une coupure entre p et $1-E$[1]). Or, en s'appuyant sur le théor. $\mathfrak{C}$ et le théor. de Janiszewski, on prouve facilement[2]) que tout continu Péanien qui est une coupure entre deux points contient une courbe simple fermée qui est aussi une coupure entre ces points. Il existe donc une courbe simple fermée S qui est une coupure entre p et $1-E$. La région-composante R de $1-S$ qui contient p est bien la région demandée, car $R(1-E)=0$, donc $R \subset E$ et selon le théor. de Jordan, S constitue la frontière de R.

[1]) ibid. p. 309, proposition A_2'.
[2]) cf. ma note de Fund. Math. VI, p. 140.

On the set of all cut points of a continuous curve.

By

G. T. Whyburn (Austin, U. S. A.).

In this paper it will be shown that as a consequence principally of the author's results on the structure of a continuous curve relative to its cyclic elements it follows that the set of all cut points of any continuous curve in a metric space is homeomorphic with a certain kind of subset of an acyclic continuous curve. It is also shown, conversely, that a subset of this kind of any acyclic continuous curve is always identically the set of all cut points of some continuous curve. Thus we obtain a complete characterization of a point set which is topologically equivalent to (i. e., homeomorphic with) the set of all cut points of a continuous curve. Incidentally, the results in this paper answer, indeed more than answer, a question raised by C. Zarankiewicz in the *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, 1926, p. 362.

The customary notation and terminology of point set theory will be used; $S(P, r)$ will denote the set of all points of the space whose distance from the point P is less than the number r, $\delta(M)$ denotes the diameter of the set M, $\varrho(X, Y)$ is the minimum distance between the sets X and Y, a continuous curve is a compact connected im kleinen continuum, etc. For definitions of the terms used in connection with the cyclic elements of a continuous curve the reader is referred to my paper *Concerning the structure of a continuous curve*, (American Journal of Mathematics, vol. 50, April, 1928, pp. 167—194). This paper will be referred to simply as „*Structure*".

Let W denote the universal acyclic continuous curve of Ważewski[1]), let $C(W)$ denote the set of all cut points, $E(W)$ the set of all end points, and $R(W)$ the set of all ramification points of W. It is well known that $R(W)$ is countable and dense on every subcontinuum of W. We proceed now to show that.

a). The set G of all the cut points of any continuous curve M whatever is homeomorphic with a subset K of $C(W)$ such that (a) *if A and B are any two points of K and AB is the arc of W from A to B, then $K \cdot AB$ is the sum of closed set and a countable set, and* (b) *K contains the center point*[2]) *in W of every group of three points of K.*

Proof. There exists a finite or countably infinite sequence $S_1, S_2, S_3, \ldots$ of simple cyclic chains of M such that

(1) both end elements of S_1 are nodes of M,

(2) if $n > 1$, one end element of S_n is a node of M and the other is a cut point of M which belongs to an interior element of some chain S_m, where $m < n$,

(3) if i and j, $i < j$, are two positive integers, then either S_i and S_j are mutually exclusive or else they have in common just one point A_j which is a cut point of M and is one end element of S_j,

(4) $\lim\limits_{n \to \infty} \delta(S_n) = 0$, and

(5) every point, if any, of $M - \Sigma S_n$ is an end point of M[3]).

By (1) it follows that the end elements of S_1 contain non-cut points A_1 and B_1 respectively of M. For each $n > 1$, it follows by (2) that the end element of S_n which is a node of M contains a non-cut point B_n of M. For each $n > 1$, let A_n be the end element of S_n which is a cut point of M. Now for each n, S_n contains an arc t_n from A_n to B_n (see Structure, Theorem 8). For each n, let K_n be the set of all those points of M which separate A_n and B_n in M, and let D_n be the set of all those points of K_n each of which cuts M into three or more components.

Let U_1 and V_1 be points of $E(W)$ and Z_1 the arc in W from U_1 to V_1. Now since $R(W)$ is dense on Z_1 and[1]) D_1 is countable, it follows by a theorem of Fréchet-Urysohn[2]) that there exists a biunivalued and bicontinuous transformation J_1 of t_1 into Z_1 such that $J_1(D_1) \subset R(W)$. Now for each maximal cyclic curve C_{1i} $(i = 1, 2, 3, \ldots)$ of M belonging to S_1 and containing two points X_{1i} and Y_{1i} of K_1 (where we have the order A_1, X_{1i}, Y_{1i}, B_1 on t_1), then since $G \cdot C_{1i}$ is countable, it follows readily with the aid of a theorem of Sierpiński's[3]) together with the Fréchet-Urysohn theorem just used that there exists a biunivalued and bicontinuous transformation J_{1i} of $G \cdot C_{1i}$ into a subset $J_{1i}(G \cdot C_{1i})$ of the interval of Z_1 from $J_1(X_{1i})$ to $J_1(Y_{1i})$ so that $J_{1i}(G \cdot C_{1i}) \subset R(W)$, $J_{1i}(X_{1i}) = J_1(X_{1i})$, and $J_{1i}(Y_{1i}) = J_1(Y_{1i})$. We now define a transformation T_1 over $G \cdot S_1$ as follows: if X is a point of $G \cdot S_1$ belonging to K_1, let $T_1(X) = J_1(X)$; if X does not belong to K_1 it belongs to C_{1i}, for some i, and we let $T_1(X) = J_{1i}(X)$; if X belongs both to K_1 and to some C_{1i}, then it is an X_{1i} or a Y_{1i} and hence $J_{1i}(X) = J_1(X) = T_1(X)$. Thus it is easily seen that T_1 as thus defined is biunivalued and bicontinuous.

Now for $n > 1$, I shall define a transformation T_n over $G \cdot S_n$ according to the following procedure. By (2), A_n is a point of $G \cdot S_m$, for some $m < n$, and hence there exists a point $T_m(A_n)$ in W. And since clearly A_n belongs to $D_m + \sum\limits_i C_{mi}$, where C_{mi}, $i = 1, 2, 3, \ldots$, is the collection of those maximal cyclic curves of M belonging to S_m and containing two points of $K_m + A_m$, then $T_m(A_n) \subset R(W)$. Since each point of $R(W)$ is a ramification point of infinite order of W, it follows that W contains an arc Z_n having one end point $T_m(A_n)$ and the other in $E(W)$ and such that $[Z_n - T_m(A_n)] \cdot \sum\limits_{i=1}^{n-1} Z_i = 0$ and $\delta(Z_n) > \delta(N_n) - 1/n$, where N_n is the

[1]) See T. Ważewski, *Sur les courbes de Jordan ne renfermant aucune courbe simple fermée de Jordan*, Ann. Soc. Polon. Math. Cracovie, 1923, vol. 2. p. 49 Also see Menger, Fund. Math. vol. 10, p. 108.

[2]) By the center point 0, relative to an acyclic curve W, of three points A, B, and C of W is meant the point 0 of W which is the limit point in AB of the component of $M - AB$ which contains C, when C is not on AB, and $0 = C$ if $C \subset AB$.

[3]) This statement is proved on the basis of the results in *Structure* by a method essentially the same as that outlined in the proof of Theorem 2 in my paper *Concerning Menger regular curves* (Fund. Math., vol. 12 (1928), pp. 264—294: also see R. L. Moore, Monatshefte für Math. u. Phys, vol. 36 (1929), p. 86, and W. L. Ayres *Arc-curves and basic subsets of a continuous curve. Second paper*, Trans. Amer. Math. Soc., vol. 30 (1929), pp. 595—612, Lemma 15A.

[1]) See Kuratowski and Zarankiewicz, Bull. Amer. Math. Soc., vol. 33 (1927), p. 571.

[2]) Cf. M. Fréchet, Math. Ann., vol. 68 (1910), p. 169, and P. Urysohn, Fund. Math., vol. 7 (1925), p. 88.

[3]) Cf. W. Sierpiński, Fund. Math., vol. 2. p. 89.

component of $W - \sum_{1}^{n-1} Z_i$ containing $Z_n - T_m(A_n)$. It follows now just as in the case of $n=1$ that biunivalued transformations J_n and J_{ni} exist such that $J_n(t_n) = Z_n$, $J_n(A_n) = T_m(A_n)$, $J_n(D_n) \subset R(W)$, and if $[C_{ni}]$ is the collection of all those maximal cyclic curves of M which belong to S_n and contain two points X_{ni} and Y_{ni} (in the order A_n, X_{ni}, Y_{ni}, B_n on t_n), of $K_n + A_n$, then $J_{ni}(G \cdot C_{ni})$ belongs to the interval of Z_n from $J_n(X_{ni})$ to $J_n(Y_{ni})$, $J_{ni}(G \cdot C_{ni}) \subset R(W)$, $J_{ni}(X_{ni}) = J_n(X_{ni})$ and $J_{ni}(Y_{ni}) = J_n(Y_{ni})$, for $i = 1, 2, 3, \ldots$ We now define the transformation T_n over $G \cdot S_n$ as follows: $T_n(A_n) = T_m(A_n)$, $T_n(X) = J_n(X)$ if X belongs to K_n, and $T_n(X) = J_{ni}(X)$ if X belongs to $\sum_i C_{ni}$. It follows just as in the case of T_1 that T_n is biunivalued and bicontinuous.

We now define the transformation T over G as follows: for each point X of G, by (4) it follows that there exists a smallest integer n such that X belongs to $G \cdot S_n$; so for each point X of G, let $T(X) = T_n(X)$, where n is the least integer such that X belongs $G \cdot S_n$. It is clear from the way the transformations T_n were defined that T is biunivalued. That T is bicontinuous is readily established with the aid of the following

Lemma on transformations. *Suppose $[H_n]$, $n = 1, 2, 3, \ldots$, is a sequence of point sets and $[T_n]$ is a corresponding sequence of univalued transformations such that*

a) *for each n, T_n is defined and continuous on H_n,*

b) *for each pair of integers i and j, $T_i(H_i \cdot H_j) = T_j(H_i \cdot H_j)$*

c) *$\lim_{n\to\infty} \delta[T_n(H_n)] = 0$, and*

d) *if for each point P of $H = \sum H_n$, H_p is the sum of all those sets $[H_{n_i}]$ of $[H_n]$ which contain P, and $X_1, X_2, X_3, \ldots$ is any sequence of points in $H - H_p$ having P as sequential limit point, then there exists a corresponding sequence of points $Y_1, Y_2, Y_3, \ldots$ in H_p such that $\lim_{n\to\infty} \varrho[X_n, Y_n] = 0 = \lim_{n\to\infty} \varrho[T_{a_n}(X_n), T_{b_n}(Y_n)]$, where for each n, a_n and b_n are integers such that $X_n \subset H_{a_n}$ and $Y_n \subset H_{b_n}$. Then the transformation T such that for each n and each point $P(n)$ of H_n, $T[P(n)] = T_n[P(n)]$ is defined and continuous on H.*

Proof. Let P be any point of H and let R be any neighborhood of $T(P)$, and let V be any sequence of points of H having P as sequential limit point. If $[H_{n_i}]$ is the collection of sets of $[H_n]$ which contain P, then by c) it follows that there exists an integer k such that $\sum_{k}^{\infty} T(H_{n_i}) \subset R$. Since each of the transformations $T_{n_1}, T_{n_2}, \ldots T_{n_k}$ is continuous, it readily follows now that all save possibly a finite number of point of $T(V \cdot H_p)$ belong to R. Now arrange the points of $V \cdot (H - H_p)$ into a sequence of $X_1, X_2, X_3, \ldots$ By c) it follows that for the corresponding sequence $Y_1, Y_2, Y_3, \ldots$ in H_p, $T(P)$ is the sequential limit point of $T(Y_1), T(Y_2), \ldots$ Then from d) it readily follows that $T(P)$ is likewise the sequential limit point of $T(X_1), T(X_2), \ldots$ Thus R contains all save a finite number of points of V, and hence T is continuous.

Proof of the continuity of T. To show that T is continuous, we let $H_n = G \cdot S_n$ for each n. Then clearly the transformations $[T_n]$ satisfy conditions a) and b) in the Lemma. They satisfy condition c), because $T_n(H_n) \subset Z_n$, for each n, and $\lim_{n\to\infty} \delta(Z_n) = 0$. To show that d) is satisfied, we pick the points of $[Y_n]$ as follows: For each n let N_n be the component of $M - H_p$ containing X_n, where $H_p = \sum H_{n_i}$, the sum of all those sets of $[H_n]$ which contain P, and let Y_n be the limit point of N_n which belongs to H_p. Then since $\lim_{n\to\infty} \delta(N_n) = 0$, therefore $\lim_{n\to\infty} \varrho(X_n, Y_n) = 0$. Now for each n, let L_n be the component of $\sum Z_n - \sum Z_{n_i}$ containing $T(X_n)$. Then since, for each n, it follows from (2) that a finite number $S_{a_1}, S_{a_2}, \ldots S_{a_k}$ of the chains $[S_n]$ exist whose sum is connected and contains both X_n and Y_n and lies in $N_n + Y_n$ and remains connected upon the omission of Y_n, it follows that $(Z_{a_1} + Z_{a_2} + \ldots + Z_{a_k}) - T(Y_n)$ is connected and contains $T(X_n)$. Hence $(Z_{a_1} + Z_{a_2} + \ldots + Z_{a_k}) - T(Y_n)$ lies in L_n, and therefore $T(Y_n)$ is the limit point of L_n belonging to $\sum Z_{n_i}$. Then since $\lim_{n\to\infty} \delta(L_n) = 0$, it follows that $\lim_{n\to\infty} \varrho[T(X_n), T(Y_n)] = 0$. Therefore all the conditions of the lemma are fullfilled, and consequently T is continuous. That T^{-1} is continuous follows by an argument highly analogus to the one just given, by applying the lemma, using $H_n = T(G \cdot S_n)$, for each n. Thus T is biunivalued and bicontinuous.

Let A and B be any two points of $K = T(G)$, and let Z be the arc in W from A to B. Let S be the simple cyclic chain in M from $T^{-1}(A)$ to $T^{-1}(B)$, and let F be the set of all those points of M which separate $T^{-1}(A)$ and $T^{-1}(B)$ in M. The transformation

T was defined in such a way that $T(G\cdot S)\supset K\cdot Z\supset T(F)$. Therefore, since (see *Structure*, Theorem 1), $G\cdot S=$ the closed set $[F+T^{-1}(A)+T^{-1}(B)]+$ a countable set H, and since T is continuous, it follows that $K\cdot Z$ is the sum of the closed set $T(F)+$ $+A+B$ and the countable set $K\cdot Z\cdot T(H)$. That K satisfies condition (b) in (α) follows immediately from the fact that K was constructed so that it contains every ramification point of the acyclic curve $\overline{\Sigma Z_n}$.

We now consider the following converse proposition.

(β). *If K is any subset of the set of all cut points of an acyclic continuous curve N (in a euclidean space of $n\geqslant 2$ dimensions) such that* (a) *for each pair of points A and B of K, $K\cdot Z$ is the sum of a closed set and a countable set, where Z is the arc in N from A to B, and* (b) *K contains the center point relative to N of each set of three points A, B and C of K, then there exists a continuous curve M such that K is identically the set of all cut points of M.*

Construction of the curve M Let A_1 and B_1 be points of K such that $\varrho(A_1,B_1)>\delta(K)-1$, and let Z_1 be the arc in N from A_1 to B_1. By hypothesis $K\cdot Z_1=K_1+H_1$ where K_1 is closed, H_1 is countable, and $K_1\cdot H_1=A_1+B_1$. Let the components (maximal segments) of Z_1-K_1 be $[S_{1i}]$, $i=1,2,3,\ldots$ and for each i, let X_{1i} and Y_{1i} be the endpoints of S_{1i} on Z in the order A_1, X_{1i}, Y_{1i}, B_1. It is readily seen that there exists a collection of arcs $[t_{1i}]$ such that (1) for each i, t_{1i} is an arc from X_{1i} to Y_{1i} and has only these points in common with N and is of diameter <2 times the diameter of the component of $N-(X_{1i}+Y_{1i})$ containing S_{1i}, (2) the arc segments $[t_{1i}-(X_{1i}+Y_{1i})]$ are mutually exclusive, and (3) $\operatorname*{Lim}_{i\to\infty}\delta(t_{1i})=0$. Let the points of H_1 be ordered $P_{11}, P_{12}, P_{13},\ldots$. Then for each i, there exists a simple closed curve J_{1i} such that $J_{1i}\cdot[N+\Sigma t_{1i}+\Sigma_1^{i-1}J_{1j}]=P_{1i}$ and $\delta(J_{1i})<1/i$. Let M_1 denote the continuum $Z_1+\Sigma t_{1i}+\Sigma J_{1i}$.

Now for each $n>1$ we shall construct a continuum M_n according to the following procedure. If $K\subset\sum_1^{n-1}M_i$, then let $M_n=0$. If not, then there exists a point B_n of K in $N-\sum_1^{n-1}Z_i$ such that if X is any point of K,

$$\text{(i)}\qquad \varrho(X,\sum_1^{n-1}Z_i)<\varrho[B_n,\sum_1^{n-1}Z_i]+1/n.$$

Let C_n be the component of $N-\sum_1^{n-1}Z_i$ containing B_n and let A_n be the limit point of C_n belonging to $\sum_1^{n-1}Z_i$, and let Z_n be the arc in N from A_n to B_n. It follows by hypothesis b) that A_n belongs to K. Hence, just as in the case of $n=1$, $K\cdot Z_n=K_n+H_n$, where K_n is closed, H_n is countable and $K_n\cdot H_n=A_n+B_n$; and there exist segments $[S_{ni}]$, points $[X_{ni}]$ and $[Y_{ni}]$, and arcs $[t_{ni}]$ such that (1) for each i, t_{ni} is an arc from X_{ni} to Y_{ni} which has only these points in common with $N+\sum_1^{n-1}M_i$ and $\delta(t_{ni})<$ two times the diameter of the component of $N-(X_{ni}+Y_{ni})$ containing S_{ni}, (2) the arc segments $[t_{ni}-(X_{ni}+Y_{ni})]$ are mutually exclusive, and (3) $\operatorname*{Lim}_{i\to\infty}\delta(t_{ni})=0$ [this follows from (1)]. Likewise if the points of H_n are $P_{n_1}, P_{n_2}, P_{n_3},\ldots$, then for each i, there exists a simple closed curve J_{ni} such that $J_{ni}\cdot[N+\sum_1^{n-1}M_i+\sum_i t_{ni}+\sum_{j=1}^{i-1}J_{nj}]=P_{ni}$, and $\delta(J_{ni})<1/ni$. Then let M_n be the continuum $Z_n+\sum_i t_{ni}+\sum_i J_{ni}$.

Now let $L=\sum_1^{\infty}M_n$. By (i) it follows that each point, if any, of $K-K\cdot L$ is a limit point of L. And since $\sum_1^{\infty}Z_n$ is connected and the condition of convergence clearly are such that $\overline{L}-L\subset\overline{\Sigma Z_n}-$ $-\Sigma Z_n$, and each point of K is a cut point of N, it follows that for each point X of $K-K\cdot L$ there exists a component C_x of $N-X$ which contains no point of $\overline{L}$. It readily follows then that the set of all such points X is countable and if they are ordered $X_1, X_2, X_3,\ldots$, then for each i there exists a simple closed curve C_i such that $C_i\cdot\overline{L}=X_i$, $C_i\cdot\sum_1^{i-1}C_j=0$, and (ii) $\delta(C_i)<1/i$. Now finally let $M=\overline{L}+\Sigma C_i$.

Then by (ii) it readily follows that M is a continuum. And since M clearly is the sum of the continuum $\overline{\Sigma Z_n}$, a subcontinuum of N, together with the elements of the countable collection composed of the mutually exclusive arc segments $[t_{ni}-(X_{ni}+Y_{ni})]$ and the „simple closed curves less one point" $[J_{ni}-P_{ni}]$ and $[C_i-X_i]$, remembering that at most a finite number of the arcs and curves are of diameter $>$ any preassigned positive number, it easily follows that M is a continuous curve. That K is identically the set of all cut points of M follows immediately from the construction of M.

It is interesting to note also that all the maximal cyclic curves of M are simple closed curves — indeed they are the simple closed curves $[S_{ni}+t_{ni}]$, $[J_{ni}]$, and $[C_i]$.

Results (α) and (β) together yield the following theorem:

Theorem 1. *In order that the point set G should be topologically equivalent to the set of all cut points of some compact continuous curve M in a metric space it is necessary and sufficient that G be homeomorphic with a subset K of $C(W)$ (the set of all cut points of she Ważewski curve W), such that 1) if A and B are any two points of K and Z is the arc in W from A to B, then $K \cdot Z$ is the sum of a closed set and a countable set, and 2) K contains the center point in W of every group of three points A, B, and C of K.*

It is clear that the transformation T set up in the proof of (α) is such that it not only preserves limit points among the point of G (i. e., is continuous) but also it preserves a certain degree of order among the points of G not belonging to the same cyclic element of M. Indeed, the methods of construction and proof given under (α) together with the method of construction of the set M under (β) contain all the essential features necessary for the demonstration of the following theorem which elucidates the simplicity of the structure of the set of cut points of a continuous curve

Theorem 2. *If M is any locally connected, compact and metric space* (i. e. *any compact continuous curve*), *then there exists, in the plane, a compact continuous curve K such that*

1) *every maximal cyclic curve of K is a simple closed curve,*

2) *if the cyclic elements* (i. e., *cut points, endpoints, and maximal cyclic curves*) *of M and of K are regarded as points, then there exists a biunivalued and bicontinuous correspondence between the cyclic elements of M and the cyclic elements of K which preserves class,* i. e., *$T(C)$ is a cut point, end point, or maximal cyclic curve of K according as C is a cut point, end point, or maximal cyclic curve of M;*

3) *there exists in K an acyclic continuous curve N which contains all the cut points and all the end points*[1]) *of K.*

[1]) For the case where M is a continuous curve, Theorem 2 contains the result, recently published by C. Zarankiewicz (cf. *Über Endpunkte*, Bull. Acad. Polon. des Sciences et des Lettres, 1928, pp. 445—453) that the set of all end points of any continuum M is homeomorphic with a subset of the set of all end points of the universal acyclic curve W of Ważewski. I note here the fact that Theorem 4 in Zarankiewicz's paper is a special case of Theorem 11 in *Structure*. This clearly is the case, since every cut point of a continuous curve is by definition a cyclic element of that curve, and since (see *Structure*, Theorem 8 and the reference therein to C. M. Cleveland) the words „constituant" and „component" are equivalent when applied to any set of cut points of a continuous curve.

To demonstrate the truth of Theorem 2, we would take the chains $]S_n]$ in M just as in (α) and select the arcs $[Z_n]$ from the curve W (which for convenience we may assume such that each of its arcs is free on one side). Then define the transformation T over G as in (α) and construct arcs $[t_{ni}]$ on the maximal segments $[S_{ni}]$ of $Z_n - T(K_n)$ for each n, after the manner used in (β), for convenience placing them on the free side of Z_n. For each maximal cyclic curve C_{ni} belonging to a chain S_n and containing a segment E_{ni} of $t_n - K_n$, we take the correspoinding segment S_{ni} of $Z_n - T(K_n)$ and let $T(C_{ni}) = S_{ni} + t_{ni}$. The correspondence can be extended so as to include the endpoints H of M as follows: if A_1 is an end point of M, let $T(A_1) = J_1(A_1)$; for each point B_n which is an end point of M, let $T(B_n) = J_n(B_n)$. For each point P of M not in ΣS_n, there exists a sequence $S_{n_1}, S_{n_2}, \ldots$ of the chains $[S_n]$ such that $S = \sum_i S_{ni}$ is connected and $S+P$ is a continuum. The corresponding sequence $[Z_{n_i}]$ of arcs also has a connected sum Z and there exists a point Q in W, in fact in $E(W)$, such that $Z+Q$ is a continuum. For each such point P of $M - \Sigma S_n$, let $T(P) = Q$. Then $K = \overline{\Sigma Z_n} + \sum_n \sum_i t_{ni}$ satisfies (1), T satisfies (2), and the acyclic curve $\overline{\Sigma Z_n}$ satisfies (3).

We note here the fact that the arcs $[t_{ni}]$ may be drawn in such a way that the interior of each of the simple closed curves $[t_{ni}+S_{ni}]$ (i. e., the maximal cyclic curves of K) is free of points of K; and as so constructed K would be the boundary of a plane domain, namely, its own unbounded complementary domain.

We note also that (α) together with the result of Zarankiewicz mentioned in the predeeding footnote, or indeed, Theorem 2 alone yields the interesting fact that *the universal acyclic continuous curve W of Ważewski is a universal space for the set $G+E$ of all cut points g and endpoints e of any compact continuous curve M, $C(W)$ being a universal space for sets G and $E(W)$ for sets E.*

It has been pointed out by Zarankiewicz (Bull. Acad. Polon., 1926, p. 362) that the set G of all the cut points of a continuous curve M is not necessarily a subset of an acyclic curve K lying in M. The following theorem giving conditions under which this is the case shows that the question, like so many questions concerning continuous curves (see *Structure* § 6) reduces to the same question about the subsets of G lying in the maximal cyclic curves of M.

Theorem 3. *In order that the continuous curve M should contain an acyclic continuous curve K containing the set G of all the cut points and the set E of all the end points of M it is necessary and sufficient that for each maximal cyclic curve C of M, $G.C$ is a subset of some acyclic continuous curve in M.*

Proof. The condition is obviously necessary. It is also sufficient. For it follows by hypothesis and by *Structure*, Theorem 30, that for each maximal cyclic curve C_i $(i=1,2,3,\ldots)$ of $M, G.C_i$ is a subset of an acyclic continuous curve K_i lying wholly in C_i. Then if $K=G+E+\Sigma K_i$, it readily follows from the results in *Structure* that K is an acyclic continuous curve.

Sur les images de Baire des ensembles linéaires.

Par

W. Sierpiński (Varsovie).

E étant un ensemble linéaire donné, on définit par l'induction les fonctions de classe $\leqslant\alpha$ sur E. Les fonctions de classe 0 sur E sont des fonctions définies sur E et continues sur E, et les fonctions de classe $\leqslant\alpha$ sur E sont des fonctions qui sont sur E limites des fonctions de classes $<\alpha$ sur E.

E étant un ensemble linéaire donné et f une fonction de classe $\leqslant\alpha$ sur E, nous appellerons l'ensemble $f(E)$ (de valeurs de $f(x)$ pour $x\in E$) *image de classe* $\leqslant\alpha$ de l'ensemble E (obtenue à l'aide de la fonction f). Nous désignerons par $\Gamma_\alpha(E)$ la famille de tous les ensembles linéaires qui sont des images de classe $\leqslant\alpha$ de l'ensemble E.

Le but de cette Note est de démontrer le suivant

Théorème[1]: *Pour tout ensemble linéaire analytique E on a l'égalité*

$$\Gamma_\alpha(E)=\Gamma_1(E),$$

quel que soit le nombre ordinal positif $\alpha<\Omega$.

Démonstration.

Lemme: *Si E est un ensemble au plus dénombrable, toute fonction définie sur E est de classe $\leqslant 1$ sur E.*

Toute fonction définie sur un ensemble fini étant évidemment continue sur cet ensemble, il suffira de traîter le cas, où l'ensemble E est dénombrable.

[1]) J'ai signalé ce théorème, pour $\alpha=2$, dans ma conférence *Sur les images continues des ensembles linéaires*, faite au I *Congrès des Mathématiciens des Pays Slaves* à Varsovie, le 24 septembre 1929.