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WEAKLY CONFORMALLY SYMMETRIC MANIFOLDS

BY

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Dedicated to the memory of Professor Witold Roter

Abstract. We study the properties of weakly conformally symmetric pseudo-Riemannian manifolds, with particular emphasis on the 4-dimensional Lorentzian case. We provide a decomposition of the conformal curvature tensor in dimensions $n \geq 5$. Moreover, some identities involving two particular covectors are stated; for example it is proven that under certain conditions the Ricci tensor and other tensors are Weyl compatible: this notion was recently introduced and investigated by Mantica and Molinari. Topological properties involving the vanishing of the first Pontryagin form are then stated. Further we study weakly conformally symmetric 4-dimensional Lorentzian manifolds (space-times); it is proven that one of the previously defined covectors is null and unique up to scaling; moreover it is shown that under certain conditions the same vector is an eigenvector of the Ricci tensor and its integral curves are geodesics. Finally, it is shown that such a space-time is of Petrov type N with respect to the same vector.

1. Introduction. Recurrent manifolds have been investigated by many geometers (see for example [AM], [Ka], [Kh] and [Wa]). In particular, Walker studied manifolds on which the Riemann curvature tensor is recurrent [Wa], while conformally recurrent manifolds were investigated by Adati and Miyazawa [AM] and others (see for example [R1]–[R4]). McLenaghan and Leroy [ML] and then McLenaghan and Thompson [MT] carried out a detailed investigation of space-times with complex-recurrent conformal curvature tensor. They showed that such spaces belong to types D and N of the Petrov classification, and found the metric forms of these spaces when the recurrence vector is real.

U. C. De and S. Bandyopadhyay [DB2] introduced the notion of weakly conformally symmetric manifold. An n-dimensional Riemannian manifold is said to be *weakly conformally symmetric*, briefly $(\mathrm{WCS})_n$, if it is not

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conformally flat and satisfies the following condition:

$$\nabla_i C_{jklm} = A_i C_{jklm} + B_j C_{iklm} + D_k C_{jilm} + E_l C_{jkim} + F_m C_{jkli}, \tag{1.1}$$

where A_i, B_i, D_i, E_i, F_i are 1-forms (not simultaneously zero) called *associated covectors* and $C_{jklp} = g_{mp} C^m_{jkl}$ for the $(1,3)$ conformal curvature tensor C^m_{jkl} whose local components are given by [Po]

$$\begin{aligned} C^m_{jkl} = R^m_{jkl} &+ \frac{1}{n-2}(\delta^m_j R_{kl} - \delta^m_k R_{jl} + R^m_j g_{kl} - R^m_k g_{jl}) \\ &- \frac{R}{(n-1)(n-2)}(\delta^m_j g_{kl} - g^m_k g_{jl}). \end{aligned} \tag{1.2}$$

In this expression the Ricci tensor is defined as $R_{kl} = -R^m_{mkl}$ [Wr] and the scalar curvature as $R = g^{ij} R_{ij}$. If in (1.1), C^m_{jkl} is replaced with R^m_{jkl}, this notion reduces to that of a weakly symmetric manifold $(\mathrm{WS})_n$ introduced by Tamassy and Binh (see [Bi] and [TB]) and investigated by several authors (see [DB1] and [Pr]). Some theorems about weakly conformally symmetric manifolds were stated in [DB2], [MHS], [SSH], mostly in the Riemannian case. Recently the present authors extended the concept of recurrent curvature tensors to the associated curvature 2-forms while generalizing some known results (see [MS2]–[MS4] and [MS7]). In particular the recurrence of the conformal curvature 2-form was introduced [MS4]: the conformal curvature 2-form $\Omega^m_{(C)l} = C^m_{jkl} dx^j \wedge dx^k$ is said to be *recurrent* if there exists a non-zero scalar 1-form α for which

$$D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}, \tag{1.3}$$

$\alpha = \alpha_i dx^i$ being the associated 1-form and D the exterior covariant derivative. It turns out (see [MS2] or [MS4, Theorem 2.1 and (4.1)]) that (1.3) is equivalent to

$$\nabla_i C^m_{jkl} + \nabla_j C^m_{kil} + \nabla_k C^m_{ijl} = \alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl}. \tag{1.4}$$

In this paper we investigate the properties of weakly conformally symmetric pseudo-Riemannian manifolds; particular attention is given to the 4-dimensional Lorentzian case.

In Section 2 we first provide an alternative proof of a well known result. Then several new others are obtained: in particular it is proven that $\varepsilon_m C^m_{jkl} = 0$ where $\varepsilon_j = 2B_j + E_j$ and some new identities involving the covector ε_j are stated; we also provide a decomposition of the conformal tensor in a $(\mathrm{WCS})_n$ pseudo-Riemannian manifold with $n \geq 5$. It is then shown that on a $(\mathrm{WCS})_n$ pseudo-Riemannian manifold the conformal curvature 2-form is recurrent. Moreover, under the condition of closedness of the covector $X_j = A_j - 2B_j$ it is proven that the Ricci tensor of a weakly conformally symmetric pseudo-Riemannian manifold is *Weyl compatible*; this notion was recently introduced and investigated in [MM2]–[MM4]). Finally, it is verified that $\nabla_i \varepsilon_j$ is also Weyl compatible.

In Section 3 we study weakly conformally symmetric 4-dimensional Lorentzian manifolds (space-times): it is proven that the covector ε_j is null and unique up to scaling; moreover, under certain conditions that vector is an eigenvector of the Ricci tensor and its integral curves are geodesics. Finally, it is proved that such a space-time is of Petrov type N with respect to the null vector ε_j.

All manifolds under consideration are assumed to be connected Hausdorff with a non-degenerate metric of arbitrary signature, i.e., n-dimensional pseudo-Riemannian manifolds; in Section 3 we specialize to a metric of signature $\mathrm{s} = +2$, i.e. to 4-dimensional Lorentz manifolds [FC]. Moreover it is always assumed that $\nabla_j g_{kl} = 0$ (the Levi Civita connection). It is also assumed that the space-matter content is described by the stress-energy tensor and related to the Ricci tensor by Einstein's equations $R_{kl} - (R/2)g_{kl} = kT_{kl}$, where k denotes the Einstein gravitational constant given by $k = 8\pi G/c^4$ (see [FC], [St], [SK$^+$]).

2. Weakly conformally symmetric pseudo-Riemannian manifolds: general properties. In this section we establish several new properties of weakly conformally symmetric n-dimensional pseudo-Riemannian manifolds. The following fact is found in the geometric literature (see for example [DB1], [DB2] and [Pr]).

THEOREM 2.1. *Let M be a non-conformally flat $(WCS)_n$ pseudo-Riemannian manifold. Then $B_k = D_k$ and $E_k = F_k$.*

Here we give an alternative proof of this theorem. Interchanging j and k in (1.1) and then adding the resulting relation to (1.1) we infer that

$$\omega_j C_{iklm} + \omega_k C_{ijlm} = 0, \tag{2.1}$$

where $\omega_k = B_k - D_k$.

Now write three versions of (2.1) with indices j, l, m cyclically permuted and sum up (some terms cancel by the first Bianchi identity); we get

$$\omega_j C_{lmik} + \omega_l C_{mjik} + \omega_m C_{jlik} = 0. \tag{2.2}$$

Change $l \leftrightarrow i$, $m \leftrightarrow k$ to infer that

$$\omega_j C_{iklm} + \omega_i C_{kjlm} + \omega_k C_{jilm} = 0. \tag{2.3}$$

Add this relation to (2.1), obtaining $2\omega_j C_{iklm} = \omega_i C_{jklm}$ and then $2\omega_i C_{jklm} = \omega_j C_{iklm}$. In this way we easily find $4\omega_j C_{iklm} = 2\omega_i C_{jklm} = \omega_j C_{iklm}$, from which $\omega_j C_{iklm} = 0$ and $\omega_j = 0$. Again interchanging l and m in (1.1) and then adding the resulting relation to (1.1) we easily infer that

$$\eta_j C_{iklm} + \eta_k C_{ijlm} = 0, \tag{2.4}$$

where $\eta_k = E_k - F_k$. By exactly the same arguments as before we conclude that $\eta_j = 0$. Now transvecting (1.1) with g^{im} and recalling Theorem 2.1 gives

$$\nabla_m C^m_{jkl} = (A_m + E_m) C^m_{jkl}. \tag{2.5}$$

On the other hand, write three versions of (1.1) with i, j, k cyclically permuted and sum up (some terms cancels by the first Bianchi identity); it follows that

$$\nabla_i C_{jklm} + \nabla_j C_{kilm} + \nabla_k C_{ijlm} = X_i C_{kilm} + X_j C_{kilm} + X_k C_{ijlm}, \tag{2.6}$$

where $X_j = A_j - 2B_j$.

Remark 2.2 ([MS4]). From (2.6) it follows that on a weakly conformally symmetric pseudo-Riemannian manifold the conformal 2-form $\Omega^m_{(C)l} = C^m_{jkl} dx^j \wedge dx^k$ is recurrent, i.e. $D\Omega^m_{(C)l} = X \wedge \Omega^m_{(C)l}$.

Transvecting (2.6) with g^{im} gives

$$\nabla_m C^m_{jkl} = X_m C^m_{jkl}. \tag{2.7}$$

From (2.7) and (2.5) we obtain

$$\varepsilon_m C^m_{jkl} = 0, \tag{2.8}$$

where $\varepsilon_j = 2B_j + E_j$.

Now expanding the relation $\nabla_i(\varepsilon_m C^m_{ikl}) = 0$ we get $(\nabla_i \varepsilon^m) C_{jklm} + \varepsilon^m(\nabla_i C_{jklm}) = 0$, and using definition (1.1) together with (2.8) and $E_k = F_k$ it is easily shown that

$$(\nabla_i \varepsilon^m) C_{jklm} + (\varepsilon^m E_m) C_{jkli} = 0. \tag{2.9}$$

From (2.8) and (2.9) it is easily inferred that

$$\varepsilon^i (\nabla_i \varepsilon^m) C_{jklm} = 0. \tag{2.10}$$

We can thus state a useful theorem.

Theorem 2.3. *Let M be a non-conformally flat $(WCS)_n$ pseudo-Riemannian manifold. Then* (2.8)–(2.10) *hold true.*

Consider now the second Bianchi identity for the Weyl curvature tensor. It can be written as [MS4]

$$\nabla_i C^m_{jkl} + \nabla_j C^m_{kil} + \nabla_k C^m_{ijl} = B^m_{ijkl}, \tag{2.11}$$

where B^m_{ijkl} is a source tensor. The definition (1.2) of the conformal curvature tensor and straightforward calculations allow us to rewrite the source term as

$$\begin{aligned}
B^m_{ijkl} = \frac{1}{n-2}&[\delta^m_j(\nabla_i R_{kl} - \nabla_k R_{il}) + \delta^m_i(\nabla_k R_{jl} - \nabla_j R_{kl}) \\
&+ \delta^m_k(\nabla_j R_{il} - \nabla_i R_{jl}) + g_{il}(\nabla_j R^m_k - \nabla_k R^m_j) + g_{jl}(\nabla_k R^m_j) \\
&+ g_{jl}(\nabla_k R^m_i - \nabla_i R^m_k) + g_{kl}(\nabla_i R^m_j - \nabla_j R^m_i)] \\
&- \frac{1}{(n-1)(n-2)}[\delta^m_j(\nabla_i R g_{kl} - \nabla_k R g_{il}) \\
&+ \delta^m_i(\nabla_k R g_{jl} - \nabla_j R g_{kl}) + \delta^m_k(\nabla_j R g_{il} - \nabla_i R g_{jl})].
\end{aligned} \tag{2.12}$$

Now taking the divergence of the conformal curvature tensor and recalling that $\nabla_m R^m_{jkl} = \nabla_k R_{jl} - \nabla_j R_{kl}$ and $2\nabla^m R_{jm} = \nabla_j R$, we infer that

$$\nabla_m C^m_{jkl} = \frac{n-3}{n-2}\left[\nabla_k R_{jl} - \nabla_j R_{kl} - \frac{1}{2(n-1)}(\nabla_k R g_{jl} - \nabla_j R g_{kl})\right]. \tag{2.13}$$

Using (2.13) in (2.12), after long but straightforward calculations the following form of the source term in the second Bianchi identity for the conformal tensor is achieved (see [AM, (3.7)] and [MS4, (4.3)]):

$$\begin{aligned}
\nabla_i C^m_{jkl} + \nabla_j C^m_{kil} + \nabla_k C^m_{ijl}& \\
= \frac{1}{n-3}[&\delta^m_j \nabla_p C^p_{kil} + \delta^m_k \nabla_p C^p_{ijl} + \delta^m_i \nabla_p C^p_{jkl} \\
&+ g_{kl}\nabla_p C^{mp}_{ji} + g_{il}\nabla_p C^{mp}_{kj} + g_{jl}\nabla_p C^{mp}_{ik}].
\end{aligned} \tag{2.14}$$

In view of (2.6) and (2.7), this can be rewritten as

$$\begin{aligned}
X_i C^m_{jkl} + X_j C^m_{kil} + X_k C^m_{ijl}& \\
= \frac{1}{n-3}[&\delta^m_j X_p C^p_{kil} + \delta^m_k X_p C^p_{ijl} + \delta^m_i X_p C^p_{jkl} \\
&+ g_{kl} X_p C^{mp}_{ji} + g_{il} X_p C^{mp}_{kj} + g_{jl} X_p C^{mp}_{ik}].
\end{aligned} \tag{2.15}$$

Transvecting with ε^i and recalling (2.8) one easily infers that

$$(\varepsilon^i X_i) C^m_{jkl} = \frac{1}{n-3}\{\varepsilon^m X_p C^p_{jkl} + \varepsilon_l X_p C^{mp}_{kj}\}. \tag{2.16}$$

On multiplying by X^l, since $X^p X^l C_{jklp} = 0$, the above expression becomes

$$\frac{n-4}{n-3}(\varepsilon^i X_i) X_p C^{pm}_{jk} = 0. \tag{2.17}$$

We have thus shown that if $n \geq 5$ then $\varepsilon^i X_i = 0$, or $X_p C^{pm}_{jk} = 0$, and by (2.7) also $\nabla_m C^m_{jkl} = 0$.

THEOREM 2.4. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold with $n \geq 5$. Then $\varepsilon^i X_i = 0$ or $\nabla_m C^m_{jkl} = 0$.*

On the other hand, transvecting (2.16) with ε^l we get

$$(\varepsilon_l\varepsilon^l)X_p C^{mp}_{kj} = 0. \tag{2.18}$$

Again we obtain $\varepsilon^i\varepsilon_i = 0$, or $X_p C^{pm}_{jk} = 0$, and thus $\nabla_m C^m_{jkl} = 0$.

Theorem 2.5. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold with $n \geq 4$. Then $\varepsilon^i\varepsilon_i = 0$ or $\nabla_m C^m_{jkl} = 0$.*

A useful result may be obtained from (2.15) written in the form

$$\begin{aligned} X_i C_{jklm} &+ X_j C_{kilm} + X_k C_{ijlm} \\ &= \frac{1}{n-3}[g_{jm}X^p C_{kilp} + g_{km}X^p C_{ijlp} + g_{im}X^p C_{jklp} \\ &\quad + g_{kl}X^p C_{jimp} + g_{il}X^p C_{kjmp} + g_{jl}X^p C_{ikmp}]. \end{aligned} \tag{2.19}$$

From Remark 2.2 and (2.6) and (2.7) we infer that (2.19) holds on a manifold with recurrent conformal 2-form:

Proposition 2.6. *Let M be a pseudo-Riemannian manifold on which the conformal 2-form is recurrent, i.e. $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$. Then* (2.19) *holds.*

The previous relation, multiplied by $X^i X^l$, after straightforward calculations yields

$$\frac{n-4}{n-3}\{(X^i X_i)X^l C_{kjml} + X_j X^i X^l C_{ikml} - X_k X^i X^l C_{ijml}\} = 0. \tag{2.20}$$

So if $n \geq 5$, under the condition $X_i X^i \neq 0$ we may conclude that

$$X^l C_{kjml} = X_k S_{jm} - X_j S_{km}, \tag{2.21}$$

where $S_{jm} = \frac{X^i X^l C_{ijml}}{X^i X_i}$ is a $(0,2)$ symmetric tensor. On multiplying (2.19) by X^i and employing (2.21), after straightforward calculations we finally obtain the following decomposition for the conformal tensor in dimension $n \geq 5$:

$$\begin{aligned} (X^i X_i)&C_{jkml} \\ &= \frac{(X^i X_i)}{n-3}\{g_{mk}S_{jl} - g_{mj}S_{kl} + g_{jl}S_{km} - g_{kl}S_{jm}\} \\ &\quad + \frac{n-2}{n-3}\{X_j X_m S_{kl} - X_k X_m S_{jl} + X_k X_l S_{jm} - X_j X_l S_{km}\}. \end{aligned} \tag{2.22}$$

Theorem 2.7. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold with $n \geq 5$. If $X_i X^i \neq 0$, then the decomposition* (2.22) *holds.*

Again from Remark 2.2 and (2.6) and (2.7) we infer that (2.22) holds on a manifold with recurrent conformal 2-form:

PROPOSITION 2.8. *Let M be an n-dimensional pseudo-Riemannian manifold with $n \geq 5$ on which the conformal 2-form is recurrent, i.e. $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$. Then (2.22) holds.*

Now we state a theorem involving the case in which the 1-form X_j happens to be closed or of concircular form.

THEOREM 2.9. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold. If*

(1) *$n \geq 5$ and X_j is a closed 1-form, or*
(2) *$n = 4$ and $\nabla_i X_j + X_i X_j = \gamma g_{ij}$ for some scalar γ,*

then

$$R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl} = 0. \tag{2.23}$$

The proof of this theorem requires some auxiliary lemmas (see also [MS4]). The first one is known as Lovelock's differential identity and may be found for example in [LR] and [MM1].

LEMMA 2.10 (Lovelock's differential identity). *Let M be an n-dimensional pseudo-Riemannian manifold. Then*

$$\nabla_i\nabla_m R^m_{jkl} + \nabla_j\nabla_m R^m_{kil} + \nabla_k\nabla_m R^m_{ijl} = -R_{im}R^m_{jkl} - R_{jm}R^m_{kil} - R_{km}R^m_{ijl}. \tag{2.24}$$

Lovelock's identity implies the following for the conformal curvature tensor (see [MM1], [MM3] and [MS4]):

$$\nabla_i\nabla_m C^m_{jkl} + \nabla_j\nabla_m C^m_{kil} + \nabla_k\nabla_m C^m_{ijl} = -\frac{n-3}{n-2}(R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl}). \tag{2.25}$$

Consider now the second Bianchi identity for the conformal curvature tensor written for a (WCS)$_n$ manifold,

$$\nabla_i C^m_{jkl} + \nabla_j C^m_{kil} + \nabla_k C^m_{ijl} = X_i C^m_{jkl} + X_j C^m_{kil} + X_k C^m_{ijl},$$

and the relation $\nabla_m C^m_{jkl} = X_m C^m_{jkl}$; the left hand side of the last displayed equation may be written in the form

$$(\nabla_i X_m)C^m_{jkl} + (\nabla_j X_m)C^m_{kil} + (\nabla_k X_m)C^m_{ijl} + X_m(X_i C^m_{jkl} + X_j C^m_{kil} + X_k C^m_{ijl}). \tag{2.26}$$

Then, taking the divergence of the relation $X_i C^m_{jkl} + X_j C^m_{kil} + X_k C^m_{ijl} = B^m_{ijkl}$, we easily see that

$$(\nabla_m X_i)C^m_{jkl} + (\nabla_m X_j)C^m_{kil} + (\nabla_m X_k)C^m_{ijl} + X_m(X_i C^m_{jkl} + X_j C^m_{kil} + X_k C^m_{ijl}) = \nabla_m B^m_{ijkl}. \tag{2.27}$$

If the covector X_j is closed, we can finally write

$$\nabla_m B^m_{ijkl} = -\frac{n-3}{n-2}(R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl}). \tag{2.28}$$

We have thus proved the following result:

Lemma 2.11. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold. If the covector X_j is closed, then* (2.28) *holds.*

Now the following lemma about the source term of the second Bianchi identity for the conformal curvature tensor can be stated [MS4]:

Lemma 2.12. *The divergence of the source term in the second Bianchi identity for the conformal curvature tensor has the form*

$$\nabla_m B^m_{ijkl} = -\frac{1}{n-2}(R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl}). \tag{2.29}$$

Proof. First we note that transvecting (2.25) with g^{il} we easily get

$$\nabla_l \nabla_m C^{lm}_{jk} = 0.$$

We then recall equation (2.14):

$$\begin{aligned}\nabla_i C^m_{jkl} + \nabla_j C^m_{kil} + \nabla_k C^m_{ijl} \\ = \frac{1}{n-3}[\delta^m_j \nabla_p C^p_{kil} + \delta^m_k \nabla_p C^p_{ijl} + \delta^m_i \nabla_p C^p_{jkl} \\ + g_{kl}\nabla_p C^{mp}_{ji} + g_{il}\nabla_p C^{mp}_{kj} + g_{jl}\nabla_p C^{mp}_{ik}].\end{aligned} \tag{2.30}$$

Taking the covariant derivative ∇_m of this equation and recalling that $\nabla_l \nabla_m C^{lm}_{jk} = 0$ we obtain

$$\nabla_m B^m_{ijkl} = \frac{1}{n-3}(\nabla_i \nabla_m C^m_{jkl} + \nabla_j \nabla_m C^m_{kil} + \nabla_k \nabla_m C^m_{ijl}). \tag{2.31}$$

Now Lovelock's identity (2.25) is used to complete the proof. ■

Proof of Theorem 2.9. From Lemmas 2.11 and 2.12, i.e. (2.28) and (2.29), we easily infer that $\frac{n-4}{n-2}(R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl}) = 0$; thus for $n \geq 5$ the theorem is proven.

If $n = 4$ and $\nabla_i X_j + X_i X_j = \gamma g_{ij}$ then (2.26) vanishes identically thanks to the first Bianchi identity for the Weyl tensor. Thus from (2.25) we infer $\frac{1}{2}(R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl}) = 0$. ■

If the Ricci tensor satisfies (2.23), it is called *Riemann compatible* (see [MM2]–[MM4]). Geometric and topological consequences of this condition were extensively studied in [MM3]. If we insert in (2.23) the local form of the Weyl tensor [Po], we obtain

$$R_{im}C^m_{jkl} + R_{jm}C^m_{kil} + R_{km}C^m_{ijl} = 0. \tag{2.32}$$

Thus the Ricci tensor is Weyl compatible. In recent works, Weyl compatibility has been extensively investigated in the Riemannian case [MM3] and in the pseudo-Riemannian case [MM4]. We mention that there are results on manifolds satisfying (2.32) published earlier than [MM3] and [MM4] and not cited in those papers: see, e.g., [AÇDE, Lemma 3.11] and [DG, Proposition 3.1(iv)]. Moreover, several examples of pseudo-symmetric structures satisfying (2.32) can be found in [DG$^+$], and applications to general relativity in [DH$^+$]. In Section 3 we will give a detailed account of the consequences of (2.32) on the structure of (WCS)$_n$ space-times. If we use Einstein's equations in (2.32), we infer an analogous condition for the stress energy tensor:

$$T_{im}C^m_{jkl} + T_{jm}C^m_{kil} + T_{km}C^m_{ijl} = 0. \tag{2.33}$$

From the above discussion we may derive the following:

THEOREM 2.13. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold. If*

(1) *$n \geq 5$ and X_j is a closed 1-form, or*
(2) *$n = 4$ and $\nabla_i X_j + X_i X_j = \gamma g_{ij}$ for some scalar γ,*

then the Ricci tensor and the stress energy tensor are Weyl compatible.

From Theorems 2.4 and 2.5 and (2.25) we can deduce the following corollaries.

COROLLARY 2.14. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold with $n \geq 5$. If $\varepsilon^i X_i \neq 0$, then $\nabla_m C^m_{jkl} = 0$ and $R_{im}C^m_{jkl} + R_{jm}C^m_{kil} + R_{km}C^m_{ijl} = 0$.*

COROLLARY 2.15. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold with $n \geq 4$. If $\varepsilon^i \varepsilon_i \neq 0$, then $\nabla_m C^m_{jkl} = 0$ and $R_{im}C^m_{jkl} + R_{jm}C^m_{kil} + R_{km}C^m_{ijl} = 0$.*

Now sum (2.9) cyclically over i, j, k, recalling the first Bianchi identity for the Weyl tensor, to obtain

$$(\nabla_i \varepsilon_m)C^m_{jkl} + (\nabla_j \varepsilon_m)C^m_{kil} + (\nabla_k \varepsilon_m)C^m_{ijl} = 0. \tag{2.34}$$

THEOREM 2.16. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold with $n \geq 4$. Then the tensor $\nabla_i \varepsilon_m$ is Weyl compatible.*

We remark here that the tensor $\nabla_i \varepsilon_m$ is not necessarily symmetric unless ε_j is closed. From (2.32) and (2.33), transvecting with ε^i and using (2.8), it is easily inferred that

$$\varepsilon^i R_{im} C^m_{jkl} = 0, \qquad \varepsilon^i T_{im} C^m_{jkl} = 0. \tag{2.35}$$

THEOREM 2.17. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold. If*

(1) *$n \geq 5$ and X_j is a closed 1-form, or*
(2) *$n = 4$ and $\nabla_i X_j + X_i X_j = \gamma g_{ij}$ for some salar γ,*

then (2.35) holds true.

Finally, we present two other interesting results. First, summing (2.16) cyclically over j, k, l, it is easily inferred that

$$\varepsilon_l X_p C_{kj}^{mp} + \varepsilon_j X_p C_{lk}^{mp} + \varepsilon_k X_p C_{jl}^{mp} = 0. \tag{2.36}$$

It turns out that the tensor $\varepsilon_i X_j$ is Weyl compatible.

THEOREM 2.18. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold with $n \geq 4$. Then the tensor $\varepsilon_i X_j$ is Weyl compatible.*

Secondly, multiply (2.19) by X^m to obtain, after some cancellations,

$$\frac{n-4}{n-3}[X_i X^m C_{kjml} + X_i X^m C_{kilm} + X_k X^m C_{ijlm}] = 0. \tag{2.37}$$

It turns out that the tensor $X_i X_j$ is Weyl compatible whenever $n \geq 5$.

THEOREM 2.19. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold with $n \geq 5$. Then $X_i X_j$ is Weyl compatible.*

Again from Remark 2.2 and (2.6) and (2.7) we infer that (2.37) holds on a manifold with recurrent conformal 2-form:

PROPOSITION 2.20. *Let M be an n-dimensional pseudo-Riemannian manifold with $n \geq 5$ on which the conformal 2-form is recurrent, i.e. $D\Omega_{(C)l}^m = \alpha \wedge \Omega_{(C)l}^m$. Then the tensor $X_i X_j$ is Weyl compatible.*

Moreover, another Weyl compatible tensor may be found. Multiplying (2.22) by S_i^l on, noting that $X^i S_i^l = 0$ we easily find

$$\begin{aligned}(X_p X^p) S_i^l C_{jkml} &= \frac{X_p X^p}{n-3}[g_{mk} S_i^l S_{jl} - g_{mj} S_i^l S_{kl} + S_{ij} S_{km} - S_{ki} S_{jm}] \\ &\quad + \frac{n-2}{n-3}[X_j X_m S_i^l S_{kl} - X_k X_m S_i^l S_{jl}].\end{aligned} \tag{2.38}$$

Sum this cyclically over i, j, k to obtain

$$(X_p X^p)[S_i^l C_{jkml} + S_j^l C_{kiml} + S_k^l C_{ijml}] = 0. \tag{2.39}$$

We thus state the following.

THEOREM 2.21. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold with $n \geq 5$. Then the tensor S_{ij} is Weyl compatible provided that $X_p X^p \neq 0$.*

Finally, we investigate some consequences of particular curvature conditions imposed on a (WCS)$_n$ pseudo-Riemannian manifold. Consider a weakly conformally symmetric manifold whose covectors B and E are subject to the conditions (see also Section 3 below and [DD], [DH], [DG]):

$$\begin{aligned}E_i C_{jklm} + E_j C_{kilm} + E_k C_{ijlm} &= 0, \\ B_i C_{jklm} + B_j C_{kilm} + B_k C_{ijlm} &= 0.\end{aligned} \tag{2.40}$$

Then from (1.1) with $B_k = D_k$ and $E_k = F_k$ we infer $E_l C_{jkim} + E_m C_{jkli} = E_l C_{imjk} + E_m C_{lijk} = E_i C_{jklm}$ and $B_j C_{iklm} + B_k C_{jilm} = B_i C_{jklm}$; thus (1.1) becomes

$$\nabla_i C_{jklm} = (A_i + B_i + E_i) C_{jklm}, \tag{2.41}$$

and the manifold turns out to be conformally recurrent.

THEOREM 2.22. *Let M be a $(WCS)_n$ pseudo-Riemannian manifold. If condition* (2.38) *holds, then the manifold is conformally recurrent.*

REMARK 2.23. Transvecting (2.40) by g^{im} we get $E^m C_{jklm} = 0$ and $B^m C_{jklm} = 0$ from which it is easily inferred that $E_i E^i = B_i B^i = E_i B^i = 0$ whenever the manifold is not conformally flat. In n-dimensional Lorentzian geometry it is well known that two null vectors are orthogonal if and only if they are collinear [SRT, Proposition 4]. Thus a $(\mathrm{WCS})_n$ Lorentzian manifold with covectors B and E satisfying (2.40) is conformally recurrent of type (2.41) with $B_k = \lambda E_k$ for some scalar λ.

3. Weakly conformally symmetric Lorentzian manifolds. In this section, we investigate in more detail the properties of $(\mathrm{WCS})_4$ Lorentzian manifolds (space-times). We begin with some auxiliary lemmas recently reviewed in [MM5]:

LEMMA 3.1 (see [LR, p. 128] and [MM5]). *Let M be a 4-dimensional pseudo-Riemannian manifold. Then the following identity involving the conformal curvature tensor holds:*

$$\begin{aligned} \delta^i_r C^{jk}_{st} + \delta^i_t C^{jk}_{rs} + \delta^i_s C^{jk}_{tr} + \delta^k_r C^{ij}_{st} + \delta^k_t C^{ij}_{rs} + \delta^k_s C^{ij}_{tr} \\ + \delta^j_r C^{ki}_{st} + \delta^j_t C^{ki}_{rs} + \delta^j_s C^{ki}_{tr} = 0. \end{aligned} \tag{3.1}$$

LEMMA 3.2 (see [FC, p. 46] and [MM5]). *Let M be a 4-dimensional pseudo-Riemannian manifold. Let X be a null vector and Y a vector orthogonal to X, i.e. $X^i Y_i = 0$. Then Y is space-like or null, and proportional to X, i.e. $Y_j = \lambda X_j$ for some $\lambda \in \mathbb{R}$.*

Now consider a non-conformally flat 4-dimensional Lorentzian manifold and two vector fields X_j and Y_j satisfying $X^m C_{jklm} = 0$ and $Y^m C_{jklm} = 0$. Transvecting (3.1) with $X_j Y^s$ we infer $(X_j Y^j) C^{ki}_{tr} = 0$; in the same way we have $(X_j X^j) C^{kj}_{tr} = 0$ and $(Y_j Y^j) C^{ki}_{tr} = 0$ [LR, p. 128]. Thus X_j and Y_j are orthogonal null vectors. Combining these results with Lemma 3.2 we obtain:

THEOREM 3.3. *Let M be a 4-dimensional non-conformally flat manifold with $X^m C_{jklm} = 0$ and $Y^m C_{jklm} = 0$. Then $X_j X^j = 0$, $Y_j Y^j = 0$ and $Y_j = \lambda X_j$ for some $\lambda \in \mathbb{R}$.*

The fact that if X_i and Y_i are orthogonal null vectors then $Y_j = \lambda X_j$ is pointed out also in Hall [Ha, p. 148].

Now if we consider a non-conformally flat (WCS)$_4$ Lorentzian manifold, then $\varepsilon_m C^m_{jkl} = 0$, and thus we have:

Corollary 3.4. *Let M be a non-conformally flat $(WCS)_4$ Lorentzian manifold. Then the covector ε_j is null and unique up to scaling.*

Now recall the Bel–Debever version of the Petrov [Pe] classification of the Weyl tensor on 4-dimensional Lorentzian manifolds (see [Be], [D], [Ha] and [Sa]); it is based on null vectors k satisfying increasingly restricted conditions as follows:

$$\begin{array}{llll} & \text{(a)} & \text{type I} & k_{[b}C_{a]rs[q}k_{n]}k^r k^s = 0, \\ & \text{(b)} & \text{type II, D} & k_{[b}C_{a]rsq}k^r k^s = 0, \\ (3.2) & \text{(c)} & \text{type III} & k_{[b}C_{a]rsq}k^r = 0, \\ & \text{(d)} & \text{type N} & C_{arsq}k^r = 0, \\ & \text{(e)} & \text{type O} & C_{arsq} = 0. \end{array}$$

When k satisfies condition (b), the Weyl tensor is called *algebraically special* (see [Ha], [Sa], [St] and [SK$^+$]). Choosing $k_i = \varepsilon_i$ in the null tetrad formalism we may assert:

Corollary 3.5. *Let M be a non-conformally flat $(WCS)_4$ Lorentzian manifold. Then the Weyl tensor is of Petrov type N with respect to the null vector ε_j.*

As a first application of Theorem 3.7 we may consider equations (2.8) and (2.35), i.e. $\varepsilon^m C_{jklm} = 0$ and $\varepsilon^i R_{im} C^m_{jkl} = 0$; for $n = 4$ the last is true under the condition $\nabla_i X_j + X_i X_j = \gamma g_{ij}$ and defines a vector $Y_m = \varepsilon^i R_{im}$ such that $Y^m C_{jilm} = 0$. Thus $\varepsilon^i R_{im} = \lambda \varepsilon_m$ and ε_j is an eigenvector of the Ricci tensor. Recalling Einstein's equations we infer $K\sigma^i T_{im} = \big(\lambda - \frac{1}{2}R\big)\sigma_m$, so that the following theorem holds.

Theorem 3.6. *Let M be a non-conformally flat $(WCS)_4$ Lorentzian manifold with $\nabla_i X_j + X_i X_j = \gamma g_{ij}$. Then the covector ε_j is an eigenvector of the Ricci tensor and the stress-energy tensor.*

Secondly, consider equations (2.8) and (2.10), i.e. $\varepsilon^m C_{jklm} = 0$ and $\varepsilon^i(\nabla_i \varepsilon^m) C_{jklm} = 0$; by the same arguments we have $\varepsilon^i(\nabla_i \varepsilon_m) = \lambda \varepsilon_m$ and thus the integral curves of the covector ε_j are geodesics [FC, p. 41].

Theorem 3.7. *Let M be a non-conformally flat $(WCS)_4$ Lorentzian manifold. Then the integral curves of the fundamental covector ε_j are geodesics.*

Thirdly, consider equation (2.9) with the condition $\varepsilon^i E_i = 0$, that is, $(\nabla_i \varepsilon_m) C^m_{jkl} = 0$. Equation (3.1) is then multiplied by $\nabla_p \varepsilon_j$ to produce

$$(\nabla_p \varepsilon_r) C^{ki}_{st} + (\nabla_p \varepsilon_t) C^{ki}_{rs} + (\nabla_p \varepsilon_s) C^{ki}_{tr} = 0. \tag{3.3}$$

If $\nabla_i \varepsilon_j = \nabla_j \varepsilon_i$ then contraction of s and p immediately gives

$$(\nabla^s \varepsilon_s) C^{ki}_{tr} = 0. \tag{3.4}$$

We have thus proved:

Theorem 3.8. *Let M be a non-conformally flat $(WCS)_4$ Lorentzian manifold. If $\varepsilon^i E_i = 0$, and ε_j is a closed 1-form, then the divergence of ε_j vanishes, i.e. $\nabla^j \varepsilon_j = 0$.*

Finally from $\varepsilon^m C_{jklm} = 0$ and $\varepsilon^i R_{im} = \lambda \varepsilon_m$ a direct calculations yields (see also Hall's theorem in [MM4] and [SK$^+$]) $\varepsilon^m \varepsilon^j R_{jklm} = (\lambda - R/6)\varepsilon_k \varepsilon_l$, from which we obtain $\varepsilon_{[p} R_{k]jm} \varepsilon^m \varepsilon^j = 0$ and so the Riemann tensor is algebraically special.

Theorem 3.9. *Let M be a non-conformally flat $(WCS)_4$ Lorentzian manifold with $\nabla_i X_j + X_i X_j = \gamma g_{ij}$. Then $\varepsilon_{[p} R_{k]jlm} \varepsilon^m \varepsilon^j = 0$.*

Now, on multiplying (3.1) by ε^j and recalling that $\varepsilon^m C_{jklm} = 0$ for a 4-dimensional metric of any signature we get

$$\varepsilon_r C^{ki}_{st} + \varepsilon_t C^{ki}_{rs} + \varepsilon_s C^{ki}_{tr} = 0. \tag{3.5}$$

Theorem 3.10. *Let M be a non-conformally flat $(WCS)_4$ pseudo-Riemannian manifold. Then $\varepsilon_r C^{ki}_{st} + \varepsilon_t C^{ki}_{rs} + \varepsilon_s C^{ki}_{tr} = 0$.*

Detailed studies have been devoted to n-dimensional pseudo-Riemannian manifolds satisfying the following condition (see for example [DD], [DH], [DG] and [MM5]):

$$A_i C_{jklm} + A_j C_{kilm} + A_k C_{ijlm} = 0, \tag{3.6}$$

A_j being a covector. Here we prove (see [DD]):

Lemma 3.11. *Let M be an n-dimensional non-conformally flat pseudo-Riemannian manifold. If $A_i C_{jklm} + A_j C_{kilm} + A_k C_{ijlm} = 0$, then:*

(1) $A^i A_i = 0$.
(2) $C_{jklm} C^{jklm} = 0$.
(3) $C^k_{lmj} C^j_{pqk} = 0$.
(4) $C_{jklm} = A_j A_m T_{kl} - A_j A_l T_{mk} - A_k A_m T_{jl} + A_k A_l T_{jm}$, *where T_{kl} is a symmetric $(0,2)$ tensor.*

Proof. Contracting with A^i one obtains $(A^i A_l) C_{jklm} = 0$, from which we infer (1); on the other hand, contracting with C^{jklm} one obtains $A_i C_{jklm} C^{jklm} = 0$, from which we deduce (2). Finally, contracting with C^{kj}_{pq} and using $A_m C^m_{jkl} = 0$ we get $A_i C_{jklm} C^{kj}_{pq} = 0$, from which (3) follows immediately.

Now let θ^i be a unit vector such that $\theta^i A_i = 1$; then contracting $A_i C_{jklm} + A_j C_{kilm} + A_k C_{ijlm} = 0$ with θ^i we infer that

$$C_{jklm} = A_j(\theta^i C_{iklm}) - A_k(\theta^i C_{ijlm}). \tag{3.7}$$

Contracting again with θ^m yields

$$\theta^m C_{mlkj} = A_j T_{kl} - A_k T_{jl},$$

where $T_{kl} = \theta^i \theta^m C_{iklm}$ is a symmetric $(0,2)$ tensor. Inserting this back in (3.7) we get (4). ■

The third relation in Lemma 3.11 was recently obtained in [MM5]; it is of importance in the study of Pontryagin forms on a pseudo-Riemannian manifold satisfying (3.6).

Consider the following $4k$-forms ω_k evaluated on an orthonormal basis of tangent vectors built with the Riemann tensor (see [DR], [DS], [MM3], [MM5] and [N]):

$$\begin{aligned} \omega_1(X_1,\ldots,X_4) &= R_{ija}^{b} R_{klb}^{a} (X_1^i \wedge X_2^j)(X_3^k \wedge X_4^l), \\ \omega_2(X_1,\ldots,X_8) &= R_{ija}^{b} R_{klb}^{c} R_{mnc}^{d} R_{pqd}^{a} (X_1^i \wedge X_2^j) \cdots (X_7^p \wedge X_8^q). \end{aligned} \tag{3.8}$$

The generating forms P_k of the Pontryagin forms (see [MM3], [MM5], [N] and also [Po, pp. 317–318] result from total antisymmetrization of ω_k:

$$P_k(X_1,\ldots,X_{4k}) = \sum_{\pi} (-1)^{\operatorname{sgn}\pi} \omega_k(X_{\pi(1)},\ldots,X_{\pi(4k)})$$

where π runs through all permutations of $(1,\ldots,4k)$ and $\operatorname{sgn}\pi = \pm 1$ is the sign of π.

In [DR] the authors considered compact manifolds admitting indefinite metrics with $\nabla_i C_{jkl}^{m} = 0$; they showed that in that case all the Pontryagin forms vanish. Topological consequences of condition (3.6) on an n-dimensional pseudo-Riemannian manifold were studied recently in [MM5]; we reproduce them here for completeness. First, from Lemma 3.11, we have $C_{lmj}^{k} C_{pqk}^{j} = 0$. Now as shown by Avez [A] (see also [DR]), in the definition of the forms ω_k one may replace the Riemann curvature tensor with the conformal curvature tensor, i.e. for example

$$\omega_1(X_1,\ldots,X_4) = C_{ija}^{b} C_{klb}^{a} (X_1^i \wedge X_2^j)(X_3^k \wedge X_4^l). \tag{3.9}$$

In the case of a pseudo-Riemannian $(\mathrm{CQR})_n$ manifold which fulfils (3.6) one thus has $\omega_1 = 0$. The following theorem was recently proven in [MM5].

THEOREM 3.12 ([MM5]). *Let M be an n-dimensional pseudo-Riemannian manifold. If $A_i C_{jklm} + A_j C_{kilm} + A_k C_{ijlm} = 0$, then the first Pontryagin form vanishes, i.e. $P_1 = 0$.*

In view of Theorem 3.10, equation (3.5) and Lemma 3.11 we infer:

COROLLARY 3.13. *Let M be a non-conformally flat $(WCS)_4$ pseudo-Riemannian manifold. Then*

$$C_{jklm} = \varepsilon_j \varepsilon_m T_{kl} - \varepsilon_j \varepsilon_l T_{mk} - \varepsilon_k \varepsilon_m T_{jl} + \varepsilon_k \varepsilon_l T_{jm}$$

where T_{kl} is a symmetric tensor.

Moreover from Theorem 3.12 we can also deduce:

COROLLARY 3.14. *Let M be a non-conformally flat $(WCS)_4$ pseudo-Riemannian manifold. Then the first Pontryagin form vanishes, i.e. $P_1 = 0$.*

Let us now consider a compact orientable 4-dimensional pseudo-Riemannian manifold. The vanishing of the first Pontryagin form has a profound topological consequence. In fact according to Hirzebruch's *Signature Theorem* (see [Hi] and [Po, pp. 229–230],

$$3\tau(M) = \int_M P_1. \tag{3.10}$$

Here $\tau(M)$ is the Hirzebruch signature, a topological invariant linked to the Euler index X by the relation $\tau = X \pmod 2$ (see [N, p. 465]). We conclude that:

THEOREM 3.15. *Let M be a compact orientable $(WCS)_4$ pseudo-Riemannian manifold. Then the Hirzebruch signature is zero.*

Let us now consider $(\mathrm{WCS})_n$ Lorentzian manifolds. The Bel–Debever classification was recently extended to the n-dimensional case and is presented for example in [C], [CMPP] and [O]. Here we refer to the classification given in [O, Table 1]. Equation (2.37) gives

$$X_i X^m C_{jklm} + X_j X^m C_{kilm} + X_k X^m C_{ijlm} = 0$$

whenever $n \geq 5$; if $X_p X^p = 0$ this matches type $\mathrm{II_d}$ space-time of [O, Table 1]. We have thus proven the following results.

THEOREM 3.16. *Let M be a $(WCS)_n$ Lorentzian manifold with $n \geq 5$. If $X_p X^p = 0$, then the Weyl tensor is of type II_d with respect to X_i.*

THEOREM 3.17. *Let M be an n-dimensional Lorentzian manifold with $n \geq 5$ on which the conformal curvature 2-form is recurrent, i.e. $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$. If $X_p X^p = 0$, then the Weyl tensor is of type II_d with respect to X_i.*

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