On a property of linear fractional sets of points.

By

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§ 1. This paper is devoted to the question of the density of s-dimensional sets of points, $0 < s < 1$. Hausdorff[1]) has defined the measure of his fractional sets as follows.

Let p be a positive integer, E a set of points in p-dimensional space, $u(\varrho, E)$ a finite or enumerable sequence of sets of points, $u_1, u_2, u_3, \ldots$, whose sum Σu_i contains the set E, the diameter of u_i being less than a positive number ϱ, for each i, $i = 1, 2, 3, \ldots$ Denoting the diameter of u_i by l_i, the s-measure $s - mE$, of the set E has been defined as the lower limit as ϱ tends to zero of $\sum_{u(\varrho, E)} l_i^s$, the summation extending over all members u_i of $u(\varrho, E)$.

We consider only s-sets of finite positive s-measure and as there will be no ambiguity we write mE instead of $s - mE$.

Now let e be any point of an s-set E in a straight line, $(e - z, e + z)$ the segment of the line whose centre is the point e and whose length is $2z$, $D^*(e, E)$ and $D_*(e, E)$ the upper and lower two-sided[2]) densities respectively of the set E at the point e. Then[3])

$$D^*(e, E) = \overline{\lim_{z \to 0}} \frac{s - m[E \times (e - z,\ e + z)]}{(2z)^s}$$

$$D_*(e, E) = \underline{\lim_{z \to 0}} \frac{s - m[E \times (e - z,\ e + z)]}{(2z)^s}.$$

In his paper Besicovitch[4]) has proved the following theorem. *At almost all points of an s-set,* $0 < s < 1$, *the upper density is*

[1]) Math. Ann. 79 (1918) pp. 157 et seq.

[2]) Since we are only concerned in this paper with *two-sided density* we shall in the sequel, refer to this property as *density*.

[3]) Besicovitch. Math. Ann. 101 (1929) p. 163.

[4]) loc. cit. pp. 175 et seq.

included between the limits $\dfrac{2}{2^s(1+h)^s}$ *and* 1, *where* h *is defined by the equation*

$$(2 + h)^s = 2.$$

If s_0 be the value of s for which

$$(1 + h)^s = \tfrac{3}{2} \qquad (s_0 = \tfrac{2}{3} \text{ approx.}),$$

the lower bound given in the theorem is exact for $s \leqslant s_0$. This result we prove in the second part of the paper by considering one of the s-sets defined by Besicovitch.

Denoting this set by E we show that the upper density at almost all points e of E is $\dfrac{2}{2^s(1+h)^s}$, $0 < s \leqslant s_0$, and for $s > s_0$, is greater than $\dfrac{2}{2^s(1+h)^s}$. The latter half of this statement is true for all s-sets i. e. for $s > s_0$ the upper density at almost all points of any s-set exceeds $\dfrac{2}{2^s(1+h)^s}$. This follows from the theorem which we prove in the first part of the paper, where we find new limits for the lower bound of the upper density in the open interval $(s_0, 1)$.

§ 2. We require the following lemma[1]).

Let δ *be a positive number,* $V(\delta)$ *a finite or enumerable set of segments lying in a given straight line, the length* l *of each segment being less than* δ. *Then corresponding to any positive number* ε_1, *and to any s-set E, contained in the given line we can find* $\delta = \delta(E, \varepsilon_1)$ *such that the inequality*

$$\sum_{V(\delta)} l^s > m[E \times V(\delta)] - \varepsilon_1$$

is satisfied for any $V(\delta)$.

§ 3. We define the functions $\varphi_1(s)$, $\psi_1(s)$ by the equations

$$\varphi_1 = \frac{2 \cdot 20^{\frac{1}{s}} (1 + h)^2}{[38(1 + h)^s + 3]^{\frac{1}{s}}}$$

$$\{\psi_1\}^s = \{1 - \tfrac{1}{2}\varphi_1\}^{-s} - 1.$$

[1]) Besicovitch. loc. cit. p. 164.

Now let s_1, s_2, and s_3 be the values of s, $1 > s > s_0$ which satisfy (1), (2) and (3) respectively.

$$\tag{1} \tfrac{1}{2}\varphi_1[1-\psi_1(1-\tfrac{1}{2}\varphi_1)] = 2^{-\frac{1}{s}}$$

$$\tag{2} [2^{\frac{2}{s}}h + 2^{\frac{1}{s}}(h+1)+1]^s = \tfrac{3}{2}$$

$$\tag{3} 2[10(2-h^s)(1+h)^s - 19]\,[2^{\frac{2}{s}}h + 2^{\frac{1}{s}}(h+1)+1]^s = 7(1+h)^s.$$

We have $s_1 = 0{\cdot}72$, $s_2 = 0{\cdot}76$, $s_3 = 0{\cdot}82$ approx. Now we define $f(s)$ as follows,

$$f(s) = \begin{cases} \dfrac{2}{2^s(1+h)^s}, & 0 < s < s_0 \\[2mm] \dfrac{38(1+h)^s+3}{20\cdot 2^s(1+h)^{2s}}, & s_0 \leqslant s < s_1 \\[2mm] \dfrac{76+3(1+h)^s}{20\cdot 2^s(1+h)^s}, & s_1 \leqslant s < s_2 \\[2mm] \dfrac{38[2^{\frac{2}{s}}h + 2^{\frac{1}{s}}(h+1)+1]^s + 7[1+h]^s}{20\cdot 2^s(1+h)^s[2^{\frac{2}{s}}h + 2^{\frac{1}{s}}(h+1)+1]^s}, & s_2 \leqslant s < s_3 \\[2mm] \dfrac{2-h^s}{2^s}, & s_3 \leqslant s < 1 \end{cases}$$

Theorem I.

At almost all points of an s-set, $0 < s < 1$, the upper density is greater than or equal to $f(s)$.

It is only necessary to prove the result for $s_0 \leqslant s < 1$.

We assume the theorem false; then there exists a positive number d and an s-set E, of positive measure for which the upper density at all points is less than $\dfrac{f(s)}{(1+d)^s}$.

We choose $\varepsilon_1, \varepsilon_2$ and ε_6 to satisfy the following conditions.

$$\tag{4} \varepsilon_2^2 = 2\varepsilon_1, \quad 28\varepsilon_2 + \varepsilon_1 < d,$$

$$\tag{5a} [1-(1-\varepsilon_6)^s]mE = 9\varepsilon_2 + 5\varepsilon_1, \quad s_0 \leqslant s < s_3,$$

$$\tag{5b} [1-(1-hd)^s]mE - \varepsilon_2 - 5\varepsilon_1 > \varepsilon_2[2(2-h^s)^{-\frac{1}{s}}-1]^{-s}, \quad s_3 \leqslant s < 1,$$

$$\tag{6a} \frac{(1-\frac{1}{2}\varphi)^s\{2\psi^s-(1-\frac{1}{2}\varphi\psi)^s\}}{2^s[1-2^{-\frac{1}{s}}+\varepsilon_6 2^{-\frac{1}{s}} - \frac{1}{2}\varphi\psi(1-\frac{1}{2}\varphi)]^s} > \frac{f(s)}{(1+d)^s} + 10\varepsilon_2, \qquad s_0 \leqslant s < s_1$$

$$\tag{6b} \frac{2\psi^s-(1-\frac{1}{2}\varphi\psi)^s}{2^s[1+\varepsilon_6 2^{-\frac{1}{s}}(1-\frac{1}{2}\varphi)^{-1}]^s} > \frac{f(s)}{(1+d)^s} + 10\varepsilon_2, \quad s_1 \leqslant s < s_2,$$

$$\tag{6c} \begin{gathered} \frac{3-3(1-\frac{1}{2}\varphi)^s-2(1-\frac{1}{2}\varphi)^s(1-\frac{1}{2}\varphi\psi)^s-(1-\frac{1}{2}\varphi)^s(1-\frac{1}{2}\varphi\psi)^s(1-\frac{1}{2}\varphi\theta)^s}{2^s[\frac{1}{2}\varphi-2^{-\frac{1}{s}}+\varepsilon_6 2^{-\frac{1}{s}}+(1-\frac{1}{2}\varphi)\{\frac{1}{2}\varphi\psi+(1-\frac{1}{2}\theta\varphi)(1-\frac{1}{2}\varphi\psi)\}]^s} > \\ > \frac{f(s)}{(1+d)^s} + 18\varepsilon_2, \qquad s_2 \leqslant s < s_3, \end{gathered}$$

$$\tag{6d} \frac{2(1-\varepsilon_2)-(1+\varepsilon_2)h^s(1+2d)^s}{(1-hd)^s} > \frac{2-h^s}{(1+d)^s}, \quad s_3 \leqslant s < 1,$$

where $\varphi(s)$, $\psi(s)$ and $\theta(s)$ are defined by the relations:

$$\begin{aligned} \{\varphi(s)\}^s &= \{f(s)\}^{-1} \\ \{\psi(s)\}^s &= \{1-\tfrac{1}{2}\varphi(s)\}^{-s} - 1 \\ \{\theta(s)\}^s &= \{\psi(s)\}^s\{1-\tfrac{1}{2}\varphi(s)\psi(s)\}^{-s} - 1 \end{aligned}$$

Since

$$\begin{aligned} &(1-\tfrac{1}{2}\varphi)^s[2\psi^s-(1-\tfrac{1}{2}\varphi\psi)^s] \geqslant \\ &2^s f(s)[1-2^{-\frac{1}{s}}-\tfrac{1}{2}\varphi\{1-(1-\tfrac{1}{2}\varphi)^s\}^{\frac{1}{s}}]^s, && s_0 \leqslant s < s_1, \\ &2\psi^s-(1-\tfrac{1}{2}\varphi\psi)^s > 2^s f(s) && s_1 \leqslant s < s_2 \end{aligned}$$

and

$$\begin{gathered} 3-3(1-\tfrac{1}{2}\varphi)^s-2(1-\tfrac{1}{2}\varphi)^s(1-\tfrac{1}{2}\varphi\psi)^s-(1-\tfrac{1}{2}\varphi)^s(1-\tfrac{1}{2}\varphi\psi)^s(1-\tfrac{1}{2}\theta\varphi)^s \\ > 2^s f(s)[\tfrac{1}{2}\varphi-2^{-\frac{1}{s}}+(1-\tfrac{1}{2}\varphi)\{\tfrac{1}{2}\varphi\psi+(1-\tfrac{1}{2}\theta\varphi)(1-\tfrac{1}{2}\varphi\psi)\}]^s, \\ s_2 \leqslant s < s_3, \end{gathered}$$

it follows that ε_2 and ε_6 can be chosen to satisfy (6a), (6b) and (6c).

§ 4. Since the upper density at all points of E is less than $\dfrac{f(s)}{(1+d)^s}$ we can find a positive number γ_1 such that the mean density of E on any segment whose centre is a point of E and whose length is less than γ_1, is less than or equal to

$$\frac{f(s)}{(1+d-\varepsilon_1)^s}.$$

Further let $\gamma_2 > 0$, and e_1 any point of E at which the mean density of E is less than $\dfrac{f(s)}{(1+d)^s}$ on any segment whose centre is e_1 and whose length is less than γ_2.

Let E_1 be the set of points e_1, then we can choose γ_2 so small that

(7) $$mE_1 > mE - \varepsilon_1.$$

Write $\gamma = \min(\gamma_1, \gamma_2)$ and let δ be a positive number less than $\frac{1}{4}\gamma$ such that the inequality of the lemma is satisfied for any $V(\delta)$.

Then by the definition of the set E and by Vitali's argument, we can find a set of non-overlapping segments $U(\delta, E)$, each of length less than δ, covering almost all points of E and such that

(8) $$mE - \varepsilon_1 < \sum_{U(\delta,E)} l^s < mE + \varepsilon_1$$

where l denotes the length of a typical segment of $U(\delta, E)$.

Let U_2 be the set of segments of $U(\delta, E)$ on which the mean density of E is less than or equal to $1-\varepsilon_2$, and write

$$U = U_1 + U_2.$$

Then from (6)

(9) $$\sum_{U_1} l^s + \sum_{U_2} l^s < mE + \varepsilon_1$$

and from the definition of U_2

(10) $$m[E \times U_2] \leqslant (1-\varepsilon_2) \sum_{U_2} l^s.$$

Writing $U_1 = V(\delta)$ in the lemma, we have

(11) $$\sum_{U_1} l^s > m[E \times U_1] - \varepsilon_1.$$

From (9), (10) and (11) it follows that

$$\sum_{U_2} l^s < \varepsilon_2$$

and from the lemma

(12) $$m[E \times U_2] < \varepsilon_1 + \varepsilon_2.$$

§ 5. We first consider the case $s_0 \leqslant s < s_2$.

Denote any segment of U_1 by ν_1 (and also its length) and its end points by a and b. From the ends of ν_1 cut off two segments ac and db of length $l_1 = (1-\frac{1}{2}\varphi)\nu_1$ (fig. 1). Next from both ends of ac and db cut off segments ae, fc and dg, hb respectively, each of length $l_2 = (1-\frac{1}{2}\varphi\psi)l_1$. Finally from the ends of ae and hb cut off segments ak, je and hm, nb respectively, each of length $l_3 = (1-\frac{1}{2}\theta\varphi)l_2$.

We suppose that all the segments ν_1 of U_1 have been operated on in this way. If there are, among the segments enumerated above, segments on which the mean density of E is greater than or equal to $1+\varepsilon_2$, we denote the largest of them by μ; in the event of there being two segments of the same length with this property we take any one of them.

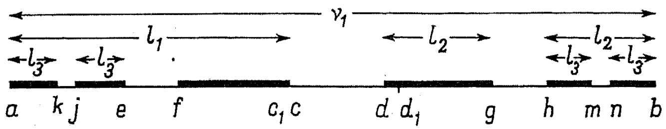

Fig. 1.

Let M denote the set of segments μ, for all ν_1 of U_1. Then

$$m[E \times M] \geqslant (1+\varepsilon_2) \sum_M \mu^s$$

and writing $M = V(\delta)$ in the lemma we have

$$m[E \times M] > (1+\varepsilon_2)\, m[E \times M] - 2\varepsilon_1$$

and hence

(13) $$\sum_M \mu^s < \varepsilon_2$$

Now write

$$U_1 = U_1' + U_1''$$

where U_1'' is the set of segments of U_1 to which belong members of M. Since $8\mu^s \geqslant \nu_1^s$, it follows from (13) that

$$\sum_{U_1''} \nu_1^s < 8\varepsilon_2.$$

Then putting $U_1'' = V(\delta)$ in the lemma we have

(13) $$m[E \times U_1''] < 8\varepsilon_2 + \varepsilon_1.$$

§ 6. We now denote by ν_1 any segment of U_1' and by e' any point of $E \times \nu_1$, and let $r\nu_1$ be the distance of e' from the centre of ν_1. Then the segment α, centre e' and length $(1+2r)\nu_1$ completely covers ν_1 and hence

$$\frac{m[E\times\alpha]}{(1+2r)^s\nu_1^s}>\frac{1-\varepsilon_2}{(1+2r)^s}.$$

Since $(1+2r)\nu_1<\gamma$ we have

$$\frac{1-\varepsilon_2}{(1+2r)^s}<\frac{f(s)}{(1+d-\varepsilon_1)^s}$$

and from (4)

$$2r>\varphi-1.$$

Now the length of the segment cd (fig. (1)), whose centre is the midpoint of ν_1 is by construction $(\varphi-1)\nu_1$, and hence cd contains no points of E. Further since

$$\frac{m[E\times\nu_1]}{\nu_1^s}>1-\varepsilon_2$$

and

$$\frac{m[E\times ac]}{(ac)^s}<1+\varepsilon_2$$

it follows that

$$\frac{m[E\times db]}{(db)^s}>\frac{(1-\varepsilon_2)\,\nu_1^s}{(db)^s}-\frac{(1+\varepsilon_2)\,(ac)^s}{(db)^s}$$
$$=\psi^s-\varepsilon_3,$$

where

$$\psi^s=(1-\tfrac{1}{2}\varphi)^{-s}-1,\quad\text{and}\quad \varepsilon_3=\varepsilon_2(\psi^s+2)<3\varepsilon_2.$$

Similarly

$$\frac{m[E\times ac]}{(ac)^s}>\psi^s-\varepsilon_3.$$

We now repeat the argument on the segments ac and db and prove that in each of these there exists a segment of length $(\varphi\psi-1)(1-\frac{1}{2}\varphi)\nu_1$, whose centre coincides with the midpoint of ac and of db respectively, which contains no points of E. In the construction of § 5 these are denoted by ef and gh (fig. (1)) respectively. Then since the mean density of E on ac is greater than $\psi^s-\varepsilon_3$, and that on fc is less than $1+\varepsilon_2$, it follows as above, that

$$\frac{m[E\times ae]}{(ae)^s}>\theta^s-\varepsilon_4,$$

where

$$\theta^s=\psi^s(1-\tfrac{1}{2}\varphi\psi)^{-s}-1,\quad\text{and}\quad \varepsilon_4=\varepsilon_3(1-\tfrac{1}{2}\varphi\psi)^{-s}+\varepsilon_2<7\varepsilon_2.$$

Similarly it may be shown that the mean density of E on the segments fc, dg and hb exceeds $\theta^s-\varepsilon_4$.

Finally we apply a similar argument to the segments ae and hb and prove that in each of these there exists a segment, denoted in our construction by kj and mn respectively, of length $(\theta\varphi-1)(1-\frac{1}{2}\varphi\psi)(1-\frac{1}{2}\varphi)\nu_1$, which contains no points of E.

Then since the mean density of E on ae exceeds $\theta^s-\varepsilon_4$, and that on ak is less than $1+\varepsilon_2$ it follows that

$$\frac{m[E\times je]}{(je)^s}>\theta^s(1-\tfrac{1}{2}\theta\varphi)^{-s}-1-\varepsilon_5,$$

where $\varepsilon_5=\varepsilon_4(1-\frac{1}{2}\theta\varphi)^{-s}+\varepsilon_2<15\varepsilon_2$.

In the same way we may show that the mean density of E on hm exceeds $\theta^s(1-\frac{1}{2}\theta\varphi)^{-s}-1-\varepsilon_5$.

§ 7. We now operate on each ν_1 of U_1' as follows; from the ends a, b of ν_1 we cut off two segments ac_1, bd_1 (fig. 1) of length $(1-\varepsilon_6)\nu_1/2^{\frac{1}{s}}$, denoting the set of such segments by N, and any member of the set by η. Let $Z=\{\xi\}$ be the set of strips of $\{\nu_1\}$ which remain.

Then

$$\sum_N\eta^s=(1-\varepsilon_6)^s\sum_{U_1'}\nu_1^s$$
$$<(1-\varepsilon_6)^s mE+\varepsilon_1,\qquad\text{from (8).}$$

Writing $N=V(\delta)$ in the lemma we have

$$m[E\times N]<\sum_N\eta^s+\varepsilon_1 \tag{15}$$
$$<(1-\varepsilon_6)^s mE+2\varepsilon_1.$$

Now $m[E\times Z]\geqslant mE-m[E\times U_1'']-m[E\times U_2]-m[E\times N]$.

Hence from (14), (12) and (15) we have

$$m[E\times Z]>[1-(1-\varepsilon_6)^s]mE-4\varepsilon_1-9\varepsilon_2$$

and from (5a)

$$m[E\times Z]>\varepsilon_1.$$

From (7) it follows that there are points of E_1 which belong to Z.

§ 8. Let ξ be any segment of Z which contains points of E_1, ν_1 the segment of U_1^{+} to which ξ belongs and consider the operations defined in §§ 5, 6 as applied to this ν_1. From § 7 it follows that one at least of the segments cc_1, dd_1 (fig. 1) contains points of E_1. Let us suppose that dd_1 has this property, and let e_1 be a point of E_1 belonging to dd_1. We write

$$r_1 = d_1 f, \quad r_2 = db, \quad r_3 = d_1 j.$$

Then

$$r_1 = [1 - 2^{-\frac{1}{s}} + \varepsilon_6 2^{-\frac{1}{s}} - \tfrac{1}{2}\varphi\psi(1 - \tfrac{1}{2}\varphi)]\nu_1,$$
$$r_2 = [1 - \tfrac{1}{2}\varphi]\nu_1,$$
$$r_3 = [\tfrac{1}{2}\varphi - 2^{-\frac{1}{s}} + \varepsilon_6 2^{-\frac{1}{s}} + (1 - \tfrac{1}{2}\varphi)\{\tfrac{1}{2}\varphi\psi + (1 - \tfrac{1}{2}\varphi\psi)(1 - \tfrac{1}{2}\theta\varphi)\}]\nu_1$$

First suppose that s lies in the interval (s_0, s_1) and consider the segment α_1 whose midpoint is e_1 and whose length is $2r_1$. Since the mean density of E on fc exceeds $\theta^s - \varepsilon_4$ and that on bd exceeds $\psi^s - \varepsilon_3$, then provided α_1 covers the segment fb, we have

$$\frac{m[E \times \alpha_1]}{(2r_1)^s} > \frac{(\psi^s - \varepsilon_3)(bd)^s + (\theta^s - \varepsilon_4)(fc)^s}{(2r_1)^s}$$
$$= \frac{(1 - \frac{1}{2}\varphi)^s[2\psi^s - (1 - \frac{1}{2}\varphi\psi)^s]\nu_1^s - \varepsilon_2 F_1(s)\nu_1^s}{(2r_1)^s}$$
$$> \frac{(1 - \frac{1}{2}\varphi)^s[2\psi^s - (1 - \frac{1}{2}\varphi\psi)^s]}{2^s[1 - 2^{-\frac{1}{s}} + \varepsilon_6 2^{-\frac{1}{s}} - \frac{1}{2}\varphi\psi(1 - \frac{1}{2}\varphi)]^s} - 10\varepsilon_2,$$

where

$$F_1(s) = 2 + 2(1 - \tfrac{1}{2}\varphi)^s + (1 - \tfrac{1}{2}\varphi)^s(1 - \tfrac{1}{2}\varphi\psi)^s < 5.$$

From (6a) it follows that

$$\frac{m[E \times \alpha_1]}{(2r_1)^s} > \frac{f(s)}{(1+d)^s}.$$

Now in order that $fb \subset \alpha_1$ it is sufficient that $d_1 f > db$,

i. e. $$\tfrac{1}{2}\{1 - \psi(1 - \tfrac{1}{2}\varphi)\} \geqslant 2^{-\frac{1}{s}}.$$

By the definition of s_1 it follows that this is satisfied for values of s in the interval, $s_0 \leqslant s < s_1$.

Thus, since $2r_1 < \gamma$ we have a contradiction and the theorem is proved for this interval of s.

Secondly suppose $s_1 \leqslant s < s_2$ and consider the segment α_2 whose length is $2(r_2 + \varepsilon_6 2^{-\frac{1}{s}}\nu_1)$ and whose midpoint is e_1. Since the segment $fb \subset \alpha_2$ we have

$$\frac{m[E \times \alpha_2]}{2^s(r_2 + \varepsilon_6 2^{-\frac{1}{s}}\nu_1)^s} > \frac{2\psi^s - (1 - \frac{1}{2}\varphi\psi)^s}{2^s[1 + \varepsilon_6(1 - \frac{1}{2}\varphi)^{-1}2^{-\frac{1}{s}}]^s} - 10\varepsilon_2$$

and from (6b) the mean density of E on α_2 exceeds $\dfrac{f(s)}{(1+d)^s}$. Again since the length of α_2 is less than γ we have obtained a contradiction for this case also.

Thirdly, we take a value of s in the interval (s_2, s_3). In order to obtain the required contradiction in this case we consider the segment α_3, whose midpoint is e_1 and whose length is $2r_3$. For the range of s under consideration it is easy to see that the segment jb has no points exterior to α_3.

Then since

$$m[E \times bd] > (\psi^s - \varepsilon_3)(1 - \tfrac{1}{2}\varphi)^s\nu_1^s$$
$$m[E \times cf] > (\theta^s - \varepsilon_4)(1 - \tfrac{1}{2}\varphi)^s(1 - \tfrac{1}{2}\varphi\psi)^s\nu_1^s$$

and

$$m[E \times je] > \{\theta^s - (1 - \tfrac{1}{2}\theta\varphi)^s(1 + \varepsilon_5)\}(1 - \tfrac{1}{2}\varphi)^s(1 - \tfrac{1}{2}\varphi\psi)^s\nu_1^s$$

we have:

$$\frac{m[E \times \alpha_3]}{(2r_3)^s}$$
$$> \frac{3 - 3(1-\frac{1}{2}\varphi)^s - 2(1-\frac{1}{2}\varphi)^s(1-\frac{1}{2}\varphi\psi)^s - (1-\frac{1}{2}\varphi)^s(1-\frac{1}{2}\varphi\psi)^s(1-\frac{1}{2}\theta\varphi)^s - \varepsilon_2 . F_2(s)}{2^s[\frac{1}{2}\varphi - 2^{-\frac{1}{s}} + \varepsilon_6 2^{-\frac{1}{s}} + (1 - \frac{1}{2}\varphi)\{\frac{1}{2}\varphi\psi + (1 - \frac{1}{2}\theta\varphi)(1 - \frac{1}{2}\psi\varphi)\}]^s}$$

where

$$F_2(s) = 3 + 3(1-\tfrac{1}{2}\varphi)^s + 2(1-\tfrac{1}{2}\varphi)^s(1-\tfrac{1}{2}\varphi\psi)^s + (1-\tfrac{1}{2}\varphi)^s(1-\tfrac{1}{2}\varphi\psi)^s(1-\tfrac{1}{2}\theta\varphi]^s.$$

Since $F_2(s) < 9$ and the denominator of the expression on the right exceeds $\frac{1}{2}$ it follows from (6c) that

$$\frac{m[E \times \alpha_3]}{(2r_3)^s} > \frac{f(s)}{(1+d)^s}.$$

Then since $2r_3 < \gamma$, we have a contradiction and the theorem is proved for $s_0 \leqslant s < s_3$.

§ 9. *Note on the determination of the function $f(s)$.*

In order to obtain the best possible value for $f(s)$, which the above method will yield it is necessary to find a function $f(s)$ which satisfies the following equations, where

$$\psi^s = \{1 - \tfrac{1}{2}\varphi\}^{-s} - 1, \quad \text{and} \quad \theta^s = \psi^s\{1 - \tfrac{1}{2}\varphi\psi\}^{-s} - 1.$$

$$\varphi(1 - \tfrac{1}{2}\varphi)\{2\psi^s - (1 - \tfrac{1}{2}\varphi\psi)^s\}^{\frac{1}{s}} = 2\{1 - 2^{-\frac{1}{s}} - \tfrac{1}{2}\varphi\psi(1 - \tfrac{1}{2}\varphi)\}, \qquad s_0 \leqslant s < s_1 \tag{16}$$

$$\varphi\{2\psi^s - (1 - \tfrac{1}{2}\varphi\psi)^s\}^{\frac{1}{s}} = 2, \qquad s_1 \leqslant s < s_2 \tag{17}$$

$$\begin{aligned}\varphi\{3 - 3(1-\tfrac{1}{2}\varphi)^s - 2(1-\tfrac{1}{2}\varphi)^s(1-\tfrac{1}{2}\varphi\psi)^s - (1-\tfrac{1}{2}\varphi)^s(1-\tfrac{1}{2}\varphi\psi)^s(1-\tfrac{1}{2}\theta\varphi)^s\}^{\frac{1}{s}} \\ = 2[\tfrac{1}{2}\varphi - 2^{-\frac{1}{s}} + (1 - \tfrac{1}{2}\varphi)\{\tfrac{1}{2}\varphi\psi + (1 - \tfrac{1}{2}\theta\varphi)(1 - \tfrac{1}{2}\varphi\psi)\}], \\ s_2 \leqslant s < s_3.\end{aligned} \tag{18}$$

Then the value of the lower limit of the upper density would be given by $\{\varphi(s)\}^{-s}$. The function $\varphi(s)$ which has been used in the theorem is such that the left hand side of equations (16), (17) and (18) exceeds the sight hand side. In fact any function $\varphi(s)$ which is less than $2(h+1)/2^{\frac{1}{s}}$ in the intervals of s stated above, and which provides definite inequality in this way will give a function $f(s)$ which exceeds $\dfrac{2}{2^s(1+h)^s}$ for $s > s_0$.

The method adopted in the interval $(s_3, 1)$ for finding the lower limit of the upper density of s-set is essentially different to that employed in the interval (s_0, s_3) and the value of $f(s)$ given by the method adopted in § 10, is the best possible function of which the method is capable.

§ 10. Finally we consider values of s lying in the interval $s_3 \leqslant s < 1$. We assume the results already proved in §§ 2, 3, 4.

The set Λ is now constructed in the following way. From each end of each segment ν_1 of U_1 we cut off a segment λ of length $(1 - hd)\nu_1 2^{\frac{1}{s}}$ and write $\Lambda = \{\lambda\}$. Then in the same way as in § 4 we prove that

$$m[E \times (U_1 - \Lambda)] > [1 - (1 - hd)^s]mE - \varepsilon_2 - 3\varepsilon_1$$

and from (5b)

$$> 2\varepsilon_1 + \varepsilon_2[2(2 - h^s)^{-\frac{1}{s}} - 1]^{-s}. \tag{19}$$

Then from (7) it follows that there are points of E_1 belonging to $U_1 - \Lambda$. Let e_1 be such a point, ν_1 the segment of U_1 to which it belongs, $r\nu_1$ the distance of e_1 from the centre of ν_1.

Then

$$0 \leqslant r \leqslant \tfrac{1}{2}h(1 + 2d)2^{-\frac{1}{s}}.$$

From the end of ν_1 remote from e_1 cut off a segment ω of length $2r\nu_1$. We suppose the operation performed on each ν_1 of U_1 which possesses the property that the corresponding member of $U_1 - \Lambda$ contains points of E_1, and write $\Omega = \{\omega\}$.

We divide Ω into two sets Ω_1 and Ω_2. The latter is the set of segments of Ω on which the mean density of E is greater than or equal to $1 + \varepsilon_2$. Then

$$m[E \times \Omega_2] \geqslant (1 + \varepsilon_2)\sum_{\Omega_2}(2r\nu_1)^s$$

Writing $\Omega_2 = V(\delta)$ in the lemma we have

$$\sum_{\Omega_2}(2r\nu_1)^s > m[E \times \Omega_2] - \varepsilon_1$$

and, since $\varepsilon_2^2 = 2\varepsilon_1$

$$m[E \times \Omega_2] < \varepsilon_2$$

and a fortiori

$$\sum_{\Omega_2}(2r\nu_1)^s < \varepsilon_2. \tag{20}$$

Then either (a) the value of r for each member of Ω_2 exceeds $\dfrac{1}{(2 - h^s)^{\frac{1}{s}}} - \dfrac{1}{2}$ or (b) there exists at least one r for which $r \leqslant \dfrac{1}{(2 - h^s)^{\frac{1}{s}}} - \dfrac{1}{2}$.

If (a) be satisfied then

$$\sum_{\Omega_2}(2r\nu_1)^s > \{2(2 - h^s)^{-\frac{1}{s}} - 1\}^s \sum_{U_1'}\nu_1^s,$$

where U_1' is the set of segments of U_1, to which Ω_2 belong.

Then from (20)

$$\sum_{U_1'} \nu_1^s < \{2(2-h^s)^{-\frac{1}{s}}-1\}^{-s}\varepsilon_2$$

and from the lemma

$$m[E\times U_1'] < \varepsilon_1 + \{2(2-h^s)^{-\frac{1}{s}}-1\}^{-s}\varepsilon_2. \tag{21}$$

In case (b) we consider the segment β of length $(1+2r)\nu_1$ whose centre is the point e_1. Since ν_1 is interior to this segment, and the mean density of E on ν_1 exceeds $1-\varepsilon_2$ we have from (4)

$$\frac{m[E\times\beta]}{(1+2r)^s\nu_1^s} > \frac{(1-\varepsilon_2)(2-h^s)}{2^s}$$
$$> \frac{2-h^s}{2^s(1+d)^s}.$$

Hence since $(1+2r)\nu_1<\gamma$ we obtain a contradiction, and so if (b) be satisfied the theorem is proved.

We return to the consideration of the other possibility.

Let Λ' be the set of segments of Λ which belong to U_1-U_1':

Then from (19) and (21) we have

$$\begin{aligned} m[E\times\{(U_1-U_1')-\Lambda'\}] &= m[E\times(U_1-\Lambda)]-m[E\times\{U_1'-(\Lambda-\Lambda')\}] \\ &\geqslant m[E\times(U_1-\Lambda)]-m[E\times U_1'] \\ &> \varepsilon_1. \end{aligned}$$

and from (7) it follows that

$$(U_1-U_1')-\Lambda'$$

contains points of E_1.

Let e_1 be the point of $E_1\times\{(U_1-U_1')-\Lambda'\}$ used in the definition of Ω, ν_1 the segment of U_1-U_1' to which it belongs, $r\nu_1$ the distance of e_1 from the centre of ν_1.

If $0\leqslant r\leqslant(2-h^s)^{-\frac{1}{s}}-\frac{1}{2}$, then as before we take a segment with centre e_1 and of length $(1+2r)\nu_1$ and show that the mean density of E on this segment exceeds $\dfrac{2-h^s}{2^s(1+d)^s}$. Suppose then that

$$(2-h^s)^{-\frac{1}{s}}-\tfrac{1}{2}<r\leqslant\tfrac{1}{2}h(1+2d)2^{-\frac{1}{s}}.$$

Let ω_1 be the segment of length $2r\nu_1$ which belongs to Ω_1, so that the mean density of E on ω_1 is less than $1+\varepsilon_2$.

We take β_1 to be the segment whose midpoint is e_1 and whose length is $(1-2r)\nu_1$. Then since the mean density of E on ν_1 exceeds $1-\varepsilon_2$, we have

$$\frac{m[E\times\beta_1]}{(1-2r)^s\nu_1^s} > \frac{1-\varepsilon_2-(2r)^s(1+\varepsilon_2)}{(1-2r)^s}.$$

Now the expression on the right decreases as r increases and takes its least value when $r=\frac{1}{2}h(1+2d)2^{-\frac{1}{s}}$.

Then from (6d) we have

$$\frac{m[E\times\beta_1]}{(1-2r)^s\nu_1^s} > \frac{2(1-\varepsilon_2)-h^s(1+\varepsilon_2)(1+2d)^s}{2^s(1-hd)^s}$$
$$> \frac{2-h^s}{2^s(1+d)^s}.$$

Since $(1-2r)\nu_1<\gamma$ it follows from the definition of E_1 that this is impossible. Thus our assumption that the upper density at all points of E is less than $\dfrac{f(s)}{(1+d)^s}$, is false and the theorem is proved.

§ 11. We now show that the lower limit of the upper density of s-sets given in theorem I is exact for values of s, $0<s\leqslant s_0$, where s_0 has bem defined by the equation

$$(1+h)^s=\tfrac{3}{2}.$$

We require the following lemma.

Let q, r and j be positive integers satisfying $q\geqslant 2$, $0\leqslant j\leqslant q-2$, and $q\geqslant r\geqslant j+2$. Then the function

$$F^q(\alpha_{j+1},\alpha_{j+2},\ldots\alpha_{r-1})=\left[h2^{\frac{q}{s}}+2^{\frac{q-j}{s}}+(h+1)\sum_{i=j+1}^{r-1}\alpha_i 2^{\frac{q-i}{s}}+2^{\frac{q-r}{s}}\right]^s$$
$$-\left[2^{q-j}+\sum_{i=j+1}^{r-1}\alpha_i 2^{q-i}+2^{q-r}\right]$$

defined in the region $0\leqslant\alpha_i\leqslant 1$, $i=(j+1),\ldots,(r-1)$, is positive for all values of s, $0<s<1$, and for every positive integer $q\geqslant 2$.

It is easily seen that

$$\frac{\partial F^q}{\partial \alpha_i} < 2^{q-i}\{2^s(1+h)^s 2^{-\frac{1}{s}} - 1\}$$
$$< 0, \quad 0 < s < 1, \quad i = (j+1), \ldots, (r-1).$$

and hence

$$F^q(\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_{r-1})$$

takes its least value for $\alpha_i = 1$.

Denoting this value of F^q by $F^q_{r,j}$ we have

$$F^q_{r,j} = 2^q[\{h + 2^{-\frac{j}{s}+1} - 2^{-\frac{r}{s}}(2^{\frac{1}{s}} - 1)\}^s - \{2^{-j+1} - 2^{-r}\}].$$

Now $F^q_{r,j_2} < F^q_{r,j_1}$, for $j_1 > j_2$ and

$$F^q_{r_1,j} < F^q_{r_2,j}, \quad \text{for} \quad r_1 > r_2.$$

Hence if $j+2 \leqslant r \leqslant q$, $0 \leqslant j \leqslant q-2$, we have

$$F^q_{r,j} \geqslant F^q_{q,0}.$$

Again since $F^{q_1}_{q_1,0} \geqslant F^{q_2}_{q_2,0}$, for $q_1 > q_2$, and $F^2_{2,0} > 0$, $0 < s < 1$, it follows that $F^q_{q,0} > 0$ for every integer $q \geqslant 2$.

This proves the lemma.

In the applications we are only concerned with the particular cases in which $\alpha_i = 0$ or 1.

§ 12. Theorem II.

The lower limit of the upper density of s-sets is exactly $\frac{2}{2^s(1+h)^s}$ *for* $0 < s \leqslant s_0$.

The previous theorem states that for $0 < s \leqslant s_0$ the lower limit of the upper density of s-sets is greater than or equal to $\frac{2}{2^s(1+h)^s}$. We show that this lower limit is attained and that there exist s-sets for which the upper density at almost all points is $\frac{2}{2^s(1+h)^s}$, $0 < s \leqslant s_0$.

Let P_0 be a segment of unit length. From each of the ends of P_0 we cut off a segment of length $l_1 = 2^{-\frac{1}{s}}$ and denote these two segments by P_1. From the ends of each segment of P_1 we cut off segments of length $l_2 = 2^{-\frac{1}{s}} l_1$ and denote the set of the 2^2 segments so obtained by P_2. In general from the ends of each segment of P_{n-1} we cut off segments of length $l_n = 2^{-\frac{1}{s}} l_{n-1}$ and denote the set of 2^n segments so obtained by P_n, and any member of P_n by p_n. Then the set

$$E = P_0 \times P_1 \times P_2 \times \ldots$$

is an s-set which has the property that, for $0 < s \leqslant s_0$ the upper density at almost all points is $\frac{2}{2^s(1+h)^s}$, and for $1 > s > s_0$ is greater than $\frac{2}{2^s(1+h)^s}$.

It is easy to see that the set E is closed.

Again since the set P_n of all segments p_n contains E, for every n and $\Sigma l_n^s = 1$ it follows that

$$(22) \qquad mE \leqslant 1.$$

Now let $U(\varrho, E)$ be a set of segments, each of length less than ϱ and such that E is completely interior to the set. To each point e of E there corresponds a sequence of segments $\{p_n\}$ to which e belongs and which is interior to $U(\varrho, E)$ i. e. interior to at least one member of $U(\varrho, E)$. Let $p(e)$ be the first segment of the sequence. Then we can select a finite number of segments $\{p(e)\}$ such that E is contained in their sum and the sum of the segments is interior to $U(\varrho, E)$. If n be the greatest suffix occurring in $\{p(e)\}$ then the set P_n possesses the same properties, and the segments of P_n form a non-overlapping set.

Let $U(E)$ be any segment of $U(\varrho, E)$, of length $\delta < \varrho$, and K_n the set of segments p_n which are interior to $U(E)$.

If K_n consists of more than one segment p_n, then there exists an integer $k \leqslant n$ such that the segments p_n of K_n belong to different segments p_k but to the same segment p_{k-1}. Let t be the number of segments p_n belonging to K_n so that $0 < t \leqslant 2^{q+1}$, where we have written $q = n - k$. There are three cases to consider. Firstly t is a power of 2, secondly t is the sum of two consecutive powers of 2 and finally t is neither a power of 2 nor the sum of two consecutive powers.

Case (I)

$$t = 2^i, \quad i = 0, 1, \ldots, \quad q+1.$$

We have

$$\delta \geqslant l_{n-\iota}$$

and

$$\delta^s \geqslant 2^\iota l_n^s$$

$$(23) \qquad = \sum_{K_n} l^s,$$

the summation being taken over all the segments p_n which are completely interior to $U(E)$.

Case (II)

$$t = 2^\iota + 2^{\iota-1}, \quad \iota = 1, 2, \ldots, q$$

We have

$$\delta \geqslant h\, l_{n-q} + l_{n-\iota} + (h+1)\, l_{n-\iota+1}$$

$$\delta^s \geqslant 2^{\iota-1} \{h\, 2^{\frac{q-\iota+1}{s}} + 2^{\frac{1}{s}+1} - 1\}^s l_n^s$$

$$(24) \qquad \geqslant \sum_{K_n} l^s,$$

since

$$(2^{\frac{2}{s}} - 1)^{\frac{1}{s}} > 3, \quad 0 < s < 1.$$

Case (III)

$$t = \alpha_0 2^q + \alpha_1 2^{q-1} + \ldots + \alpha_\iota 2^{q-\iota} + \ldots + \alpha_q$$

where

$$\alpha_\iota = 0 \text{ or } 1, \quad \iota = 0, 1, 2, \ldots, q.$$

Let α_j be the first α_ι, α_r the last α_ι of the sequence

$$\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_q$$

which does not vanish for the particular t under consideration. We have

$$\delta \geqslant h\, l_{n-q} + l_{n-q+j} + (h+1) \sum_{i=j+1}^{r-1} \alpha_i\, l_{n-q+i} + l_{n-q+r}$$

$$= \left[h 2^{\frac{q}{s}} + 2^{\frac{q-j}{s}} + (h+1) \sum_{i=j+1}^{r-1} \alpha_i 2^{\frac{q-i}{s}} + 2^{\frac{q-r}{s}}\right] l_n.$$

From the lemma of § 11 it follows that, since $r \geqslant j+2$

$$\delta^s \geqslant \left\{2^{q-j} + 2^{q-r} + \sum_{i=j+1}^{r-1} \alpha_i\, 2^{q-i}\right\} l_n^s$$

$$(25) \qquad = \sum_{K_n} l^s.$$

From (23), (24) and (25) it follows that for all values of t

$$\delta^s \geqslant \sum_{K_n} l^s$$

and hence

$$(26) \qquad \sum_{U(\varrho, E)} \delta^s \geqslant \sum_{P_n} l_n^s = 1.$$

Combining (22) and (26) we have

$$m E = 1.$$

In a similar way it may be shown that

$$m[E \times p_n] = l_n^s.$$

§ 13. We now prove that, at almost all points of the set E the upper density is greater than or equal to $\dfrac{2}{2^s(1+h)^s}$, $0 < s \leqslant s_0$ where s_0 has been defined by the equation

$$(1+h)^s = \tfrac{3}{2}.$$

Consider any set P_n of segments. In each p_{n-1} of P_{n-1} there are two segments p_n which are symmetrical with respect to the centre of p_{n-1}. From those ends of each p_n which are nearest to the centre of the segment p_{n-1}, in which they lie, cut off segments of length l_{2n}, denoting the set of segments cut off by R_n. Then

$$m[E - (E \times R_n)] = \left(1 - \frac{1}{2^n}\right).$$

Again, from each of the segments $p_{2n} \subset P_{2n} - R_n$, cut off segments of length l_{3n}, the strips cut off being taken from those ends of p_{2n} which are nearest to the centre of the corresponding p_{2n-1}. This new set of segments which have been cut off we denote by R_{2n}. We have

$$m[E - E \times (R_n + R_{2n})] = \left(1 - \frac{1}{2^n}\right)^2.$$

In general we define the set of segments R_{in} as follows. From those ends of the segments $p_{in} \subset P_{in} - R_n - R_{2n} - \ldots - R_{(i-1)n}$, which are nearest to the centre of the corresponding p_{in-1} we cut off segments of length $l_{(i+1)n}$. The set of these segments we denote by R_{in}.

Clearly the sets of segments R_{in}, R_{jn} are non-overlapping for $i \neq j$. We write

$$ {}_iE_n = E \times R_{in}, \quad i = 1, 2, 3, \ldots, $$

$$ E_n^j = \sum_{i=1}^{j} {}_iE_n. $$

We have

$$ m[{}_iE_n] = \frac{1}{2^n}\left(1 - \frac{1}{2^n}\right)^{i-1} $$

$$ m[E_n^j] = \sum_{i=1}^{j} \frac{1}{2^n}\left(1 - \frac{1}{2^n}\right)^{i-1} $$

and hence

$$ m[E - E_n^j] = \left(1 - \frac{1}{2^n}\right)^j. $$

Let η be an arbitrary positive number, then we can find $j_n = j(n, \eta)$ such that

$$ m[E - E_n^{j_n}] < \eta. $$

Now suppose that the operation defined above has been carried out for $n = 1, 2, 3, \ldots$ Then writing $\eta = \frac{\alpha}{2^n}$ and

$$ B_n = E - E_n^{j_n} $$

we have

$$ m[B_1 + B_2 + B_3 + \ldots] \leqslant m B_1 + m B_2 + m B_3 + \ldots < \alpha. $$

and if we put

$$ E_1 = E - B_1 - B_2 - B_3 - \ldots $$

then

$$ m E_1 > 1 - \alpha, $$

Thus corresponding to each point e of E_1, there exists a sequence

$$ q_1 < q_2 < q_3 < \ldots < q_i < \ldots $$

tending to ∞ with i, with the property that e belongs to R_{tq_i}, where t is a positive integer.

Let p_{tq} be the segment of P_{tq} to which e belongs and let x be the distance from e to the end, remote from e, of the other member of P_{tq} which belongs to the same p_{tq-1} as p_{tq}. Then

$$ x = (h+1)\, l_{tq} + \theta\, l_{(t+1)q}; \quad 0 \leqslant \theta \leqslant 1. $$

and

$$ \frac{m[E \times (e - x,\ e + x)]}{(2x)^s} = \frac{2}{2^s\{h + 1 + \theta 2^{-\frac{q}{s}}\}^s}. $$

Since $q \to \infty$ it follows that, at almost all points e of E

(27) $$ D^*(e, E) \geqslant \frac{2}{2^s(1+h)^s}. $$

§ 14. We now show that for $0 < s \leqslant s_0$ the upper density, at points of E cannot exceed $\frac{2}{2^s(1+h)^s}$.

Consider any segment p_{k-1} and denote the two segments p_k which belong to it by p_k', p_k'' and let A be the end point of p_k' which is nearer to p_k''. Further we take X (fig. 2) to denote the position of a point e^* of $E \times p_k'$, and write $c(e^*, r)$ for a segment whose centre is the point e^* and whose length is $2r$.

Then corresponding to every point e^* and to every value r_1 of r lying in the interval

$$ (h+1)(l_{k+2} + l_{k+1}) \leqslant r \leqslant (h+1)(l_{k+1} + l_k) $$

there exists a value r_2 of r lying in the same interval and possessing the property that for $0 < s < 1$

$$ \frac{m[E \times c(A, r_2)]}{(2r_2)^s} \geqslant \frac{m[E \times c(e^*, r_1)]}{(2r_1)^s}. $$

Let s'' be the value of s defined by the equation

$$ h = (1-h)^s, \quad (0{\cdot}69 < s'' < 0{\cdot}7). $$

We suppose that e^* belongs to the same segment p_{k+1} $(= AB)$ to which A belongs — the other case in which e^* belongs to CD may be treated in a similar way. For the purpose of the theorem it is sufficient to prove the result for $0 < s \leqslant s''$ since $s_0 < s''$ The result is however true $1 > s > s''$; the proof is very tedious and we do not give it.

We write

$$AX = x\, l_k, \quad r_1 = t\, l_k.$$

Case (I)

$$(h+1)(l_{k+1}+l_{k+2}) \leqslant r_1 \leqslant l_k.$$

We have

$$\frac{m[E \times c(e^*, r_1)]}{(2r_1)^s} \leqslant \frac{1+(t-h-x)^s}{(2t)^s}$$
$$= f(t) \qquad \text{(say)}.$$

where we interpret the expression $(t-h-x)^s$ as being 0 if $t \leqslant h+x$.

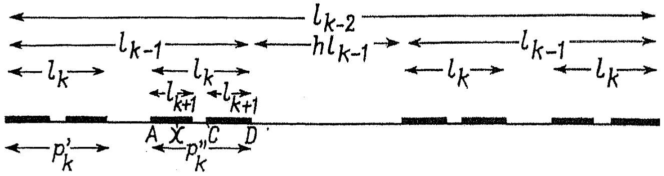

Fig. 2.

Since

$$(h+1)(2^{-\frac{1}{s}}+2^{-\frac{2}{s}}) \leqslant t \leqslant 1$$

we have

$$f(t) \leqslant \frac{2\{1+(1-h)^s\}}{2^s(h+1)^s\{1+2^{-\frac{1}{s}}\}^s} \qquad (h \leqslant 1)$$

and

$$f(t) \leqslant \frac{2}{2^s(1+h)^s\{1+2^{-\frac{1}{s}}\}^s}, \qquad (h \geqslant 1)$$
$$< \frac{2}{2^s(1+h)^s}, \qquad (h \geqslant 1).$$

Now let s' be the value of s for which

$$1+(1-h)^s = \{1+2^{-\frac{1}{s}}\}^s, \qquad (0{\cdot}64 < s' < 0{\cdot}65).$$

Then for $0 < s \leqslant s'$

$$\frac{m[E \times c(e^*, r_1)]}{(2r_1)^s} \leqslant \frac{2}{2^s(1+h)^s}.$$

Now for $s' < s \leqslant s''$, $f(t)$ is an increasing function of t.

Hence

$$\frac{m[E \times c(e^*, r_1)]}{(2r_1)^s} \leqslant \frac{1+(1-h)^s}{2^s} \leqslant F(s)$$

where

$$F(s) = \max\left\{\frac{2}{2^s(1+h)^s}, \quad \frac{3}{2^s(1+h)^{2s}}\right\}.$$

We take

$$r_2 = \begin{cases} (h+1)\, l_k & 0 < s \leqslant s_0 \\ h\, l_k + l_{k+1}, & s_0 < s \leqslant s'' \end{cases}$$

and it follows that

$$\frac{m[E \times c(A, r_2)]}{(2r_2)^s} = \max.\left\{\frac{2}{2^s(1+h)^s}, \quad \frac{3}{2^s(1+h)^{2s}}\right\}, \qquad 0 < s \leqslant s''.$$

Case (II)

$$l_k < r_1 \leqslant \min.\begin{Bmatrix} (h+1)(l_k+l_{k+1}), \\ h\, l_{k-1} + l_k - x\, l_k \end{Bmatrix}.$$

We take $r_1 = r_2$ and we have the required inequality.

Case (III).

If $h\, l_{k-1} + l_k - x\, l_k < (h+1)(l_k + l_{k+1})$ we consider next

$$h\, l_{k-1} + l_k - x\, l_k \leqslant r_1 \leqslant \min.\begin{Bmatrix} (h+1)(l_k+l_{k+1}), \\ 2(h+1)\, l_{k+1} + h\, l_{k+1} \end{Bmatrix}.$$

We have

$$\frac{m[E \times c(e^*, r_1)]}{(2r_1)^s} \leqslant \frac{5/2}{2^s\{h\, 2^{\frac{1}{s}} + 1 - 2^{-\frac{1}{s}}\}^s}$$

and we write

$$r_2 = (h+1)\, l_k.$$

Since for $0 < s \leqslant s''$ we have

$$(h+1)(l_k + l_{k+1}) < 2(h+1)\, l_{k+1} + h\, l_{k-1},$$

our statement is proved.

§ 15. It remains to consider, for what value of r

$$(h+1)(l_{k+1}+l_{k+2}) \leqslant r \leqslant (h+1)(l_k+l_{k+1})$$

the expression

$$\frac{m[E \times c(A, r)]}{(2r)^s}$$

has its maximum value.

It is easily seen that for $0 < s \leqslant s_0$ we must take $r = (h+1)\, l_k$, and for $s_0 \leqslant s \leqslant s''$, $r = h\, l_k + l_{k+1}$.

We then have the following result.

At almost all points e of E

$$D^*(e, E) = \begin{cases} \dfrac{2}{2^s(1+h)^s}, & 0 < s \leqslant s_0 \\ \dfrac{3}{2^s(1+h)^{2s}}, & s_0 \leqslant s \leqslant s''. \end{cases}$$

§ 16. We now give without proof, the values of the upper density, at almost all points, of the set E, for values of s greater than s_0.

The following lemma is required:

Let $\alpha \geqslant 0$, $\gamma > \beta \geqslant 0$, $\delta_2 > \delta_1 > 0$, be numbers depending on a parameter s, $0 < \bar{s} \leqslant s \leqslant \bar{s}' < 1$, and define

$$g(t, s) = \frac{\alpha + (t-\beta)^s + (t-\gamma)^s}{(2t)^s}, \quad \delta_1 \leqslant t \leqslant \delta_2,$$

where $\alpha, \beta, \gamma, \delta_1, \delta_2$ are to be such that $g(t, s)$ is less than or equal to unity for all values of s and t, and the expression $(t-\beta)^s$ and $(t-\gamma)^s$ are to be interpreted as meaning zero if $t-\beta \leqslant 0$, or $t-\gamma \leqslant 0$. We then have the following results:

a) $\beta \geqslant 0$, $\delta_1 \geqslant \gamma$, *$g(t, s)$ is an increasing function of t in the closed interval* (δ_1, δ_2);

b) $\beta > 0$, $\delta_1 \geqslant \beta$, $\delta_2 \leqslant \gamma$, *$g(t, s)$ is an increasing function of t in the closed interval* (δ_1, δ_2), *for all values of s in the open interval* $(0, \bar{s}')$ *provided*

$$\delta_2 - \beta \leqslant \delta_1 / 2^{\frac{s}{1-s}}, \quad 0 < s \leqslant \bar{s}';$$

c) $\beta = 0$, $\delta_2 \leqslant \gamma$, *$g(t, s)$ is a decreasing function of t in* (δ_1, δ_2), *for all values of s in* $(0, 1)$;

d) $\beta > 0$, $\delta_2 \leqslant \beta$, *$g(t, s)$ has the same properties as in* (c).

Now let s_{2i-1}, s_{2i}, $i = 1, 2, 3, \ldots$, be the values of s in the interval $(s_0, 1)$ which are defined by equations (28) and (29) respectively:

$$2^{\frac{i}{s}} = (1+h)(2^{\frac{i}{s}} - 1) \tag{28}$$

$$(1+h)^s (2^{\frac{i+1}{s}} - 1)^s = 2^i (2^{i+2} - 1)/(2^{i+1} - 1). \tag{29}$$

Then at almost all points e of E we have

$$D^*(e, E) = \frac{2^{i+2} - 1}{2^s (1+h)^s (2^{\frac{i+1}{s}} - 1)^s}, \quad s_{2i} \leqslant s \leqslant s_{2i+1},$$

$i = 0, 1, 2, 3, \ldots$, and

$$D^*(e, E) = \frac{2^{i+1} - 1}{2^{i+s}}, \quad s_{2i-1} \leqslant s \leqslant s_{2i},$$

$i = 1, 2, 3, 4, \ldots$