Properties of the intersection ideal $\mathcal{M} \cap \mathcal{N}$ revisited

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Summary. We investigate various properties of the intersection ideal $\mathcal{M} \cap \mathcal{N}$ of subsets of the reals that are related to the translations of its members. We are also concerned with cardinal coefficients associated with the ideal $\mathcal{M} \cap \mathcal{N}$.

1. Introduction. In this paper, we are interested in the ideal $\mathcal{M} \cap \mathcal{N}$ of subsets of the Cantor space $2^\omega$. This subject was initiated by the author \cite{4} and \cite{5}. In the latter article, it was proved that the class $(\mathcal{M} \cap \mathcal{N})^*$ coincides with $\mathcal{N}^*$. Throughout the paper we use rather standard terminology. We assume that $+$ is the modulo 2 coordinatewise addition in $2^\omega$ and $I$, $J$ are $\sigma$-ideals of subsets of $2^\omega$ with $I \subseteq J$.

Definition 1.1. We say that $X \subseteq 2^\omega$ is $I$-additive, and write $X \in I^*$, if $X + A = \{x + a : x \in X, a \in A\} \in I$ for any $A \in I$; and we write $X \in (I, J)^*$ if $X + A \in J$ for every $A \in I$.

Definition 1.2. Let $I$, $J$ be proper $\sigma$-ideals of subsets of $X$ containing all singletons, with $I \subseteq J$. Following [1] we define

$\text{add}(I) = \min\{|A| : A \subset I \text{ and } \bigcup A \notin I\},$

$\text{add}(I, J) = \min\{|A| : A \subset I \text{ and } \bigcup A \notin J\},$

$\text{cov}(I) = \min\{|A| : A \subset I \text{ and } X = \bigcup A\},$

$\text{non}(I) = \min\{|A| : A \subset X \text{ and } A \notin I\},$

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and
\[ \text{cof}(I) = \min \{|A| : A \subseteq I \text{ and } \forall_{A \in I} \exists_{B \in A} A \subseteq B \}. \]

We denote by \(\mathcal{M}\) the \(\sigma\)-ideal of meager subsets of \(2^\omega\), by \(\mathcal{N}\) the \(\sigma\)-ideal of measure zero subsets of \(2^\omega\), and by \(\mathcal{E}\) the \(\sigma\)-ideal generated by the \(F_\sigma\) measure zero subsets of \(2^\omega\). In \([1]\), the authors prove that \(\mathcal{E}\) is strictly contained in \(\mathcal{M} \cap \mathcal{N}\).

In the definition below we introduce the following notation. Let \(I\) be a \(\sigma\)-ideal of subsets of \(2^\omega\).

**Definition 1.3.**

\[(I, 2^\omega) = \{X \subseteq 2^\omega : X + A \neq 2^\omega \text{ for every } A \in I\}.\]

**Observation 1.4.** \(\mathcal{SN}\) (strongly measure zero sets) = \((\mathcal{M}, 2^\omega)\), and \(\mathcal{SM}\) (strongly meager sets) = \((\mathcal{N}, 2^\omega)\).

2. Main theorems. In the first part of this section, we shall be interested in relationships between some classes of sets of the form \((I, 2^\omega)\), where \(I\) is a \(\sigma\)-ideal of subsets of \(2^\omega\). This was motivated by \([4, \text{Proposition 19}]\) where the author presents a diagram of inclusions between some families of subsets of \(2^\omega\) defined in terms of translation which are smaller than \((I, 2^\omega)\).

Assume now that \(\rightarrow\) denotes inclusion and \(\not\rightarrow\) means that inclusion cannot be proved in ZFC.

**Theorem 2.1.** The following diagram of inclusions holds:

\[
\begin{array}{c}
N, 2^\omega \rightarrow \mathcal{SM} \\
E, 2^\omega \not\rightarrow \mathcal{M} \cap \mathcal{N}, 2^\omega \\
\mathcal{M}, 2^\omega \not\rightarrow \mathcal{N}, 2^\omega
\end{array}
\]

*Proof.* We prove the non-trivial part. To see that \((E, 2^\omega) \not\rightarrow (\mathcal{M} \cap \mathcal{N}, 2^\omega)\) apply Theorem 2.2 below. It is well-known that in \(M[c]\), where \(M\) is a model of ZFC, and \(c\) is a Cohen real over \(M\), \(X = 2^\omega \cap M\) is an \(SN\) set and \(X \notin SM\) by Carlson’s result (see \([1, \text{Lemma 3.3.12}]\)). This proves that \((\mathcal{M} \cap \mathcal{N}, 2^\omega) \not\rightarrow (\mathcal{N}, 2^\omega)\). To show that \((\mathcal{M} \cap \mathcal{N}, 2^\omega) \not\rightarrow (\mathcal{M}, 2^\omega)\) consider \(L * B_{\aleph_1}\), where \(L\) is the countable support \(\aleph_2\)-iteration of Laver forcing over a model of ZFC + GCH followed by the random algebra \(B_{\aleph_1}\). In the resulting model, there is a Sierpiński set (thus strongly meager) \(Y \in (\mathcal{M} \cap \mathcal{N}, 2^\omega)\). On the other hand, by Woodin’s argument (see \([1, \text{Theorem 8.3.7}]\)), \(Y \notin (\mathcal{M}, 2^\omega)\). Crossed arrows in \((\mathcal{M}, 2^\omega) \not\rightarrow (\mathcal{N}, 2^\omega)\) follow easily from the previous considerations. ■
Theorem 2.2. Assume CH. Then there exists $X \in (\mathcal{E}, 2^\omega)$ such that $X \notin (\mathcal{M} \cap \mathcal{N}, 2^\omega)$.

Proof. Let $\{z_\alpha\}_{\alpha<\omega}$ be an enumeration of $2^\omega$, and let $\{F_\alpha\}_{\alpha<\omega}$ be a list of all Borel sets in $\mathcal{E}$. Fix a set $A$ which belongs to $\mathcal{M} \cap \mathcal{N} \setminus \mathcal{E}$. We build $X = \{x_\alpha\}_{\alpha<\omega}$ and a sequence $\{r_\alpha\}_{\alpha<\omega}$ by induction. Suppose that we have already constructed $\{x_\alpha\}_{\alpha<\omega}$ and $\{r_\alpha\}_{\alpha<\omega}$. We will define $x_\lambda$ and $r_\lambda$. Choose $r_\lambda \notin \bigcup_{\alpha<\omega}(F_\lambda+x_\alpha)$, and let $B_\lambda = 2^\omega \setminus \bigcup_{\alpha<\omega}(F_\alpha+r_\alpha)$. Consider $z_\lambda$.

It is easy to see that Theorem 2.2 can be proved in a more general setting. Namely, if $I, J$ are Borel supported $\sigma$-ideals of subsets of $2^\omega$ with $I \subsetneq J$, then the following holds.

Theorem 2.3. Assume CH. Then $(J, 2^\omega) \subsetneq (I, 2^\omega)$.

Proof. Similar to the proof of Theorem 2.2. ■

A well-known theorem of Carlson (see [2]) states that $(\mathcal{E}, 2^\omega)$ forms a $\sigma$-ideal. The author of this article has not been able to answer the following question.

Problem 2.4. Is it consistent with ZFC that $(\mathcal{M} \cap \mathcal{N}, 2^\omega)$ is not closed under taking finite unions?

Theorem 2.5. It is consistent with ZFC that every member of $(\mathcal{M} \cap \mathcal{N}, 2^\omega)$ is at most countable.

Proof. There is a model of ZFC (see [3] Lemmas 5.1 and 5.2) in which for every uncountable $X \subseteq 2^\omega$, there are $A \in \mathcal{M}$ and $B \in \mathcal{N}$ such that $X + A = 2^\omega$ and $X + B = 2^\omega$. Assume that $X + (A \cap B) \neq 2^\omega$. Then there is $t \in 2^\omega$ such that $X + t \subseteq (2^\omega \setminus A) \cup (2^\omega \setminus B)$. Thus there are uncountable $X', X'' \subseteq X$ with $X' + t \subseteq 2^\omega \setminus A$ or $X'' + t \subseteq 2^\omega \setminus B$. But this is impossible by [3] Lemmas 5.1 and 5.2. ■

To finish this article we consider cardinal invariants related to the intersection ideal $\mathcal{M} \cap \mathcal{N}$. The idea of applying the property of two $\sigma$-ideals, defined below, was pointed out to the author by A. Krawczyk.

Definition 2.6. We say that two (proper) $\sigma$-ideals $I, J$ of subsets of a set $X$ are orthogonal if there is a set $\overline{X} \in I$ such that $\overline{X}^c \in J$.

Lemma 2.7. Suppose that $I$ and $J$ are orthogonal. Then $\text{add}(I \cap J, I) = \text{add}(I)$. 

Proof. Clearly, $\text{add}(I \cap J, I) \geq \text{add}(I)$. Suppose that $\kappa = \text{add}(I) < \text{add}(I \cap J, I)$. Then there is $\{G_\xi\}_{\xi<\kappa} \subset I$ such that $\bigcup_{\xi<\kappa} G_\xi \notin I$. This implies that $\bigcup_{\xi<\kappa} (G_\xi \cap X^c) \notin I$. 

The following lemma holds for arbitrary $\sigma$-ideals.

Lemma 2.8.

\begin{align*}
\text{cov}(I \cap J) &= \max\{\text{cov}(I), \text{cov}(J)\}, \\
\text{non}(I \cap J) &= \min\{\text{non}(I), \text{non}(J)\}.
\end{align*}

Proof. Left to the reader. $lacksquare$

Lemma 2.9. Suppose that $I$ and $J$ are orthogonal $\sigma$-ideals. Then

$$\text{cof}(I \cap J) = \max\{\text{cof}(I), \text{cof}(J)\}.$$ 

Proof. Assume that $\text{cof}(I) \leq \text{cof}(J)$ and consider $\mathcal{F} = \{G \cap X : G \in J\} \subseteq I \cap J$. Obviously, $\text{cof}(\mathcal{F}) \leq \text{cof}(I \cap J)$. Since $\text{cof}(\mathcal{F}) = \text{cof}(J)$, we conclude that $\text{cof}(I \cap J) = \text{cof}(J)$. $lacksquare$

As a corollary we obtain the following.

Proposition 2.10. We have

\begin{align*}
\text{add}(\mathcal{M} \cap \mathcal{N}, \mathcal{N}) &= \text{add}(\mathcal{N}), \\
\text{add}(\mathcal{M} \cap \mathcal{N}, \mathcal{M}) &= \text{add}(\mathcal{M}), \\
\text{cov}(\mathcal{M} \cap \mathcal{N}) &= \max\{\text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})\}, \\
\text{non}(\mathcal{M} \cap \mathcal{N}) &= \min\{\text{non}(\mathcal{M}), \text{non}(\mathcal{N})\}, \\
\text{cof}(\mathcal{M} \cap \mathcal{N}) &= \max\{\text{cof}(\mathcal{M}), \text{cof}(\mathcal{N})\}.
\end{align*}

Proof. This follows from the fact that $\mathcal{M}$ and $\mathcal{N}$ are orthogonal and from the above lemmas. $lacksquare$

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