ACTA ARITHMETICA
180.4 (2017)

On the behavior close to the unit circle of power series with additive coefficients

by

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1. Introduction. In this paper we study power series with completely additive coefficients. Power series with coefficients that have some arithmetical structure have interesting properties. All known power series with non-trivial arithmetical coefficients have no continuation beyond the unit circle. Moreover they have interesting properties when z tends to the unit circle along a radius.

For the following classes of power series there exist classical theorems stating that the series have the unit circle as the natural boundary:

- Lacunary series
$$f(z)=\sum_{n=1}^{\infty} a_{\lambda_n} z^{\lambda_n},$$
where
$$\lambda_n \in \mathbb{N}, \qquad \lim_{n\to\infty} \frac{n}{\lambda_n} = 0.$$
This result was proved by E. Fabry in 1896 (see [B, p. 80], [F]).
- Power series whose coefficients take a finite number of values and are not ultimately periodic (see [B, p. 165], [S]).
- Power series with integer coefficients which are not rational functions. This was proved by F. Carlson in 1921 (see [C]).

In 1981 L. G. Lucht [L] proved that for an extensive set of multiplicative functions $\alpha(n)$ the unit circle is the natural boundary of the series $\sum_{n=1}^{\infty} \alpha(n) z^n$.

2010 *Mathematics Subject Classification*: Primary 11N37; Secondary 30B30.
Key words and phrases: power series, additive functions, growth conditions, omega-estimates, behavior, arithmetic functions.
Received 28 May 2016; revised 31 March 2017.
Published online 1 September 2017.

DOI: 10.4064/aa8536-4-2017

Denote $e^{2\pi i\beta}$ by $e(\beta)$. In [P1] we studied the power series $\mathfrak{M}(z) = \sum_{n=1}^{\infty}\mu(n)z^n$ where $\mu(n)$ is the Möbius function and proved that for each $\beta\in\mathbb{Q}$,

$$\mathfrak{M}(re(\beta)) = \Omega((1-r)^{-a})$$

as $r\to 1-$, for some $a>0$ depending on β.

In [P2] we obtained non-trivial estimates for $\mathfrak{M}_0(z) = \sum_{n=1}^{\infty}\mu^2(n)z^n$. The behavior of this series when $z = e(\beta)r$, $r\to 1-$, depends on Diophantine approximation properties of β. We proved that if the irrationality exponent of β equals 2 then

$$\mathfrak{M}_0(re(\beta)) = O((1-r)^{-1/2-\varepsilon}), \qquad r\to 1-.$$

An arithmetical function $\alpha(n)$ is *additive* if

$$\alpha(mn) = \alpha(m)+\alpha(n) \quad \text{for coprime integers } m \text{ and } n.$$

An arithmetical function $\alpha(n)$ is *completely additive* if

$$\alpha(mn) = \alpha(m)+\alpha(n) \quad \text{for all } m \text{ and } n.$$

As usual for $s\in\mathbb{C}$ we denote $\sigma = \Re s$, $t = \Im s$.

In 2000 L. G. Lucht and A. Schmalmack [LS] found an extensive class of additive functions $\alpha(n)$ such that the unit circle is the natural boundary of the series $\sum_{n=1}^{\infty}\alpha(n)z^n$. Their class was defined by the asymptotic equality

$$\sum_{n<x}\alpha(n)\chi_0(n) = c_q x^s l(x) + o(x^\sigma|l(x)|), \qquad x\to\infty,$$

where $l(x)$ is a slowly oscillating function, χ_0 is a principal character and the sum has some other properties. But for arbitrary complex additive $\alpha(n)$ we do not know how to apply their method.

In this paper we prove that some conditions on the growth of the coefficients of power series with additive coefficients give us a class of power series that have the unit circle as the natural boundary. Moreover we prove some omega-estimates of such series with z tending to roots of unity.

For an arbitrary sequence $\alpha(n)$, a real β and Dirichlet character χ we set

$$(1.1)\qquad \begin{aligned} F(s) &= \sum_{n=1}^{\infty}\frac{\alpha(n)}{n^s},\\ F[\beta](s) &= \sum_{n=1}^{\infty}\alpha(n)e(\beta n)n^{-s},\\ F(s,\chi) &= \sum_{n=1}^{\infty}\alpha(n)\chi(n)n^{-s},\\ A(x,\beta) &= \sum_{n<x}\alpha(n)e(\beta n). \end{aligned}$$

Denote by $\mathfrak{A}(z)$ with $z \in \mathbb{C}$ the power series

$$\mathfrak{A}(z) = \sum_{n=1}^{\infty} \alpha(n) z^n.$$

For a Dirichlet character χ modulo q, $\bar{\chi}$ is the character conjugate to χ, $\tau(\chi, l) = \sum_{k=1}^{q} \chi(k) e(lk/q)$, and $L(s, \chi)$ is a Dirichlet L-series.

In our paper we study the Dirichlet series $F[\beta](s)$ and from its properties we deduce the properties of $\mathfrak{A}(z)$.

We will denote by p prime numbers.

We consider the class $\mathfrak{B}$ of additive functions defined by

$$\alpha(p^m) = m f(p, m) \ln p \tag{1.2}$$

for p prime and m an integer, where $f(p, m)$ is a complex function such that $f(p, m) \to 0$ as $p \to \infty$, uniformly in m. We prove the following theorems.

THEOREM 1.1. *Let $\alpha \in \mathfrak{B}$. Let $\beta = l/p^k$, where $(l, p) = 1$. Then*

$$\overline{\lim_{r \to 1-}} |\mathfrak{A}(e(\beta) r)(1 - r)| \geq \left| \sum_{m=k}^{\infty} \frac{\alpha(p^m) - \alpha(p^{m-1})}{p^m} \right|, \tag{1.3}$$

$$\overline{\lim_{x \to \infty}} \left| \frac{A(x, \beta)}{x} \right| \geq \left| \sum_{m=k}^{\infty} \frac{\alpha(p^m) - \alpha(p^{m-1})}{p^m} \right|. \tag{1.4}$$

THEOREM 1.2. *Let $\alpha \in \mathfrak{B}$. Suppose $\mathfrak{A}(z) = \sum_{n=1}^{\infty} \alpha(n) z^n$ has non-singular points on the unit circle. Then $\mathfrak{A}(z)$ is a rational function, and α satisfies the following conditions: for some $P \in \mathbb{N}$,*

$$\alpha(p^k) = 0 \qquad \text{for } p > P \text{ and each } k, \tag{1.5}$$

$$\alpha(p^k) = C(p) \qquad \text{for } p \leq P \text{ and } k > k_0(p), \tag{1.6}$$

where $C(p)$, $k_0(p)$ do not depend on k.

Hence all series $\mathfrak{A}(z)$ with additive coefficients that satisfy (1.2) and do not satisfy (1.5) or (1.6) have the unit circle as the natural boundary.

In particular, we have

THEOREM 1.3. *Let $\alpha(n)$ be a completely additive function with*

$$\alpha(p) = o(\ln p). \tag{1.7}$$

Let $\beta = l/p^k$, where $(l, p) = 1$. Then

$$\overline{\lim_{r \to 1-}} |\mathfrak{A}(e(\beta) r)(1 - r)| \geq \frac{|\alpha(p)| p^{1-k}}{p - 1}, \tag{1.8}$$

$$\overline{\lim_{x \to \infty}} \left| \frac{A(x, \beta)}{x} \right| \geq \frac{|\alpha(p)| p^{1-k}}{p - 1}. \tag{1.9}$$

Moreover if $\mathfrak{A}(z)$ has non-singular points on the unit circle then $\alpha(n) \equiv 0$.

Hence all series $\mathfrak{A}(z)$ with completely additive coefficients that satisfy (1.7) and are not identically zero have the unit circle as the natural boundary.

Theorems 1.1–1.3 are proved in Sections 2–4. In Section 5 we also note that the growth conditions (1.2), (1.7) in Theorems 1.1–1.3 cannot be significantly weakened.

2. Preliminary results. From the definition (1.1) of $F(s,\chi)$ we easily obtain the following properties.

LEMMA 2.1. *Suppose the Dirichlet series F and G are absolutely convergent at the point s. Then*

$$(F+G)(s,\chi) = F(s,\chi) + G(s,\chi),$$
$$(FG)(s,\chi) = F(s,\chi)G(s,\chi).$$

LEMMA 2.2. *Let $G_p(s) = \sum_{k=1}^{\infty} c_{p^k} p^{-ks}$. Suppose the double series $F(s) = \sum_p G_p(s)$ is absolutely convergent. Then*

$$F(s,\chi) = \sum_p G_p(s,\chi).$$

The following lemma detects the structure of the Dirichlet series with additive coefficients.

LEMMA 2.3. *Let $\alpha(n)$ be an additive function with $\alpha(n) = O(\ln n)$. Then*

$$F(s) = \zeta(s) \sum_p G_p(s), \tag{2.1}$$

where $G_p(s) = \sum_{k=1}^{\infty} (\alpha(p^k) - \alpha(p^{k-1}))p^{-ks}$ for $\Re s > 1$.

Proof. If the series $\sum_p G_p(s)$ is considered as a double series, then for $\Re s > 1$ the series is absolutely convergent. We have

$$\sum_p G_p(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}, \tag{2.2}$$

where

$$b_n = \begin{cases} \alpha(p^k) - \alpha(p^{k-1}) & \text{if } n = p^k, \\ 0 & \text{if } n \neq p^k. \end{cases}$$

The double series on the left-hand side of (2.2) and the series on the right-hand side of (2.2) are absolutely convergent for $\Re s > 1$. In this half-plane

$$\zeta(s) \sum_p G_p(s) = \sum_{n=1}^{\infty} \frac{\gamma(n)}{n^s},$$

where for $n = p_1^{l_1} \dots p_r^{l_r}$ we have

$$\gamma(n) = \sum_{d|n} b_d = \sum_{p^k|n} [\alpha(p^k) - \alpha(p^{k-1})] = \sum_{i=1}^{r} \alpha(p^{l_i}) = \alpha(p_1^{l_1} \dots p_r^{l_r}) = \alpha(n). \blacksquare$$

Let $I(s)$ be a Dirichlet integral,

$$I(s) = \int_0^\infty u^{s-1} f(u)\, du, \qquad s \in \mathbb{C},$$

where $f(u) \in L^1[r, R]$ for any $0 < r < R < \infty$. Let

$$I_1(s) = \int_0^1 u^{s-1} f(u)\, du, \qquad I_2(s) = \int_1^\infty u^{s-1} f(u)\, du.$$

The following theorem relates $I(s)$ to $f(u)$.

Theorem 2.4. *Suppose that $I(s)$ has the following properties:*

(1) $I_2(s) = \int_1^\infty u^{s-1} f(u)\, du$ *is convergent for all* $s \in \mathbb{C}$.
(2) $I(s)$ *is convergent for* $\Re s > \sigma_0$.
(3) *There exist* $c, \alpha > 0$ *and* $t_0 \in \mathbb{R}$ *such that*

$$\overline{\lim_{x \to 0+}} \frac{|I(\sigma_0 + it_0 + x)|}{x^{-\alpha}} \ge c. \tag{2.3}$$

Then

$$\overline{\lim_{u \to 0+}} \frac{|f(u)|}{(-\ln u)^{\alpha-1} u^{-\sigma_0}} \ge \frac{c}{\Gamma(\alpha)}. \tag{2.4}$$

Proof. Assume that

$$\overline{\lim_{u \to 0+}} \frac{|f(u)|}{(-\ln u)^{\alpha-1} u^{-\sigma_0}} < \frac{c}{\Gamma(\alpha)}.$$

Then for $u < u_0$ we have $|f(u)| < \frac{c_1}{\Gamma(\alpha)} (-\ln u)^{\alpha-1} u^{-\sigma_0}$ where $c_1 < c$. Since $I_2(s)$ is convergent for all $s \in \mathbb{C}$, it is an entire function and $I_2(\sigma_0 + it_0 + x) = O(1)$ for $x \to 0+$. Hence for $x \to 0+$,

$$\begin{aligned} |I(\sigma_0 + it_0 + x)| &\le O(1) + \left| \int_0^1 u^{\sigma_0 + it_0 + x - 1} f(u)\, du \right| \\ &\le C + \frac{c_1}{\Gamma(\alpha)} \int_0^1 u^{-\sigma_0} (-\ln u)^{\alpha-1} u^{\sigma_0 + x - 1}\, du \\ &\le C + \frac{c_1}{\Gamma(\alpha)} \int_0^1 u^{x-1} (-\ln u)^{\alpha-1}\, du. \end{aligned}$$

Since

$$\int_0^1 u^{x-1}(-\ln u)^{\alpha-1}\,du = \int_0^\infty e^{-ux}u^{\alpha-1}\,du = \Gamma(\alpha)x^{\alpha},$$

we obtain

$$(2.5)\qquad |I(\sigma_0+it_0+x)| \le C + \frac{c_1}{\Gamma(\alpha)}\Gamma(\alpha)x^{-\alpha} = C + c_1x^{-\alpha}, \qquad x\to 0+\,.$$

The inequality (2.5) contradicts condition (3) of the theorem. ■

Recall the definitions (1.1).

PROPOSITION 2.5. *Let*

$$(2.6)\qquad A(x) = \sum_{n\le x}\alpha(n).$$

If for some $c>0$,

$$(2.7)\qquad \overline{\lim_{r\to 1-}}\, |\mathfrak{A}(r)|(1-r) > c,$$

then

$$\overline{\lim_{x\to\infty}}\, |A(x)|x^{-1} > c.$$

Proof. Using the Abel transform we obtain

$$\mathfrak{A}(r) = (1-r)\sum_{n=1}^{\infty} A(n)r^n.$$

Assume that for some c_1 satisfying $0<c_1<c$ we have $|A(x)|<c_1x$ for $x>x_0$. Then

$$|\mathfrak{A}(r)| \le O(1) + (1-r)c_1\sum_{n>x_0} nr^n \le O(1) + (1-r)c_1\sum_{n=1}^{\infty} nr^n$$
$$\le C + c_1(1-r)^{-1}, \qquad r\to 1-\,.$$

This contradicts (2.7). ■

The following lemma relates $F[l/q](s)$ to $\mathfrak{A}(e(l/q)r)$.

LEMMA 2.6. *Let* $\alpha(n)$ *be an arbitrary sequence of complex numbers, and* $l\in\mathbb{Z}$. *Suppose the Dirichlet series* $F(s)=\sum_{n=1}^{\infty}\alpha(n)e(ln/q)n^{-s}$ *is convergent for* $\sigma=\Re s>\sigma_0>0$. *Then for each* s *with* $\Re s>\sigma_0$,

$$\Gamma(s)\sum_{n=1}^{\infty}\alpha(n)e(ln/q)n^{-s} = \int_0^\infty t^{s-1}\mathfrak{A}(e(l/q)e^{-t})\,dt.$$

Proof. This follows from the result of [H]. ■

The following lemma relates the behavior of $\mathfrak{A}(e(l/q)r)$ as $r\to 1-$ to the behavior of $F[l/q](s)$ near its singular point.

THEOREM 2.7. *Let $\alpha(n)$ be an arbitrary sequence of complex numbers, and $l,q\in\mathbb{N}$, $q>1$, $(l,q)=1$. Suppose the series $F[l/q](s)$ is convergent in $\{\Re s>\sigma_0\}$ for some $\sigma_0>0$. Suppose that there exist $c_1,\alpha>0$ and $t_0\in\mathbb{R}$ such that*

$$\overline{\lim_{x\to 0+}}\frac{|F[l/q](\sigma_0+it_0+x)|}{x^{-\alpha}}\ge c_1. \tag{2.8}$$

Then

$$\overline{\lim_{u\to 0+}}\frac{|\mathfrak{A}(e(l/q)e^{-u})|}{(-\ln u)^{\alpha-1}u^{-\sigma_0}}\ge c_1\frac{|\Gamma(\sigma_0+it_0)|}{\Gamma(\alpha)}. \tag{2.9}$$

Proof. By Lemma 2.6,

$$\Gamma(s)F[l/q](s)=\int_0^\infty u^{s-1}\mathfrak{A}(e(l/q)e^{-u})\,du$$

where the integral is also convergent in $\{\Re s>\sigma_0\}$. The assumptions of Theorem 2.4 are satisfied with $I(s)=\int_0^\infty u^{s-1}\mathfrak{A}(e(l/q)e^{-u})\,du$, $f(u)=\mathfrak{A}(e(l/q)e^{-u})$ and $c=c_1|\Gamma(\sigma_0+it_0)|$. By Theorem 2.4 we obtain (2.9). ∎

Let q be a fixed positive integer, $\beta=l/q$, $(l,q)=1$; let $q=p_1^{h_1}\dots p_r^{h_r}$ be the prime factorization. Denote by $K(q)$ the set $\{n:n=p_1^{l_1}\dots p_r^{l_r}\}$ where l_i are arbitrary positive integers. Each $n\in\mathbb{N}$ has a unique representation $n=mk$ where $k\in K(q)$ and $(m,q)=1$. For $k\in K(q)$ denote by A_k the set $\{n:n=mk,\ (m,q)=1\}$. From the above it follows that $\mathbb{N}=\bigsqcup_k A_k$. Let $u(n)=e(ln/q)$ if $(n,q)=1$, and $u(n)=0$ if $(n,q)\ne 1$. From the orthogonality relations for Dirichlet characters we derive

$$u(n)=\frac{1}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}\tau(\bar\chi,l)\chi(n). \tag{2.10}$$

Let $\alpha(n)$ be an additive function of an integer argument.

Let us represent $F[\beta](s)$ in terms of $F(s,\chi)$. For $k\in K(q)$ consider

$$S_k=\sum_{n\in A_k}\alpha(n)n^{-s}e(\beta n)=\sum_{(m,q)=1}\alpha(mk)(mk)^{-s}e(mk\beta).$$

Additivity of $\alpha(n)$ yields

$$\begin{aligned}S_k&=\sum_{(m,q)=1}\frac{\alpha(m)+\alpha(k)}{(mk)^s}e(km\beta)\\&=\frac{1}{k^s}\sum_{(m,q)=1}\alpha(m)e(mk\beta)m^{-s}+\frac{\alpha(k)}{k^s}\sum_{(m,q)=1}e(km\beta)m^{-s}.\end{aligned}$$

From (2.10) we obtain

$$S_k = \frac{1}{\phi(q)}\left(\frac{1}{k^s}\sum_{\chi\,(\mathrm{mod}\,q)}\tau(\bar\chi, lk)\sum_{m=1}^{\infty}\frac{\alpha(m)\chi(m)}{m^s} + \frac{\alpha(k)}{k^s}\sum_{\chi\,(\mathrm{mod}\,q)}\tau(\bar\chi, lk)\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}\right)$$
$$= \frac{1}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}\left(\frac{\tau(\bar\chi, lk)}{k^s}F(s,\chi) + \tau(\bar\chi, lk)\frac{\alpha(k)}{k^s}L(s,\chi)\right).$$

Summing S_k with respect to $k \in K(q)$ we obtain

LEMMA 2.8. *Let $\alpha(n)$ be an additive function. Then*

$$F[\beta](s) = \frac{1}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)} C_\chi(s)F(s,\chi) + D_\chi(s)L(s,\chi), \tag{2.11}$$

where

$$C_\chi(s) = \sum_{k\in K(q)}\frac{\tau(\bar\chi, lk)}{k^s}, \tag{2.12}$$

$$D_\chi(s) = \sum_{k\in K(q)}\frac{\tau(\bar\chi, lk)\alpha(k)}{k^s}. \tag{2.13}$$

This lemma relates the behavior of $F[\beta](s)$ to the behavior of $F(s,\chi)$ and $L(s,\chi)$.

3. The behavior of some useful Dirichlet series. In this section for an arbitrary function A and positive B, $A \ll B$ means $A = O(B)$.

Let us recall the properties of the Ramanujan sum. For a prime p, integers k, l, a with $(l,p)=1$ and the principal character χ_0 modulo p^k, we have

$$\tau(\chi_0, lp^{k-1}) = -p^{k-1},$$
$$\tau(\chi_0, lp^a) = p^{k-1}(p-1), \qquad a \ge k,$$
$$\tau(\chi_0, lp^a) = 0, \qquad a < k-1.$$

Consider $C_\chi(s)$ and $D_\chi(s)$ in the case $q = p_0^k$, where p_0 is a fixed prime. Let χ_0 be the principal character modulo $q = p_0^k$. Then from (2.12) and (2.13),

$$C_{\chi_0}(s) = -\frac{p_0^{k-1}}{p_0^{(k-1)s}} + p_0^{k-1}(p_0-1)\sum_{n=k}^{\infty}\frac{1}{p_0^{ns}}$$
$$= -\frac{p_0^{k-1}}{p_0^{(k-1)s}} + p_0^{k-1}(p_0-1)\frac{p_0^{-ks}}{1-p_0^{-s}},$$

$$D_{\chi_0}(s) = -\frac{p_0^{k-1}}{p_0^{(k-1)s}}\alpha(p_0^{k-1}) + p_0^{k-1}(p_0-1)\sum_{n=k}^{\infty}\frac{\alpha(p_0^n)}{p_0^{ns}}. \tag{3.1}$$

Note that $\tau(\chi, lp_0^a) = 0$ if $a > k$ for a nonprincipal character χ modulo q. Hence if $\chi \neq \chi_0 \pmod q$, the sums in (2.12) and (2.13) are finite. Thus $C_\chi(s) \ll 1$ and $D_\chi(s) \ll 1$ as $s \to 1+$ if $\chi \neq \chi_0 \pmod q$. Further we will use the simple asymptotic equality

$$L(1+\sigma, \chi_0) \sim \frac{p_0 - 1}{p_0}\sigma^{-1}, \qquad \sigma \to 0+\,. \tag{3.2}$$

Consider the function $F(s, \chi)$. By Lemma 2.3, for $\Re s > 1$,

$$F(s) = \zeta(s) \sum_p G_p(s)$$

where $G_p(s) = \sum_{k=1}^{\infty} (\alpha(p^k) - \alpha(p^{k-1}))p^{-ks}$. Hence by Lemmas 2.1 and 2.2,

$$\begin{aligned} F(s, \chi_0) &= L(s, \chi_0) \sum_p G_p(s, \chi_0) \\ &= L(s, \chi_0) \sum_{p \neq p_0} G_p(s). \end{aligned} \tag{3.3}$$

The Abel transform yields

$$G_p(s) = (1 - p^{-s}) \sum_{m=1}^{\infty} \alpha(p^m) p^{-ms}.$$

Since $\alpha(p^k) = kf(p,k)\ln p$ where $f(p,k) \to 0$ uniformly with respect to k, for $\Re s > 1$ we obtain

$$|G_p(s)| \le (1 + p^{-\sigma}) \sum_{m=1}^{\infty} g(p) \frac{m}{p^\sigma} \ln p,$$

where $g(p) = \max_k f(p,k)$. From (1.2), $g(p) \to 0$ as $p \to \infty$. Hence for $\sigma \to 0+$,

$$\begin{aligned} |G_p(1+\sigma)| &\le 2g(p) \Big(\sum_{m=1}^{\infty} \frac{m}{p^{(1+\sigma)m}} \Big) \ln p = 2 \frac{p^{-1-\sigma}}{(1 - p^{-1-\sigma})^2} g(p) \ln p \\ &\le 8p^{-1-\sigma} g(p) \ln p, \end{aligned}$$

where $g(p) \to 0$ as $p \to \infty$.

Hence for arbitrary $c > 0$ there exists a number $P(c)$ such that $g(p) \le c/8$ if $p > P(c)$, and we obtain

$$\begin{aligned} |F(1+\sigma, \chi_0)| &\le L(1+\sigma, \chi_0) \Big(O(1) + c \sum_{p > P(c)} p^{-1-\sigma} \ln p \Big) \\ &= L(1+\sigma, \chi_0) \Big(O(1) + c \sum_p p^{-1-\sigma} \ln p \Big) \\ &\le L(1+\sigma, \chi_0) \Big(O(1) + c \frac{\zeta'(1+\sigma)}{\zeta(1+\sigma)} \Big) \sim c \frac{p_0 - 1}{p_0} \sigma^{-2}. \end{aligned}$$

Since $C_{\chi_0}(s)$ has a simple zero at $s = 1$, we get

$$|C_{\chi_0}(1+\sigma)F(1+\sigma,\chi_0)| \le c_1\sigma^{-1}, \qquad \sigma \to 0+,$$

where $c_1 > 0$ is an arbitrarily small number. Hence

$$C_{\chi_0}(1+\sigma)F(1+\sigma,\chi_0) = o(\sigma^{-1}), \qquad \sigma \to 0+. \tag{3.4}$$

Let $\chi \ne \chi_0 \pmod{p_0^k}$. From Lemmas 2.1 and 2.2,

$$F(s,\chi) = L(s,\chi)\sum_p G_p(s,\chi).$$

Hence

$$|F(s,\chi)| \le |L(s,\chi)|\sum_p |G_p(s,\chi)|.$$

Let us estimate $G_p(s,\chi)$. From the definition of the class $\mathfrak{B}$ it follows that $|\alpha(p^m) - \alpha(p^{m-1})| \le mg(p)\ln p + (m-1)g(p)\ln p \le 2mg(p)\ln p$, where $g(p) \to 0$. Hence

$$\begin{aligned} G_p(s,\chi) &= \sum_{m=1}^{\infty}(\alpha(p^m) - \alpha(p^{m-1}))p^{-ms}\chi(p)^m \\ &= \alpha(p)p^{-s}\chi(p) + 2\ln p\sum_{m=2}^{\infty} mp^{-ms}O(g(p)). \end{aligned}$$

Since

$$\begin{aligned} \sum_{m=2}^{\infty} mp^{-ms}\ln p &= -\Big(\sum_{m=2}^{\infty} p^{-ms}\Big)' = -\left(\frac{p^{-2s}}{1-p^{-s}}\right)' \\ &= \frac{2(\ln p)p^{-2s}(1-p^{-s}) + p^{-2s}(\ln p)p^{-s}}{(1-p^{-s})^2} = O(|p^{-2s}\ln p|), \qquad s \to 1+, \end{aligned}$$

we have

$$G_p(1+\sigma,\chi) = \alpha(p)\chi(p)p^{-1-\sigma} + O(p^{-2(1+\sigma)}\ln p). \tag{3.5}$$

Since $|\alpha(p)| < c\ln p$ if $p > P(c)$, we obtain

$$|G_p(1+\sigma,\chi)| \le cp^{-(1+\sigma)}\ln p + O(p^{-2(1+\sigma)}\ln p) \qquad \text{for } p > P. \tag{3.6}$$

Since $\sum_p p^{-2(1+\sigma)}\ln p \ll 1$ when $\sigma \to 0+$, from (3.6) we have

$$\begin{aligned} \Big|\sum_p G_p(1+\sigma,\chi)\Big| &\le \sum_{p\le P(c)} p^{-1-\sigma}\ln p + c\sum_{p>P(c)} p^{-1-\sigma}\ln p + H \\ &\le H + H_1 + c\sum_p p^{-1-\sigma}\ln p \\ &\le H_2 + c\frac{\zeta'(1+\sigma)}{\zeta(1+\sigma)} \le c_1\sigma^{-1}, \end{aligned}$$

where H, H_1, H_2 are positive constants, $\sigma \to 0+$ and $c_1 > 0$ is an arbitrarily small number. Hence if $\chi \neq \chi_0 \pmod q$ then

$$C_\chi(1+\sigma)F(1+\sigma,\chi) = o(\sigma^{-1}). \tag{3.7}$$

Thus from (3.4) and (3.7) we obtain

$$\frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} C_\chi(1+\sigma)F(1+\sigma,\chi) = o(\sigma^{-1}). \tag{3.8}$$

Let us estimate $D_{\chi_0}(s)$. Using (1.2) we obtain

$$\begin{aligned}|D_{\chi_0}(s)| &\le p_0^{k-1} p_0^{-(k-1)\sigma} |\alpha(p_0^{k-1})| + p_0^{k-1}(p_0-1) \sum_{m=k}^{\infty} |\alpha(p_0^m)| p_0^{-m\sigma} \\ &\le c(p_0)\biggl(\frac{p_0^{k-1}}{p_0^{(k-1)\sigma}}(k-1)\ln p_0 + p_0^{k-1}(p_0-1) \sum_{m=k}^{\infty} \frac{m}{p_0^{m\sigma}} \ln p_0 \biggr) \\ &\le c_1(p_0) \sum_{m=k}^{\infty} m p_0^{-m\sigma} \ll_{p_0,\delta} 1\end{aligned}$$

for each $\delta > 0$ when $\Re s > \delta$.

Hence $D_{\chi_0}(s)$ is an analytic function in the half-plane $\{\Re s > 0\}$.

Since for nonprincipal χ the functions $L(s,\chi)$ are holomorphic at $s = 1$ and $D_\chi(s)$ are bounded when $s \to 1$, we get

$$\sum_{\chi \neq \chi_0 \,(\mathrm{mod}\, q)} D_\chi(1+\sigma)L(1+\sigma,\chi) \ll 1, \qquad \sigma \to 0+. \tag{3.9}$$

From (3.1) we deduce

$$\begin{aligned}D_{\chi_0}(1) &= -\alpha(p_0^{k-1}) + p_0^{k-1}(p_0-1) \sum_{m=k}^{\infty} \frac{\alpha(p_0^m)}{p_0^m} \\ &= p_0^{k-1}\biggl(-\frac{\alpha(p_0^{k-1})}{p_0^{k-1}} + (p_0-1) \sum_{m=k}^{\infty} \frac{\alpha(p_0^m)}{p_0^m} \biggr) \\ &= p_0^{k-1} \sum_{m=k}^{\infty} \frac{\alpha(p_0^m) - \alpha(p_0^{m-1})}{p_0^{m-1}} = p_0^k \sum_{m=k}^{\infty} \frac{\alpha(p_0^m) - \alpha(p_0^{m-1})}{p_0^m}.\end{aligned} \tag{3.10}$$

If $\sum_{m=k}^{\infty} (\alpha(p_0^m) - \alpha(p_0^{m-1}))/p_0^m \neq 0$, from (3.2) and (3.10) we obtain

$$\frac{1}{\phi(q)} D_{\chi_0}(1+\sigma)L(1+\sigma,\chi_0) \sim \sum_{m=k}^{\infty} \frac{\alpha(p_0^m) - \alpha(p_0^{m-1})}{p_0^m} \sigma^{-1}. \tag{3.11}$$

From (3.9) and (3.11) we derive

$$(3.12)\qquad \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} D_\chi(1+\sigma)L(1+\sigma,\chi) \sim \sum_{m=k}^{\infty} \frac{\alpha(p_0^m) - \alpha(p_0^{m-1})}{p_0^m}\sigma^{-1}$$

when $\sigma \to 0+$.

Thus if $\sum_{m=k}^{\infty} (\alpha(p_0^m) - \alpha(p_0^{m-1}))/p_0^m \neq 0$, from (3.8), (3.12) and Lemma 2.8 we deduce

$$(3.13)\qquad F[l/p_0^k](1+\sigma) \sim \sum_{m=k}^{\infty} \frac{\alpha(p_0^m) - \alpha(p_0^{m-1})}{p_0^m}\sigma^{-1}.$$

4. Proofs of main theorems

Proof of Theorem 1.1. If $\sum_{m=k}^{\infty} (\alpha(p^m) - \alpha(p^{m-1}))/p^m = 0$ the assertion is trivial. Consider the case $\sum_{m=k}^{\infty} (\alpha(p^m) - \alpha(p^{m-1}))/p^m \neq 0$. Then from (3.13) it follows that

$$\lim_{\sigma \to 0+} |F[l/p^k](1+\sigma)| \geq \left| \sum_{m=k}^{\infty} \frac{\alpha(p^m) - \alpha(p^{m-1})}{p^m} \right|,$$

Applying Theorem 2.7 to $F(s)$, l, $q = p_0^k$, $t_0 = 0$, $\sigma_0 = 1$, $\alpha = 0$ we obtain (1.3). Using Proposition 2.5 we deduce (1.4). ■

Proof of Theorem 1.2. Since $\mathfrak{A}(z)$ has non-singular points on the unit circle, the singular points are not dense there. Hence for each sufficiently large q there exists an l with $(l,q) = 1$ such that $\mathfrak{A}(e(l/q)r)$ is bounded when $r \to 1-$.

Applying this argument and Theorem 1.1 to sufficiently large $q = p^k$, and an integer l with $(l,p) = 1$, we deduce that there exists an integer P such that for any $p > P$ and any $k \geq 1$,

$$(4.1)\qquad \sum_{m=k}^{\infty} \frac{\alpha(p^m) - \alpha(p^{m-1})}{p^m} = 0.$$

From (4.1) we find that $\alpha(p^k) - \alpha(p^{k-1}) = 0$ for all $k \geq 1$ and $p > P$. Thus $\alpha(p^k) = 0$ for $k > 0$ and $p > P$, and the assertion (1.5) is proved. Similarly for any $p < P$ there exists an integer $k(p)$ such that for $k > k(p)$ (4.1) holds. Hence $\alpha(p^k) - \alpha(p^{k-1}) = 0$ for $k > k(p)$, hence $\alpha(p^m) = \alpha(p^{k(p)})$ if $m \geq k(p)$, and the assertion (1.6) is proved. From (1.5) and (1.6) and the representation (2.1) we find that $F(s) = \zeta(s)P(s)$ where $P(s)$ is a Dirichlet polynomial, $P(s) = \sum_{n \leq X} a(n)/n^s$. Hence

$$\mathfrak{A}(z) = \sum_{n<X} a(n)\frac{z^n}{1-z^n},$$

and represents a rational function. The theorem is proved. ■

Proof of Theorem 1.3. Since $\alpha(n)$ is completely additive, we have $\alpha(p^k)-\alpha(p^{k-1})=\alpha(p)$ for each p. Hence

$$\sum_{m=k}^{\infty}\frac{\alpha(p^m)-\alpha(p^{m-1})}{p^m}=\alpha(p)\sum_{m=k}^{\infty}p^{-m}=\alpha(p)p^{1-k}/(p-1). \tag{4.2}$$

From Theorem 1.1 using (4.2) we obtain (1.8) and (1.9). Theorem 1.2 implies $\alpha(p)=0$ for each p. Hence $\alpha(n)=0$ for each p. ∎

5. Some examples. Using Theorem 1.1 we obtain

PROPOSITION 5.1. *Let $\alpha(n)$ be an additive function with $\alpha(p)=o(\ln p)$, and $\alpha(p^k)=\alpha(p)$ if $k>1$. Then*

$$\overline{\lim_{r\to 1-}}\,|\mathfrak{A}(e(l/p)r)(1-r)|\ge\frac{|\alpha(p)|}{p},\qquad \overline{\lim_{x\to\infty}}\,\frac{|A(x,e(l/p))|}{x}\ge\frac{|\alpha(p)|}{p}.$$

Let $n=p_1^{l_1}\dots p_r^{l_r}$ be the prime factorization of n. Let $\omega(n)=r$ and $\Omega(n)=l_1+\dots+l_r$. Let $W_1(x,\beta)=\sum_{n<x}\omega(n)e(n\beta)$, $\mathfrak{W}_1(z)=\sum_{n=1}^{\infty}\omega(n)z^n$, $W_2(x,\beta)=\sum_{n<x}\Omega(n)e(n\beta)$ and $\mathfrak{W}_2(z)=\sum_{n=1}^{\infty}\Omega(n)z^n$. Applying Theorems 1.1, 1.3 and Propositions 5.1, 2.5 to $\omega(n)$, $\Omega(n)$ we obtain the following estimates. For each prime p and integer $k>0$,

$$\overline{\lim_{r\to 1-}}\,|\mathfrak{W}_1(e(l/p)r)(1-r)|\ge\frac{1}{p},$$
$$\overline{\lim_{x\to\infty}}\,\frac{|W_1(x,e(l/p))|}{x}\ge\frac{1}{p},$$
$$\overline{\lim_{r\to 1-}}\,|\mathfrak{W}_2(e(l/p^k)r)(1-r)|\ge\frac{p^{1-k}}{p-1},$$
$$\overline{\lim_{x\to\infty}}\,\frac{|W_2(x,e(l/p^k))|}{x}\ge\frac{p^{1-k}}{p-1}.$$

We note that the growth condition (1.2) cannot be much weakened.

EXAMPLE 5.2. If $\alpha(n)=c\ln n$ then $\mathfrak{A}(z)$ has an analytic continuation to $\mathbb{C}\setminus[1,\infty)$.

Proof. Note that $\sum_{n=1}^{\infty}(\ln n)z^n$ is the s-derivative of the polylogarithm function $\sum_{n=1}^{\infty}n^sz^n$ at $s=0$ (see [LS]). ∎

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