$$G(\overline{\overline{\beta}}) - G(\overline{\overline{\alpha}}) = \int_{\overline{\overline{\alpha}}}^{\overline{\overline{\beta}}} G'(x)\,dx = \sum \int_{\overline{\overline{\alpha}}_i}^{\overline{\overline{\beta}}_i} G'(x)\,dx + \int_{\overline{\overline{P}}} G'(x)\,dx =$$

$$= \sum \{G(\overline{\overline{\beta}}_i) - G(\overline{\overline{\alpha}}_i)\} + \int_{\overline{\overline{P}}} G'(x)\,dx,$$

c'est à dire

$$F(\overline{\overline{\beta}}) - F(\overline{\overline{\alpha}}) = \sum \{F(\overline{\overline{\beta}}_i) - F(\overline{\overline{\alpha}}_i)\} + \int_{\overline{\overline{P}}} f(x)\,dx.$$

C. q. f. d.

Moscou, Février 1931.

On the imbedding of subsets of a metric space in Jordan continua[1]).

By

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In the present paper I propose to consider the following problems:

Let R be a metric space, and let M be an arbitrary compact subset of R. Do there exist necessary and sufficient conditions under which M is a subset of a Jordan continuum[2]) *of R?*

In case M is a subset of a Jordan continuum J of R, what is the minimum dimension (Menger-Urysohn) for which such curves J exist?

These and related problems have already received some attention by various authors. Moore and Kline[3]) gave conditions under which a bounded closed set in the euclidean plane is a subset of a simple arc, and E. W. Miller[4]) extended the same conditions to any euclidean space E_n. Gehman found[5]) that in the euclidean plane every bounded continuum K is imbedded in a Jordan continuum obtained by adding to K a denumerable infinity of arcs. Whyburn and Ayres showed[6]) recently that if K is a bounded

[1]) Presented to the American Mathematical Society, Mar. 29, 1929.

[2]) By a *Jordan continuum* is meant a compact, connected im kleinen continuum; i. e., a one-way continuous mapping of the straight line interval $0 \leqq x \leqq 1$.

[3]) R. L. Moore and J. R. Kline, *On the most general plane closed point-set through which it is possible to pass a simple continuous arc*, Annals of Math., (2), v. 20, pp. 218—223.

[4]) Cf. Bull. Amer. Math. Soc., v. 35, p. 152, abstract n° 6.

[5]) H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of. Math., v. 27, pp. 29—46, Th. II.

[6]) G. T. Whyburn and W. L. Ayres, *On continuous curves in n dimensions*. Bull. Amer. Math. Soc., v. 34, pp. 349—360, Th. 1.

subcontinuum of a continuous curve M in E_n, then K is contained in a Jordan continuum obtained by adding to K a denumerable infinity of arcs of M, and the present author found [7]) that if K is a closed and bounded subset of a component, Q, of an open subset of a continuous curve in E_n, then K is a subset of a Jordan continuum J such that J is itself a subset of Q. Then Stepanoff and Tumarkin have obtained the following result [8]): If M is a closed and compact subset of positive dimension of a *streckenweise connected* metric space R, then M is a subset of a Jordan continuum in R, of the same dimension as M, and obtained from M by the addition of a denumerable infinity of arcs [9]). It is noteworthy that in all of these results, except that of the present authors quoted above, the extension of the given set to a Jordan continuum has been accomplished by the addition of a denumerable infinity of arcs.

In connection with the general result of Stepanoff and Tumarkin, it is to be noted that „streckenweise connected" implies considerably more than „arcwise connected". A space R is arcwise connected if every two points of it are joined by an arc; i. e., by a set which is homeomorphic with a straight line interval. A space R is *streckenweise* connected if every two points of it are joined by an arc that is *isometric* [10]) with a straight line interval; in other words, the condition imposes a uniformity which is not implied in the former condition indeed, it imposes uniform arcwise connectedness im kleinen on the space R.

This uniformity is unnecessary, and the condition may be replaced, as shown below, by the condition „connected and arcwise connected im kleinen". However, even this condition is too strong to be necessary, as the following example shows: In the cartesian plane, let the space R consist of the x-axis, which we shall denote

[7]) R. L. Wilder, *On connected and regular point sets*, Bull. Amer. Math. Soc., v. 34, pp. 649—655, Th. 5.

[8]) W. Stepanoff and L. Tumarkin, *Über eine Erweiterung abgeschlossenen Mengen zu Jordanschen Kontinuen derselben Dimension*, Fund. Math., v. 12, pp. 43—46, Th. I.

[9]) It should be remarked here that in quoting these results I have not tried to include all of the properties stated by the authors, as, for instance, the fact that the arcs added have diameters converging to zero, etc.

[10]) Two metric spaces are called *isometric* if there exists a (1—1) continuous correspondence between them which preserves distances. Cf. F. Hausdorff, *Mengenlehre*, 2 Aufl., 1927, p. 94.

by X, together with the set of all points both of whose coordinates are rational. Then every compact subset of R of dimension 1 which lies on X is a subset of a Jordan continuum of dimension 1, although R itself is neither connected, nor arcwise connected im kleinen at any point.

It is to be noted in this example, however, that any compact set F which lies on X is arcwise connected *through* X as well as arcwise connected im kleinen *through* X. For the sake of clarity in this connection and in what follows, the following definitions are introduced:

Definition. If M is a subset of a metric space R, then M is *arcwise connected through* R if every two points of M are joined by an arc of R.

Definition. If M is a subset of a metric space R, then M is *arcwise connected im kleinen through* R provided that if P is a point of M and ε is any positive number, there exists a positive number δ_ε such that if Q is a point of M whose distance from P is less than δ_ε, then P and Q are joined by an arc of R every point of which is at a distance less than ε from P.

We may now proceed to develop the results of the paper.

Theorem 1. *Let M be a compact and closed subset of a metric space R. Then in order that M should be a subset of a Jordan continuum J in R, it is necessary and sufficient that M should be arcwise connected through R and arcwise connected im kleinen through R. Moreover, if these conditions are satisfied, J may by so chosen that the complement of M (relative to J) is a denumerable set of mutually exclusive open arcs, $t_1, t_2, t_3, \dots$, such that the diameter of t_n converges to zero as n increases, and such that if M is of positive dimension h, the dimension of J is likewise h, and if the dimension of M is zero, then the dimension of J is one.*

Proof. That the conditions stated are necessary is quite obvious, since every Jordan continuum itself possesses these properties.

The conditions are sufficient. Consider the number $1/n$, where n is a positive integer. If p is a point of M, there exists a number δ_{np} such that if q is a point of M such that

$$\varrho(p, q) < \delta_{np},$$

then p and q are joined by an arc of R every point of which lies at a distance from p less than $1/n$. Let

(1) $$\epsilon_{np} = \tfrac{1}{2}\delta_{np},$$

and let s_{np} denote the set of all points of R which are at a distance less than ϵ_{np} from p; i. e., the „sphere" of R of „radius" ϵ_{np} and center p.

Since M is compact and closed, there exists, by the Borel theorem, a finite set of spheres,

(2) $$s_{np_1}, s_{np_2}, \ldots, s_{np_k},$$

of the set of all spheres $\{s_{np}\}$ covering M, where for each i, p_i is the center of s_{np_i}. Let ϵ_n denote the minimum radius of the spheres (2), and let p and q now denote two points of M such that

$$\varrho(p, q) < \epsilon_n.$$

There exists a sphere s_{np_m} of (2) which contains p. It is clear, from the triangular law of distance, and from (1), that

$$\varrho(q, p_m) < \delta_{np_m}.$$

Accordingly there exist arcs from p to p_m and from q to p_m, and consequently an arc pq from p to q, such that every point of these arcs is at a distance from p_m less than $1/n$.

To summarize, we have shown that for every $1/n$, where n is a positive integer, there exists a positive number ϵ_n such that if p and q are two points of M whose distance from one another is less than ϵ_n, then there exists in R an arc from p to q whose diameter is less than $1/n$.

By a theorem of Alexandroff[11]) the set M can be regarded as a continuous mapping of the Cantor terniary set, C, on the interval (0, 1) of the linear continuum, by means of a continuous function $f(x)$, (x in C). Denote the complementary intervals of C on the linear continuum by (a_i, b_i), ($i = 1, 2, 3, \ldots$). We shall proceed to define a new function $F(x)$, defined over the interval (0, 1), which is continuous, and such that for every x, $F(x)$ is a point of R, and for x in C, $F(x) = f(x)$.

[11]) P. Alexandroff, *Über stetige Abbildungen kompakter Raume*, Math. Ann., v. 96, pp. 555—571. See pp. 563 and 567. Also see F. Hausdorff, *loc. cit.*, p. 197, Th. V.

To every pair of points a_i, b_i, there corresponds in M the pair of points $\alpha_i = f(a_i)$ and $\beta_i = f(b_i)$, which may or may not be identical. If α_i and β_i are identical, we may let $F(x)$, for $a_i \leqq x \leqq b_i$, denote the point $\alpha_i \equiv \beta_i$. In the case where $\alpha_i \neq \beta_i$, we shall proceed as follows: Since $f(x)$ is continuous on a compact closed set, it is uniformly continuous, and there exist at most a finite number of point pairs (a_i, b_i) such that

$$\varrho(\alpha_i, \beta_i) \geqq \epsilon_1.$$

Since M is arcwise connected through R, there exists, for every such pair of points α_i, β_i, an arc s_i of R whose end-points are α_i and β_i. Since s_i is homeomorphic with (a_i, b_i), there exists a continuous function $\varphi_i(x)$ such that

$$\varphi_i(a_i) = f(a_i) = \alpha_i,$$
$$\varphi_i(b_i) = f(b_i) = \beta_i,$$
$$\varphi_i(x) = \xi, \ (a_i < x < b_i, \ \xi \subset s_i).$$

Then over the interval (a_i, b_i) we shall define $F(x) = \varphi_i(x)$.

For any positive integer n, there exist at most a finite number of point pairs, (α_i, β_i) such that

$$\epsilon_n > \varrho(\alpha_i, \beta_i) \geqq \epsilon_{n+1}.$$

For every such point pair there exists, as shown above, an arc s_i whose end-points are α_i and β_i, and whose diameter is less than $1/n$. The function $F(x)$ may be defined over the intervals (a_i, b_i) in a manner similar to that indicated above, so that $F(x)$ maps (a_i, b_i) into the arc s_i.

For any x in C, the function $F(x)$ is defined so that $F(x) = f(x)$.

That the function $F(x)$ is continuous over the interval (0, 1) is easily shown, and the set of points of R defined by $F(x)$ is a Jordan continuum J.

That the diameters of the arcs $\{s_n\}$ converge to zero is obvious. However, these arcs may have points other than their end-points in common with M. To obviate this feature, we may proceed, as follows:

If s_1 contains points which are not also points of M, then the set of all such points is the sum of a denumerable set of mutually exclusive open arcs,

$$w_1^1, w_2^1, w_3^1, \ldots,$$

which have no point in common with F, and such that the diameter of w_n^1 converges to zero as n increases.

In general, if s_n contains points which are not also points of

$$F_{n-1} = M + \sum_{i=1}^{n-1} s_i,$$

then the set of all such points is the sum of a denumerable set of open arcs

$$w_1^n, w_2^n, w_3^n, \ldots,$$

which are mutually exclusive, have no point in common with F_{n-1}, and such that the diameter of w_i^n converges to zero as i increases.

By the ordinary diagonal procedure, the arcs $\{w_i^n\}$ $(i = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots)$ may be arranged in a sequence of open arcs

$$t_1, t_2, t_3, \ldots,$$

which satisfy the conditions stated in the theorem.

That J has the same dimension as M, in case M is of positive dimension, and dimension one, in case the dimension of M is zero, is an immediate consequence of a theorem of Urysohn and Menger[12]).

Corollary 1. *Let M be a compact and closed subset of a connected and arcwise connected im kleinen metric space R. Then M is a subset of a Jordan continuum, J, of R, satisfying the conditions stated in the latter part of Theorem 1.*

Corollary 2. *If M is a compact and closed subset of a component, Q, of an open subset of a continuous curve*[13]), *then M is a subset of a Jordan continuum J which is itself a subset of Q, and such that J satisfies the conditions stated in the latter part of Theorem 1.*

The proofs of these corollaries should be quite evident. The second corollary is of course a strengthening, so far as the character of J is concerned, of the theorem of the authors referred to in the introductory paragraphs.

[12]) P. Urysohn, *Sur les multiplicités Cantoriennes*, Fund. Math., v. 8, p. 337; K. Menger, *Über die Dimensionalität von Punktmengen*, Monatsh. f. Math. u. Phys., v. 34, p. 147.

[13]) A continuous curve may be non-compact. We shall define it as a locally compact, connected im kleinen continuum that lies in a metric space.

In order to state a theorem analogous to Theorem 1 for the case of sets not necessarily closed, we make the following definition:

Definition: If M is a point set, then the set of all limit points of M that do not belong to M, together with the set of all limit points of the latter set, will be called the *outer shell* of M. In symbols, if $H = \overline{M} - M$[14]), the outer shell of M is $\overline{H}$.

Theorem 2. *In order that a compact set M in a metric space R should be a subset of a Jordan continuum J in R, it is necessary and sufficient that $\overline{M}$ should be arcwise connected through R and arcwise connected im kleinen through R. Moreover, if these conditions are satisfied, J may be so chosen that*

$$J = M + (\overline{M} - M) + \sum_{n=1}^{\infty} t_n,$$

where M, $(\overline{M} - M)$ and $\sum_{n=1}^{\infty} t_n$ are mutually exclusive, and every t_n is an open arc such that no two arcs t_i and t_k $(i \neq k)$ have a point in common, and the diameter of t_n converges to zero as n increases. Moreover, if the dimension of S is s (where S is the outer shell of M), and the dimension of M is m, then the dimension of J is the greater (or common) value of the numbers s, m, where at least one of these numbers is positive, or the dimension of J is one if $s = m = 0$.

Proof. The proof is mainly the same as that of Theorem 1, except of course that in the latter proof M is replaced by $\overline{M}$.

That the last sentence of the theorem holds follows from the following considerations: Since J is compact and metric, it is separable. Also, S is closed, and hence both an F_σ and a G_δ. By a theorem of Hurewicz[15]), the set $M + S = \overline{M}$ has a dimension equal to the greater (or common value) of the numbers s and m (as defined in the theorem). The statement concerning the dimension of J follows at once.

In answer to the second of the questions proposed in the introductory paragraphs, we can now state the following theorem:

[14]) By $\overline{M}$ is meant the set consisting of M together with its limit points.

[15]) W. Hurewicz, *Normalbereiche und Dimensionstheorie*, Math. Ann., v. 96, pp. 736–764, Th. XXVII.

Theorem 3. *If a compact set M is a subset of a continuous curve N in a metric space R, then there exists in N a Jordan continuum J containing M such that the dimension of J is the greater (or common) value of the numbers s and m (defined as in Theorem 2) if either s or m is positive; and if $s = m = 0$, then the dimensions of J is one.*

Each of Theorems 1 and 2 impose, for the imbedding of the set M, the condition that all of the points of $\overline{M}$ should be arcwise connected through R and arcwise connected im kleinen through R. I shall now show that this condition can be weakened in such a way that we do not have to suppose the points of M even arcwise connected through R.

Definition. If a set M is a subset of a metric space R, then M is said to be *uniformly arcwise connected im kleinen through* R if for every $\epsilon > 0$ there exists a positive number δ such that if P and Q are points of M for which $\varrho(P, Q) < \delta$, then P and Q are joined by an arc of R whose diameter is less than ϵ.

Lemma 1. *If M is any compact subset of a metric space R, and M is uniformly arcwise connected im kleinen through R, then $\overline{M}$ is arcwise connected im kleinen through R.*

Proof. Let ϵ be a positive number, and P a point of $\overline{M}$. There exists a positive number δ such that if p and q are any two points of M whose distance apart is less than δ, then p and q are joined by an arc of R of diameter less than $\epsilon/2$. Of the two numbers δ, $\epsilon/2$, let η be the smaller (or common) value.

Let $S(P, \epsilon)$ and $S(P, \eta)$ be spheres with center P and radii ϵ and η, respectively. Let Q be any point of $\overline{M}$ distinct from P in $S(P, \eta)$. If P and Q are both points of M, they are joined in $S(P, \epsilon)$ by an arc of R. Suppose P is not a point of M. Then it is a sequential limit point of a set of points $P_1, P_2, P_3, \ldots$, of M, all of which lie in $S(P, \eta)$.

Let $S_1, S_2, S_3, \ldots$, be a series of spheres with centers at P, having only P in common, and such that 1) $S_1 = S(P, \eta)$, 2) for every n, any two points of M that lie in S_{n+1} are joined by an arc of R in S_n. It is easy to see that with this situation at hand, there exists, in $S(P, \epsilon)$, an arc of R from P_1 to P.

In a similar fashion, it can be shown that there exists, in $S(P, \epsilon)$, an arc of R from P_1, to Q, and consequently P and Q are joined in $S(P, \epsilon)$ by an arc of R, and the lemma is proved[16].

On the basis of this lemma we can first state a theorem which imposes a condition on the points of M, instead of $\overline{M}$.

Theorem 4. *In order that a compact subset M of a metric space R should be a subset of a Jordan continuum J in R, it is necessary and sufficient that M should be arcwise connected through R and uniformly arcwise connected im kleinen through R. Under these conditions, J may be so chosen as to have the properties stated in the latter part of Theorem 2.*

Proof. It is evident that the condition stated in the theorem is necessary.

The condition is sufficient. That $\overline{M}$ is arcwise connected im kleinen through R follows from the above lemma. It remains to show that $\overline{M}$ is arcwise connected through R. Let P and Q be points of $\overline{M}$. If P and Q are isolated points of $\overline{M}$, they are accordingly in M, and by hypothesis joined by an arc of R. If P is isolated and Q a point of $\overline{M} - M$, then P is a point of M and Q a limit point of M. Hence, as $\overline{M}$ is arcwise connected im kleinen through R, there is an arc of R joining some point of M, say x to Q. The remainder of the proof, to show that the set $\overline{M}$ is arcwise connected through R, should be obvious. The theorem follows as a consequence of Theorem 2.

Theorem 5. *In order that an arbitrary compact subset, M, of a metric space R should be a subset of a Jordan continuum in R, it is necessary and sufficient that there exist, in R, a sequence of points, $P_1, P_2, P_3, \ldots$, dense[17]) on M, such that the set of points $P_1, P_2, P_3, \ldots$ is both arcwise connected through R and uniformly arcwise connected im kleinen through R.*

Proof. Let K denote the set $P_1 + P_2 + P_3 + \ldots$. We can assume, without loss of generality, that K is compact, since M is compact. Then, by the above lemma, $\overline{K}$ is arcwise connected im kleinen through R, and, as shown in the proof of Theorem 4, $\overline{K}$ is arcwise connected through R. Then, since $\overline{M}$ is a subset of $\overline{K}$, $\overline{M}$

[16]) Compare this lemma with Theorem 1 of R. L. Moore, *Concerning connectedness im kleinen and a related property*, Fund. Math., v. 3, pp. 232—237.

[17]) I. e., if P is a point of M, then either P is a point of this sequence, or a limit point of the set of all points in the sequence.

has these properties, and the sufficiency of the condition stated in the theorem follows from Theorem 2.

It is clear that the condition stated in the theorem is necessary.

Applications.

We shall now consider some applications to general continuous curve [13]) theory. Thus far we have considered only the imbedding of compact sets, and we shall first obtain some theorems concerning the imbedding of non-compact sets in continuous curve spaces. We shall then be able to obtain some fundamental theorems concerning such spaces.

Theorem 6. *Let M be a continuous curve, and K a closed subset (not necessarily compact) of an open connected subset Q of M. Also, let the boundary, B, of Q be compact. Then there exists in Q a continuous curve N containing K.*

Proof. For every point P of B there is a neighborhood $R(P)$ such that $\overline{R}(P)$ is compact and contains no point of K. By the Borel theorem there is a finite number of the neighborhoods $R(P)$ covering B. Their sum is an open set, D, whose boundary, β, is closed and compact. Let $\beta \cdot Q = \beta_1$; then β_1 is also closed and compact.

By Corollary 2 there is a Jordan continuum C that lies in Q and contains β_1. Let the set of all points of Q that do not lie in $\overline{D}$ be denoted by Q_1. Then $Q_1 + C$ is a continuous curve N satisfying the theorem.

To show this, we note first that N is certainly connected im kleinen at any point of Q_1, and also at any point of C that is not a limit point of Q_1. Let x be a point of C that is a limit point of Q_1 but which is not itself a point of Q_1 (that is, x is a point of β_1). It is easy to see that x is a point of Q. Let R_x be any neighborhood of x. Then there is a neighborhood $V_x \subset R_x$ such that if y is a point of C in V_x, there is an arc from x to y that lies in $C \cdot R_x$. Also, there is a neighborhood $W_x \subset V_x$ such that if z is a point of Q in W_x, there is an arc of Q from x to z in V_x. Let z be a point of N in W_x. If z is a point of C, there is clearly an arc of N from z to x in R_x. If z is a point of Q_1 not in C, there is an arc t of Q from z to x in V_x. Let w be the first point of C on t in the order from z to x, and denote that portion of t from z to w by t_1. Clearly t_1 is an arc of N. Since w is a point of C in V_x, there is an arc of C from w to x lying in R_x; denote this arc by u. Clearly the set $t_1 + u$ is an arc of N from z to x that lies in R_x. Thus N is connected im kleinen.

That N is connected follows at once from the fact that every component of Q_1 has at least one limit point in β_1 and hence in C. It is clear that N is closed. Hence the theorem is proved.

The necessity for requiring, in the statement of Theorem 6, that the set B be compact is made apparent by the following example: In the cartesian plane, let Y denote the set of all points on the y-axis; let C denote the set of all points on the curve $y = \frac{1}{x} \sin \frac{1}{x}$ for which $0 < x \leqslant 1$, and for every positive integer n let L_{np} denote the set of all points on the line $y = p/2^n$ for which $0 < x \leqslant 1/2^n$ and p takes on the odd values $\pm 1, \pm 3, \ldots, \pm (2n-1)$. Let

$$M = Y + C + \sum_{n=1}^{\infty} \Big(\sum_p L_{np} \Big).$$

It is clear that M is a continuous curve.

Every line segment L_{np} is separated into a denumerable infinity of open intervals by its intersections with the curve C; in each one of these that has two endpoints in C, let one point not on C be selected and assigned to a set B. Also, let B include the set Y. Then $M - B = Q$ is a connected open subset of M. Let K denote the set of all points on C which are the positions of the relative maxima of C. Then K is a closed subset of Q which is not contained in any continuous curve of Q.

Although in this example we note that B is not connected, and that there are points of K arbitrarily close to B, it is not difficult to construct examples, in the plane, to show that the requirement that these conditions shall not be present is not sufficient to ensure K lying in a continuous curve of Q.

For the case where the continuous curve M lies in a euclidean space, E_n, of n dimensions, we can obtain theorems more analogous to Theorem 1, as follows:

Theorem 6a. *In E_n, let K be any closed set (not necessarily compact). Then there exists, in E_n, a continuous curve M which con-*

tains K and which is obtained by adding to K a denumerable infinity of mutually exclusive open arcs none of which contains a point of K.

Proof. Since the case where $K = E_n$ is of no interest, and the case where $n = 1$ is obvious, we let P be a point of $E_n - K$, where $n > 1$. Let $S_1, S_2, S_3, \ldots$ be a system of spherical neighborhoods of P such that 1) S_1 contains no point of K, 2) if r_k represents the radius of S_k, then $r_k < r_{k+1}$ $(k = 1, 2, 3, \ldots)$ and 3) every point of E_n is in some S_k. For every $k > 1$ let

$$F_k = K \cdot (\overline{S}_k - S_{k-1}),$$

$$R_k = \overline{S}_k - S_{k-1}.$$

Then F_k is a closed subset of R_k, and since R_k is arcwise connected and arcwise connected im kleinen, it follows from Theorem 1 that F_k is a subset of a Jordan continuum J_k in R_k, such that $J_k - F_k$ is the sum of a denumerable infinity of mutually exclusive open arcs. Furthermore, it is evident from the proof of Theorem 1 that the open arcs of $J_k - F_k$ may all be selected in the set $S_k - \overline{S}_{k-1}$. Consequently, if $i \neq j$,

$$(J_i - F_i) \cdot (J_j - F_j) = 0.$$

In order to complete the proof it is only necessary to join, for each k, the continua J_k and J_{k-1} by an arc which lies in $\overline{S}_k - S_{k-2}$. If F_k and J_k are vacuous, we join the first non-vacuous J_i such that $i > k - 1$ to J_{k-1} by an arc in $\overline{S}_i - S_{k-2}$. Overlappings are easily eliminated as in the proof of Theorem 1.

Theorem 6b. *If K is a closed subset (not necessarily compact) of a connected open subset Q of a continuous curve M in E_n, such that the boundary, B, of Q, is compact, then there exists, in Q, a continuous curve J containing K such that 1) $J - K$ is the sum of a countable infinity of mutually exclusive open arcs, and 2) if K is compact, then J is compact.*

Proof. If K is compact, then the theorem is a corollary of Theorem 1. We shall therefore suppose that K is not compact. Also, since if $M = E_n$ the proof may be handled (by virtue of the fact that B is compact) much as in the proof of the preceding theorem, we shall suppose that there is a point P not in M.

We begin by making an inversion about P, denoting the inverse of any set of points A by $\varphi(A)$. Then $\varphi(M)$ is a compact set, and $\varphi(M) + P$ is a Jordan continuum of which P is a non-cut point; we shall denote this Jordan continuum by M'.

The distance from P to $\varphi(B)$ is a positive number d. Also, $\varphi(Q) + P$ is an arcwise connected and arcwise connected im kleinen point set, and by virtue of Theorem 1, $\varphi(K) + P$ is a subset of a continuous curve E_1 that lies in $\varphi(Q) + P$, and which satisfes the conditions of Theorem 1. If $E_1 - P$ is connected, we let $\varphi^{-1}(E_1 - P) = J$.

Suppose $E_1 - P$ is not connected; then for any positive number ϵ, it is easy to show, by virtue of the fact that E_1 is a continuous curve, that there is only a finite number of components in $E_1 - P$ of diameter $> \epsilon$.

Let the component of $M' \cdot S(P, d/2^i)$ $(i = 1, 2, 3, \ldots)$ determined by P be denoted by $M'(P, d/2^i)$. Then $M'(P, d/2) - P$ contains only a finite number of components. For let M_1 and M_2 be distinct components of this set. There exist points P_1 and P_2 of M_1 and M_2, respectively, such that $\varrho(P, P_i) < d/4$ $(i = 1, 2)$. As $M' - P$ is connected, there is an arc $P_1 P_2$ joining P_1 and P_2 in $M' - P$[18]. It is clear that the arc $P_1 P_2$ cannot lie wholly in $S(P, d/2)$, since if it did it would lie in $M'(P, d/2) - P$. Let x be the first point of $P_1 P_2$, in the order from P_1 to P_2, such that $\varrho(P, x) = d/4$. The point x lies in M_1. In the same way it follows that every component of $M'(P, d/2) - P$ contains such a point x. Now if the set of all such components were infinite, the set of points $\{x\}$ (*one* from each component) would have a limit point, L, in some component of $M'(P, d/2) - P$. A contradiction follows at once by virtue of the connectedness im kleinen of $M'(P, d/2)$.

Denote those components of $M'(P, d/2) - P$ that lie in $\varphi(Q)$ by $C_1, C_2, \ldots, C_k$. Similarly, those components of $M'(P, d/4) - P$ that lie in C_i $(i = 1, 2, \ldots, k)$ denote by $C_{i1}, C_{i2}, \ldots, C_{in_i}$. Similarly, in $M'(P, d/8) - P$ let the components that belong to C_{ij} $(i = 1, 2, \ldots, k;$ $j = 1, 2, \ldots, n_i)$ be denoted by $C_{ij1}, C_{ij2}, \ldots, C_{ijn_{ij}}$. And so on.

Let P_i denote a point of C_i $(i = 1, 2, \ldots, k)$. Then the points $\{P_i\}$ are joined by a finite set of arcs in $\varphi(Q)$; denote the resulting

[18]) Cf. R. L. Moore, *Concerning continuous curves in the plane*, Math. Zeit. v. 15, pp. 254—260, Th. 1.

set by S_1. Let P_{ij} $(i=1, 2, \ldots, k;\ j=1, 2, \ldots, n_i)$ denote a point of C_{ij}; then for a fixed i, all points P_{ij} are joined in C_i by a finite set of arcs t_i, to the point P_i; let S_2 denote the set of all points in arcs of the sets t_i. Then, if P_{ijr} is a point of C_{ijr}, all points P_{ijr} for a fixed pair of values i and j can be joined in C_{ij} to P_{ij} by a finite set of arcs, and a set S_3 is obtained; and so on. The set $S=S_1+S_2+S_3+\ldots$ is a „skeleton" set upon which we can now construct the continuous curve desired.

Only a finite number of components of E_1-P fail to lie wholly in $M'(P, d/4)$. If A is such a component, then A has points in some C_i in which it can be joined by an arc to P_i. After all such joinings have been made, denote the resulting set, obtained by adding to S the set of all such components and arcs by J_1.

Only a finite number of components of E_1-P that lie in $M'(P, d/4)$ fail to lie in $M'(P, d/8)$. If A is such a component, then A has points in some C_{ij}, and can be joined to P_{ij} by an arc of this C_{ij}. Then the set consisting of all such components and joining arcs we shall denote by J_2.

We proceed in this manner ad infinitum, and let $\varphi(J)=\sum_{i=1}^{\infty} J_i$. It is easy to show that J is a continuous curve which satisfies the conditions of the Theorem.

Theorem 7. *In any locally compact metric space, in order that a continuum M should be a continuous curve, it is necessary and sufficient that if K is any closed subset of M that is compact, and ϵ any positive number, then there exists in M an open subset D containing K, such that every point of D is at a distance less than ϵ from K, and $M-D$ contains only a finite number of components.*

Proof. The condition is necessary. For each point P of K there is a positive number $d_P<\epsilon$ such that $\overline{S}(P, d_P)$ is compact. By the Borel theorem, there is a finite number of the neighborhoods $S(P, d_P)$ covering K, and the set of all points in this finite number of neighborhoods let us denote by R. By virtue of the connectedness im kleinen, there is only a finite number of components of $M-K$ that have points not in R; denote these by $C_1, C_2, \ldots, C_n$. For each i $(i=1, 2, \ldots, n)$, denote by E_i the set of all points of C_i that are not in R. By Theorem 6 there exists in C_i a continuous curve N_i that contains E_i. The set

$$D=M-\sum_{i=1}^{n} N_i$$

is an open subset of M containing K, every point of which is at a distance less than ϵ from K. Since the set $M-D$ is the sum $\sum_{i=1}^{n} N_i$, where each set N_i is a continuous curve, the theorem is proved.

The condition is sufficient. For suppose M is not a continuous curve. Then there exist[19]) two positive numbers d_1 and d_2 and a denumerable infinity of continua M_i such that 1) each of these continua is a subset of M and contains points whose distances from some point P are d_1 and d_2, respectively, and if x is any point of one of these continua, then $d_1 \geqslant (P, x) \geqslant d_2$, 2) no two of these continua have a point in common and no one of them is a proper subset of any connected subset of M which is such that all of its points x satisfy the distance relation in 1), and 3) $\overline{S}(P, d_1)$ is compact. Let K denote the set of all points y of M such that $\varrho(P, y)=d_1$ or $\varrho(P, y)=d_2$, and let $\epsilon=\frac{1}{4}(d_1-d_2)$. Then if D is an open subset of M as defined in the statement of the theorem, no finite number of continua of $M-D$ can contain all points of $(M-D)\cdot\sum_{i=1}^{\infty} M_i$.

It will be noted that the above proof also shows that *if M is any continuous curve, K is any closed subset of M that is compact, and ϵ any positive number, then there exists in M an open subset D containing K, such that every point of D is at a distance less than ϵ from K, and $M-D$ consists of a finite number of continuous curves.* In this connection it is of interest to note that it has been shown by H. M. Gehman [20]) that if M is a *plane* continuous curve and P is a non-cut point of M, then for any $\epsilon>0$ there is an M-domain D containing P whose diameter is less than ϵ and such that $M-D$

[19]) Cf. R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bull. Amer. Math. Soc., v. 29, pp. 289—302, § 3. Condition (4) as stated at the top of p. 297 is incorrect; the correct statement may be found in Fund. Math., v. 7, p. 371.

[20]) *Concerning certain types of non-cut points, with an application to continuous curves*, Proc. Nat. Acad. Sci., v. 14, pp. 431—432; cf. Ths. 4 and 5, and the concluding paragraph of the paper.

is a continuous curve [21]); also, that if P is a cut-point of M, then $M-D$ is the sum of a finite number of continuous curves. Our next theorem will generalize, and extend to general metric spaces these results of Gehman. First it will be necessary to establish a lemma.

Lemma 2. *Let $M_1, M_2, \dots, M_k$ be mutually exclusive continuous curves (any of which may reduce to single points) which are subsets of a continuous curve M, and let C be a component of $M-\sum_{i=1}^{k} M_i$. Then $M-C$ consists of a finite number, at most k, of continuous curves* [22]).

Proof. If Q is a component of the set $M-\sum_{i=1}^{k} M_i$ that is distinct from C, then the boundary points of Q form a non-vacuous set that lies in $\sum_{i=1}^{k} M_i$, and if M_j contains a boundary point of Q the component of $M-C$ that contains M_j is a continuum which contains Q. Consequently the number of components of $M-C$ is at most k.

Let K be a component of $M-C$, and let P be a point of K It is clear that if P is not in any M_i, the set K is connected im kleinen at P. Suppose P is in M_j. Let ϵ be any positive number, which we may assume, without loss of generality, to be less than all the distances between sets M_h, M_i $(h \neq i)$. Since M_j is a continuous curve, there is a positive number d_1 such that any point of M_j whose distance from P is less than d_1 is joined to P by an arc of M_j every point of which is at a distance from P less than ϵ. Also, there is a positive number d_2 such that every point of M whose distance from P is less than d_2 is joined to P by an arc of M every point of which is at a distance from P less than d_1. Of course $d_2 \leqslant d_1 \leqslant \epsilon$.

Let x be a point of K whose distance from P is less than d_2. If x is in M_j, then certainly x is joined to P by an arc of M_j, and hence of K, every point of which is at a distance from P less than ϵ. Suppose x is not a point of M_j. Then, because of the above

[21]) By an M-domain is meant an open connected subset of M.

[22]) A special case of this lemma was proved by Gehman, *Some relations between a continuous curve and its subsets*, Annals of Math., v. 28, pp. 108—111. Th. 8.

stipulations concerning ϵ, x is a point of some component, Q_1 of $M-\sum_{i=1}^{k} M_i$. There is an arc of M joining x to P every point of which is at a distance from P less than d_1; let y be the first point of this arc in M_j, in the order from x to P, and denote the portion of it from x to y by t. Then t lies, except for y, wholly in Q_1, and must also be an arc of K.

Since the distance from y to P is less than d_1, there is an arc, u, of M_j, joining y to P, every point of which is at a distance from P less than ϵ. Then $t+u$ is an arc of K from x to P every point of which is at a distance from P less than ϵ. Consequently, K is connected im kleinen at P.

As it is therefore connected im kleinen at all points, the continuum K is a continuous curve. This completes the proof of the Lemma.

Theorem 8. *Let K be a compact subcontinuum, (which may reduce to a single point) of a continuous curve M. Then if ϵ is any positive number there is an M-domain D containing K such that every point of D is at a distance less than ϵ from K, and such that $M-D$ is the sum of a finite number of continuous curves; indeed, if $M-K$ is connected, then $M-D$ is itself a continuous curve.*

Proof. Case 1. The set $M-K$ connected. Let E denote the set of points of $M-K$ whose distances from K are $\geqslant \epsilon$. Then E is a closed subset of $M-K$, and by virtue of Theorem 6 there is a continuous curve J lying in $M-K$ and containing E. Let D denote that component of $M-J$ that contains K. Then D is an M-domain, and every point of D is at a distance from K less than ϵ. By lemma 2, $M-D$ is a continuous curve.

Case 2. The set $M-K$ not connected. Since M is locally compact, and K is compact, there exists, as in the proof of Theorem 7, an open set R which contains K, every point of which is at a distance from K less than ϵ, and such that $\overline{R}$ is compact. Let E denote the set of points $M-R$. Then E is contained in a finite number of components, $C_1, C_2, \dots, C_n$, of $M-K$. By Theorem 6, each set $E \cdot C_i$ is contained in a continuous curve J_i of C_i. Let D be that component of $M-\sum_{i=1}^{n} J_i$ that contains K. Then by Lemma 2 $M-D$ consists of a finite number (at most n) of continuous curves.

It is obvious that every point of D is at a distance from K less than ϵ, since D is a subset of R.

It has been shown by W. L. Ayres[23]) that if P is a non-cut point of a continuous curve M in E_n, and ϵ is any positive number, there exists in M a continuous curve N_ϵ and a positive number δ_ϵ such that N_ϵ is of diameter less than ϵ, contains every point of M whose distance from P is less than δ_ϵ and such that $M - N_\epsilon$ is connected. We shall first prove two preliminary theorems and then generalize this result of Ayres.

Theorem 9a. *Let M be a continuous curve, and P a non-cut point of M. Then M contains a Jordan continuum N such that 1) N contains all points of M whose distances from P are less than a certain positive number d, and 2) $N - P$ is connected.*

Proof. Let r be a positive number such that the set of points $S(P, r)$ is compact. Then by a theorem of Hahn[24]) there exists a Jordan continuum $M(P, r)$ which is a subset of M, contains every point of M which is less than a certain distance $d(< r)$ from P, and is such that all of its points lie in $S(P, r)$.

If $M(P, r) - P$ is connected, our proof is complete. Suppose, however, that $M(P, r) - P$ is not connected. Let M_1 and M_2 be distinct components of $M(P, r) - P$. There exist points P_1 and P_2 in M_1 and M_2, respectively, such that $\varrho(P, P_i) < d\ (i = 1, 2)$. As $M - P$ is connected, there is an arc $P_1 P_2$ joining P_1 and P_2 in $M - P$. It is obvious that the arc $P_1 P_2$ cannot lie wholly in $S(P, d)$. Let Q_1 be the first point of $P_1 P_2$, in the order from P_1 to P_2, such that $\varrho(P, Q_1) = d$. Then Q_1 is a point of M_1. Thus, every component of $M(P, r) - P$ contains at least one point whose distance from P is exactly d. The set, K, of all such points, is compact and closed, and by Theorem 1 is a subset of a Jordan continuum, J, which lies in $M - P$. The sum, $M(P, r) + J$ is a Jordan continuum which is not disconnected by the omission of P.

Theorem 9b. *Let M be a continuous curve, and K a compact subcontinuum of M such that $M - K$ is connected. Then M contains a Jordan continuum N such that 1) N contains all points of M whose distances from K are less than a certain positive number d, and 2) $N - K$ is connected.*

Proof. Let P be a point of K. Then there exist r, $S(P, r)$, $M(P, r)$ and d as defined in the proof of Theorem 9a. The component, $M'(P, d)$, determined by P in $M \cdot S(P, d)$, is a subset of $M(P, d)$, and constitutes a *region* of M as defined in my paper *The non-existence of a certain type of regular point set*[25]. For every point P of K we obtain in a like manner a region $M'(P, d)$ (the number d is variable with P, of course). By applying the Borel theorem to the regions $M'(P, r)$ covering K, we obtain a finite set of regions covering K, and the sets $M(P, r)$ associated with these constitute a Jordan continuum $M(K, r')$ which contains K as well as every point of M whose distance from K is less than some positive number r'. As in the proof of Theorem 9b it may be shown that the set of all points of $M'(K, r')$ whose distance from K is equal to r' lies in a Jordan continuum J of $M - K$, and the Jordan continuum $N = J + M'(K, r')$ satisfies the conditions of the Theorem.

Theorem 10. *Let K be a compact subcontinuum (which may be a single point) of a continuous curve M. Then for any positive number ϵ there exist a Jordan continuum J and a positive number δ_ϵ such that J is a subset of M every point of which is at a distance less than ϵ from K, and which contains every point of M whose distance from K is less than δ_ϵ, and such that $M - J$ contains only a finite number of components; and, indeed, if $M - K$ is connected, then $M - J$ is connected.*

Proof Case 1. The set $M - K$ connected. By Theorem 9b M contains a Jordan continuum N such that 1) N contains all points of M whose distances from K are less than a certain positive number d, and 2) $N - K$ is connected. We may assume $d < \epsilon$.

By Theorem 8 there is an N-domain D containing K every point of which is at a distance from K less than $d/2$ and such that $N - D$ is a Jordan continuum. Since $N - D$ is a continuum that does not disconnect N, there exists by Theorem 8 an N-domain

[23]) *On continua that are disconnected by the omission of any point and some related problems*, Mon. f. Math. u. Phys., v. 36, pp. 135—148, Th. 2.

[24]) H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, Wien. Akad. Sitz, v. 123, Part IIa, pp. 2433—2489; cf. Th. XXI. p. 2475.

[25]) Bull. Amer. Math. Soc., v. 33, pp. 439—446. Although the terminology employed in this paper is that of the plane, the extension to metric spaces in general is obvious.

D' containing $N-D$, such that the distance between D' and K is positive, and $N-D'$ is a Jordan continuum J. The set $N-J=D'$ is connected, and it is obvious that every point of J is at a distance from K less than $d/2$. Also, since the distance between D' and K is positive, J contains all points of N (and hence of M) that lie within a distance less than some positive number $\delta_\epsilon < d/2$ from K.

Consider, now, the set

$$M-J=(M-N)+(N-J).$$

We have already noted that $N-J$ is connected. Furthermore, every component of $M-N$ has its boundary points in $N-J$. Consequently $M-J$ is connected, and the theorem is proved for this case.

Case 2. The set $M-K$ not connected. As in the proof of Theorem 7, we obtain the open set R containing K, every point of R being at a distance from $K<\epsilon$ and $\bar{R}$ being compact. Let P be any point of K. By the theorem of Hahn quoted in the proof of Theorem 9a, there is a Jordan continuum J_P which is a subset of $M\cdot R$ and contains every point of M less than a certain distance d from P. Applying the Borel Theorem, there is a finite number of such sets J_P covering K, and their sum let us denote by J_1. It is easy to see that J_1 contains all points of M that lie within a certain distance δ_ϵ from K. As $\bar{R}$ is compact, only a finite number of components of $M-J_1$ contain points of $M-R$; all other components we add to J_1 and call the resulting set J. Then J is a Jordan continuum[26]) satisfying the conditions of the theorem.

[26]) Cf. H. M. Gehman, *Some relations between a continuous curve and its subsets,* Annals of Math. v. 28. pp. 103—111, Th. 8. Although Gehman's proof depends upon properties of continuous curves that hold only in the plane, an independent proof such as the following establishes it for the general space we are considering. Theorem: *If M and N are continuous curves and N is a subset of M, and if K is a set consisting of N and any collection of components of $M-N$, then K is a continuous curve.* That K is connected im kleinen at any point of $K-N$ is obvious. Let P be a point of K in N, and let ϵ be any positive number. There is a positive number d_1 such that if x is a point of N whose distance from P is less than d_1, then x and P are joined by an arc of N every point of which is at a distance from P less than ϵ; and there is a positive number d_2 such that if y is a point of M whose distance from P is less than d_2, then y and P are joined by an arc of M every point of which is at a distance from P less than d_1. The remainder of the proof is similar to that of Lemma 2 above.

Sur un problème concernant les types de dimensions.

Par

W. Sierpiński (Varsovie).

M. Kuratowski et moi, nous avons démontré [1]) que, E étant un ensemble linéaire de puissance du continu, il existe toujours un ensemble Z de puissance du continu, tel que $dZ<dE$.

Or, le problème se pose: *E et H étant deux ensembles linéaires de puissance du continu, existe-t-il toujours un ensemble Z de puissance du continu, tel que $dZ<dE$ et $dZ<dH$?*

Le but de cette Note est de prouver (à l'aide de l'axiome du choix) que la réponse y est *négative*.

On voit sans peine qu'il s'agit ici des ensembles E et H *totalement imparfaits*. En effet, admettons que E contient un sous-ensemble parfait P. L'ensemble H, dont la puissance est celle du continu, contient, comme on sait, un sous-ensemble ponctiforme de puissance du continu, soit Q, et, d'après le théorème mentionné, il existe un ensemble Z de puissance du continu, tel que $dZ<dQ$. Or, on a, comme on sait, $dQ<dP$ (P étant parfait et Q ponctiforme), donc $dZ<dQ<$ $<dP\leqslant dE$ (puisque $P\subset E$) et, d'autre part, $dZ<dQ\leqslant dH$ (puisque $Q\subset H$). On a donc $dZ<dE$ et $dZ<dH$.

D'abord je démontrerai, en m'appuyant sur le théorème de M. Zermelo, ce

Lemme[2]). *Il existe un ensemble N de puissance du continu, tel que deux sous-ensembles disjoints de N de puissance du continu ne sont jamais homéomorphes.*

[1]) Fund. Math. t. VIII (1926), p. 200.

[2]) Notre lemme, dont nous donnons ici une démonstration directe, peut être déduit sans peine d'un théorème plus général de M. Banach: ce volume p. 14. (Théorème 2).