

Sums of four polygonal numbers with coefficients

by

XIANG-ZI MENG and ZHI-WEI SUN (Nanjing)

1. Introduction. Let $m \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$. The *polygonal numbers of order $m+2$* (or *$(m+2)$ -gonal numbers*), which are constructed geometrically from regular polygons with $m+2$ sides, are given by

$$(1.1) \quad p_{m+2}(n) := m \binom{n}{2} + n = \frac{mn^2 - (m-2)n}{2}$$

for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. Clearly,

$$p_{m+2}(0) = 0, \quad p_{m+2}(1) = 1, \quad p_{m+2}(2) = m+2, \quad p_{m+2}(3) = 3m+3,$$

and $p_{m+2}(x)$ with $x \in \mathbb{Z}$ are called *generalized $(m+2)$ -gonal numbers*. It is easy to see that generalized hexagonal numbers coincide with triangular numbers (i.e., $p_3(n) = n(n+1)/2$ with $n \in \mathbb{N}$). Note that

$$p_4(n) = n^2, \quad p_5(n) = n(3n-1)/2, \quad p_6(n) = n(2n-1) = p_3(2n-1).$$

Fermat's claim that each $n \in \mathbb{N}$ can be written as the sum of $m+2$ polygonal numbers of order $m+2$ was proved by Lagrange in the case $m=2$, by Gauss in the case $m=1$, and by Cauchy in the case $m \geq 3$ (cf. [9, pp. 3–35] and [7, pp. 54–57]). In 1830 Legendre refined Cauchy's polygonal number theorem by showing that any integer $N \geq 28m^3$ with $m \geq 3$ can be written as

$$p_{m+2}(x_1) + p_{m+2}(x_2) + p_{m+2}(x_3) + p_{m+2}(x_4) + \delta_m(N)$$

where $x_1, x_2, x_3, x_4 \in \mathbb{N}$, $\delta_m(N) = 0$ if $2 \nmid m$, and $\delta_m(N) \in \{0, 1\}$ if $2 \mid m$. Nathanson ([8] and [9, p. 33]) simplified the proofs of Cauchy's and Legendre's theorems.

In 1917 Ramanujan [10] listed 55 possible quadruples (a, b, c, d) of positive integers with $a \leq b \leq c \leq d$ such that any $n \in \mathbb{N}$ can be written as

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$ax^2 + by^2 + cz^2 + dw^2$ with $x, y, z, w \in \mathbb{Z}$, and 54 of them were later confirmed by Dickson [2] while the remaining one on the list was actually wrong.

Recently, Sun [12] showed that any positive integer can be written as the sum of four generalized octagonal numbers one of which is odd. He also proved that for many triples (b, c, d) of positive integers (including $(1, 1, 3)$, $(1, 2, 2)$ and $(1, 2, 4)$) we have

$$\{p_8(x_1) + bp_8(x_2) + cp_8(x_3) + dp_8(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{Z}\} = \mathbb{N}.$$

In [12, Conjecture 5.3], Sun conjectured that any $n \in \mathbb{N}$ can be written as $p_6(x_1) + p_6(x_2) + 2p_6(x_3) + 4p_6(x_4)$ with $x_1, x_2, x_3, x_4 \in \mathbb{N}$.

Motivated by the above work, for $(a, b) = (1, 1), (2, 2), (1, 3), (2, 4)$ and $m \in \{3, 4, 5, \dots\}$, we study whether any sufficiently large integer can be written as

$$p_{m+2}(x_1) + p_{m+2}(x_2) + ap_{m+2}(x_3) + bp_{m+2}(x_4) \quad \text{with } x_1, x_2, x_3, x_4 \in \mathbb{N}.$$

Now we state our main results.

THEOREM 1.1. *Let $m \in \mathbb{Z}^+$ with $4 \mid m$.*

(i) *Any integer $N \geq 28m^3$ can be expressed as*

$$(1.2) \quad p_{m+2}(x_1) + p_{m+2}(x_2) + p_{m+2}(x_3) + p_{m+2}(x_4) \quad (x_1, x_2, x_3, x_4 \in \mathbb{N}).$$

(ii) *There are infinitely many positive integers not of the form*

$$p_{m+4}(x_1) + p_{m+4}(x_2) + p_{m+4}(x_3) + p_{m+4}(x_4)$$

with $x_1, x_2, x_3, x_4 \in \mathbb{N}$.

REMARK 1.1. This can be viewed as a supplement to Legendre’s theorem. By Theorem 1.1(ii), there are infinitely many positive integers none of which is the sum of four octagonal numbers; in contrast, Sun [12] showed that any $n \in \mathbb{N}$ is the sum of four generalized octagonal numbers.

COROLLARY 1.1. *We have*

$$(1.3) \quad \{p_6(x_1) + p_6(x_2) + p_6(x_3) + p_6(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} \\ = \mathbb{N} \setminus \{5, 10, 11, 20, 25, 26, 38, 39, 54, 65, 70, 114, 130\},$$

and hence any $n \in \mathbb{N}$ can be written as the sum of a triangular number and three hexagonal numbers. Also, any integer $n > 2146$ can be written as the sum of four decagonal numbers, and thus

$$(1.4) \quad \{p_{10}(x_1) + p_{10}(x_2) + p_{10}(x_3) + p_{10}(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{Z}\} \\ = \mathbb{N} \setminus \{5, 6, 26\}.$$

Proof. Via a computer, we can easily verify that

$$5, 10, 11, 20, 25, 26, 38, 39, 54, 65, 70, 114, 130$$

are the only natural numbers smaller than 28×4^3 which cannot be written as the sum of four hexagonal numbers, but all these numbers can be expressed as the sum of a triangular number and three hexagonal numbers. Also, every $n = 2147, \dots, 28 \times 8^3 - 1$ is the sum of four decagonal numbers, and 5, 6 and 26 are the only natural numbers smaller than 2147 which cannot be written as the sum of four generalized decagonal numbers. Now it suffices to apply Theorem 1.1 with $m = 4, 8$. ■

REMARK 1.2. Sun [11, Conjecture 1.10] conjectured that any $n \in \mathbb{N}$ can be written as the sum of two triangular numbers and a hexagonal number. Krachun [6] proved that

$$\begin{aligned} & \{p_6(-w) + p_6(-x) + p_6(y) + p_6(z) : w, x, y, z \in \mathbb{N}\} \\ & = \{p_6(-w) + 2p_6(-x) + p_6(y) + 2p_6(z) : w, x, y, z \in \mathbb{N}\} = \mathbb{N}, \end{aligned}$$

as conjectured by the second author [12].

THEOREM 1.2. *Let $m \geq 3$ be an integer.*

(i) *Suppose that $2 \nmid m$ or $4 \mid m$. Then any integer $N \geq 1628m^3$ can be written as*

$$(1.5) \quad p_{m+2}(x_1) + p_{m+2}(x_2) + 2p_{m+2}(x_3) + 2p_{m+2}(x_4) \quad (x_1, x_2, x_3, x_4 \in \mathbb{N}).$$

(ii) *If $m \equiv 2 \pmod{4}$, then there are infinitely many positive integers not representable by $p_{m+2}(x_1) + p_{m+2}(x_2) + 2p_{m+2}(x_3) + 2p_{m+2}(x_4)$ with $x_1, x_2, x_3, x_4 \in \mathbb{N}$.*

REMARK 1.3. Actually our proof of Theorem 1.2(i) given in Section 3 allows us to replace $1628m^3$ by $418m^3$ in the case $m \equiv 1 \pmod{2}$. By Theorem 1.2(ii), there are infinitely many positive integers not representable by $p_8(x_1) + p_8(x_2) + 2p_8(x_3) + 2p_8(x_4)$ with $x_1, x_2, x_3, x_4 \in \mathbb{N}$; in contrast, Sun [12] proved that any $n \in \mathbb{N}$ can be written as $p_8(x_1) + p_8(x_2) + 2p_8(x_3) + 2p_8(x_4)$ with $x_1, x_2, x_3, x_4 \in \mathbb{Z}$.

COROLLARY 1.2. *We have*

$$(1.6) \quad \{p_5(x_1) + p_5(x_2) + 2p_5(x_3) + 2p_5(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} = \mathbb{N},$$

$$(1.7) \quad \{p_6(x_1) + p_6(x_2) + 2p_6(x_3) + 2p_6(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} \\ = \mathbb{N} \setminus \{22, 82, 100\},$$

$$(1.8) \quad \{p_7(x_1) + p_7(x_2) + 2p_7(x_3) + 2p_7(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} \\ = \mathbb{N} \setminus \{13, 26, 31, 65, 67, 173, 175, 215, 247\}.$$

Also, for each $k = 9, 10, 11$, any integer $n > C_k$ can be written as

$$p_k(x_1) + p_k(x_2) + 2p_k(x_3) + 2p_k(x_4)$$

with $x_1, x_2, x_3, x_4 \in \mathbb{N}$, where $C_9 = 925$, $C_{10} = 840$ and $C_{11} = 1799$. There-

fore,

$$(1.9) \quad \{p_3(w) + p_6(x) + 2p_6(y) + 2p_6(z) : w, x, y, z \in \mathbb{N}\} = \mathbb{N},$$

$$(1.10) \quad \{2p_3(w) + p_6(x) + p_6(y) + 2p_6(z) : w, x, y, z \in \mathbb{N}\} = \mathbb{N},$$

$$(1.11) \quad \{p_7(w) + p_7(x) + 2p_7(y) + 2p_7(z) : w \in \mathbb{Z} \ \& \ x, y, z \in \mathbb{N}\} = \mathbb{N},$$

$$(1.12) \quad \{p_9(w) + 2p_9(x) + p_9(y) + 2p_9(z) : w, x \in \mathbb{Z} \ \& \ y, z \in \mathbb{N}\} = \mathbb{N},$$

$$(1.13) \quad \{p_{10}(w) + 2p_{10}(x) + p_{10}(y) + 2p_{10}(z) : w, x \in \mathbb{Z} \ \& \ y, z \in \mathbb{N}\} = \mathbb{N},$$

$$(1.14) \quad \{p_{11}(w) + 2p_{11}(x) + p_{11}(y) + 2p_{11}(z) : w, x \in \mathbb{Z} \ \& \ y, z \in \mathbb{N}\} \\ = \mathbb{N} \setminus \{7\}.$$

Proof. Note that $\{p_6(w) : w \in \mathbb{Z}\} = \{p_3(w) : w \in \mathbb{N}\}$. It suffices to apply Theorem 1.2(i) with $m \in \{3, 4, 5, 7, 8, 9\}$ and check those $n \in \mathbb{N}$ with $n < 1628m^3$ via a computer. ■

REMARK 1.4. (1.6) appeared as part of [12, Conjecture 5.2(ii)], and it indicates that the set $\{p_5(x)+2p_5(y) : x, y \in \mathbb{N}\}$ is an additive base of order 2. For positive integers a, b, c with $\{ap_5(x) + bp_5(y) + cp_5(z) : x, y, z \in \mathbb{Z}\} = \mathbb{N}$, see [11] and [4].

THEOREM 1.3. *Let $m \geq 3$ be an integer. Then each integer $N \geq 924m^3$ can be expressed as*

$$(1.15) \quad p_{m+2}(x_1) + p_{m+2}(x_2) + p_{m+2}(x_3) + 3p_{m+2}(x_4) \quad (x_1, x_2, x_3, x_4 \in \mathbb{N}).$$

COROLLARY 1.3. *We have*

$$(1.16) \quad \{p_5(x_1) + p_5(x_2) + p_5(x_3) + 3p_5(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} = \mathbb{N} \setminus \{19\},$$

$$(1.17) \quad \{p_6(x_1) + p_6(x_2) + p_6(x_3) + 3p_6(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} \\ = \mathbb{N} \setminus \{14, 23, 41, 42, 83\},$$

$$(1.18) \quad \{p_7(x_1) + p_7(x_2) + p_7(x_3) + 3p_7(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} \\ = \mathbb{N} \setminus \{13, 16, 27, 31, 33, 49, 50, 67, 87, 178, 181, 259\}.$$

Also, for each $k = 8, 9, 10$, any integer $n > M_k$ can be written as $p_k(x_1) + p_k(x_2) + p_k(x_3) + 3p_k(x_4)$ with $x_1, x_2, x_3, x_4 \in \mathbb{N}$, where $M_8 = 435$, $M_9 = 695$ and $M_{10} = 916$. Therefore

$$(1.19) \quad \{p_7(w) + p_7(x) + p_7(y) + 3p_7(z) : w \in \mathbb{Z} \ \& \ x, y, z \in \mathbb{N}\} = \mathbb{N},$$

$$(1.20) \quad \{p_9(x_1) + p_9(x_2) + p_9(x_3) + 3p_9(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{Z}\} = \mathbb{N} \setminus \{17\},$$

$$(1.21) \quad \{p_{10}(x_1) + p_{10}(x_2) + p_{10}(x_3) + 3p_{10}(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{Z}\} \\ = \mathbb{N} \setminus \{16, 19\}.$$

Proof. It suffices to apply Theorem 1.3 with $m \in \{3, 4, 5, 6, 7, 8\}$ and check those $n \in \mathbb{N}$ with $n < 924m^3$ via a computer. ■

REMARK 1.5. Guy [5] thought that 10, 16 and 76 might be the only natural numbers which cannot be written as the sum of three generalized

heptagonal numbers. The second author [12, Remark 5.2 and Conjecture 1.2] conjectured that $\{p_7(x) + p_7(y) + p_7(z) : x, y, z \in \mathbb{Z}\} = \mathbb{N} \setminus \{10, 16, 76, 307\}$ and

$$\{p_8(x) + p_8(y) + 3p_8(z) : x, y, z \in \mathbb{Z}\} = \mathbb{N} \setminus \{7, 14, 18, 91\}.$$

THEOREM 1.4. *Let $m \geq 3$ be an integer. Then any integer $N \geq 1056m^3$ can be written as*

$$(1.22) \quad p_{m+2}(x_1) + p_{m+2}(x_2) + 2p_{m+2}(x_3) + 4p_{m+2}(x_4) \quad (x_1, x_2, x_3, x_4 \in \mathbb{N}).$$

COROLLARY 1.4. *We have*

$$(1.23) \quad \{p_5(x_1) + p_5(x_2) + 2p_5(x_3) + 4p_5(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} = \mathbb{N},$$

$$(1.24) \quad \{p_6(x_1) + p_6(x_2) + 2p_6(x_3) + 4p_6(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} = \mathbb{N},$$

$$(1.25) \quad \{p_7(x_1) + p_7(x_2) + 2p_7(x_3) + 4p_7(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} \\ = \mathbb{N} \setminus \{17, 51\},$$

$$(1.26) \quad \{p_8(x_1) + p_8(x_2) + 2p_8(x_3) + 4p_8(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} \\ = \mathbb{N} \setminus \{19, 30, 39, 59, 78, 91\},$$

$$(1.27) \quad \{p_9(x_1) + p_9(x_2) + 2p_9(x_3) + 4p_9(x_4) : x_1, x_2, x_3, x_4 \in \mathbb{N}\} \\ = \mathbb{N} \setminus \{17, 21, 34, 41, 44, 67, 89, 104, 119, 170, 237, 245, 290\}.$$

Also, for each $k = 10, 11, 12$ any integer $n > N_k$ can be written as

$$p_k(x_1) + p_k(x_2) + 2p_k(x_3) + 4p_k(x_4)$$

with $x_1, x_2, x_3, x_4 \in \mathbb{N}$, where $N_{10} = 333$, $N_{11} = 734$ and $N_{12} = 1334$. Therefore,

$$(1.28) \quad \{p_k(w) + p_k(x) + 2p_k(y) + 4p_k(z) : w \in \mathbb{Z} \text{ \& } x, y, z \in \mathbb{N}\} = \mathbb{N}$$

for $k = 7, 9$, and

$$(1.29) \quad \{p_k(w) + p_k(x) + 2p_k(y) + 4p_k(z) : w, x, y, z \in \mathbb{Z}\} = \mathbb{N}$$

for $k = 8, 10, 11, 12$.

Proof. It suffices to apply Theorem 1.4 with $m \in \{3, \dots, 10\}$ and check those $n \in \mathbb{N}$ with $n < 1056m^3$ via a computer. ■

REMARK 1.6. (1.23) and (1.24) were conjectured by the second author [12, Conjectures 5.2(ii) and 5.3]. Sun [12, Remark 5.2] also conjectured that

$$\{p_7(x) + 2p_7(y) + 4p_7(z) : x, y, z \in \mathbb{Z}\} = \mathbb{N} \setminus \{131, 146\}.$$

We will prove Theorems 1.1–1.4 in Sections 2–5 respectively.

Throughout this paper, for a prime p and $a, n \in \mathbb{N}$, by $p^a \parallel n$ we mean $p^a \mid n$ and $p^{a+1} \nmid n$. For example, $4 \parallel n$ if and only if $n \equiv 4 \pmod{8}$.

2. Proof of Theorem 1.1. We first give a lemma which is a slight variant of [9, Lemma 1.10].

LEMMA 2.1. *Let $l, m, N \in \mathbb{Z}^+$ with $N \geq 7l^2m^3$. Then the length of the interval*

$$(2.1) \quad I_1 = \left[\frac{1}{2} + \sqrt{\frac{6N}{m} - 3}, \frac{2}{3} + \sqrt{\frac{8N}{m}} \right]$$

is greater than lm .

Proof. Let L_1 denote the length of I_1 . Then $L_1 = \sqrt{8x} - \sqrt{6x - 3} + 1/6$, where $x = N/m \geq 7l^2m^2$. Let $l_0 = lm - 1/6$. Then

$$\begin{aligned} L_1 > lm &\Leftrightarrow \sqrt{8x} > \sqrt{6x - 3} + l_0 \\ &\Leftrightarrow 2x + 3 - l_0^2 > 2l_0\sqrt{6x - 3} \\ &\Leftrightarrow 4x(x + 3 - 7l_0^2) + (l_0^2 - 3)^2 + 12l_0^2 > 0. \end{aligned}$$

As $x \geq 7l_0^2$, by the above we have $L_1 > lm$. ■

The following lemma is a slight modification of [9, Lemma 1.11].

LEMMA 2.2. *Let $a, b, m, N \in \mathbb{Z}^+$ with $m \geq 3$ and*

$$N = \frac{m}{2}(a - b) + b \geq \frac{2}{3}m.$$

Suppose that b belongs to the interval I_1 given by (2.1). Then

$$(2.2) \quad b^2 < 4a \quad \text{and} \quad 3a < b^2 + 2b + 4.$$

Proof. Observe that

$$a = \left(1 - \frac{2}{m}\right)b + \frac{2N}{m}$$

and

$$\begin{aligned} b^2 - 4a &= b^2 - 4\left(1 - \frac{2}{m}\right)b - \frac{8N}{m} \\ &= \left(b - 2\left(1 - \frac{2}{m}\right)\right)^2 - 4\left(\left(1 - \frac{2}{m}\right)^2 + \frac{2N}{m}\right). \end{aligned}$$

As $m \geq 3$ and $b \in I_1$, we have

$$b \leq \frac{2}{3} + \sqrt{\frac{8N}{m}} < 2\left(1 - \frac{2}{m}\right) + 2\sqrt{\left(1 - \frac{2}{m}\right)^2 + \frac{2N}{m}},$$

and hence $b^2 - 4a < 0$. On the other hand, since $(1/2 - 3/m)^2 < 1$ and $b \in I_1$, we have

$$b \geq \frac{1}{2} + \sqrt{\frac{6N}{m} - 3} > \frac{1}{2} - \frac{3}{m} + \sqrt{\left(\frac{1}{2} - \frac{3}{m}\right)^2 - 4 + \frac{6N}{m}},$$

and hence

$$\begin{aligned} b^2 + 2b + 4 - 3a &= b^2 - \left(1 - \frac{6}{m}\right)b + \left(4 - \frac{6N}{m}\right) \\ &= \left(b - \left(\frac{1}{2} - \frac{3}{m}\right)\right)^2 - \left(\left(\frac{1}{2} - \frac{3}{m}\right)^2 - 4 + \frac{6N}{m}\right) > 0. \end{aligned}$$

This proves (2.2). ■

LEMMA 2.3. *Let a, b, c be positive integers and let x, y, z be real numbers. Then*

$$(2.3) \quad (ax + by + cz)^2 \leq (a + b + c)(ax^2 + by^2 + cz^2).$$

Proof. By the Cauchy–Schwarz inequality (cf. [9, p. 178]),

$$\begin{aligned} (\sqrt{a}(\sqrt{ax}) + \sqrt{b}(\sqrt{by}) + \sqrt{c}(\sqrt{cz}))^2 \\ \leq ((\sqrt{a})^2 + (\sqrt{b})^2 + (\sqrt{c})^2)((\sqrt{ax})^2 + (\sqrt{by})^2 + (\sqrt{cz})^2). \end{aligned}$$

This yields (2.3). ■

The following lemma with $2 \nmid ab$ is usually called Cauchy's Lemma (cf. [9, pp. 31–34]).

LEMMA 2.4. *Let a and b be positive integers satisfying (2.2). Suppose that $2 \nmid ab$, or $2 \parallel a$ and $2 \mid b$. Then there exist $s, t, u, v \in \mathbb{N}$ such that*

$$(2.4) \quad a = s^2 + t^2 + u^2 + v^2 \quad \text{and} \quad b = s + t + u + v.$$

Proof. By the Gauss–Legendre theorem (cf. [9, Section 1.5]), we have

$$(2.5) \quad \{x^2 + y^2 + z^2 : x, y, z \in \mathbb{Z}\} = \mathbb{N} \setminus \{4^k(8l + 7) : k, l \in \mathbb{N}\}.$$

We claim that there are $x, y, z \in \mathbb{Z}$ with $4a - b^2 = x^2 + y^2 + z^2$ such that all the numbers

$$(2.6) \quad \begin{aligned} s &= \frac{b + x + y + z}{4}, & t &= \frac{b + x - y - z}{4}, \\ u &= \frac{b - x + y - z}{4}, & v &= \frac{b - x - y + z}{4} \end{aligned}$$

are integers.

CASE 1: $2 \nmid ab$. In this case, $4a - b^2 \equiv 3 \pmod{8}$ and hence $4a - b^2 = x^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{Z}$ with $2 \nmid xyz$. Without loss of generality, we may assume that $x \equiv y \equiv z \equiv b \pmod{4}$. (If $x \equiv -b \pmod{4}$ then $-x \equiv b \pmod{4}$.) Thus the numbers in (2.6) are all integers.

CASE 2: $2 \parallel a$ and $2 \mid b$. Write $a = 2a_0$ and $b = 2b_0$ with $a_0, b_0 \in \mathbb{Z}$ and $2 \nmid a_0$. Since $2a_0 - b_0^2 \equiv 1, 2 \pmod{4}$, we have $2a_0 - b_0^2 = x_0^2 + y_0^2 + z_0^2$ for some $x_0, y_0, z_0 \in \mathbb{Z}$. Without loss of generality, we may assume that $x_0 \equiv b_0 \pmod{2}$ and $y_0 \equiv z_0 \pmod{2}$. Set $x = 2x_0$, $y = 2y_0$ and $z = 2z_0$. Then $4a - b^2 = 4(2a_0 - b_0^2) = x^2 + y^2 + z^2$, and the numbers in (2.6) are all integers.

In either case, there are $x, y, z \in \mathbb{Z}$ for which $4a - b^2 = x^2 + y^2 + z^2$ and $s, t, u, v \in \mathbb{Z}$, where s, t, u, v are as in (2.6). Obviously, $s + t + u + v = b$ and

$$\begin{aligned} s^2 + t^2 + u^2 + v^2 &= 2\left(\frac{s+t}{2}\right)^2 + 2\left(\frac{s-t}{2}\right)^2 + 2\left(\frac{u+v}{2}\right)^2 + 2\left(\frac{u-v}{2}\right)^2 \\ &= 2\left(\frac{b+x}{4}\right)^2 + 2\left(\frac{y+z}{4}\right)^2 + 2\left(\frac{b-x}{4}\right)^2 + 2\left(\frac{y-z}{4}\right)^2 \\ &= \frac{b^2 + x^2 + y^2 + z^2}{4} = a. \end{aligned}$$

In view of Lemma 2.3 and the second inequality in (2.2), we have

$$(|x| + |y| + |z|)^2 \leq 3(x^2 + y^2 + z^2) = 3(4a - b^2) < (b + 4)^2.$$

Therefore

$$s, t, u, v \geq \frac{b - |x| - |y| - |z|}{4} > -1,$$

and hence $s, t, u, v \in \mathbb{N}$. ■

Now we need one more lemma which is well known (cf. [1, p. 59]).

LEMMA 2.5. *For any $n \in \mathbb{Z}^+$, we have*

$$(2.7) \quad r_4(n) = 8 \sum_{d|n, 4 \nmid d} d$$

where

$$r_4(n) := |\{(w, x, y, z) \in \mathbb{Z}^4 : w^2 + x^2 + y^2 + z^2 = n\}|.$$

Proof of Theorem 1.1. (i) Let $I_1 = [\alpha, \beta]$ be the interval given by (2.1). As $N \geq 7 \times 2^2 m^3$, by Lemma 2.1 the length of I_1 is greater than $2m$. Choose $b_0 \in \{[\alpha] + r : r = 0, \dots, m - 1\}$ with $b_0 \equiv N \pmod{m}$. Then $b_1 = b_0 + m \leq [\alpha] + 2m - 1 < \alpha + 2m < \beta$. Thus both b_0 and b_1 lie in I_1 . Note that

$$\frac{2}{m}(N - b_1) + b_1 - \left(\frac{2}{m}(N - b_0) + b_0\right) = m - 2 \equiv 2 \pmod{4}.$$

So, for some $b \in \{b_0, b_1\}$ and $a = \frac{2}{m}(N - b) + b$ we have $2 \nmid ab$, or $2 \parallel a$ and $2 \mid b$. Obviously,

$$b \geq \min I_1 > 0, \quad a = \frac{2}{m}N + \left(1 - \frac{2}{m}\right)b > 0, \quad N = \frac{m}{2}(a - b) + b.$$

Applying Lemmas 2.2 and 2.4, we see that there are $s, t, u, v \in \mathbb{N}$ satisfy-

ing (2.4). Therefore,

$$\begin{aligned} N &= \frac{m}{2}(a-b) + b \\ &= \frac{m}{2}(s^2 - s + t^2 - t + u^2 - u + v^2 - v) + s + t + u + v \\ &= p_{m+2}(s) + p_{m+2}(t) + p_{m+2}(u) + p_{m+2}(v). \end{aligned}$$

(ii) Write $m = 4l$ with $l \in \mathbb{Z}^+$. Let φ denote Euler's totient function. We want to show that none of the positive integers

$$4l^2 \times \frac{4^{k\varphi(2l+1)} - 1}{2l+1} \quad (k = 1, 2, \dots)$$

can be written as $\sum_{i=1}^4 p_{m+4}(x_i)$ with $x_1, x_2, x_3, x_4 \in \mathbb{N}$.

Suppose that for some $n \in \mathbb{Z}^+$ divisible by $\varphi(2l+1)$ there are $w, x, y, z \in \mathbb{N}$ such that

$$\begin{aligned} 4l^2 \times \frac{4^n - 1}{2l+1} &= p_{m+4}(w) + p_{m+4}(x) + p_{m+4}(y) + p_{m+4}(z) \\ &= \frac{4l+2}{2}(w^2 + x^2 + y^2 + z^2 - w - x - y - z) + w + x + y + z. \end{aligned}$$

Then

$$\begin{aligned} 4^{n+1}l^2 &= ((2l+1)w - l)^2 + ((2l+1)x - l)^2 \\ &\quad + ((2l+1)y - l)^2 + ((2l+1)z - l)^2. \end{aligned}$$

As $r_4(4^{n+1}l^2) = r_4(4l^2)$ by Lemma 2.5, there are $w_0, x_0, y_0, z_0 \in \mathbb{Z}$ with $w_0^2 + x_0^2 + y_0^2 + z_0^2 = 4l^2$ such that

$$\begin{aligned} (2l+1)w - l &= 2^n w_0, & (2l+1)x - l &= 2^n x_0, \\ (2l+1)y - l &= 2^n y_0, & (2l+1)z - l &= 2^n z_0. \end{aligned}$$

Since $2^n \equiv 1 \pmod{2l+1}$ by Euler's theorem, we have

$$w_0 \equiv x_0 \equiv y_0 \equiv z_0 \equiv -l \pmod{2l+1}.$$

As $w_0^2 + x_0^2 + y_0^2 + z_0^2 = 4l^2$, we must have $w_0 = x_0 = y_0 = z_0 = -l$. Thus $(2l+1)w = 2^n w_0 + l = l(1 - 2^n) < 0$, which contradicts $w \in \mathbb{N}$. ■

3. Proof of Theorem 1.2

LEMMA 3.1. *Let $l, m, N \in \mathbb{Z}^+$ with $N \geq 11lm^2(lm+1)$. Then the length of the interval*

$$(3.1) \quad I_2 = \left[\frac{3}{2} + \sqrt{\frac{10N}{m}} - 3, 1 + \sqrt{\frac{12N}{m}} \right]$$

is greater than lm .

Proof. The length L_2 of I_2 is $\sqrt{12x} - \sqrt{10x - 3} - 1/2$, where $x = N/m \geq 11lm(lm + 1)$. Let $l_0 = lm + 1/2$. Then

$$\begin{aligned} L_2 > lm &\Leftrightarrow \sqrt{12x} > \sqrt{10x - 3} + l_0 \\ &\Leftrightarrow 2x + 3 - l_0^2 + 3 > 2l_0\sqrt{10x - 3} \\ &\Leftrightarrow 4x(x - 11l_0^2 + 3) + (l_0^2 - 3)^2 + 12l_0^2 > 0. \end{aligned}$$

As

$$x \geq 11lm(lm + 1) > 11l^2m^2 + 11lm + 11/4 - 3 = 11l_0^2 - 3,$$

we have $L_2 > lm$ as desired. ■

LEMMA 3.2. Let $a, b, m, N \in \mathbb{Z}^+$ with $m \geq 3$ and

$$N = \frac{m}{2}(a - b) + b \geq \frac{3}{5}m.$$

Suppose that b belongs to the interval I_2 given by (3.1). Then

$$(3.2) \quad b^2 < 6a \quad \text{and} \quad 5a < b^2 + 2b + 6.$$

Proof. Note that

$$a = \left(1 - \frac{2}{m}\right)b + \frac{2N}{m}.$$

Thus

$$\begin{aligned} b^2 - 6a &= b^2 - 6\left(1 - \frac{2}{m}\right)b - \frac{12N}{m} \\ &= \left(b - 3\left(1 - \frac{2}{m}\right)\right)^2 - 9\left(1 - \frac{2}{m}\right)^2 - \frac{12N}{m}. \end{aligned}$$

As $b \in I_2$, we have

$$b \leq 1 + \sqrt{\frac{12N}{m}} < 3\left(1 - \frac{2}{m}\right) + \sqrt{9\left(1 - \frac{2}{m}\right)^2 + \frac{12N}{m}},$$

and hence $b^2 - 6a < 0$. On the other hand,

$$\begin{aligned} b^2 + 2b + 6 - 5a &= b^2 - \left(3 - \frac{10}{m}\right)b + \left(6 - \frac{10N}{m}\right) \\ &= \left(b - \left(\frac{3}{2} - \frac{5}{m}\right)\right)^2 - \left(\frac{3}{2} - \frac{5}{m}\right)^2 + \left(6 - \frac{10N}{m}\right) > 0 \end{aligned}$$

since $(3/2 - 5/m)^2 \leq 9/4 < 3$ and

$$b \geq \frac{3}{2} + \sqrt{\frac{10N}{m} - 3} > \frac{3}{2} - \frac{5}{m} + \sqrt{\left(\frac{3}{2} - \frac{5}{m}\right)^2 - 6 + \frac{10N}{m}}.$$

This proves (3.2). ■

LEMMA 3.3. *Let a and b be positive integers with $a \equiv b \pmod{2}$ satisfying (3.2). Suppose that $2 \nmid a$ or $4 \parallel a$, and $3 \mid a$ or $3 \nmid b$. Then there exist $s, t, u, v \in \mathbb{N}$ such that*

$$(3.3) \quad a = s^2 + t^2 + 2u^2 + 2v^2 \quad \text{and} \quad b = s + t + 2u + 2v.$$

Proof. If $n \in \mathbb{N}$ is not of the form $4^k(8l + 7)$ with $k, l \in \mathbb{N}$, then by (2.5) there are $x, u, v \in \mathbb{Z}$ with $u \equiv v \pmod{2}$ such that

$$n = x^2 + u^2 + v^2 = x^2 + 2\left(\frac{u-v}{2}\right)^2 + 2\left(\frac{u+v}{2}\right)^2.$$

We claim that there are $x, y, z \in \mathbb{Z}$ with $6a - b^2 = x^2 + 2y^2 + 2z^2$ such that all the numbers

$$(3.4) \quad \begin{aligned} s &= \frac{b+x+2y+2z}{6}, & t &= \frac{b-x-2y+2z}{6}, \\ u &= \frac{b-x+y-z}{6}, & v &= \frac{b+x-y-z}{6} \end{aligned}$$

are integers.

CASE 1: $3 \nmid b$. If $a \equiv b \equiv 1 \pmod{2}$, then $6a - b^2 \equiv 1 \pmod{4}$. When $4 \parallel a$ and $2 \mid b$, we have $6a - b^2 \equiv 4, 8 \pmod{16}$. Thus $6a - b^2 = x^2 + 2y^2 + 2z^2$ for some $x, y, z \in \mathbb{Z}$. Clearly, $x \equiv b \pmod{2}$ and $y \equiv z \pmod{2}$. Note that

$$x^2 + 2y^2 + 2z^2 \equiv 6a - b^2 \equiv 2 \pmod{3}.$$

As $x^2 \not\equiv 2 \pmod{3}$, we have $3 \nmid y$ or $3 \nmid z$. Without loss of generality, we assume that $3 \nmid z$ and moreover $z \equiv b \pmod{3}$. (If $z \equiv -b \pmod{3}$ then $-z \equiv b \pmod{3}$.) As $x^2 - y^2 \equiv x^2 + 2y^2 \equiv 0 \pmod{3}$, without loss of generality we may assume that $x \equiv y \pmod{3}$. Now it is easy to see that all the numbers in (3.4) are integers.

CASE 2: $3 \mid a$ and $3 \mid b$. In this case, $a = 3a_0$ and $b = 3b_0$ for some $a_0, b_0 \in \mathbb{Z}$. If $a \equiv b \equiv 1 \pmod{2}$, then $2a_0 - b_0^2 \equiv 1 \pmod{4}$. When $4 \parallel a$ and $2 \mid b$, we have $2a_0 - b_0^2 \equiv 4, 8 \pmod{16}$. Thus $2a_0 - b_0^2 = x_0^2 + 2y_0^2 + 2z_0^2$ for some $x_0, y_0, z_0 \in \mathbb{Z}$. It follows that $x_0 \equiv b_0 \pmod{2}$ and $y_0 \equiv z_0 \pmod{2}$ since $a_0 \equiv b_0 \pmod{2}$. Set $x = 3x_0$, $y = 3y_0$ and $z = 3z_0$. Then

$$6a - b^2 = 9(2a_0 - b_0^2) = x^2 + 2y^2 + 2z^2$$

and all the numbers in (3.4) are integers.

By the above, in either case, there are $x, y, z \in \mathbb{Z}$ for which $x^2 + 2y^2 + 2z^2 = 6a - b^2$ and $s, t, u, v \in \mathbb{Z}$, where s, t, u, v are as in (3.4). Observe that

$$s + t + 2(u + v) = \frac{b + 2z}{3} + 2 \times \frac{b - z}{3} = b$$

and also

$$\begin{aligned}
 s^2 + t^2 + 2u^2 + 2v^2 &= 2\left(\frac{s+t}{2}\right)^2 + 2\left(\frac{s-t}{2}\right)^2 + (u+v)^2 + (u-v)^2 \\
 &= 2\left(\frac{b+2z}{6}\right)^2 + 2\left(\frac{x+2y}{6}\right)^2 + \left(\frac{b-z}{3}\right)^2 + \left(\frac{x-y}{3}\right)^2 \\
 &= \frac{b^2 + x^2 + 2y^2 + 2z^2}{6} = a.
 \end{aligned}$$

In view of Lemma 2.3 and (3.2),

$$(|x| + 2|y| + 2|z|)^2 \leq 5(x^2 + 2y^2 + 2z^2) = 5(6a - b^2) < (b + 6)^2,$$

and hence

$$b - |x| - 2|y| - 2|z| > -6.$$

So we have

$$s, t, u, v \geq \frac{b - |x| - 2|y| - 2|z|}{6} > -1,$$

and so $s, t, u, v \in \mathbb{N}$.

In view of the above, we have thus completed the proof of Lemma 3.3. ■

REMARK 3.1. For s, t, u, v given in (3.4), the identity:

$$\begin{aligned}
 6(s^2 + t^2 + 2u^2 + 2v^2) &= b^2 + x^2 + 2y^2 + 2z^2 \\
 &= (s + t + 2u + 2v)^2 + (s - t - 2u + 2v)^2 \\
 &\quad + 2(s - t + u - v)^2 + 2(s + t - u - v)^2
 \end{aligned}$$

is a special case of the following general identity:

$$\begin{aligned}
 (3.5) \quad (a + b)(c + d)(w^2 + abx^2 + cdy^2 + abcdz^2) \\
 &= ac(w + bx + dy + bdz)^2 + ad(w + bx - cy - bcz)^2 \\
 &\quad + bc(w - ax + dy - adz)^2 + bd(w - ax - cy + acz)^2.
 \end{aligned}$$

We have also found another similar identity:

$$\begin{aligned}
 (3.6) \quad (3b + 4)(w^2 + 2x^2 + (b + 1)y^2 + 2bz^2) \\
 &= (w + 2x + (b + 1)y + 2bz)^2 + 2(w - (b + 1)y + bz)^2 \\
 &\quad + (b + 1)(w - 2x + y)^2 + 2b(w + x - 2z)^2.
 \end{aligned}$$

Proof of Theorem 1.2. (i) Let $I_2 = [\alpha, \beta]$ be the interval given by (3.1). As $N \geq 1628m^3 = (12+1/3)m \times 132m^2 \geq 11m^2 \times 12(12m+1)$, by Lemma 3.1 the length of I_2 is greater than $12m$. We distinguish two cases to construct integers $b \in I_2$ and $a \equiv b \pmod{2}$ for which $N = \frac{m}{2}(a - b) + b$, and $2 \nmid a$ or $4 \parallel a$, and $3 \mid a$ or $3 \nmid b$.

CASE 1: $3 \nmid m$ or $3 \nmid N$. Choose $b_0 \equiv N \pmod{m}$ with $b_0 \in \{[\alpha] + r : r = 0, \dots, m-1\}$, and let $b_j = b_0 + jm$ for $j = 1, \dots, 7$. Since $[\alpha] + 8m - 1 < \alpha + 8m < \beta$, we have $b_i \in I_2$ for all $i = 0, \dots, 7$.

If $2 \nmid m$, then we choose $i \in \{0, 1\}$ with b_i odd. When $4 \mid m$, we may choose $i \in \{0, 1, 2, 3\}$ with

$$a_i := \frac{2}{m}(N - b_i) + b_i \equiv 4 \pmod{8}$$

since $a_i - a_0 = -2i + im = 2i(m/2 - 1)$ with $m/2 - 1$ odd. If $3 \mid m$ and $3 \nmid N$, then $b = b_i \equiv N \not\equiv 0 \pmod{3}$. When $3 \nmid m$, we choose $j \in \{i, i+4\}$ with $b = b_j \not\equiv 0 \pmod{3}$, and note that

$$a_{i+4} - a_i = 4m - 8 \equiv \begin{cases} 0 \pmod{2} & \text{if } 2 \nmid m, \\ 0 \pmod{8} & \text{if } 4 \mid m. \end{cases}$$

As $N \equiv b \pmod{m}$, we see that $a = 2(N - b)/m + b$ is an integer with $a \equiv b \pmod{2}$. By our choice, $2 \nmid b$ if $2 \nmid m$, and $a \equiv 4 \pmod{8}$ if $4 \mid m$. Note also that $3 \nmid b$.

CASE 2: $m \equiv N \equiv 0 \pmod{3}$. Choose $b_0 \in \{[\alpha] + r : r = 0, 1, \dots, 3m-1\}$ with $b_0 \equiv N \pmod{3m}$. If $2 \nmid m$, then we choose $b \in \{b_0, b_0 + 3m\}$ with b odd and hence $a = \frac{2}{m}(N - b) + b \equiv b \equiv 1 \pmod{2}$. When $4 \mid m$, we may choose $b \in \{b_0 + 3jm : j = 0, 1, 2, 3\}$ with $a = \frac{2}{m}(N - b) + b \equiv 4 \pmod{8}$, for

$$\frac{2}{m}(N - b_0 - 3jm) + b_0 + 3jm - \left(\frac{2}{m}(N - b_0) + b_0 \right) = 6j \left(\frac{m}{2} - 1 \right)$$

with $m/2 - 1$ odd. Note that

$$\alpha \leq b_0 < b_0 + 9m \leq [\alpha] + 3m - 1 + 9m < \alpha + 12m < \beta,$$

and hence $b \in I_2$. Obviously, $a \equiv b \equiv N \equiv 0 \pmod{3}$.

Now we have constructed positive integers $b \in I_2$ and $a \equiv b \pmod{2}$ with $N = \frac{m}{2}(a - b) + b$ such that $2 \nmid a$ or $4 \parallel a$, and $3 \mid a$ or $3 \nmid b$. So (3.2) holds by Lemma 3.2. In view of Lemma 3.3, (3.3) holds for some $s, t, u, v \in \mathbb{N}$. Therefore,

$$\begin{aligned} N &= \frac{m}{2}(s^2 + t^2 + 2u^2 + 2v^2 - s - t - 2u - 2v) + s + t + 2u + 2v \\ &= m \binom{s}{2} + m \binom{t}{2} + 2m \binom{u}{2} + 2m \binom{v}{2} + s + t + 2u + 2v \\ &= p_{m+2}(s) + p_{m+2}(t) + 2p_{m+2}(u) + 2p_{m+2}(v). \end{aligned}$$

This proves part (i) of Theorem 1.2.

(ii) Now assume that $m = 2l$ with $l \in \mathbb{Z}^+$ odd. We want to show that none of the positive integers

$$(l-1)^2 \times \frac{4^{k\varphi(l)} - 1}{l} \quad (k = 1, 2, 3, \dots)$$

can be written as $p_{m+2}(w) + p_{m+2}(x) + 2p_{m+2}(y) + 2p_{m+2}(z)$ with w, x, y, z in \mathbb{N} .

Let $n \in \mathbb{Z}^+$ be a multiple of $\varphi(l)$. Then $2^n \equiv 1 \pmod{l}$ by Euler's theorem. Suppose that there are $w, x, y, z \in \mathbb{N}$ for which

$$\begin{aligned} (l-1)^2 &\times \frac{4^n - 1}{l} \\ &= p_{m+2}(w) + p_{m+2}(x) + 2p_{m+2}(y) + 2p_{m+2}(z) \\ &= \frac{2l}{2}(w^2 + x^2 + 2y^2 + 2z^2 - w - x - 2y - 2z) + w + x + 2y + 2z. \end{aligned}$$

Then

$$\begin{aligned} 4^{n+1}(l-1)^2 &= (2lw - (l-1))^2 + (2lx - (l-1))^2 + 2(2ly - (l-1))^2 + 2(2lz - (l-1))^2 \\ &= (2lw - (l-1))^2 + (2lx - (l-1))^2 + (2l(y+z) - 2(l-1))^2 + (2l(y-z))^2, \end{aligned}$$

and hence

$$4^n(l-1)^2 = \left(lw - \frac{l-1}{2}\right)^2 + \left(lx - \frac{l-1}{2}\right)^2 + (l(y+z-1)+1)^2 + (l(y-z))^2.$$

As $4 \mid (l-1)^2$, by Lemma 2.5 we have $r_4(4^n(l-1)^2) = r_4((l-1)^2)$. So there are $w_0, x_0, y_0, z_0 \in \mathbb{Z}$ with

$$(3.7) \quad w_0^2 + x_0^2 + y_0^2 + z_0^2 = (l-1)^2$$

such that

$$lw - \frac{l-1}{2} = 2^n w_0, \quad lx - \frac{l-1}{2} = 2^n x_0, \quad l(y+z-1)+1 = 2^n y_0, \quad l(y-z) = 2^n z_0.$$

As $2^n \equiv 1 \pmod{l}$, we see that

$$w_0 \equiv \frac{1-l}{2} \pmod{l}, \quad x_0 \equiv \frac{1-l}{2} \pmod{l}, \quad y_0 \equiv 1 \pmod{l}, \quad z_0 \equiv 0 \pmod{l}.$$

Observe that $l = m/2 \geq 2$, and hence $w_0 x_0 \neq 0$. Thus $y_0^2 + z_0^2 \leq (l-1)^2 - 2$. Since $y_0 \equiv 1 \pmod{l}$ and $z_0 \equiv 0 \pmod{l}$, we must have $y_0 = 1$ and $z_0 = 0$. Now (3.7) yields $w_0^2 + x_0^2 = (l-1)^2 - 1 = l^2 - 2l$. As $w_0 \equiv x_0 \equiv (1-l)/2 \pmod{l}$, we must have $\{w_0, x_0\} \subseteq \{(1-l)/2, (1+l)/2\}$. It is easy to verify directly that none of the numbers

$$\left(\frac{1-l}{2}\right)^2 + \left(\frac{1+l}{2}\right)^2, \quad 2\left(\frac{l-1}{2}\right)^2, \quad 2\left(\frac{1+l}{2}\right)^2$$

is equal to $(l-1)^2 - 1 = l^2 - 2l$. This contradiction concludes the proof of Theorem 1.2(ii). ■

4. Proof of Theorem 1.3

LEMMA 4.1. *Let a and b be positive integers satisfying (3.2) for which $a \equiv b \pmod{2}$, and $a \equiv 3 \pmod{9}$ or $3 \nmid b$. Then there exist $s, t, u, v \in \mathbb{N}$ such that*

$$(4.1) \quad a = s^2 + t^2 + u^2 + 3v^2 \quad \text{and} \quad b = s + t + u + 3v.$$

Proof. It is known (cf. Dickson [3, pp. 112–113]) that

$$(4.2) \quad \{x^2 + y^2 + 3z^2 : x, y, z \in \mathbb{Z}\} = \mathbb{N} \setminus \{9^k(9l + 6) : k, l \in \mathbb{N}\}.$$

If $3 \nmid b$, then $6a - b^2 \equiv 2 \pmod{3}$. If $a \equiv 3 \pmod{9}$ and $3 \mid b$, then $6a - b^2 \equiv \pm 9 \pmod{27}$. By (4.2), $6a - b^2 = x^2 + y^2 + 3z^2$ for some $x, y, z \in \mathbb{Z}$. Clearly, $x^2 + y^2 \equiv 2b^2 \pmod{3}$. Without loss of generality, we may assume that $x \equiv y \equiv b \pmod{3}$. (If $x \equiv -b \pmod{3}$ then $-x \equiv b \pmod{3}$.) If a and b are both odd, then $x^2 + y^2 + 3z^2 = 6a - b^2 \equiv 1 \pmod{4}$, and hence one of x and y is odd. If a and b are both even, then $x^2 + y^2 + 3z^2 = 6a - b^2 \equiv 0 \pmod{4}$, and hence one of x and y is even. Without loss of generality, we may assume that $x \equiv a \equiv b \pmod{2}$ and $y \equiv z \pmod{2}$. Thus all the numbers

$$(4.3) \quad s = \frac{b + x + y + 3z}{6}, \quad t = \frac{b + x + y - 3z}{6}, \quad u = \frac{b + x - 2y}{6}, \quad v = \frac{b - x}{6}$$

are integers. Observe that

$$s + t + u + 3v = \frac{b + x}{2} + 3 \times \frac{b - x}{6} = b$$

and

$$(4.4) \quad \begin{aligned} s^2 + t^2 + u^2 + 3v^2 &= 2\left(\frac{s+t}{2}\right)^2 + 2\left(\frac{s-t}{2}\right)^2 + u^2 + 3v^2 \\ &= 2\left(\frac{b+x+y}{6}\right)^2 + 2\left(\frac{z}{2}\right)^2 + \left(\frac{b+x-2y}{6}\right)^2 + 3\left(\frac{b-x}{6}\right)^2 \\ &= \frac{b^2 + x^2 + y^2 + 3z^2}{6} = a. \end{aligned}$$

In view of Lemma 2.3 and (3.2),

$$(|x| + |y| + 3|z|)^2 \leq 5(x^2 + y^2 + 3z^2) = 5(6a - b^2) < (b + 6)^2,$$

and hence

$$b - |x| - |y| - 3|z| > -6.$$

So we have

$$s, t, u, v \geq \frac{b - |x| - |y| - 3|z|}{6} > -1,$$

and hence $s, t, u, v \in \mathbb{N}$. ■

Proof of Theorem 1.3. As

$$N \geq 924m^3 = 99m^3 \left(9 + \frac{1}{3}\right) \geq 99m^3 \left(9 + \frac{1}{m}\right) = 99m^2(9m + 1),$$

the length of the interval $I_2 = [\alpha, \beta]$ defined in Lemma 3.1 is greater than $9m$.

Let $b_0 \in \{[\alpha] + r : r = 0, 1, \dots, m - 1\}$ with $b_0 \equiv N \pmod{m}$. If $3 \nmid m$ or $3 \nmid N$, then we may choose $b \in \{b_0, b_0 + m\}$ with $b \not\equiv 0 \pmod{3}$. When $3 \mid m$ and $3 \mid N$, we let $c_0 \in \{[\alpha] + r : r = 0, \dots, 3m - 1\}$ with $c_0 \equiv N \pmod{3m}$, and set $b = c_0 + j3m$ with $j \in \{0, 1, 2\}$ such that

$$\frac{2}{m}(N - c_0) + c_0 - 6j \equiv 3 \pmod{9}.$$

Note that $b \in I_2$ since

$$\alpha \leq b \leq [\alpha] + 3m - 1 + 6m < \alpha + 9m < \beta.$$

Let

$$a = \frac{2}{m}(N - b) + b, \quad \text{i.e.,} \quad N = \frac{m}{2}(a - b) + b.$$

Then

$$a = \frac{2}{m}N + \left(1 - \frac{2}{m}\right)b > 0 \quad \text{and} \quad a \equiv b \pmod{2}.$$

If $3 \mid b$, then $3 \mid m$ and

$$a = \frac{2}{m}(N - b) + b = \frac{2}{m}(N - c_0 - 3jm) + c_0 + 3jm \equiv \frac{2}{m}(N - c_0) - 6j \equiv 3 \pmod{9}.$$

By Lemmas 3.2 and 4.1, there are $s, t, u, v \in \mathbb{N}$ satisfying (4.1). Therefore,

$$\begin{aligned} N &= \frac{m}{2}(s^2 + t^2 + u^2 + 3v^2 - s - t - u - 3v) + s + t + u + 3v \\ &= p_{m+2}(s) + p_{m+2}(t) + p_{m+2}(u) + 3p_{m+2}(v). \quad \blacksquare \end{aligned}$$

5. Proof of Theorem 1.4

LEMMA 5.1. *Let $l, m, N \in \mathbb{Z}^+$ with $lm \geq 20$ and $N \geq 3lm^2(5lm + 12)$. Then the length of the interval*

$$(5.1) \quad I_3 = \left[\frac{5}{2} + \sqrt{\frac{14N}{m} - 1}, \frac{4}{3} + 4\sqrt{\frac{N}{m}} \right]$$

is greater than lm .

Proof. The length of I_3 is $4\sqrt{x} - \sqrt{14x - 1} - 7/6$, where $x = N/m$. Set $l_0 = lm + 7/6$. Then

$$\begin{aligned} 4\sqrt{x} - \sqrt{14x - 1} - 7/6 > lm &\Leftrightarrow 4\sqrt{x} > \sqrt{14x - 1} + l_0 \\ &\Leftrightarrow 2x + 1 - l_0^2 > 2l_0\sqrt{14x - 1} \\ &\Leftrightarrow 4x(x + 1 - 15l_0^2) + (l_0^2 - 1)^2 + 4l_0^2 > 0. \end{aligned}$$

Note that

$$x \geq 15l^2m^2 + 36lm > 15l^2m^2 + 35lm + \frac{245}{12} - 1 = 15l_0^2 - 1.$$

So the desired result follows. ■

LEMMA 5.2. *Let a, b, m, N be positive integers with $m \geq 3$,*

$$N = \frac{m}{2}(a - b) + b \geq \frac{4}{7}m$$

and $b \in I_3$, where I_3 is the interval given by (5.1). Then

$$(5.2) \quad b^2 < 8a \quad \text{and} \quad 7a < b^2 + 2b + 8.$$

Proof. Clearly,

$$a = \left(1 - \frac{2}{m}\right)b + \frac{2N}{m}.$$

Thus

$$\begin{aligned} b^2 - 8a &= b^2 - 8\left(1 - \frac{2}{m}\right)b - \frac{16N}{m} \\ &= \left(b - 4\left(1 - \frac{2}{m}\right)\right)^2 - 16\left(\left(1 - \frac{2}{m}\right)^2 + \frac{N}{m}\right). \end{aligned}$$

As $b \in I_3$, we have

$$b \leq \frac{4}{3} + 4\sqrt{\frac{N}{m}} < 4\left(1 - \frac{2}{m}\right) + 4\sqrt{\left(1 - \frac{2}{m}\right)^2 + \frac{N}{m}},$$

and hence $b^2 - 8a < 0$. On the other hand,

$$\begin{aligned} b^2 + 2b + 8 - 7a &= b^2 - \left(5 - \frac{14}{m}\right)b + \left(8 - \frac{14N}{m}\right) \\ &= \left(b - \left(\frac{5}{2} - \frac{7}{m}\right)\right)^2 - \left(\left(\frac{5}{2} - \frac{7}{m}\right)^2 + \frac{14N}{m} - 8\right) > 0 \end{aligned}$$

since $(5/2 - 7/m)^2 \leq 25/4 < 7$ and

$$b \geq \frac{5}{2} + \sqrt{\frac{14N}{m} - 1} > \frac{5}{2} - \frac{7}{m} + \sqrt{\left(\frac{5}{2} - \frac{7}{m}\right)^2 + \frac{14N}{m} - 8}.$$

Therefore (5.2) holds. ■

LEMMA 5.3. *Let a and b be positive integers satisfying (5.2). Then there are $s, t, u, v \in \mathbb{N}$ such that*

$$(5.3) \quad a = s^2 + t^2 + 2u^2 + 4v^2 \quad \text{and} \quad b = s + t + 2u + 4v,$$

under one of the following conditions:

- (i) $2 \nmid ab$.
- (ii) $2 \mid a$ and $2 \parallel b$.
- (iii) $4 \mid a$ and $4 \parallel b$, or $a \equiv b + 4 \pmod{16}$ and $8 \mid b$.

Proof. It is known (cf. [3, pp. 112–113]) that

$$(5.4) \quad \{x^2 + 2y^2 + 4z^2 : x, y, z \in \mathbb{Z}\} = \mathbb{N} \setminus \{4^k(16l + 14) : k, l \in \mathbb{N}\}.$$

We claim that if one of (i)–(iii) holds then $8a - b^2 = x^2 + 2y^2 + 4z^2$ for some $x, y, z \in \mathbb{Z}$ such that all the numbers

$$(5.5) \quad u = \frac{b + x - 2y}{8}, \quad v = \frac{b - x}{8}, \quad s = u + \frac{y + z}{2}, \quad t = u + \frac{y - z}{2}$$

are integers.

CASE 1: $2 \nmid ab$. Since $8a - b^2 \equiv -1 \pmod{8}$, we have $8a - b^2 = x^2 + 2y^2 + 4z^2$ for some $x, y, z \in \mathbb{Z}$. As $x^2 + 2y^2 \equiv -1 \pmod{4}$, we have $2 \nmid xy$. Since $4z^2 \equiv -b^2 - x^2 - 2y^2 \equiv -1 - 1 - 2 \pmod{8}$, we also have $2 \nmid z$. Note that

$$x^2 = 8a - b^2 - 2y^2 - 4z^2 \equiv 8 - b^2 - 2 - 4 = 2 - b^2 \equiv b^2 \pmod{16},$$

and hence $x \equiv \pm b \pmod{8}$. Without loss of generality, we may assume that $x \equiv b \pmod{8}$ and $y \equiv b \pmod{4}$. (If $y \equiv -b \pmod{4}$ then $-y \equiv b \pmod{4}$.) Thus all the four numbers in (5.5) are integers.

CASE 2: $2 \mid a$ and $2 \parallel b$. Write $a = 2a_0$ and $b = 2b_0$ with $a_0, b_0 \in \mathbb{Z}$ and $2 \nmid b_0$. Since $4a_0 - b_0^2 \equiv 3 \pmod{4}$, by (5.4) we have $4a_0 - b_0^2 = x_0^2 + 2y_0^2 + 4z_0^2$ for some $x_0, y_0, z_0 \in \mathbb{Z}$. As $x_0^2 + 2y_0^2 \equiv 3 \pmod{4}$, both x_0 and y_0 are odd. Without loss of generality, we may assume that $x_0 \equiv y_0 \equiv b_0 \pmod{4}$. Set $x = 2x_0$, $y = 2y_0$ and $z = 2z_0$. Then

$$8a - b^2 = 4(4a_0 - b_0^2) = x^2 + 2y^2 + 4z^2$$

and all the numbers in (5.5) are integers.

CASE 3: $4 \mid a$ and $4 \parallel b$, or $a \equiv b + 4 \pmod{16}$ and $8 \mid b$. Write $a = 4a_0$ and $b = 4b_0$ with $a_0, b_0 \in \mathbb{Z}$. Then

$$2a_0 - b_0^2 \equiv \begin{cases} 1 \pmod{2} & \text{if } 4 \parallel b \text{ (i.e., } 2 \nmid b_0), \\ 2(b_0 + 1) - b_0^2 \equiv 2 \pmod{8} & \text{if } a \equiv b + 4 \pmod{16} \text{ and } 8 \mid b. \end{cases}$$

Thus, by (5.4) we have $2a_0 - b_0^2 = x_0^2 + 2y_0^2 + 4z_0^2$ for some $x_0, y_0, z_0 \in \mathbb{Z}$. Obviously $x_0 \equiv b_0 \pmod{2}$. Set $x = 4x_0$, $y = 4y_0$ and $z = 4z_0$. Then all the numbers in (5.5) are integers.

Now assume that one of (i)–(iii) holds. By the claim we have proved, there are $x, y, z \in \mathbb{Z}$ such that $8a - b^2 = x^2 + 2y^2 + 4z^2$ and also $s, t, u, v \in \mathbb{Z}$, where s, t, u, v are given by (5.5). Clearly,

$$s + t + 2u + 4v = y + 2u + 2u + 4v = \frac{b + x}{2} + \frac{b - x}{2} = b$$

and

$$\begin{aligned}
 (5.6) \quad s^2 + t^2 + 2u^2 + 4v^2 &= 2\left(u + \frac{y}{2}\right)^2 + 2\left(\frac{z}{2}\right)^2 + 2\left(\frac{b+x-2y}{8}\right)^2 + 4\left(\frac{b-x}{8}\right)^2 \\
 &= 2\left(\frac{b+x+2y}{8}\right)^2 + 2\left(\frac{b+x-2y}{8}\right)^2 + \frac{z^2}{2} + \left(\frac{b-x}{4}\right)^2 \\
 &= \left(\frac{b+x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 + \frac{z^2}{2} + \left(\frac{b-x}{4}\right)^2 \\
 &= \frac{b^2 + x^2 + 2y^2 + 4z^2}{8} = a.
 \end{aligned}$$

In view of Lemma 2.3 and (5.2),

$$(|x| + 2|y| + 4|z|)^2 \leq 7(x^2 + 2y^2 + 4z^2) = 7(8a - b^2) < (b + 8)^2,$$

and hence

$$b - |x| - 2|y| - 4|z| > -8.$$

So we have

$$u, v, s, t \geq \frac{b - |x| - 2|y| - 4|z|}{8} > -1,$$

and hence $s, t, u, v \in \mathbb{N}$.

In view of the above, we have completed the proof of Lemma 5.3. ■

Proof of Theorem 1.4. As

$$N \geq 96m^2 \times 11m \geq 96m^2(10m + 3) = 24m^2(40m + 12),$$

applying Lemma 5.1 with $l = 8$ we find that the interval $I_3 = [\alpha, \beta]$ given by (5.1) has length greater than $8m$.

CASE 1: $4 \nmid m$ or $8 \nmid N$. Choose $b_0 \in \{\lceil \alpha \rceil + r : r = 0, \dots, m-1\}$ with $N \equiv b_0 \pmod{m}$. Set $b_1 = b_0 + m$. Then

$$\alpha \leq b_0 < b_1 \leq \lceil \alpha \rceil + 2m - 1 < \alpha + 8m < \beta,$$

and thus $b_j \in I_3$ for each $j = 0, 1$. Note that

$$a_j := \frac{2}{m}(N - b_j) + b_j = \frac{2}{m}(N - b_0) + b_0 + (m - 2)j.$$

If $2 \nmid m$, then $a_j \equiv b_j \equiv 1 \pmod{2}$ for some $j \in \{0, 1\}$. If $2 \mid m$ and $2 \nmid N$, then $a_0 \equiv b_0 \equiv 1 \pmod{2}$.

If $2 \parallel m$ and $2 \mid N$, then for some $j \in \{0, 1\}$ we have $b_j \equiv 2 \pmod{4}$ and $2 \mid a_j$. When $4 \mid m$ and $2 \parallel N$, we have $b_0 \equiv 2 \pmod{4}$ and $a_0 \equiv b_0 \equiv 0 \pmod{2}$. If $4 \parallel m$ and $4 \parallel N$, then $4 \mid b_0$, and for some $j \in \{0, 1\}$ we have $b_j \equiv 4 \pmod{8}$ and $a_j \equiv b_j \equiv 0 \pmod{4}$. When $8 \mid m$ and $4 \parallel N$, we have

$b_0 \equiv 4 \pmod{8}$ and $a_0 \equiv b_0 \pmod{2}$, hence for some $j \in \{0, 1\}$ we have $a_j \equiv 0 \pmod{4}$ and $b_j \equiv b_0 \equiv 4 \pmod{8}$.

CASE 2: $4 \mid m$ and $8 \mid N$. Choose $b \in \{\lceil \alpha \rceil + r : r = 0, \dots, 8m - 1\}$ such that $b \equiv N - 2m \pmod{8m}$. Since $\alpha \leq b \leq \lceil \alpha \rceil + 8m - 1 < \alpha + 8m < \beta$, we have $b \in I_3$. Clearly, $8 \mid b$ since $8 \mid N$ and $4 \mid m$. Note that

$$\frac{2}{m}(N - b) + b \equiv 4 + b \pmod{8}.$$

By the above, in either case we can always find $b \in I_3$ and $a \in \mathbb{Z}$ for which one of (i)–(iii) in Lemma 5.3 holds and also

$$a = \frac{2}{m}(N - b) + b, \quad \text{i.e.,} \quad N = \frac{m}{2}(a - b) + b.$$

By Lemmas 5.2 and 5.3, there are $s, t, u, v \in \mathbb{N}$ satisfying (5.3). Therefore,

$$\begin{aligned} N &= \frac{m}{2}(s^2 + t^2 + 2u^2 + 4v^2 - s - t - 2u - 4v) + s + t + 2u + 4v \\ &= p_{m+2}(s) + p_{m+2}(t) + 2p_{m+2}(u) + 4p_{m+2}(v). \end{aligned}$$

In view of the above, we have finished the proof of Theorem 1.4. ■

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Xiang-Zi Meng, Zhi-Wei Sun (corresponding author)
Department of Mathematics
Nanjing University
Nanjing 210093, People's Republic of China
E-mail: xzmeng@smail.nju.edu.cn
zwsun@nju.edu.cn

