Concerning continuous curves [1].

By

R. L. Wilder (Austin, U. S. A.).

It has been shown by Hahn [2], and independently by Mazurkiewicz [3] that in order that a closed, bounded and connected [4] point-set should be a continuous curve (une ligne de Jordan) it is necessary and sufficient that it be connected im kleinen at every point. A point-set M is connected im kleinen at a point P if for every $\varepsilon > 0$ there corresponds an $\eta > 0$, such that if K_1 and K_2 are circles with radii ε and η, respectively, and centers at P, every point X of M interior to K_2 lies with P in a connected [5] subset of M that lies wholly interior to K_1. A point-set that is connected im kleinen at every point is said to be connected im kleinen.

The present paper has three main objects, viz.,

(1) To study the analogy between ordinary two-dimensional space and a plane continuous curve,

(2) To characterize and analyze the boundaries of the domains complementary to a plane continuous curve, and

[1]) Dissertation offered to the Department of Pure Mathematics, University of Texas, U. S. A., in partial fulfillment of the requirements for the degree of Doctor of Philosophy, June, 1923.

[2]) *Mengentheoretische Charakterisierung der stetigen Kurve,* Wiener Akademie Sitzungsberichte, CXXIII Band. Abt. IIa, pp. 2433—2489.

[3]) *Sur les lignes de Jordan,* Fund. Math., Tom I, (1920), pp. 166—209.

[4]) A point-set is said to be *connected* if it is not the sum of two point-sets neither of which contains a limit point of the other. A point-set is *bounded* if it lies wholly in a finite portion of the space under consideration.

[5]) According to Hahn's definition X and P must lie together in a *closed* and connected subset of M that lies wholly interior to K_1. The word „closed" is unnecessary, however, when dealing with closed point sets.

(3) To give a new characterization of continuous curves, suitable for any number of dimensions.

I wish to thank Professor R. L. Moore for many valuable suggestions and criticisms, and to express my gratitude to him for first interesting me in a field of mathematics that is as fascinating as it is fruitful.

I. A continuous curve in the role of a space [1]).

For the present, I shall consider a space S, which consists of all the points of a plane continuous curve. I shall define a *region* in that space as follows: If P is a point of S, and k a circle with center at P, then the set of all points of S which (1) lie interior to k, and (2) lie with P in a connected subset of S that lies wholly interior to k, constitutes a region R. A point P is a *limit point* of a point-set M in space S if and only if every region that contains P contains at least one point of M distinct from P. This definition of limit point is equivalent to the ordinary definition of limit point for two-dimensional space, in that a point P which is a limit point of a point-set M in space S is also a limit point of M in the ordinary sense, and vice versa. The set of all limit points of R that do not belong to R constitute the *boundary* of R. Every boundary point of R lies on k. A *domain with respect to* S [2]), or an *S-domain* is a connected subset D of S having the property that if P is a point of D, P lies in some region that is a subset of D. The set of points, that are limit points of D but that do not belong to D, constitute the *boundary* of D.

Using the above definitions, it is interesting to note that many of the properties of ordinary two-dimensional space are also properties of space S. In particular, Theorems 1—16, inclusive, (with the exception of the latter part of Theorem 16) of R. L. Moore's paper *On the foundations of plane analysis situs* [3]) all hold true for space S with no change in the wording, except that the word „domain" should be replaced by „S-domain".

[1]) Presented to the American Mathematical Society, in part, April 15, 1922, and in more complete and generalized form, Feb. 24, 1923.

[2]) Kuratowski has used an analogous definition for what he calls »domaine connexe par rapport a C«. Cf. C. Kuratowski, *Une définition topologique de la ligne de Jordan*, Fund. Math. I, (1920), pp. 40—43.

[3]) Transactions of the Amer. Math. Soc., Vol. XVII, (1916), pp. 131—164.

Two of these theorems which are of fundamental importance are:

If A and B are distinct points of a domain M, there exists a simple continuous arc [1]) from A to B that lies wholly in M.

Every two points of a region R can be joined by an arc lying entirely in R.

The close analogy between space S and ordinary two-dimensional space can be further exhibited by a consideration of accessibility conditions [2]). For this purpose, I shall establish the following theorem:

Theorem 1. *In order that a boundary point x of an S-domain D should be accessible from any point y of D, it is sufficient either that (1) there exist a circle C_1 with center at x, such that the set of all points of the boundary, B, that lie interior to C_1 is a subset of a connected im kleinen subset of $S-D$, or that (2) x belong to no continuum of condensation* [3]) *of B.*

Proof. I shall consider these two conditions separately.

(1) There exists a circle C_1 with center at x, such that if L is the set of all points of B that lie within C_1, L is a subset of a connected im kleinen subset K of $S-D$. The circle C_1 may be taken so small that it does not enclose y or contain y.

Since x is a boundary point of D, and therefore a limit point of D, there exists in D a sequence of distinct points $y_1, y_2, y_3, \dots$ having x as a sequential limit point [4]). C_1 encloses some point of this sequence. Call one such point P_1. Let C_2 be a circle concentric with C_1, of radius $< r/2$, (where r is the radius of C_1) and such that P_1 is exterior to C_2. C_2 encloses some point of the above

[1]) A *simple continuous arc*, or an *arc*, from A to B is a closed, bounded and connected set of points that is disconnected by the omission of any one point except A and B. A and B are called the *end-points*, or *extremities*, of this arc, and all other points of the arc are *interior* points. Two points are said to be *joined* by an arc if they are the end-points of that arc.

[2]) If x is a boundary point of a domain (in the ordinary sense, or an S-domain) D, and y a point of D, x is said to be *accessible* from y provided there exists an arc from x to y which lies, except for x, wholly in D.

[3]) A *continuum* is a closed and connected point-set consisting of more than one point. A point-set C is a *continuum of condensation* of a point-set M if C is a sub-continuum of M such that every point of C is a limit point of $M-C$.

[4]) A point P is a *sequential limit point* of a sequence of points $P_1, P_2, P_3, \dots$ provided that if R is a region containing P there exists a number N such that if $n > N$, P_n lies in R.

sequence; call one such point P_2. Let C_3 be a circle concentric with C_2, of radius $< r/3$, and such that P_2 is exterior to it. C_3 encloses a point P_3 of the above sequence. Continuing this process indefinitely, there is obtained a sequence of circles, $C_1, C_2, C_3 \ldots$ with centers at x and such that for every positive integer $n > 1$, C_n is of radius $< r/n$; and a sequence of points, $P_1, P_2, P_3, \ldots$ belonging to D, such that for every positive integer n, P_n lies interior to C_n, but exterior to C_{n+1}.

Since, as pointed out above, any two points of D are the extremities of an arc lying wholly in D, y and P_1 are the extremities of an arc a_1, P_1 and P_2 of an arc t_2, P_2 and P_3 of an arc t_3, etc., such that $a_1, t_2, t_3, t_4 \ldots$ all lie wholly in D. Let x_2 be the last point of a_1 on the arc from P_1 to P_2, in the order from P_1 to P_2. (See Fig. 1). That portion of t_2 from x_2 to P_2 constitutes an arc a_2. Since $a_1 + a_2$ is a closed set of points, there will be a last point of it on the arc t_3 in the order from P_2 to P_3; call this point x_3. That portion of t_3 from x_3 to P_3 constitutes an arc a_3. The set $a_1 + a_2 + a_3$ is closed, and there therefore exists a last point of it, x_4, on the arc t_4 in the order from P_3 to P_4. That portion of t_4 from x_4 to P_4 constitutes an arc $a_4 \ldots$. Continuing this process indefinitely, there is obtained a sequence of arcs, $a_1, a_2, a_3, \ldots$ such that for every positive integer n, a_n has only one point, an end-point x_n, in common with the set $a_1 + a_2 + a_3 + \ldots + a_{n-1}$.

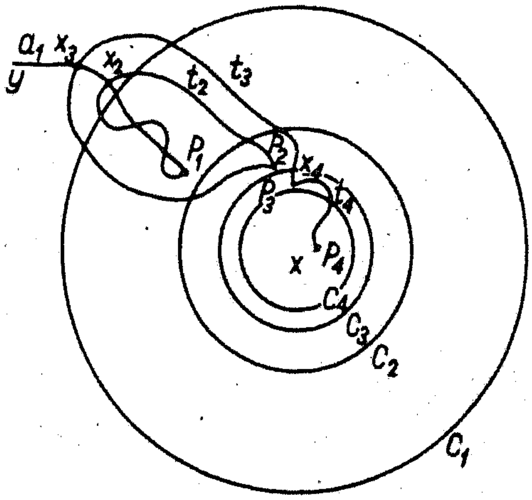

Fig. 1

There are two cases to consider: Either (1) for every value of n there exists a positive integer k, such that a_k is the last arc of the sequence $a_1, a_2, a_3, \ldots$ having points on C_n; or (2) there exists an n, such that an infinite number of arcs of the sequence $a_1, a_2, a_3, \ldots$ have points on C_n.

Case 1. For every value of n there exists a positive integer k, such that a_k is the last arc of the sequence $[a_i]$ [1]) having points

[1]) Hereafter, in the proof of this theorem, I shall denote the sequence a_1, a_2, $a_3, \ldots$ by the symbol $[a_i]$.

on C_n. Then the point-set constituted by all the arcs of the sequence $[a_i]$ together with the point x is a continuous curve and contains an arc from x to y [1]) every point of which, except x, belongs to D. In this case, then, the theorem is true.

Case 2. There exists an n, such that an infinite number of the arcs of $[a_i]$ have points on C_n. It is certain that an infinite number of arcs of $[a_i]$ have points exterior to C_{n+1}, and points interior to C_{n+2}. For every such arc, a_i, let A_i be the first point of C_{n+1} on a_i in the order from P_i to x_i. (See Fig. 2). Then, let B_i be the first point of C_{n+2} on that portion of a_i from A_i to P_i, in the order from A_i to P_i. Then from A_i to B_i there exists an arc which is a subset of a_i, and such that if I_{n+1} is the set of all points of the plane between C_{n+1} and C_{n+2}, $A_i B_i$ is a subset of I_{n+1} except for the points A_i and B_i. The set of all such arcs call $[a_i^*]$.

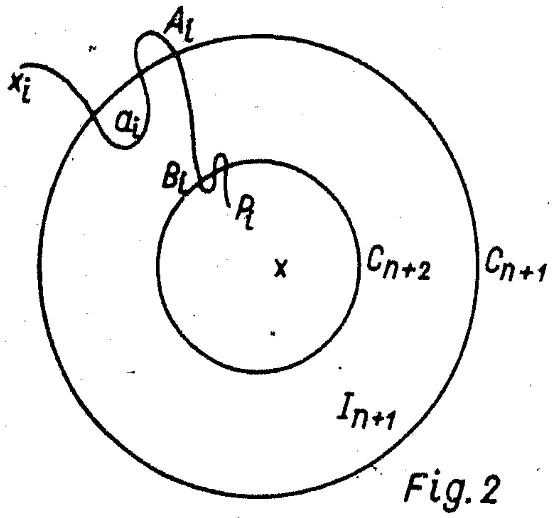

Fig. 2

No two arcs of the set $[a_i^*]$ have any points in common. For, suppose $A_k B_k$ and $A_m B_m$ are two arcs of the set $[a_i^*]$ having a point, x^*, in common, and that $k < m$. Then a_m was taken subsequently to a_k, and can therefore have at most one end-point, x_m, in common with a_k. Other than x_m, a_k and a_m can have no points in common. Hence x^* must be identical with x_m. But A_m is the first point of C_{n+1} on a_m in the order from P_m to x_m, and $A_m B_m$ is therefore a subset of that portion of a_m from P_m to A_m, and unless A_m is identical with x_m, $A_m B_m$ can have no point in common with $A_k B_k$. But if A_m is identical with x_m, a_m can have no points exterior to C_{n+1}, which is contradictory, since

[1]) Every two points of a continuous curve M can be joined by a simple continuous arc which is a subset of M. For a proof of this, see R. L. Moore, *A theorem concerning continuous curves*, Bull. Amer. Math. Soc. 2d. series, XXIII (Feb. 1917), S. 233—236. See also R. Tietze, *Über stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven*, Math. Zeitschr. V (1919), S. 284–291; and S. Mazurkiewicz, *Sur les lignes de Jordan*, *loc. cit.* In this article, Mazurkiewicz establishes numerous results and indicates that some of them were published in a journal (C. R. Soc. Sc. Varsovie) to which, I have not had access.

a_m was taken as one of the arcs of $[a_i]$ having points exterior to C_{n+1}. Hence $A_m B_m$ and $A_k B_k$ have no point in common.

The infinite set of points of the type A_i has at least one limit point on C_{n+1}. Let $\overline{A}$ be one such point. If z is any other point on C_{n+1}, then at least one of the arcs into which $\overline{A}$ and z divide C_{n+1} must contain an infinite set of points of the type A_i having $\overline{A}$ as a limit point; call this arc $\overline{A}z$. Then from the points of the type A_i can be selected (See Fig. 3) an infinite sequence $\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots$ in the order from z to $\overline{A}$ on $\overline{A}z$, and having $\overline{A}$ as a sequential limit point. The set of points $\overline{B}_1, \overline{B}_2, \overline{B}_3, \ldots$ where $\overline{B}_n$ is the other end-point of the arc of $[a_i^*]$ to which $\overline{A}_n$ belongs has a sequential limit point $\overline{B}$. There is obtained thus a sequence $\overline{A}_1\overline{B}_1$, $\overline{A}_2\overline{B}_2$, $\overline{A}_3\overline{B}_3, \ldots$ of arcs of the set $[a_i^*]$ arranged in a definite order. Call this sequence the set $[\overline{A}_i\overline{B}_i]$.

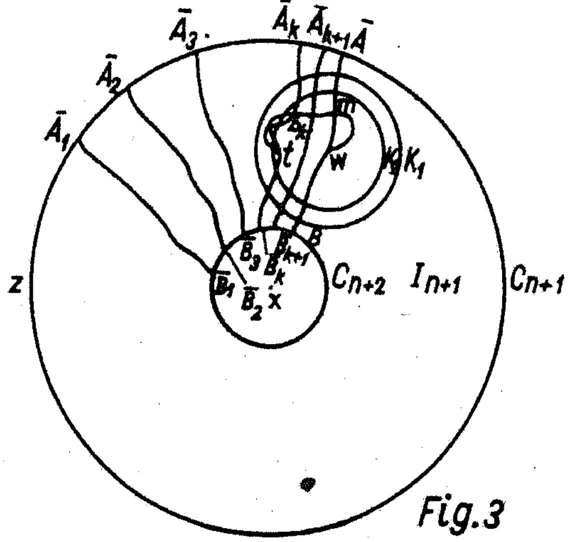

Fig. 3

The set $[\overline{A}_i\overline{B}_i]$ has a closed and connected limiting set [1]), M_1, which has at least one point $(\overline{A})$ on C_{n+1}, and at least one point $(\overline{B})$ on C_{n+2}. Let w be a point of M_1 within I_{n+1}, and let K_1 be a circle with center w and lying wholly in I_{n+1}, but not enclosing C_{n+2}. Because of the property of connectivity im kleinen of S, there exists, concentric with, and lying interior to K_1, a circle K_2, such that every point of S interior to K_2 is joined to w by an arc [2]) of S lying wholly interior to K_1. Let $\overline{A}_k\overline{B}_k$ be the first arc of the set $[\overline{A}_i\overline{B}_i]$ having points interior to K_2.

It is certain that no point on any arc $\overline{A}_n\overline{B}_n$ can be joined to w by an arc lying wholly interior to K_1 unless this arc contains points of all arcs of the set $[\overline{A}_i\overline{B}_i]$ of subscript $> n$.

[1]) Cf. S. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, (2), XVI, (1912), p. 109. Th. 1.

[2]) It is permissible to say »arc« here instead of »connected set« since all points of S lying interior to K_2 belong to that region of S determined by K_1, and since every two points of a region are the extremities of an arc lying wholly in that region.

There are two casses:

(a) If w is a point of D, K_1 can be taken so small as to contain no points of $S-D$, in which event all arcs of the set $[\overline{A_i}\,\overline{B_i}]$ of subscript $\geqq K$ are connected by an arc of D lying wholly interior to K_1.

(b) If w is a point of B:

For each n, $\overline{A_n}\,\overline{B_n}$ and $\overline{A_{n+1}}\,\overline{B_{n+1}}$, together with those arcs $\overline{A_n}\,\overline{A_{n+1}}$ and $\overline{B_n}\,\overline{B_{n+1}}$ of C_{n+1} and C_{n+2}, respectively, that contain no end-points of the set $[\overline{A_i}\,\overline{B_i}]$, form a simple closed curve $\overline{K_n}$ which cannot enclose w. Let t be a point of $\overline{A_k}\,\overline{B_k}$ interior to K_2. An arc of S from t to w lying wholly interior to K_1, must contain, for every $\overline{K_n}$ (where $n \geqq k$) an arc b_n. such that b_n lies wholly interior to $\overline{K_n}$ except for its end-points; but for each $\overline{K_n}$ there can be at most a finite number of arcs such as b_n. If w is not the first point of M, on the arc from t to w, in the order from t to w, let m be that first point. Let $[b_n]$ denote the set of all arcs of the type b_n on the arc from t to m.

If only a finite number of arcs of $[b_n]$ contain points of $S—D$, then there exists an n $(n \geqq k)$ such that if $i > n$, $\overline{A_i}\,\overline{B_i}$ and $\overline{A_{i+1}}\,\overline{B_{i+1}}$, are connected by an arc of D lying wholly interior to K_1.

I shall show next that it is impossible that an *infinite* number of the arcs $[b_n]$ contain points of $S—D$.

If any arc, b_n, contains points of $S-D$, it must, since its end-points are in D, contain points of B So, if an infinite number of arcs of $[b_n]$ contain points of $S-D$, the same arcs contain points of B; that is, every such b_n contains at least one point, d_n, of B. Call the set of such points $[d_n]$.

Clearly, m is the limit point in M_1 of the set of end-points of the set $[b_n]$. For the end-points belong to the set $[\overline{A_i}\,\overline{B_i}]$, so that their limit point must lie in M_1, and if any other point, m', of M_1, were this limit point, it would mean that a set of points in the arc from t to m has a limit point exterior to the arc (m is the first point of M_1 on the arc), an impossibility, since an arc is a closed set. But if m is the limit point of the end-points of the set $[b_n]$ it is the limit point of the end points of those arcs to which the set $[d_n]$ belongs, and hence also the limit point of the set $[d_n]$. Therefore m must belong to L since B is a closed set. But L is a subset of K; thus m is a point of K and the set $[d_n]$ belongs to K.

Let K_1^* be a circle with center at m and lying in I_{n+1}, but not enclosing C_{n+2}. Since K is connected im kleinen, there exists, concentric with, and lying interior to K_1^*, a circle K_2^*, such that all points of K lying interior to K_2^* can be joined to m by a connected subset of K lying interior to K_1^*. But K_2^* must enclose points of $[d_n]$, and a connected set containing a point of $[d_n]$ and m and lying interior to K_1^* must contain points of D; that is, points not belonging to K. Hence the contradiction.

In any case, then, there must exist a number δ_1, such that for $i > \delta_1$ and $j > \delta_1$, any arc $\overline{A_i}\,\overline{B_i}$ is connected to any other arc $\overline{A_j}\,\overline{B_j}$ by an arc of D lying entirely interior to K_1. This means, then, that those arcs of the set $[a_i]$ of which the arcs of $[\overline{A_i}\,\overline{B_i}]$ of subscript $> \delta_1$ are subsets, can be joined by arcs of D in the same manner. Call the set of these arcs S_1.

For each arc a_i of S_1, that contains points interior to C_{n+3} let D_i be the last point of C_{n+2} on a_i in the order from x_i to P_i and E_i the first point of C_{n+3} on that portion of a_i from D_i to P_i, in the order from D_i to P_i. Then each such a_i contains an arc $D_i E_i$, such that $D_i E_i$ lies wholly in I_{n+2} (where I_{n+2} is the set of all points of the plane between C_{n+2} and C_{n+3}) except for its end-points. The set of all such arcs call $[D_i E_i]$ It can be proved, by the method used above, that the arcs of the set $[D_i E_i]$ can be ordered in a sequence $\overline{D_1}\,\overline{E_1}$. $\overline{D_2}\,\overline{E_2}$, $\overline{D_3}\,\overline{E_3}, \dots$ having the property that there exists a δ_2, such that for $i > \delta_2$ and $j > \delta_2$, any arc $\overline{D_i}\,\overline{E_i}$ is connected to any other arc $\overline{D_j}\,\overline{E_j}$ by an arc of D lying wholly in I_{n+2}. Those arcs of S_1 which contain arcs of $[\overline{D_i}\,\overline{E_i}]$ of subscript $> \delta_2$ call S_2.

This process may be repeated indefinitely, since S_2 contains an infinite number of arcs of the set $[a_i]$ and thereafter any S_i will have the same property. Furthermore, for any j, (1) S_j will be a subset of S_{j-1}, (2) if a_k and a_m are any two arcs of $[a_i]$ which belong to S_j, there exists an arc $a\,b$ or D such that (*i*) one end-point of this arc, a, is a point of a_k, and the other end-point, b, is a point of a_m, (*ii*) the arc $a\,b$ lies wholly interior to some circle which lies in I_{n+j} (the annular domain bounded by C_{n+j} and C_{n+j+1}) but which does not enclose C_{n+j+1}, and (*iii*) the arcs $a\,P_k$ and $b\,P_m$, subsets of a_k and a_m, respectively, lie wholly interior to C_{n+j}

The first arc of the sequence $[a_i]$ belonging to S_1 call c_1. The first arc of $[a_i]$ after c_1 which belongs to S_2 call c_2. Then there exists, interior to K_1, an arc of D from a point x' on c_1 to a point y' on c_2, and such that the arc $y' P_t$, a subset of the arc $c_2 \equiv a_t$, lies wholly interior to C_{n+1} (See Fig. 4). The first arc of $[a_i]$ after c_2 which belongs to S_3 call c_3. Then there exists, interior to I_{n+2}, an arc of D from a point x'' of the arc $y' P_t$ to a point y'' of the arc c_3, and such that the arc $y'' P_m$ (where $c_3 \equiv a_m$) subset of c_3, lies wholly interior to C_{n+2}. Continue this process indefinitely. Then, if $a_k \equiv c_1$, the set $a_1 + a_2 + \dots + a_k + x'y' + y'P_t + x''y'' + y''P_m + \dots + x$ can be shown, as in Case 1, to contain an arc from x to y, every point of which, except x, is a point of D. This completes the proof for Case 2.

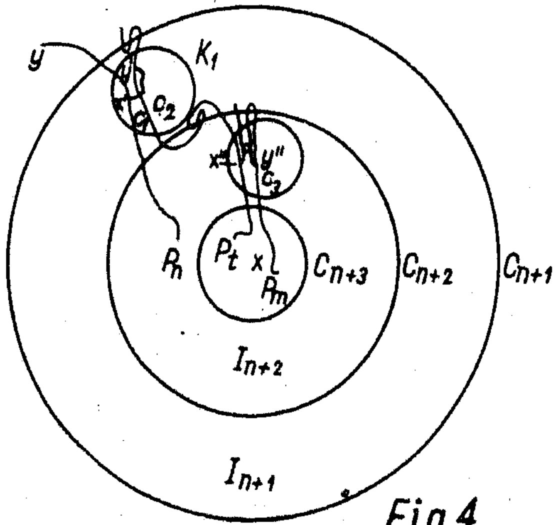

Fig. 4

(2) x belongs to no continuum of condensation of B.

Choose the sequence of circles C_1, C_2, $C_3, \dots$, the sequence of points P_1, P_2, $P_3, \dots$ and the sequence of arcs a_1, a_2, $a_3, \dots$ as in (1) starting, however, with C_1 as any circle with center at x. As in (1), there are two cases to consider:

Case 1. Suppose that for every value of n there exists a positive integer k, such that a_k is the last arc of the sequence $a_1, a_2, a_3 \dots$ having points on C_n. This situation is handled exactly as in Case 1 of (1).

Case 2. Suppose there exists an n, such that an infinite number of arcs of $[a_i]$ have points on C_n. Select the set $[a_i^*]$ as in Case 2 of (1), with the difference that to each $A_k B_k$ has been added the arc $B_k P_k$ forming the arc $A_k P_k \equiv a_k^*$. As above, it can be shown that no two arcs of $[a_i^*]$ have points in common.

The points of the type A_i have at least one limit point $\overline{A}$ on C_{n+1}. Let z be any point of C_{n+1} distinct from $\overline{A}$. Then at least one of the arcs into which z and $\overline{A}$ divide C_{n+1} contains an infinite number of points of the type A_i; call this arc $\overline{A} z$. Choose on $\overline{A} z$, in the order from z to $\overline{A}$, an infinite sequence of points of type A_i, namely. $A_1^*, A_2^*, A_3^*, \ldots$ having $\overline{A}$ as a sequential limit point. The set of points, $P_1^*, P_2^*, P_3^*, \ldots$, where for every positive integer n, P_n^* is the other end-point of the arc of $[a_i^*]$ to which A_n^* belongs, has x as a sequential limit point, since x is a sequential limit point of the set $P_1, P_2, P_3, \ldots$. The sequence of arcs $A_1^* P_1^*$, $A_2^* P_2^*$, $A_3^* P_3^*, \ldots$ has a continuum M_2 as a sequential limiting set; M_2 contains $\overline{A}$ and x

Either there exists in M_2 an infinite set of points of D having x as a sequential limit point, in which event a proof similar to that used in Case 2 (a) of (1) can be used to show the accessibility of x from y, *or* there exists no such sequence. In the latter event we can consider C_{n+1} to be so small that all points of M_2 belong to $S-D$. Then M_2 is a subset of B, since each point of it is a limit point of D and does not belong to D.

Let e_1 be any point of M_2 interior to C_{n+1} and distinct from x. Let K_1' be a circle with e_1 as center, not enclosing x or containing x, and lying entirely interior to C_{n+1}. K_1' determines a concentric circle K_2' such that every point of S interior to K_2' is joined to e_1 by an arc of S that lies wholly interior to K_1'. There exists a positive number N such that for $n > N$, K_2' cuts off at least one segment of $A_n^* P_n^*$. Let t be a point of $A_k^* P_k^*$ $(K > N)$ interior to K_2', and let m be the first point of M_2 on an arc of S from t to e_1 that lies entirely interior to K_1' (in the order from t to e_1). For every $n > N$ let $p_n q_n$ and $p_{n+1} q_{n+1}$ be arcs of $A_n^* P_n^*$ and $A_{n+1}^* P_{n+1}^*$, respectively, cut off by K_1' (i. e. $p_n q_n$ lies entirely interior to K_1' except for the points p_n and q_n, etc.) and such that one of the arcs into which p_n and p_{n+1} divide K_1' does not contain q_n or q_{n+1}. Let $p_n q_{n+1}$ be that simple closed curve formed by the arcs $p_n q_n$, $p_{n+1} q_{n+1}$, that arc $p_n p_{n+1}$ of K_1' which contains neither q_n nor q_{n+1}, and that arc $q_n q_{n+1}$ of K_1' which contains neither p_n nor p_{n+1}.

Two possible situations have to be considered. I shall say that e_1 possesses *property F* if, no matter how K_1' is selected, subject to the conditions noted above, or how tm is selected, the arc tm contains, for an infinite number of distinct values of n, a point of $S-D$ that lies interior to $p_n q_{n+1}$. Either e_1 possesses *property F* or it does not; in the latter event, e_1 will be said to possess *property G*.

If e_1 possesses *property G*, let C_{n+2}^* be a circle concentric with C_{n+1}, not enclosing e_1 and lying interior to C_{n+2}. There exists a subcontinuum M_3 of M_2, which contains x and at least one point on C^*_{n+2}, but no points exterior to C^*_{n+2}. Either a point e_2 of M_3 interior to C^*_{n+2} and distinct from x can be found possessing *property* G or such a point cannot be found. If such a point can be found, let C^*_{n+3} be a circle concentric with C^*_{n+2}, not enclosing e_2, and lying interior to C_{n+3}. There exists a subcontinuum M_4 of M_3 which contains x and at least one point on C^*_{n+3}, but no points exterior to C^*_{n+3}. A point e_3 of M_4 interior to C^*_{n+3} and distinct from x can be found possessing *property* G, or no such point can be found. If such a point can be found, continue as before. In the event that this process can be kept up indefinitely, that is, if for every new C^*_{n+i} lying within C_{n+i} a point e_i interior to C^*_{n+i} can be found possessing *property* G, then there exists a circle K^i_1 with center at e_i and lying within C^*_{n+i} such that an infinite number of arcs of $[a^*_i]$ that have points interior to K^i_1 can be connected by an arc of D lying wholly interior to K^i_1. The method used in Case 2 (b) of (1) can now be used to show the accessibility of x.

In the event that this process can not be kept up indefinitely; that is, if finally a circle C^*_{n+i} is found such that every point of M_i interior to C^*_{n+i} possesses *property* F, we may proceed as follows: Consider C_{n+1} taken so that every point of M_2 other than x interior to C_{n+1} possesses *property* F. But then every point of M_2 other than x is a limit point of boundary points of B that do not belong to M_2, and since M_2 is a continuum, x itself is a limit point of such points. That is, x belongs to a continuum of condensation, M_2, of B. contrary to hypothesis.

Hence, in every case, x is accessible from y if it belongs to no continuum of condensation of the boundary.

This completes the proof of Theorem 1.

It will be noticed that in the above proof no properties of space S are made use of, that are not also properties of ordinary two-dimensional space. The arguments used. apply, therefore, in the latter space. Hence the following theorem:

Theorem 2. *In order that a boundary point x of a connected domain D in ordinary two-dimensional space should be accessible from all points of D, it is sufficient either that (1) there exist a circle C with centre at x such that the set of all points of the boundary, B, interior to C is a subset of a connected im kleinen subset of $S-D$ (S being* *the set of all points in the space considered), or that* (2) x *belong to no continuum of condensation of* B [1]).

It is clear that the conditions of Theorem 2 are satisfied if any one of the following simpler conditions are satisfied, viz., that (a) B be connected im kleinen, (b) B a continuous curve, since in this case (a) is fulfilled [2]), (c) x belong to a connected im kleinen subset, K, of B, such that x is not a limit point of $B-K$.

The accompanying figures are given to show that the two conditions of the above theorem are independent, and that they are only sufficient, not necessary.

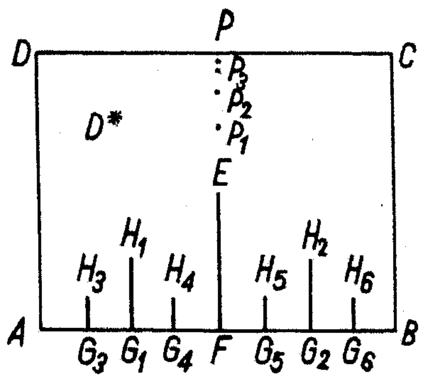

Fig.5

Figure 5 is an illustration showing the independence of the conditions $ABCD$ is a square, E the center point of its interior, and P and F the mid-points of DC and AB, respectively, P_1, P_2, $P_3, \ldots$ is a sequence of points such that if the distance $EP=1$. P_n lies on the straight line EP and at a distance $1/n$ from P. G_1 and G_2 are the mid-points of AF and FB, respectively, and $G_1 H_1$ and $G_2 H_2$ are perpendiculars to AB of length $1/2$; G_3, G_4, G_5, and G_6 are mid-points of $A_1 G_1$, $G_1 F$, FG_2, $G_2 B$, respectively, and $H_3 G_3$, $H_4 G_4$, $H_5 G_5$, $H_6 G_6$, are perpendiculars to AB of length $1/4$; and so on. Consider the bounded domain D^* whose boundary is the set of points consisting of the square $ABCD$, the sequence of points P_1, P_2. $P_3, \ldots$ and the straight line intervals EF, $G_1 H_1$, $G_2 H_2$, $G_3 H_3, \ldots$ Every boundary point of R is accessible from any point of R; every point of the side AB of $ABCD$ is on

[1]) I will remark here that, if P is a point of D, and arcs are taken from P to two points A and B' of B in such a way that D is divided into two domains D_1 and D_2. x being on the boundary of D_1, as well as on B, that x is accessible from D_1 provided that the conditions of the theorem are satisfied for D, at x. That is, the conditions of the theorem are sufficient for accessibility from all sides of the point x. I shall not, however, make use of the idea of accessibility from all sides in this paper.

[2]) Schoenflies showed that all boundary points of an ordinary two-dimensional connected domain are accessible from the domain if its boundary is a continuous curve. Cf. A. Schoenflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Zweiter Teil, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 2 (1908), p. 215

a continuum of condensation of the boundary of R, yet condition (1) of Theorem 2 is satisfied for all points of AB; condition (1) is not satisfied at P, yet P is accessible from any point of R by condition (2).

It can be shown that the above conditions are not necessary by taking as part of the boundary set of isolated points of the domain having F as sequential limit point, somewhat as P is the sequential limit point of the sequence $P_1, P_2, P_3, \dots$

A point of the boundary may be a limit point of continua of condensation of the boundary and yet be accessible from any point of the domain provided the point does not itself belong to a continuum of condensation of the boundary, as in Figure 6. The points $P_1, P_2, P_3, \dots$ are situated on the straight line interval OP_1 in such a way that for every positive integer n. P_n is at a distance $1/n$ from O, and $A_n B_n$ is a straight line interval perpendicular to OP_1, of length $1/n$ and mid-point P_n. The lines $A_n B_n$ are continua of condensation of curves of the type $\sin 1/x$, and O is then a limit point of continua of condensation, but is accessible from any point of D.

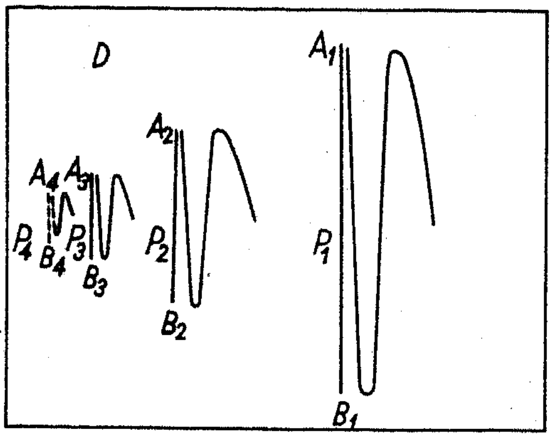

Fig. 6

An extension of space S can be made by considering S to consist of any connected im kleinen, closed and bounded set of points. For this purpose, the following theorem is introduced:

Theorem 3. *In order that a closed and bounded point-set should be connected im kleinen, it is necessary and sufficient that it should be the sum of a finite (or vacuous) set of mutually exclusive continuous curves, together with a finite (or vacuous) set of isolated points.*

Proof. 1. The condition is necessary. Let K be a closed, bounded and connected im kleinen set of points. By virtue of the property of connectivity im kleinen, every point P of K is the center of two circles C_1 and C_2 such that every point of K interior to C_2 lies with P in some continuum of K that lies entirely interior to C_1. If L denotes the set of all those points of K interior to C_1 which lie with P in some continuum of K that lies interior to C_1, together with the limit points of such points, then L is either the isolated point P, or a continuous curve which contains every point of K interior to C_2 [1]). Since K is closed and bounded, there exists, by virtue of the Heine-Borel Theorem, a finite number of circles of the type C_2, namely $C_2^1, C_2^2, C_2^3, \dots C_2^n$, such that if x is any point of K, there exists a positive integer i such that x lies interior to C_2^i. To each of these circles corresponds a set, L, which is either a continuous curve or a single point P, and which contains all points of K interior to C_2. Hence, K consists of a finite (or vacuous) set of continuous curves together with a finite (or vacuous) set of isolated points. Since a finite number of continuous curves which have points in common form one continuous curve, the continuous curves in the theorem can be considered to be mutually exclusive [2]).

2. The condition is sufficient. Let K be a closed, bounded set of points which is the sum of a finite set of mutually exclusive continuous curves $L_1, L_2, L_3, \dots, L_i$ and a finite set of isolated points $P_1, P_2, \dots P_j$. If x is a point of the continuous curve L_n $(n = 1, 2, \dots i)$ there exists a circle C with center at x which contains no points of any other continuous curve of K, nor any point of the sequence $P_1, P_2, \dots P_j$. Since L_n is connected im kleinen at x, it follows that K is connected im kleinen at x. If $x = P_n$, $(n = 1, 2, \dots j)$, there exists a circle C with center at x containing or enclosing no points of the set $L_1 + L_2 + \dots + L_n$. and no point of the sequence $P_1, P_2, \dots P_j$. The conclusion is obvious.

II. An analysis of the point-set which constitutes the boundary of a complementary domain of a plane continuous curve [3]).

In the present section I shall analyze the boundary of a complementary domain of a plane continuous curve in terms of the elementary notions of point, arc and simple closed curve, and prove certain fundamental properties about such sets:

[1]) Hahn has shown that L is connected im kleinen. L is obviously what Hahn calls $M^*(P, r)$. Cf. pg. 2448, *loc. cit.*

[2]) Point-sets are said to be mutually exclusive when they have no points in common.

[3]) Presented to the American Math. Soc., Dec. 29, 1922.

If S is a continuous curve in ordinary two-dimensional space S', and P a point of $S'-S$, that maximal [1]) connected subset of $S'-S$ determined by P is a *complementary domain* D of the continuous curve S. One of the complementary domains of S is unbounded, and the unbounded complementary domain always exists; S may or may not have bounded complementary domains. The *boundary*, β, of D, consists of all limit points of D that do not belong to D. Evidently β is a subset of S.

Since S is bounded, there exists a circle C, which encloses all points of S but contains no points of S. If D is a bounded complementary domain of S, then the *outer boundary* of D is the boundary of the point-set composed of all points $[x]$ such that x can be joined to some point of C by an arc which contains no point of $D+\beta$. R. L. Moore has shown [2]) that B, the outer boundary of D, is a simple closed curve. By a theorem due to Miss Torhorst [3]) β is itself a continuous curve.

Theorem 4. *If S_1 is the set of all simple closed curves (excluding the outer boundary in case the domain is bounded) contained in the boundary, β, of a complementary domain of a continuous curve S, then (1) S_1 is countable, (2) if C_k and C_j are two distinct simple closed curves of the set S_1, C_k and C_j have at most one point in common, and their interiors have no point in common; (3) in case D is bounded and C is any simple closed curve of S_1, C lies interior to B, the outer boundary of D, or has at most one point P' in common with B, such that $C-P'$ is interior to B.*

Proof. In case D is bounded, no points of S_1 can lie exterior to B, since all points of β must be accessible from D by Theorem 2.

If C is any closed curve of S_1, C cannot have more than one point in common with B. For, suppose C has two points, A and E, in common with B. Then C will contain an arc, $A'xE'$, which lies, except for A' and E', wholly interior to B. $A'xE'$ will divide the

[1]) If M is a point-set and P a point of M, the *maximal* connected subset of M determined by P is the set of all points of M that lie, with P, in a connected subset of M.

[2]) R. L. Moore, *Concerning continuous curves in the plane*, Math. Zeitschr. Band 15, (1922) pp. 254–260.

[3]) *Ueber den Rand der einfach zusammen hängenden ebenen Gebiete.* Math. Zeitschr. 9 (1921), S. 64 (73).

interior of B into two domains, R_1 and R_2, which are mutually exclusive [1]), and such that if I is the interior of B,

$$I = R_1 + R_2 + A'xE' - A' - F'.$$

D must lie wholly in either R_1 or R_2. Clearly points on one of the arcs into which A' and E' divide B are not accessible from D, a contradiction of Theorem 2. Hence C can have at most one point, P, in common with B, and the set $C-P$ must lie interior to B.

In any case, D being bounded or unbounded, if C_k and C_j are two distinct simple closed curves of the set S_1, the interiors of C and C_j can have no point in common; for if their interiors have a point in common, points of C_k lie interior to C_j, or vice versa, and a contradiction of Theorem 2 results.

Furthermore, C_k and C_j can not have more than one point in common. For suppose they have two points, A and E, in common. C_j must contain at least one point, x which is exterior to C_k, since C_k and C_j are not identical and their interiors have no point in common, and there will exist two points, A' and E', common to C_k and C_j, such that the arc $A'xE'$ is exterior to C_k. From Theorem 27, of R. L. Moore's *Foundations of Plane Analysis Situs*, it follows that one of the arcs into which A' and E' divide C_k is interior (except for end-points) to the simple closed curve formed by the other arc of C_k and $A'xE'$, a contradiction of Theorem 2 again.

S_1, then, is a totality of simple closed curves whose interiors have no point in common, and such that any two have not more than one point in common. The closed curves of this set must therefore form the boundaries of a set of mutually exclusive domains. As every set of mutually exclusive domains is countable, S_1 must be a countable set $C_1, C_2, C_3 \dots$

Theorem 5. *The boundary, β, of a complementary domain, D, of a continuous curve, S. cannot contain an uncountable infinity of simple continuous arcs no two of which have a point in common.*

Proof. Suppose that β does contain an uncountable set, T, of simple continuous arcs, such that no two arcs of T have a point in common. Then there exists some positive number ε, such that there

[1]) Cf. R. L. Moore, *Foundations of plane Analysis Situs, loc. cit.* p. 141.

are uncountably many arcs of T of diameter [1]) greater than ε. Call the totality of these T'.

In each arc of T' there are two points x and y such that

$$(1) \qquad \delta(x, y) > \varepsilon.$$

Assign x to a set $[A]$ and y to a set $[B]$. The set $[A]$ will have at least one limit point, A, since it is bounded. Then from $[A]$ can be selected a sequence of points $x_1, x_2, x_3, \ldots$ which has A as a sequential limit point. Let $y_1, y_2, y_3, \ldots$ be points of $[B]$ such that for every positive integer n, x_n and y_n are points of the same arc of T'.

The set $y_1, y_2, y_3, \ldots$ must have at least one limit point, B^*, and contains a subsequence $B_1, B_2, B_3, \ldots$ which has B^* as a sequential limit point. Let $A_1, A_2, A_3, \ldots$ be a subsequence of the sequence $x_1, x_2, x_3, \ldots$ such that for every positive integer n, A_n and B_n belong to the same arc of the set T'. Then A and B^* are sequential limit points of the sequences $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$, respectively, since if a sequence of points has a sequential limit point, every subsequence of it has the same sequential limit point. A and B^* are distinct points, since, as a result of (1),

$$\delta(A, B^*) \geqq \varepsilon.$$

Let

$$\delta(A, B^*) = 8\eta.$$

Hahn has shown [2]) that a connected im kleinen continuum is uniformly connected im kleinen. That is, for every positive number ε there exists a positive number δ_ε, such that if P' and P'' are two points of the continuum such that

$$\delta(P', P'') < \delta_\varepsilon$$

then P' and P'' are the extremities of an arc, L, such that if x' is any point of L.

$$\delta(x', P'') < \varepsilon,$$

and

$$\delta(x', P'') < \varepsilon.$$

[1]) The *diameter* of a point-set M is the upper limit of $\delta(x, y)$, where x and y are any two points of M and δ means distance; i. e., $\delta(x, y)$ stands for the *distance from x to y.*

[2]) *Loc. cit.*

Since β is a continuous curve, and therefore a connected im kleinen continuum, it is uniformly connected im kleinen.

Therefore, corresponding to the number η there exists a number δ_η satisfying the conditions for uniform connectivity im kleinen.

There exists a positive number N such that if $n > N$, A_n is interior to a circle of diameter δ_η with A as center, and B_n is interior to a circle of diameter δ_η with B^* as center. Hence, for any such value of n,

$$\delta(A_n, A_{n+1}) < \delta_\eta$$

and

$$\delta(B_n, B_{n+1}) < \delta_\eta.$$

Then A_n and A_{n+1} are the extremities of an arc of β such that if P is a point of this arc,

$$\delta(P, A_n) < \eta.$$

and

$$\delta(P, A_{n+1}) < \eta,$$

and, since $\delta_\eta \leqq \eta$,

$$\delta(P, A) < 2\eta$$

Similarly, for $n > N$, B_n and B_{n+1} are the extremities of an arc of β satisfying similar conditions. These two arcs can have no point in common, and the arcs $A_n B_n$ and $A_{n+1} B_{n+1}$ have no point in common by hypothesis. From the point set composed of these four arcs can be selected a simple closed curve which consists of intervals of the four arcs. This simple closed curve bounds a domain which is bounded. The totality of such domains form a sequence $D_1, D_2, D_3, \ldots$, each of whose boundaries belongs to S_1 (except for one which may be the outer boundary, B, and which, for convenience, may be considered as omitted from the sequence). By Theorem 4 (2), these domains are mutually exclusive; furthermore, they constitute an infinite set of domains complementary to β, each of which is of diameter greater than 4η. But this is a contradiction of a theorem due to Schoenflies [1]) to the effect that if ε is any given positive number, there do not exist infinitely many domains complementary to a continuous curve, each of which is of diameter greater than ε

[1]) *Loc. cit.*, p. 221, IX.

Hence β cannot contain uncountably many arcs no two of which have a point in common.

Theorem 6. *The boundary, β, of a complementary domain of a continuous curve, cannot contain an infinite set of simple continuous arcs of diameter greater than any positive number ε, such that no two of these arcs have an interior point of both in common.*

Proof. If such a set exists, select from it a sequence $t_1, t_2, t_3, \ldots$ For every positive integer n, t_n, being of diameter greater than ε, must contain an arc t'_n which does not contain the end points of t_n and is itself of diameter greater than ε. Then the set $t'_1, t'_2, t'_3, \ldots$ is an infinite sequence of arcs no two of which have a point in common, and each of which is of diameter greater than ε An argument similar to that used in the proof of Theorem 5 may be used to complete the proof of this Theorem.

Definition: If M is a continuous curve, I shall define an end-point of M to be any point P of M, such that if a is an arc of M whose end-points are P and any other point P' of M, the set $M - (a - P)$ contains no connected subset consisting of more than one point which contains P. If a point P' can be found such that this condition is not satisfied, then P is not an end-point of M.

Definition: If M is a continuous curve, and P a point of M, then if $M - P$ is not connected, P is called a cut-point [1]) of M. If $M - P$ is connected, I shall call P a *non-cut-point* of M.

Theorem 7. *In order that a point of a continuous curve that contains no simple closed curves should be an end-point, it is necessary and sufficient that it be a non-cut-point.*

Proof. (a) The condition is necessary. For, let P be an end-point of a continuous curve M which contains no simple closed curve, and let P' be any other point of M. P and P' are the extremities of an arc a of M. Since P is an end point, $M - (a - P)$ contains no connected subset consisting of more than one point that contains P. Suppose that P is a cut-point of M Then

$$M \cdot P = M_1 + M_2$$

[1]) This definition was given by R. L. Moore in *Concerning the cut-points of continuous curves and of other closed and connected point-sets*. Proc. Nat. Acad. Sci vol. IX. (1923), pp. 101—106

where M_1 and M_2 are two non-vacuous mutually separated [1]) sets. Now P, being an end-point of a, does not disconnect a. And $a-P$, being connected, must lie wholly in M_1 or M_2. Suppose it lies in M_1. Since M_1 has no limit points in M_2, M_2 contains a domain d with respect to M, whose boundary with respect to M is the point P. If x is a point of M_2, x and P are the extremities of an arc a' which lies wholly in d and hence in M_2, except for the point P, by Theorem 1. But since $a-P$ lies wholly in M_1,

$$M-(a-P) \supseteq M_2+P \supseteq a' \text{ }^{2)}$$

That is, $M-(a-P)$ contains a connected subset consisting of more than one point which contains P. Hence under the supposition that P is a cut-point of M a contradiction results. Therefore P must be a non-cut-point of M.

(b) The condition is sufficient. Let P be a non-cut-point of a continuous curve M that contains no simple closed curve. Suppose P is not an end-point of M. Then there exists a point P' of M, such that if a be an arc of M whose extremities are P and P', $M-(a-P)$ contains a connected subset M_2, consisting of more than one point, which contains P.

Since P is a non-cut-point, $M-P$ is connected. Divide $M-P$ into two sets, $a-P$ and M_2'. Then

$$M_2' \supseteq M_2 - P.$$

Let d be that maximal connected domain with respect to M such that

$$M_2' \supseteq d$$

and

$$d \supseteq M_2 - P.$$

Then d must have some limit point x distinct from P in a.

Let y be a point of d. There exists an arc a' whose extremities are x and y and which lies wholly in d except for x (see Theorem 1); also an arc a'' whose extremities are y and P, and which lies wholly in d except for P. Then a, a' and a'' contain a simple closed curve. But M contains no simple closed curve by hypothesis.

[1]) Two point-sets are said to be mutually separated when they are mutually exclusive and neither contains a limit point of the other.

[2]) The symbol $\supseteq$ should be read „contains or is identical with".

Hence the supposition that P is not an end-point of M leads to a contradiction, and P must therefore be an end-point of M.

Theorem 8. *If M is a continuous curve that contains no simple closed curve, then M cannot contain, for any given positive number ε, an infinite number of arcs of diameter greater than ε, and such that no two of these arcs have an interior point of both in common.*

Proof: The proof of this theorem is nearly identical with the proofs of Theorems 5 and 6, except that a contradiction is obtained as soon as it is demonstrated that M contains a single simple closed curve. Or, it may be considered a special case of Theorem 6, since such a continuous curve is the boundary of only one domain, namely, its unbounded complementary domain.

Theorem 9. *If M is a continuous curve, and N is a closed proper* [1]) *subset of M, then $M-N$ is a countable set of domains with respect to M whose boundary points with respect to M are contained in N.*

Proof: Let P be a point of $M-N$. Then that maximal connected subset, d, of $M-N$ determined by P is a domain with respect to M. To show this, let

$$R = M - (N + d). \tag{1}$$

Then R and d are mutually separated. For, R can contain no limit point, x, of d, since $d+x$ would be connected and x therefore a point belonging to d, contrary to (1). On the other hand, if d contains a limit point, y, of R, there exist two circles, k_1 and k_2, with centers at y, neither of which encloses a point of N, and such that every point of M interior to k_2 lies with x in a connected subset of M that lies wholly interior to k_1, and furthermore such that at least one point, x, of R lies within k_2. Then x and y lie in a connected subset L of M that lies wholly interior to k_1. But no points of N lie interior to k_1, and therefore

$$M - N \supset L.$$

As d is a maximal connected subset of $M-N$, y must belong to d. This is impossible, since R and d are mutually exclusive according to (1). Hence no points of R can lie interior to k_2 and y

[1]) If N is a subset of a point-set M, then N is a *proper* subset of M if $M-N$ is not vacuous.

cannot be a limit point of R. As R contains no limit points of d, and d contains no limit points of R, and as, moreover, the two sets are mutually exclusive by (1), R and d are mutually separated.

Hence, if P' is any point of d, there exists a circle C_1 which encloses P', but no points of R or N. It follows at once that d is a domain with respect to M. The boundary of d with respect to M must be a subset of N, since d can have no limit points in R.

Since every point P of $M-N$ determines a maximal connected subset of $M-N$, i. e., a domain with respect to M, $M-N$ is a totality of domains, $[d]$, with respect to M, whose boundaries with respect to M belong to N.

To show that these domains form a countable set, it is only necessary to make use of the fact that every uncountable point-set contains at least one of its limit points. For, if Q be a point set which consists of points of $[d]$ such that one and only one point of each domain of $[d]$ belongs to Q, Q contains at least one limit point of itself. Call such a point A. Let d be that domain of $[d]$ of which A is a point. If R be defined as in (1), then

(2) $$R \supset [d]-d.$$

But

(3) $$[d]-d \supset Q-A.$$

From (2) and (3) it follows that A is a limit point of R, which is impossible, as shown above. Hence the set of domains $[d]$ is countable.

Theorem 10. *If M is a continuous curve that contains no simple closed curve, and N a closed proper subset of M, then for any positive number ε, $M-N$ contains at most a finite number of maximal domains with respect to M of diameter greater than ε.*

This theorem is a direct consequence of Theorems 8 and 9, and the fact that every two points of a domain with respect to a continuous curve are the extremities of an arc of that domain.

Theorem 11. *Every closed and connected subset of the boundary of a complementary domain of a continuous curve is itself a continuous curve.*

Proof: Let β be the boundary of a complementary domain of a continuous curve S, and N a closed and connected subset of β. To show that N is a continuous curve, it is necessary to prove it connected im kleinen.

Let P be a point of N at which it is not connected im kleinen. Then there will exist [1]) two concentric circles k_1 and k_2, and a countable infinity of closed and connected point-sets, $M, M_1, M_2, M_3, \ldots$ such that (1) each of these point-sets is a subset of N and contains at least one point on k_1 and at least one point on k_2, but contains no point exterior to k_1 or interior to k_2, (2) no two of these point-sets have a point in common, and indeed, no one of them is a proper subset of any other connected subset of N which contains no point without k_1 or within k_2, (3) the set M is the sequential limiting set [2]) of the sequence of sets $M_1, M_2, M_3, \ldots$

Let ϱ be the lower limit of the distance xy, where x is any point of M_2 and y is a point of $M_1 + M_3$. Since β is a continuous curve and therefore uniformly connected im kleinen, there exists a number η such that if a and b are any two points of β, such that

$$\delta(a, b) < \eta$$

then a and b are the extremities of an arc ab of β, such that every point c of the arc ab satisfies the relations

$$\delta(a, c) < \varrho$$
$$\delta(b, c) < \varrho$$

Since M_2 is a continuum, there exists in M_2 a chain of points $x_1, x_2, \ldots x_k$, such that

$$\delta(x_i, x_{i+1}) < \eta, \qquad (i = 1, 2, \ldots k-1)$$

x_1 is a point of k_1, and x_k a point of k_2. Then there exists an arc $x_i x_{i+1}$ of β, such that if x' is any point of this arc,

$$\delta(x_i, x') < \varrho$$
$$\delta(x_{i+1}, x') < \varrho.$$

[1]) Cf. R. L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries*, Proc. Nat. Acad. Sci., IV (1918), pp. 364–370.

[2]) A point-set t is said to be the sequential limiting set of a sequence of point-sets $t_1, t_2, t_3, \ldots$ provided that (a) each point of t is the sequential limit point of an infinite sequence of points, $P_1, P_2, P_3, \ldots$ such that, for every n, P_n belongs to t_n, and (b) if $P_1, P_2, P_3, \ldots$ is a sequence of points such that, for every n, P_n belongs to t_n, then t contains the sequential limit point of every subsequence of $P_1, P_2, P_3, \ldots$ which has a sequential limit point.

The set $x = \Sigma x_i x_{i+1}$ is a continuous curve, since any connected set consisting of a finite number of arcs is a continuous curve, and therefore contains an arc whose end-points are x_1 and x_k. Let x_1' be the last point of this arc on k_1 in the order from x_1 to x_k, and x_k', the first point of this arc on k_2 in the order from x_1' to x_k. Then $x_1' x_k' = t_2$ is an arc of β which lies, except for its end-points, wholly interior to k_1 and wholly exterior to k_2.

Now let ϱ' be the lower limit of the distance xy, where x is any point of M_3, and y any point of $M_4 + t_2$. There exists a number η' such that if a and b are any two points of β such that

$$\delta(a, b) < \eta'$$

then a and b are the extremities of an arc ab of β, such that every point c of the arc ab satisfies the relations

$$\delta(a, c) < \varrho'$$
$$\delta(b, c) < \varrho',$$

As M_3 is a continuum, there exists in M_3 a chain of points $\overline{x}_1$, $\overline{x}_2, \dots \overline{x}_n$, such that

$$\delta(\overline{x}_i, \overline{x}_{i+1}) < \eta', \quad (i = 1, 2, \dots n-1)$$

$\overline{x}_1$ is a point of k_1, and $\overline{x}_n$ a point of k_2. By a discussion similar to that used above in showing the existence of the arc t_2, it can be proved that there exists an arc t_3, having properties similar to those of t_2.

Continuing in this manner indefinitely, it can be shown that there exist an infinite sequence of arcs, $t_2, t_3, t_4, \dots$ such that for every positive integer $n > 1$,

$$\beta \supset t_n,$$

the diameter of t_n is $> r/2$, where r is the numerical difference of the radii of k_1 and k_2 and such that t_n has no point in common with any other arc of the sequence $t_2, t_3, t_4, \dots$ Clearly this is a contradiction of Theorem 6. Hence N must be connected im kleinen at P, and as P is any point of N, N must be connected im kleinen.

It might be noted in passing that Theorem 11 is a generalization of a result obtained by Mazurkiewicz [1]) to the effect that every

[1]) S. Mazurkiewicz, *Un théorème sur les lignes de Jordan,* Fund. Math., II, (1921) pp. 123–125.

closed and connected subset of a continuous curve that contains no simple closed curve is itself a continuous curve.

Theorem 12. *If M is a continuous curve that contains no simple closed curve, N a closed and connected proper subset of M, and d a maximal connected domain which is a subset of $M-N$, then d has only one boundary point with respect to M, and that point belongs to N.*

Proof: The boundary points of d all lie in N as a consequence of Theorem 9. Suppose there exist two boundary points of d with respect to M. By Theorem 11 N is itself a continuous curve and therefore contains an arc from P_1 to P_2. But by application of Theorem 1 there exists another arc from P_1 to P_2 which lies, except for P_1 and P_2, wholly in d. As d and N have no points in common, these two arcs have in common only the points P_1 and P_2, and their sum is therefore a simple closed curve. But this is a contradiction of the hypothesis that M contains no simple closed curve. Hence d has only one boundary point with respect to M.

For convenience, the major results of Theorem 9, 10 and 12 are embodied in one Theorem as follows:

Theorem 13. *If M is a continuous curve that contains no closed curve, and N is a closed and connected proper subset of N, then $M-N$ consists of a countable set of domains with respect to M, viz., d_1. d_2, d_3,... such that (1) for every positive integer n, d_n has one and only one boundary point with respect to M, and this point is a point of N, (2) no two of these domains have a point in common, and (3) if ε is any positive number, at most a finite number of these domains are of diameter greater than ε*

Theorem 14. *If x and y are two distinct points of a continuous curve M that contains no simple closed curve, then no point of the arc xy of M, excepting the points x and y, is an end-point of M.*

Proof: As shown by Mazurkiewicz [1]), there exists in M only one arc from x to y. Let P be a point of xy, distinct from x and y. Then

$$xy-P=M_1+M_2,$$

where M_1 and M_2 are mutually separated sets. As a consequence of Theorem 13 $M-xy$ consists of a set of domains with respect to M each of which has one and only one boundary point with

[1]) See *Un théorème sur les lignes de Jordan. Loc. cit.*

respect to M, that point being a point of xy. Let $[d']$ be the collection of all those domains of $M-N$ whose boundary points with respect to M lie in M_1, and $[d'']$ the collection of all those domains of $M-N$ whose boundary points lie in M_2.

If P is the boundary with respect to M of a domain d_i, then d_i and $M-(d_i+P)$ are two mutually separated sets (see Theorem 13), whence by Theorem 7 P is not an end-point.

If P is not the boundary with respect to M of any domain, then the sets $[d']+M_1$ and $[d'']+M_2$ are mutually separated, and P is not an end-point of M.

In any case then, P cannot be an end-point.

Theorem 15. *A continuous curve M that contains no simple closed curve consists of (1) a sequence of arcs $c_1, c_2, c_3, \ldots$ no two of which have in common an interior point of both, and such that (i) if n is any positive integer, $c_1+c_2+c_3+\ldots+c_n$ is a continuous curve M_n and a proper subset of M, (ii) for $\varepsilon>0$, there exists a number ϱ such that if $n>\varrho$,*

$$\delta(c_n)<\varepsilon,$$

and the diameter of any one of the countable set of maximal domains with respect to M lying in $M-M_n$ is less than ε; and (2) a totally disconnected set of end- (or non cut-) points, P_ω, each of which is a limit point of the sequence $M_1, M_2, M_3, \ldots$ and which contains all the end- (or non-cut-) points of M.

Proof: The continuous curve M is of diameter greater than some positive number, say 1. Then there exist two points x and y belonging to M, that are not end-points of M, and whose distance apart is greater than 1 [1]). Let c_1 be that arc of M which has x and y as end-points. As a result of Theorem 13, $M-c_1$ is a set of domains with respect to M, whose boundary points with respect to M lie on c_1, and such that only a finite number of these domains, $d_2, d_3, d_4, \ldots d_k$ are of diameter greater than 1. Let P_i be the boundary point of d_i with respect to M $(i=2, 3 \ldots k)$. Since the

[1]) That two such points can be found is a direct consequence of a theorem proved by R. L. Moore, viz., in order that a closed, connected and bounded point-set M should be a continuous curve which contains no simple closed curve, it is necessary and sufficient that every closed and connected subset of M should contain uncountably many cut-points; (See R. L. Moore, *Concerning the cut-points of continuous curves and of other closed and connected point-sets, loc. cit.* Th. E). and of Theorem 7.

diameter of d_i is greater than 1, there exist in d_i two points x_i and y_i such that

$$\delta(x_i, y_i) > 1.$$

It is clear, then, that either

$$\delta(P_i, x_i) > \tfrac{1}{2}$$

or

$$\delta(P_i, y_i) > \tfrac{1}{2}.$$

Suppose that

$$\delta(P_i, x_i) > \tfrac{1}{2}.$$

Then the set composed of d_i and P_i contains an arc from x_i to P_i whose diameter is greater than $\frac{1}{2}$.

Now the set

$$M_1^* = c_1 + c_2 + \ldots + c_k$$

where

$$c_i = \text{arc}\, x_i\, P_i \qquad (i = 2, 3, \ldots k)$$

is a continuum. Then $M - M_1^*$ consists of a set of domains [1]) only a finite number of which can be of diameter greater than 1; let these be $d_{k+1}, d_{k+2}, \ldots d_{k+j}$. Since

$$\delta(d_{k+i}) > 1, \qquad (i = 1, 2, 3, \ldots, j)\ ^{2})$$

there exist in d_{k+i} two points x_i and y_i, which are not end points, such that

$$\delta(x_i, y_i) > 1.$$

Then, if P_i be the boundary point of d_{k+i} with respect to M,

$$\delta(P_i, x_i) > \tfrac{1}{2}$$

or

$$\delta(P_i, y_i) > \tfrac{1}{2}.$$

[1]) $M - M_1^*$ is not vacuous, since M_1^* contains no end-points by Theorem 14 (the points x_i and P_i being also non-cut points) and it has been shown by R. L. Moore that every bounded continuum contains at least two non-cut points. Cf. R. L. Moore, *Concerning simple continuous curves*. Transactions of the American Mathematical Society, vol. 21 (1920), pp. 340—341, Theorem 2; *Concerning the cut-points of continuous curves and of other closed and connected point-sets, loc. cit.* Also see S. Mazurkiewicz, *Un théorème sur les lignes de Jordan loc. cit.* pp. 119—130.

[2]) Where the notation for a point-set is placed in the parenthesis, the δ should be read „the diameter of".

Suppose that

$$\delta(P_i, x_i) > \tfrac{1}{2}.$$

Then by virtue of Theorem 1, $d_{k+i} + P_i$ contains an arc c_{k+i} whose extremities are x_i and P_i. Thus a continuum

$$M_2^* = M_1^* + c_{k+1} + \dots + c_{k+j}$$

is obtained.

Continuing this process, there is obtained, eventually, a continuum

$$M_m^* = c_1 + c_2 + \dots + c_h$$

such that $M - M_m^*$ consists of a set of domains only a finite number of which are of diameter $> \frac{1}{2}$, and none of which is of diameter > 1. For if domains of diameter > 1 could be obtained indefinitely, plainly an infinite set of arcs of diameter $> \frac{1}{2}$ would also be obtained, a situation contradictory to Theorem 8. $M - M_m^*$ being non-vacuous, let its domains of diameter $> \frac{1}{2}$, if any exist, be denoted by $d_{h+1}, d_{h+2}, \dots d_{h+q}$. Let the boundary point of d_{h+i} $(i = 1, 2, \dots q)$ be P_i. Since

$$\delta(d_{h+i}) > \tfrac{1}{2},$$

it can be shown that there exists in d_{h+i} a point x_i such that

$$\delta(P_i, x_i) > \tfrac{1}{4},$$

and hence $d_{h+i} + P_i$ contains an arc c_{h+i} from P_i to x_i.

If the process indicated above be continued indefinitely, there is obtained a connected set of arcs $c_1, c_2, \dots$, such that (1) none of these arcs contains any end-point of M by virtue of Theorem 14, and no two of them have in common a point which is an interior point of both, (2) if ε is any positive number, there exists a number η such that if n is any positive integer $> \eta$,

$$\delta(c_n) < \varepsilon,$$

and the diameter of every one of the countable set of maximal domains with respect to M lying in $M - M_n$, where

$$M_n = c_1 + c_2 + \dots + c_n$$

(a continuous curve since it consists of a finite and connected set of arcs) is less than ε.

If the set of points consisting of the totality of arcs in the sequence $c_1, c_2, c_3, \dots$ be denoted by M^*, then M^* is non-vacuous.

For M^* contains no end-points of M by (1) above, and M must contain at least two end-points, as already indicated.

Every point of $M-M^*$ is a limit point of M^*. For, suppose P is a point of $M-M^*$ that is not a limit point of M^*. Then there exists a circle k with center P that contains no point of M^*. But P must belong to some proper connected subset s of $M-M^*$ that lies interior to k, by virtue of the properties of connectivity and connectivity im kleinen of M. Let ϱ denote a positive number such that

$$\delta(s) > \varrho.$$

Now there exists a number η, such that if n is any positive number $>\eta$, $M-M_n$ consists of a set of maximal domains with respect to M no one of which is of diameter greater than ϱ. However, s must belong to some domain of $M-M_n$, since

$$M-M_n \supset M-M^* \supset s,$$

and this domain cannot be of diameter greater than ϱ. Therefore the supposition that P is not a limit point of M^* leads to a contradiction; hence all points of $M-M^*$ must be limit points of M^*.

The set $M-M^*$ is a totally disconnected set, since if it contains a connected subset s a contradiction will result as in the preceding paragraph.

Since M^* contains no end-points of M, and since M must have at least two end-points, all end-points of M must lie in $M-M^*$. It remains to show that every point of $M-M^*$ is an end-point of M.

Suppose there exists in $M-M^*$ a point P that is not an end-point of M, then P is a cut-point of M by virtue of Theorem 7. Hence

$$M-P=H_1+H_2,$$

where H_1 and H_2 are mutually separated sets. M^* is a connected set that does not contain P, and therefore P cannot disconnect M^*. Hence M^* must lie wholly in H_1 or H_2, say in H_1. As H_2 cannot contain any limit points of H_1. and as all points of $M-M^*$ are limit points of M^*, the set $M-(M^*+P)$ must be a subset of H_1. But

$$M-P=M^*+M-(M^*+P).$$

Hence, if

$$H_1 \supset M^*+M-(M^*+P)$$

H_2 must be a vacuous set. That is, $M-P$ does not allow of division into mutually separated sets, and is therefore connected. P is therefore a non-cut point of M, and by Theorem 7 an end-point of M. This completes the proof of the Theorem, the set $M-M^*$ being the set P_ω.

The following theorem is a generalization of Theorem 13:

Theorem 16. *If M is the boundary of a complementary domain of a continuous curve, and N a closed and connected proper subset of M, then $M-N$ consists of a countable set of domains with respect to M, viz, $d_1, d_2, d_3, \ldots$ satisfying the conditions of Theorem 13, except that (1) should be made to read, „for every positive integer n, d_n has at most two boundary points with respect to M, and these points belong to N“.*

The proof is very similar to the proof of Theorem 13.

Theorem 17. *The boundary of a complementary domain of a continuous curve is the sum of three mutually exclusive point-sets S'_1, S_2 and $[P]$, where (1) S'_1 is a countable set of simple closed curves no two of which have more than one point in common and whose interiors have no point in common (unless one of these simple closed curves be the outer boundary in the case of a bounded complementary domain), (2) S_2 is a countable set of simple continuous arcs no two of which have in common an interior point of both, and (3) $[P]$ is a totally disconnected set of limit points of the set S'_1+S_2.*

Proof. (1) has been established in Theorem 4.

Let β be the boundary of a complementary domain of a continuous curve, and P a point of $\beta-S'_1$. That maximal connected subset of $\beta-S'_1$ determined by R I shall call a set of type Q, provided it consists of more than one point. A set of type Q, together with its limit points, I shall call a set of type N.

By Theorem 11, a set of type N is a continuous curve. Furthermore, it is a continuous curve that contains no simple closed curve. For, suppose N is a set of type N that contains a simple closed curve C. Then

$$S'_1 \supset C.$$

and as

$$\beta - S'_1 \supset Q$$

where Q represents that set of type Q which determines N, then must

$$N - Q \supset C.$$

That is, every point of C is a limit point of Q that does not belong to Q.

Let d be that maximal domain with respect to β which is a subset of $\beta - C$ and of which Q is a subset. Then every point of C is a boundary point of d with respect to β. Clearly this is a contradiction of Theorem 16. Hence no set of type N can contain a simple closed curve.

Applying Theorem 15, every set of type N is the sum of a countable set of arcs, no two of which have in common an interior point of both, together with a totally disconnected set of limit points of these arcs. All arcs of β so determined assign to a set S_2.

The sets S_1' and S_2 are mutually exclusive. For, suppose they have in common a point x. Then x belongs to some arc, a, of some set N of type N, and, as determined, no points of a are end-points of N (see proof of Theorem 15). It follows, that x is a cut-point of N. Hence $N - x$ is the sum of two mutually separated point-sets, N_1 and N_2. Both N_1 and N_2 must contain points of Q, the set of type Q which determines N. But this is impossible, since Q is a connected subset of $\beta - S_1'$, and hence of $\beta - x$. It follows that the sets S_1' and S_2 are mutually exclusive, and it also follows that every arc of S_2 is a subset of some set of type Q.

If a and b are two arcs of S_2, then (1) if a and b belong to the same set of type Q, they have in common no point which is an interior point of both, and (2) if a and b do not belong to the same set of type Q they have no points in common. It follows that the arcs of S_2 are countable (as a consequence of Theorem 5).

The set of points

$$[P] = \beta - (S_1' + S_2)$$

is a totally disconnected set of points. For, if there exists a connected subset, t, of $[P]$, then t is a subset of some set of type Q, as

$$\beta - S_1' \supset t.$$

If N is the set of type N determined by this set of type Q, and $[a]$ the set of arcs common to N and S_2, then $N - [a]$ is totally disconnected by Theorem 15. But

$$N - [a] \supset t.$$

It follows that t cannot be a connected set, and that $[P]$ is totally disconnected.

This completes the proof of the Theorem.

III. Characterizations of continuous curves, and of the boundaries of the complementary domains of plane continuous curves [1]).

In this section I shall establish a condition which continua that are not connected im kleinen must satisfy. By means of this condition I shall characterize continuous curves for any number of dimensions, and the boundaries of the complementary domains of plane continuous curves.

Lemma I. *If a bounded continuum M is not a continuous curve, then there exist two concentric circles K_1 and K_2, and a sequence of sub-continua of M.*

$$M_\infty, M_1, M_2, M_3, \ldots$$

such that (1) each of these sub-continua contains at least one point on K_1 and K_2, respectively, but no points exterior to K_1 or interior to K_2, (2) no two of these sub-continua have a point in common, and no two of them contain points of any connected subset of M which lies wholly in the set $K_1 + K_2 + I$, (where I is the annular domain bounded by K_1 and K_2), (3) M_∞ is the sequential limiting set of the sequence $M_1, M_2, M_3, \ldots$, (4) if K is that maximal sub-continuum of M containing M_∞ and lying wholly in the set $K_1 + K_2 + I$, then all of the continua $M_1, M_2, M_3, \ldots$, lie in a connected subset of $M-K$.

Proof: R. L. Moore has established conditions (1), (2) and (3) of this Lemma [2]). It remains to establish condition (4).

R_{K_i} being the radius of the circle K_i, Let K_3 and K_4 be two circles concentric with K_1, and such that

$$R_{K_1} > R_{K_3} > R_{K_4} > R_{K_2}.$$

For every value of n, $(n = \infty, 1, 2, 3, \ldots)$ there exists a continuum M_n^* such that

$$M_n \supset M_n^*,$$

and such that if in the statement of this Lemma each M_i is replaced by M_i^*, K_1 and K_2 replaced by K_3 and K_4, respectively, and K replaced by T, statements (1), (2), and (3) hold true, but (4)

[1]) The theorems and lemmas in this section were presented to the American Mathematical Society, Dec. 29, 1922.

[2]) *A characterization of Jordan regions by properties having no reference to their boundaries, loc. cit.*

may or may not hold true. If statement (4) does hold true, the proof is complete. If statement (4) does not hold true, let K_5 and K_6 be circles concentric with K_1 and such that

$$R_{K_4} > R_{K_5} > R_{K_6} > R_{K_2}.$$

For every value of n, $(n = \infty, 1, 2, 3, \ldots)$, there exists a continuum $\overline{M}_n$ such that

$$M_n \supset \overline{M}_n,$$

and such that if in the statement of this Lemma each M_i is replaced by $\overline{M}_i$, K_1 and K_2 replaced by K_5 and K_6, respectively, and K replaced by W, statements (1), (2), and (3) hold true, but (4) may or may not hold true. I shall prove that (4) must hold true.

There exists an infinite sequence of distinct sets, $L_1, L_2, L_3, \ldots$ such that (1) for each n. L_n is a maximal connected subset of $M - T$, and contains at least one set, but at most a finite number of sets of the sequence $M_1, M_2, M_3, \ldots$, (2) every set of the sequence $M_1, M_2, M_3, \ldots$ belongs to some set of the sequence $L_1, L_2, L_3, \ldots$

For every value of n, $(n = 1, 2, 3, \ldots)$, L_n has a limit point in T. For, suppose L_i is a set of this sequence that has no limit point in T. Then, since L_i is a maximal connected subset of $M - T$, L_i is closed. L_i and T are then two mutually separated continua.

There exists a connected domain D_1 containing L_i, but containing no point of T nor having a point of T on its boundary.

If A is a point of L_i and B a point of T, then M is a continuum containing A and B. That is, M contains a point A interior to D_1 and a point B exterior to D_1. Then there exists a sub continuum Q of M, which contains A and at least one point x on the boundary of D_1, but no point exterior to D_1 [1]).

x cannot belong to T and hence belongs to $M - T$; furthermore x is joined to A by a subset Q of $M - T$, and must therefore belong to L_i. But no points of L_i lie on the boundary of D_1. Therefore the supposition that L_i has no limit point in T leads to a contradiction. Hence every set L_n $(n = 1, 2, 3, \ldots)$ has a limit point in T.

Not more than one set of the sequence $L_1, L_2, L_3, \ldots$ has a li-

[1]) Cf. Anna M. Mulliken, *Certain theorems relating to plane connected point-sets*, Trans. Amer. Math. Soc., XXIV (1923) Th. 1.

mit point in W. For if one of these sets, say L_j, has a limit point in W, then must

$$L_j \supset W,$$

and no two of these sets have a point in common. We can consider L_j as being omitted from the sequence $L_1, L_2, L_3, \ldots$ We have, then, an infinite sequence of connected sets. $L_1, L_2, L_3, \ldots$ such that for every n $(n = 1, 2, 3, \ldots)$,

$$M - W \supset L_n$$

and L_n has a limit point in T. Since T is connected, the set

$$U = L_1 + L_2 + L_3 + \ldots + T$$

is a connected subset of $M - W$. Since every set of the sequence $\overline{M}_1, \overline{M}_2, \overline{M}_3, \ldots$ belongs to some set of the sequence $L_1, L_2, L_3, \ldots$, and therefore to U, condition (4) of the theorem is satisfied by the sequence $W, \overline{M}_1, \overline{M}_2, \overline{M}_3, \ldots$ as also are conditions (1), (2), and (3), if, as already pointed out, K is replaced by W, each M_i by $\overline{M}_i$, and K_1 and K_2 by K_5 and K_6, respectively.

Although the above proof is given for two dimensions, it should be observed that a similar proof can be given to show that the Lemma holds for any number of dimensions.

Lemma 2. *If x and y are two points of the boundary, β, of a complementary domain of a continuous curve, there exist at most two distinct arcs from x to y; i. e., arcs that have in common at most their end-points, x and y, and if two such arcs exist every arc from x to y which belongs to β is identical with one of these arcs.*

(This Lemma is a corollary of Theorem 4).

It has recently been shown by R. L. Moore that every two points that lie together in a connected subset of an open [1]) subset of a continuous curve, lie in a sub-continuum of that open subset [2]). Using Lemma 1, it is easy to show that any bounded continuum which has this property is a continuous curve.

Theorem 18. *In order that a bounded continuum M should be a continuous curve, it is necessary and sufficient that every two points that lie in a connected subset of an open subset of M lie in a subcontinuum of that open subset.*

[1]) If L is a closed proper subset of a continuum, M, then $M - L$ is an open subset of M.

[2]) R. L. Moore, *Concerning continuous curves in the plane, loc. cit.* Theorem 1.

[The proof of this theorem for the two-dimensional case is given below. An analogous proof may be given to show that the theorem holds in m dimensions]

(1) The condition stated in the theorem is necessary, as shown by R. L. Moore.

(2) The condition stated in the theorem is sufficient. For, suppose M is a bounded continuum satisfying the condition stated in the theorem, but which is not a continuous curve. Then the conditions of Lemma 1 are satisfied.

Let A and B be two points of M_∞ that lie in I, and let C_1 and C_2 be two circles with centers A and B, respectively, such that (i) C_1 and C_2 have no point in common nor have their interiors any point in common, and (ii) C_1 and C_2 lie wholly in I, but do not enclose K_2.

If Q_1 an Q_2 denote the sets of all points of K interior to C_1 and C_2, respectively, let

$$T = K - (Q_1 + Q_2).$$

T is closed, and therefore $M - T$ is an open subset of M. Since

$$K \supset T,$$

all of the continua $M_1, M_2, M_3, \ldots$ lie in a connected subset, U, of $M - T$.

Since A and B belong to M_∞, they are limit points of the sequence $M_1, M_2, M_3, \ldots$ and therefore limit points of U. The set $U + A + B$ is therefore a connected subset of $M - T$.

By hypothesis every two points that lie in a connected subset of an open subset of M lie in a sub-continuum of that open subset. Then A and B must lie in a continuum N, such that

$$M - T \supset N.$$

Since N is a continuum containing a point A interior to C_1 and a point B exterior to C_1, there exists a sub-continuum N_1 of N which contains A and at least one point P on C_1, but no point exterior to C_1. Since C_1 lies wholly in I, and does not enclose K_2, P is joined to A by a continuum of M that lies in I and must therefore be a point of the set K, but not of the set Q_1 or the set Q_2. That is, P must be a point of the set T. But P also is a point of the set N, which is a subset of the set $M - T$. Hence the supposition that M is not a continuous curve has led to a contradiction, and M must therefore be a continuous curve.

Theorem 19. *In order that the boundary of a simply connected domain should be a continuous curve, it is necessary and sufficient that every connected subset of it be connected in the strong sense*[1].

A. The condition stated in the theorem is necessary. Let M be a continuous curve which is the boundary of a simply connected domain D, and let N be any connected subset of M. I shall show that any two points x and y of N are the extremities of an arc that lies wholly in N. There are three cases to consider:

(1) x and y the extremities of two distinct (in the sense defined in Lemma 2) arcs, a_1 and a_2, of M.

If N contains no arc from x to y, there exists at least one point, P_1, on a_1, and at least one point, P_2, on a_2, such that neither P_1 nor P_2 belongs to N.

Add to N its limit points, and call the resulting set N'. By virtue of Theorem 11, N' is a continuous curve, and hence contains an arc, a, from x to y. As a result of Lemma 2 either $a \equiv a_1$, or $a \equiv a_2$. Suppose that $a \equiv a_1$. Then P_1 belongs to N', but not to N. But the set $N' - P_1$ must then contain a connected subset N, containing x and y, and therefore an arc from x to y[2]. By Lemma 2 this arc must be the arc a_2. However, the set $N' - (P_1 + P_2)$ also contains a connected subset, N, containing x and y, and hence an arc from x to y; call this arc a_3. We have, then, two distinct arcs, a_1 and a_2, from x to y, and a third arc, a_3, which is identical with neither a_1 nor a_2. But this is a contradiction of Lemma 2. In this case, then, N must contain an arc from x to y.

(2) x and y the extremities of only one arc, a, of M.

In this case, if P is a point of a not belonging to N, the set $N' - P$ contains an arc from x to y which cannot be identical with a, thus giving an immediate contradiction of the hypothesis that M contains only one arc from x to y. In this case also, then, must N contain an arc from x to y.

(3) x and y neither the extremities of two distinct arcs of M, nor of only one arc of M, but the extremities of a totality of arcs, T, such that if a_1 and a_2 are any two arcs of T, a_1 and a_2 have

[1] A point-set M is said to be *connected in the strong sense* if every two points that lie in a connected subset N of M lie also in a sub-continuum of N.

[2] Cf. R. L. Moore, *Concerning continuous curves in the plane. loc. cit.*; Th. 1.

interior points in common, and each has interior points that do not belong to the other.

Let F denote a set of points such that (1) if P is any point of F, P lies on every arc of T, and (2) if P is a point that lies on every arc of T, P belongs to F. That such points exist, distinct from x and y, can be easily shown as follows: Let a_1 be any arc of T. Suppose a_2 is any other arc of T, and let P be a point of a_2 not on a_1. P is an interior point of an arc a_2' which is a subset of a_2, and which has in common with a_1 only its end-points; call these end-points A and B. Either A is distinct from x and y or B is distinct from x and y. Let A be distinct from x and y. Then A is a point of F. For, suppose A is not a point of F. Then some arc, a_3, of T, does not contain A. Let A' be the last point of a_3 on the arc xA of a_1 in the order from x to A; on the arc $A'y$ of a_3, in the order from A' to y, let B' be the first point that a_3 has in common with the point-set consisting of the arc Ay of a_1 and the arc a_2'. That arc of a_3 from A' to B' call a_3'. The set $a_1 + a_2' + a_3'$ contains two points that are the end-points of three distinct arcs, a contradiction of Lemma 2. Therefore A must be a point of F distinct from x and y. Similarly, all points common to a_1 and a_2 are common to all arcs of T and hence belong to F.

Either N contains all points of F or it does not.

Suppose N contains all points of F. If P is any point of a_1 that is not a point of F, let a_2 be an arc of T that does not contain it. In the order from x to y let x' be the last point a_2 has in common with the arc xP of a_1, and y' the first point after x' that a_2 has in common with the arc Py of a_1. Since, as shown above, x' and y' are points of F, they belong to N, and as they are the end-points of two distinct arcs of M, the proof of case (1) of this theorem shows that N contains one of these arcs. Similarly, every point P of a_1 that does not belong to F determines two points x' and y' of F which are the end-points of an arc that belongs to N. Let $\overline{L}$ denote the point-set obtained by adding together the points of all the arcs of the set L.

The point-set $\overline{L} + F$ contains an arc t from x to y. This fact can be established by showing that $\overline{L} + F$ is a continuous curve. $\overline{L} + F$ is a closed set. For, (i) F is closed. If P is a limit of F, then, since F is a subset of a_1, and a_1 is closed, P must belong to a_1. If a_2 is any other arc of T, P lies on a_2 for the same reason. That is, P is common to all arcs of T, and hence belongs to F. Also, (ii) all limit points of $\overline{L}$, if they do not belong to $\overline{L}$, belong to F. For, let P be a limit point of $\overline{L}$ that does not belong to $\overline{L}$. Let k be a circle of arbitrary radius r and center P, and k' a circle of radius $r/2$, concentric with k. k' must contain points of an infinite set of arcs of L. But since L is a subset of M and M cannot contain an infinite set of arcs of diameter greater than $r/2$ that have only end-points in common (Theorem 6) then must an infinite set of arcs of L lie wholly interior to k. But the end-points of each such arc of L must then lie interior to k, and as the radius of k is arbitrary it follows that P is a limit point of such end-points; i. e., of F. Therefore if P is not a point of $\overline{L}$ it is a point of F. Since F is closed, and $\overline{L}$ is either closed or those limit points that it does not contain belong to F, it follows that $F+\overline{L}$ is closed. As $\overline{L}+F$ is connected and a subset of M, $\overline{L}+F$ is a bounded continuum, and by virtue of Theorem 16 a continuous curve, therefore containing an arc from x to y. As $\overline{L}+F$ is a subset of N, it follows that N contains an arc from x to y.

Suppose N does not contain all points of F; that is, that there exists some point P of F that N does not contain. If, as in case (1) of this theorem, N' denotes the set N together with its limit points, then N' is a continuous curve and therefore contains an arc from x to y; this arc must belong to T, and hence contains P. That is, P belongs to N', but not to N. Then $N'-P$ contains a connected subset, N, containing x and y, and as shown in case (1) of this theorem must therefore contain an arc from x to y that does not contain P. But this is impossible, since P is a point of F and therefore common to all arcs from x to y. Hence N must contain all points of F, and therefore, as shown above, an arc from x to y.

B. The condition is sufficient by Theorem 18.

It will be noted that the above proof also establishes the following theorem:

Theorem 20. *In order that the boundary of a simply connected domain should be a continuous curve, it is necessary and sufficient that every connected subset of it be arc-wise* [1]) *connected.*

[1]) A point-set M is said to be arc-wise connected when every two points of M are the extremities of a simple continuous arc that lies wholly in M.