pp-wave space-time of dimension $d + 2 \geq 4$ with the metric (2.5). If $\sum_{\rho=1}^d \partial_\rho^2 H$ depends only on u , then the metric solves the field equations of the generic theory of gravitation with pure radiation source.*

We give two examples where $\nabla^2 R_{kl} = 0$:

- (1) The pp-wave metric is conformally recurrent, with potential H given by (3.6).
- (2) The pp-wave metric is two-symmetric, i.e. $\nabla_p \nabla_i R_{jklm} = 0$. This occurs if and only if the potential function of the pp-wave has the form

$$(4.4) \quad H(\vec{x}, u) = \sum_{\mu, \nu=1}^d (ua_\mu \delta_{\mu\nu} + b_{\mu\nu}) x^\mu x^\nu$$

where $0 \leq a_1 \leq \dots \leq a_d$ and $b_{\mu\nu} = b_{\nu\mu}$ are real numbers [2, 3, 13]. One evaluates $\psi(u) = -\sum_{\mu=1}^d (ua_\mu + b_{\mu\mu})$. Thus for two-symmetric pp-wave space-times the divergence of the conformal tensor vanishes, and as one may directly calculate, $\nabla^2 R_{kl} = 0$.

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