part. L'ensemble $E_1 - (p_0, u_1, u_2, \ldots)$ est encore un G_δ condensé, donc (d'après ce que nous avons démontré plus haut) il est en homéomorphie généralisée de 1^{re} classe avec H_1. On voit sans peine que l'ensemble E sera encore en homéomorphie généralisée de 1^{re} classe avec H.

Pareillement on traite le cas où E_2 est dénombrable et H_2 est fini ou vide, et le cas, où E_2 et H_2 sont finis.

Notre théorème est ainsi démontré complètement.

Planar Graphs [1].

By

Hassler Whitney [2] (Cambridge, U. S. A.).

Introduction. Kuratowski [3] has shown that a topological graph is planar, i. e. it can be mapped in a 1—1 continuous manner on the surface of a sphere, if and only if it contains neither of two certain graphs within it. The author has shown [4] (I, Theorem 29) that a graph is planar if and only if it has a dual as defined in I. It is the main purpose of the present paper to give a proof of the purely combinatorial theorem (Theorem 12) that a graph has a dual if and only if it contains neither of Kuratowski's graphs as a subgraph [5]. This together with the above mentioned theorem of the author gives a proof (for graphs) of Kuratowski's theorem which involves little of a point set nature.

[1]) Presented to the American Mathematical Society, Dec. 28, 1931.

[2]) National Research Fellow.

[3]) Fund. Math. vol 15 (1930), pp. 271—283. He considers a more general point set than a graph.

[4]) We shall refer to the following papers by the author.

I. *Non-separable and planar graphs.* Trans. Amer. Math. Soc. Vol. 34 (1932), pp. 339—362.

II. *Congruent graphs and the connectivity of graphs*, Amer. Journ. of Math. Vol. 54 (1932), pp. 150—168.

III. 2-*isomorphic graphs*, Amer. Journ. of Math. Vol. 55 (1933), pp. 245—254. Note that we now use the term *isomorphic* instead of *congruent*.

[5]) Kuratowski's theorem was proved independently by Orrin Frink and P. A. Smith. Prof. Frink has been kind enough to show me their (unpublished) proof; our proof (of Theorem 12) has many points in common with theirs.

In § 1 we show that, to determine whether two graphs are duals or not, it is not necessary to regard all their subgraphs, but merely a part of them. This section will not be used in the sequel. In § 2 cut sets of arcs are discussed. In §§ 3, 4 and 5 some properties of planar graphs are described which correspond to common point set theorems in the plane. The rest of the paper is devoted to the proof of Theorem 12. In this proof we need only Theorems 4, 9 and 11.

We recall the following definitions and theorems from the paper I. A *graph* G consists of a set of *vertices* $a, b, \dots, f$, and *arcs* $ab, ac, \dots, df$; the end vertices a and b of each arc ab must be in the graph. A *chain* consists of distinct vertices $a, b, c, \dots, d, e$, and arcs $ab, bc, \dots, de$. A *suspended chain* is a chain such that the first and last vertices and only those, are on at least three arcs of the graph. A *circuit* is a set of distinct vertices $a, b, c, \dots, d, e$, and arcs $ab, bc, \dots, de, ea$; a *k-circuit* is a circuit of k arcs. A graph is *separable* if it is the union of two graphs, each containing at least one arc, and having at most a single common vertex. A *component* of a graph is a maximal non-separable subgraph.

If a graph G has E arcs and V vertices and is in P connected pieces, then its *rank* R and *nullity* N are defined by the equations $R = V - P$, $N = E - R = E - V + P$. A *forest* is a graph containing no circuit, i. e. a graph of nullity 0. A *subgraph* H of G contains some arcs of G and those vertices which are on these arcs. The *complement* of H in G is the subgraph of G containing those arcs not in H; the complement of G in G is the null graph. Let R, R', r, r', etc. be the ranks of G, G', H, H', etc. Suppose there is a $1-1$ correspondence between the arcs of G and G' so that if H is any subgraph of G and H' is the complement of the corresponding subgraph of G', then $r' = R' - n$; then G' is a *dual* of G.

I, Theorem 2. *If an arc ab is added to a graph, the rank (nullity) is increased if and only if a and b were not (were) formerly connected.*

I, Theorem 9. *Let G be a graph of nullity 1 containing no isolated vertices, such that removing any arc reduces the nullity to 0. Then G is a circuit.*

I, Theorem 11. *Every non-separable subgraph of G is contained wholly in one of the components of G.*

I, Theorem 18. *If G is a non-separable graph of nullity $N > 1$, we can remove an arc or suspended chain, leaving a non-separable graph of nullity $N - 1$.*

A simple proof of this theorem is given in III, footnote on p. 247.

I, Theorems 20 and 21. *If G' is a dual of G, then G is a dual of G', and $R' = N$, $N' = R$.*

I, Theorem 23. *Let $G_1, \dots, G_m$ and $G'_1, \dots, G'_m$ be the components of G and G' respectively. and let G'_i be a dual of G_i $(i = 1, \dots, m)$. Then G' is a dual of G.*

I, Theorem 25. *Let G and G' be dual graphs, and let $H_1, \dots, H_m$ be the components of G. Let $H'_1, \dots, H'_m$ be the corresponding subgraphs of G'. Then $H'_1, \dots, H'_m$ are the components of G', and H'_i is a dual of H_i $(i = 1, \dots, m)$.*

I, Theorem 28. *Let ab and $a'b'$ be corresponding arcs of the dual graphs G and G'. Form G_1 from G by dropping out ab, and form G'_1 from G' by dropping out $a'b'$ and letting a' and b' coalesce. Then G_1 and G'_1 are duals.*

I, Theorem 30. *If the non-separable graph G has a (non-separable) dual G', then we can map G and G' together on a sphere so that (1) corresponding arcs, and only such arcs, cross each other, and (2) inside each region of one graph there is just one vertex of the other graph.*

1. Theorem 1. *Suppose there is a $1-1$ correspondence between the arcs of the two graphs G and G' so that a set of arcs in G' form a circuit if and only if the corresponding arcs of G form a circuit. Then any two corresponding subgraphs H and H' of G and G' are of the same rank and nullity.*

The conclusion holds if the word „circuit“ is replaced by „forest“ or „subgraph of nullity > 0“ or „subgraph of nullity 1“, or „cut set of arcs“ (see § 2) *or „subgraph containing no cut set of arcs“ or „subgraph containing at least one cut set of arcs“ or „subgraph containing exactly one cut set of arcs“.*

Let H and H' be any corresponding subgraphs. We shall build them up arc by arc: whenever we add an arc of H, we add also the corresponding arc of H'. To begin with, the two graphs contain no arcs, and are of rank 0, nullity 0. When we add a given arc ab to H, the nullity increases if and only if a and b were already connected, i. e. if and only if ab together with some other arcs already added form a circuit P; similarly for H'. But by hypothesis, if the circuit P is present, then the corresponding arcs of H' form a circuit P', and conversely, if P' is present, so is P. Hence the nullity of H' increases each time if and only if the nullity of H increases. It follows that H and H' are of the same nullity, and therefore also of the same rank.

The next statement follows from the easily proved fact that if any forest in one graph corresponds to a forest in the other, then also circuits correspond to circuits (use I, Theorem 9); similarly for the next two statements.

To prove the last statements, we form H and H' by dropping out the remaining arcs of G and G' one by one, dropping out corresponding arcs at the same time. The proof runs just as before, cut sets of arcs etc. taking the place of circuits etc.

Theorem 1'. *Under the same conditions as Theorem* 1, *G and G' are* 2-*isomorphic.*

This follows from the theorem of III.

Theorem 2. *Suppose there is a* $1-1$ *correspondence between the arcs of the two graphs G and G' so that a set of arcs in G' form a cut set of arcs if and only if the corresponding arcs of G form a circuit. Then G and G' are duals.*

The conclusion holds if the words „cut set of arcs" and „circuit" are replaced by „subgraph containing no cut set of arcs" and „forest", or „subgraph containing a cut set of arcs" and „subgraph of nullity >0", or „subgraph containing exactly one cut set of arcs" and „subgraph of nullity 1".

Suppose cut sets of arcs in G' correspond to circuits in G. Let H, of nullity n, be any subgraph of G, and let H', of rank r', be the complement of the corresponding subgraph of G'. We must show that $r' = R' - n$.

We form H and H' together: We begin with no arcs of G and all the arcs of G'. Each time we add an arc of H, we drop out the corresponding arc of G'; H and H' are then formed at the same time. The nullity of the subgraph of G increases if and only if the last arc added together with arcs already present form a circuit. The rank of the subgraph of G' decreases if and only if the last arc dropped out together with arcs already dropped out form a cut set of arcs. As circuits in G correspond to cut sets of arcs in G', the nullity of the first subgraph increases if and only if the rank of the other decreases, and the first statement follows.

The proof of the other statements follows the lines of the proof of Theorem 1.

Theorem 3. *If G' is a dual of G, then G'' is a dual of G if and only if G' and G'' are* 2-*isomorphic.*

This follows from Theorem 2 and the Theorem of III.

2. *Cut sets of arcs.* Suppose that dropping out a certain set of arcs from a graph G increases the number of connected pieces in the graph, while dropping out no proper subset of these arcs does; we then say these arcs form a *cut set of arcs*. If a single arc forms a cut set of arcs, we call it a *cut arc*. Note that no cut set of arcs contains a 1-circuit.

Theorem 4. *If G and G' are duals, then any circuit in one graph corresponds to a cut set of arcs in the other, and conversely.*

This follows from the definition of dual graphs and I, Theorem 9.

Theorem 5. *If a cut set of arcs is dropped out of a graph G, then the resulting graph G' contains two connected pieces H_1 and H_2 such that each arc of the cut set joined H_1 and H_2.*

Each arc of the cut set joined two distinct connected pieces of G', as putting it back reduces the number of connected pieces in the graph. Say an arc ab of the cut set joins the two pieces H_1 and H_2. Having put back ab, putting back any other arc cd of the cut set leaves the number of connected pieces the same; hence c and d are in the same connected piece in $G' + ab$, while they are not connected in G'. This can only be if cd joins H_1 to H_2 also.

The next two theorems are immediate consequences of this theorem.

Theorem 6. *Any two arcs of a cut set are contained in a circuit in the graph.*

Theorem 7. *Any circuit has an even number of arcs in common with any cut set of arcs.*

Theorem 6, and I, Theorem 11 give.

Theorem 8. *All the arcs of a cut set in a graph lie in a single component of the graph.*

3. *The sides of a circuit.* Suppose a topological planar graph G^*, containing a simple closed curve P^*, is mapped on the surface of a sphere. This surface is divided into two regions by P^*, the two sides of P^*, and all of G^* not in P^* lies in one of these regions.

Let G be a non-separable planar graph containing a circuit P, and let G' be a dual of G; G' is non-separable, by I, Theorem 26. Then, with reference to G', we can define the two sides of P in G as follows. If P' is the subgraph of G' corresponding to P, then P' is a cut set of arcs (Theorem 4). G' minus the arcs of P' is in two connected pieces, say H_1' and H_2'. Let the corresponding subgraphs of G be H_1 and H_2; the arcs of these two graphs, together with the vertices of these graphs which are not in P, form the *two sides of P in the graph G, with reference to the dual G'*

To show that this definition is admissible, we must show that there is no vertex not on P which lies in both H_1 and H_2; i. e. *if an arc of H_1 and an arc of H_2 have a common vertex c, then c lies in P.*

Let C be the arcs on c; the corresponding arcs C' of G' form a circuit (Theorem 4), containing an arc of H_1' and an arc of H_2'. Hence C' contains arcs of the cut set P', and C contains arcs of P; therefore c lies in P. We can now state.

Theorem 9. *If the vertices a and b are on opposite sides of a circuit P in a (non-separable) graph G, with reference to a dual G', then a and b are distinct, and every chain from a to b in G contains a vertex of P.*

The converse of this theorem is not true: Two vertices may be on the same side of a circuit, while there is no chain joining them which does not contain vertices of the circuit.

The different duals a non-separable graph which is not triply connected may have correspond to the different ways of mapping this graph on a sphere (see I, Theorem 30, and II, Theorem 11).

4. *Boundaries.* We consider in this section a pair of dual graphs G, G', neither of which contains a 1-circuit. (Hence also neither contains a cut arc). Consider a vertex a' of one of the graphs, say G', and let A' be the arcs on a'. The corresponding arcs A of G we shall say form the *boundary of a' in G*. If G and G' are non-separable, and the corresponding topological graphs are mapped on a sphere as in I, Theorem 30, then A is a simple closed curve, and forms the boundary of the region of G which contains a'. If G and G' are separable, the theorem referred to does not hold in general, and this interpretation falls down.

Theorem 10. *The boundary of a vertex a' as above described consists of a set of circuits, each of which lies in a different component of G* [1]).

Let $a'b'$ be one of the arcs on a'. If we drop out all the arcs A', a' and b' are disconnected. If we put back as many as we can without connecting these vertices, the remaining arcs A_1' of A' form

[1]) Recall that neither graph contains a 1-circuit or cut arc.

a cut set of arcs. (If the graphs are non-separable. $A_1' = A'$). The corresponding arcs A_1 of G form a circuit; thus we see that each arc of A is contained in a circuit in A.

Consider now an arc $a'c'$ of A' not in A_1', if there is one; it lies in a different component of G' from $a'b'$. For otherwise, there would be a chain joining b' and c' and not passing through a', by I, Theorem 6, which together with $a'b'$ and $a'c'$ forms a circuit P'. But this circuit contains only the arc $a'b'$ of the cut set A_1', contrary to Theorem 7. Hence the arc of A corresponding to $a'c'$ lies in a different component of G from that containing the circuit A_1, by I, Theorem 25, and is contained in a circuit A_2 of A lying in this component. Continuing in this manner, we see that A consists of circuits $A_1, \dots, A_m$, as required.

5. *Construction of dual graphs.* We prove here some theorems similar to I, Theorem 28, and II, Theorem 8. The present theorems correspond to theorems in the plane on how cross cuts may be drawn in a region, dividing it into two regions, etc.

Theorem 11. *Let G and G' be duals. Let a be a vertex of G, and let A be the arcs on a. Let A', the corresponding arcs of G', consist of two chains B' and C', each joining the vertices b' and c'. Let B and C be the corresponding arcs of G. Then:*

(1) *If G_1' is formed from G' by letting b' and c' coalesce into the vertex a', and G_1 is formed from G by replacing the vertex a by the two vertices b and c, and letting the arcs of B and C end on b and c respectively, then G_1 and G_1' are duals, preserving the correspondence between their arcs.*

(2) *If G_2' is formed from G' by adding the arc $b'c'$, and G_2 is formed from G_1 by adding a corresponding arc bc, then G_2 and G_2' are duals, preserving the correspondence between their arcs.*

(1) is a slight generalization of II, Theorem 8; no change in the proof of that theorem is necessary.

To prove (2), let H_2 be a subgraph of G_2, and H_2', the complement of the corresponding subgraph of G_2'. Suppose first H_2 contains bc; then H_2' does not contain $b'c'$. Form H from H_2 by dropping out the arc bc and letting the vertices b and c coalesce; G is formed similarly from G_2. Thus $r = r_2 - 1$, $R = R_2 - 1$. As H_2' is also the complement of the subgraph of G' corresponding to the subgraph H of G, and G and G' are duals, $r' = R' - n_2'$; therefore $r_2' = R_2' - n_2'$, as required. If H_2 does not contain bc while H_2' contains $b'c'$, we employ the graphs G_1 and G_1' instead of G and G'.

The last theorem can be generalized in the following one; we make use of the notation of combinatorial analysis situs.

Theorem 11'. *Theorem* 11 *holds if we assume of the arcs A' merely that they contain no* 1-*circuit, provided we choose the sets of arcs B' and C' and the vertices b' and c' so that B' and C' together make up A', and they are each bounded by $a' + b'$ (mod* 2).

We note first that A' is a cycle (we always understand mod 2), being a sum of circuits (see Theorem 10); hence such a division of its arcs into two groups B' and C' is always possible for some two vertices b' and c'.

The theorem follows as before as soon as we have proved that b' and c' are connected in H' if and only if b and c are not connected in H_1 (see the proof of II, Theorem 8). The only difficulty is in showing that if b and c are connected in H_1, then b' and c' are not connected in H'. We shall now prove this fact.

As B' is bounded by $b' + c'$, there is a chain E' joining b' and c' in B'; similarly there is a chain F' joining b' and c' in C'. $E' + F'$ has no boundary, is thus a cycle, and consists of one or more circuits $P_1', \dots, P_p'$ [1]. Similarly $B' - E'$ and $C' - F'$ consist of circuits $Q_1', \dots, Q_q'$ and $R_1', \dots, R_r'$ respectively. As the circuits of A' are in different components of G' (Theorem 10), A' can be expressed as the union of circuits in but a single manner; these circuits are therefore those above named.

Let D_1 be a chain joining b and c in H_1; the corresponding arcs in G form a circuit D, containing an arc of B and an arc of C. The corresponding subgraph D' of G' is a cut set of arcs, containing an arc of B' and an arc of C', and lying in a single component of G' (Theorem 8). Hence the cut set contains arcs of some circuit P_i', and thus contains an arc of E' and an arc of F'. The required fact now follows exactly as in II, Theorem 8, Case 2.

[1]) For these facts, see for instance Whitney, *A characterization of the closed 2-cell*, Trans. Amer. Math. Soc., vol. 35 (1933), Lemmas G and H.

6. We turn now to the main theorem of the paper.

Theorem 12. *A necessary and sufficient condition that a graph have a dual is that it contain neither of the two following types of graphs as a subgraph:*

K_1. *This graph is formed by taking five vertices, and joining each two by an arc or suspended chain.*

K_2. *This graph is formed by taking two sets of three vertices each, and joining each vertex in one set to each vertex in the other set by an arc or suspended chain.*

The necessity of the condition was proved in I, Theorem 32 [1]).

We must prove the sufficiency of the condition. We shall assume the theorem holds for all graphs containing fewer than E arcs, and shall prove it for any graph containing E arcs. As a graph of a single arc has a dual, the theorem will be proved in general.

Let G be a graph of E arcs therefore containing neither a graph K_1 nor a graph K_2 as subgraph. Suppose first G is separable. Then none of its components contain K_1 or K_2, hence each component has a dual, and hence G has a dual, by I, Theorem 23.

Suppose next G is non-separable, and suppose it contains a vertex b on but two arcs ab and bc. Dropping out the vertex b and replacing ab and bc by the single arc ac gives a graph G_1 containing neither K_1 nor K_2; it has therefore a dual G_1'. If $a'b'$ is the arc of G_1' corresponding to the arc ac of G_1, then adding another arc $a'b'$ to G_1' gives a graph G' which is a dual of G, as is easily seen (compare the proof of Theorem 11).

7. Suppose finally G is non-separable, and each vertex is on at least three arcs. By I, Theorem 18, we can drop out an arc ab, leaving a non-separable graph G_0. As G_0 contains neither K_1 nor K_2 and has fewer than E arcs, it has a non-separable dual G_0' (I, Theorem 26). Let A and B be the arcs of G_0 on a and b respectively. The corresponding subgraphs A' and B' of G_0' are circuits (Theorem 4). Suppose A' and B' have a common vertex c'; then if C' is the

[1]) We proved it by proving that neither of two certain graphs G_3 and G_4 had a dual. Note that we proved there a more general result: *There is no graph G_3' (or G_4') which, for any two numbers i and j, has the same number of subgraphs of rank i, nullity j, that a dual of G_3 (or G_4) should have.*

arcs of G_0' on c', C is a circuit in G, containing arcs of both A and B, and containing therefore a and b. A dual of G is now constructed as in Theorem 11.

The rest of the proof is concerned with the case that A' and B' have no common vertex. If we drop out just enough arcs P' from G' to disconnect A' and B', then P' is a cut set of arcs; hence the corresponding arcs P of G form a circuit.

8. We show in this section that *there are two vertices c and d in P, dividing P into the two chains C and D, such that G_0 can be expressed as the union of two graphs G_1^* and G_2^*, the first containing a and C, and the second containing b and D*; *moreover, G_1^* and G_2^* have only the vertices c and d in common.*

We note first that as the arcs P' separate A' and B', the vertices a and b are on opposite sides of P relative to G', and hence every chain from a to b in G_0 passes through P (Theorem 9). Say P consists of the arcs $a_1 a_2$, $a_2 a_3, \dots, a_{n-1} a_n$, $a_n a_1$. If there is a chain in G_0 from a to a vertex a_l which does not contain any other vertex of P, we shall say that a_l is *accessible* from a. Let $a_{i_1}, a_{i_2}, \dots, a_{i_p}$ be the vertices of P accessible from a, named in cyclic order, and let $a_{j_1}, a_{j_2}, \dots, a_{j_q}$ be those accessible from b. There are at least two vertices in each set, as G_0 is non-separable.

To prove the statement, we need merely show that *for some two vertices c and d and corresponding chains C and D, all the vertices $a_{i_1}, \dots, a_{i_p}$ lie in C, and all the vertices $a_{j_1}, \dots, a_{j_q}$ lie in D*. For then we can let G_1^* contain C, and all those arcs of G_0, some end vertex of which can be joined to a by a chain containing neither c nor d, and let G_2^* be the complementary subgraph of G_1^* in G_0. G_1^* and G_2^* evidently have the required properties.

To find the vertices c and d, suppose first some vertex a_{j_t} does not coincide with any vertex a_{i_s}, but lies in P between two vertices $a_{i_h} = c$ and $a_{i_{h+1}} = d$. Then these last two vertices are the required vertices. For if not, then there is some vertex a_{j_u} lying in P on the other side of c and d from a_{j_t}. From a, draw chains to c and d having only their end vertices in P; from these we pick out a chain E from c to d. Using b, we find a similar chain F joining a_{j_t} and a_{j_u}. Of course E and F have no vertex in common. Using the arc ab, we find a chain in G having no vertices

in P, and joining a vertex e of E and a vertex f of F. In G there are theretore two sets of vertices e, a_{j_t}, a_{j_u}, and f, c, d, there are chains joining each vertex of the first set to each vertex of the second, and no two of these chains have a common vertex, except perhaps for their end vertices. Thus G contains a graph K_2, contrary to hypothesis.

Suppose next every vertex a_{j_t} coincides with some vertex a_{i_s}. If there are only two vertices in the set $a_{j_1}, \dots, a_{j_q}$, we can call these c and d (use the reasoning above). Otherwise let c, d and e be three of these vertices. Draw chains from a to these three vertices, each having only an end vertex on P. From these chains we can pick out a vertex f and three chains fc, fd, fe, which have only the vertex f in common, and have only their other end vertices in P. Using b, we find similar chains gc, gd, ge. Using the arc ab, we find in G a chain joining a vertex x of the first set of chains to a vertex y of the second set; this chain does not touch P. If x is f and u is g, the graph thus constructed (including P) is a graph K_1 in G, contrary to hypothesis. Suppose this is not the case; say x lies on the chain fc between f and c. From the graph we have constructed, drop out the chain de of P, and the chain gc if y is not an inner vertex of this chain, otherwise that much of gc between y and c. The resulting graph is a graph K_2 in G (the two sets of vertices are x, e, d, and f, c, g or y), again a contradiction; thus the supposition that there were three vertices in the set $a_{j_1}, \dots, a_{j_q}$, was impossible. The statement of § 8 is now proved.

9. Form the graph G_1 from the graph G_1^* by renaming the vertices c and d c_1 and d_1 (and renaming the arcs on these vertices accordingly), and adding a new vertex e_1 and arcs $e_1 a$, $e_1 c_1$, $e_1 d_1$; form G_2 from G_2^* by renaming c and d c_2 and d_2, and adding a vertex e_2 and arcs $e_2 b$, $e_2 c_2$, $e_2 d_2$. We shall show that G_1 and G_2 have duals, and shall then reconstrust G from them, at the same time forming a dual of G.

G_1 can be constructed from G as follows. From chains in G_2^* joining b to the vertices c and d, together with the arc ab, we find in G a vertex e_1 and three chains $e_1 a$, $e_1 c$, $e_1 d$, which have only the vertex e_1 in common and only their other ends in G_1^*. We now drop out all other arcs of G_2^* (note that we drop out at least one arc, namely an arc of P), replace each of the above chains by a single arc. and rename c and d c_1 and d_1. Thus we see that G_1 contains fewer arcs than G, and contains neither a graph K_1 nor a graph K_2, as G contains neither. Hence G_1 has a dual G_1'; similarly G_2 has a dual G_2'.

Let $c_1'd_1'$, $d_1'a_1'$, $a_1'c_1'$, and $c_2'd_2'$, $d_2'b_2'$, $b_2'c_2'$, be the arcs of G_1' and G_2' corresponding to the arcs e_1a, e_1c_1, e_1d_1, and e_2b, e_2c_2, e_2d_2, of G_1 and G_2 respectively (these arcs must form circuits). Form G_3 by letting the vertices c_1 and c_2 of G_1 and G_2 coalesce into the vertex c; form G_3' by letting the vertices a_1' and b_2' of G_1' and G_2' coalesce into the vertex a'. By I, Theorem 23, G_3 and G_3' are duals (preserving the correspondence between their arcs — we shall understand these words without mention in the future).

Evidently G_1 and G_2 are non-separable; hence the sets of arcs P_1 and P_2 of G_1 and G_2 corresponding to the sets of arcs of G_1' and G_2' on a_1' and b_2' respectively form circuits. P_1 is formed of two chains E_1 and F_1 where $E_1 = c_1e_1 + e_1d_1$, and $P_2 = E_2 + F_2$, where $E_2 = c_2e_2 + e_2d_2$. Thus in G_3, the arcs corresponding to the arcs of G_3' on a' consist of two chains, $E_1 + E_2$ and $F_1 + F_2$; each of these joins d_1 to d_2. By Theorem 11, if we let d_1 and d_2 coalesce into the vertex d, forming the graph G_4, and replace a' by the two vertices a_3' and a_4', letting the arcs corresponding to E_1 and E_2 end on a_3', and those corresponding to F_1 and F_2, on a_4', forming the graph G_4', G_4 and G_4' are duals.

By the same theorem, if we let e_1 and e_2 coalesce in G_4 into the vertex e, forming the graph G_5, and replace the vertex a_3' in G_4' by the vertices a_5' and a_6', letting $a_3'd_1'$ and $a_3'd_2'$ end on a_5', and $a_3'c_1'$ and $a_3'c_2'$, on a_6', forming the graph G_5', G_5 and G_5' are duals. By I, Theorem 28, if we drop out both arcs ec and both arcs ed and the vertex e, and replace the arcs ae and eb by the single arc ab, the resulting graph has a dual. But this graph is just the graph G, and the theorem is proved.

Sur un problème concernant les transformations continues.

Par

B. Knaster et S. Mazurkiewicz (Warszawa).

1. Appelons *α-connexe* [1]) tout continu C qui contient pour chaque couple de ses points p, q un arc simple A (image homéomorphe du segment rectiligne) ayant ces points pour extrémités. On sait que α-connexité est une propriété invariante par rapport aux transformations continues f de C, car les images continues du segment étant α-connexes [2]), il existe déjà dans l'image $f(A)$ de A, donc à plus forte raison dans $f(C)$, un arc simple aux extrémités $f(p)$ et $f(q)$ toutes les fois que $f(p) \neq f(q)$.

Les généralisations de la notion d'arc simple conduisent d'une façon naturelle aux généralisations parallèles de celle de α-connexité. Ainsi nous appellerons *λ-connexe* tout continu C qui contient pour tout couple de ses points p, q un continu K irréductible du type λ [3]) entre p et q. Le problème se pose: *λ-connexité est-elle un invariant des transformations continues?* La réponse est *négative*, ce que nous allons montrer sur deux exemples différents et dont la discussion nous conduira à préciser d'autres problèmes dans le même ordre d'idées.

2. **Continu** $\mathfrak{L}_1$. Soient x, y, z les coordonnées cartésiennes dans l'espace euclidien à 3 dimensions et C l'ensemble parfait ponctiforme de Cantor sur le segment $0 \leqslant x \leqslant 1$ de l'axe des x. Considérons le continu indécomposable $\mathfrak{B}$, décrit par B. Knaster et C. Kura-

1) *„arcwise connected“* des auteurs américains.
2) Cf. p. ex. S. Mazurkiewicz Fundam. Math. I, p. 201, th. IX.
3) C. Kuratowski, Fundam. Math. X, p. 225—276, en particulier p. 256 et 262.