# On the Theory of Trigonometric Series V [1]).

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## I. Introduction.

1. Let $\sum_1^\infty a_n$ be a series. The upper and lower sums $(R, 3)$ of this series are the upper and lower limits as $\theta \to 0$ of

$$\text{(i)} \qquad \sum_1^\infty a_n \left(\frac{\sin n\theta}{n\theta}\right)^3.$$

We suppose that for relevant $\theta$, the series (i) is convergent. This will always happen in the cases with which we are concerned in this paper.

We write

$$A_n(x) = a_n \cos nx + b_n \sin nx, \quad B_n(x) = b_n \cos nx - a_n \sin nx,$$

and consider a trigonometric series

$$\text{(ii)} \qquad \sum_1^\infty A_n(x). \qquad a_n = o(n), \quad b_n = o(n).$$

We seek to determine what knowledge of its sums $(R, 3)$ will enable us to infer that it is a Fourier series. If we write

$$F(x) = \sum_1^\infty \frac{B_n(x)}{n^3},$$

[1]) The first four papers of this series have appeared in *Proc. Lond. Math. Soc.* 34, 35. This paper can be read independently. We shall have occasion to quote one or two results of the second paper, *Proc. Lond. Math. Soc.* 34 (457—491). It will be referred to as II.



then the upper and lower sums $(R, 3)$ of (ii) are the upper and lower limits as $\theta \to 0$ of

$$\frac{F(x+3\theta) - 3F(x+\theta) + 3F(x-\theta) - F(x-3\theta)}{(2\theta)^3}.$$

They are denoted by $\overline{D}^3F(x)$, $\underline{D}^3F(x)$ respectively. We are thus led to consider what knowledge of these limits will enable us to infer that $F(x)$ is of the form

$$\int^x dy \int^y dt \int^t \varphi(z)\,dz. \qquad \text{(iii)}$$

All that is already known on this subject, is contained in the following theorem of Saks [1]): If $F(x)$ is continuous and $F'(x)$ is finite in an interval, and $\underline{D}^3F(x) > 0$ in that interval, then $F'(x)$ is continuous and convex in the interval.

In Theorem I we replace „$\underline{D}^3F(x) > 0$" by „$\overline{D}^3F(x) > 0$, $\underline{D}^3F(x) > -\infty$", and we replace „$F'(x)$ is finite" by „$F'(x)$ is unique, finite or infinite", the latter generalisation being required for application to trigonometric series. From Theorem I we deduce Theorem II, which qives a condition enabling us to assert that $F(x)$ has the form (iii). We are then able to prove in Theorem III that when $\overline{R}(x)$, $\underline{R}(x)$, the upper and lower sums $(R, 3)$ of (ii) satisfy $\overline{R}(x) \geqslant f(x)$, where $f(x)$ is integrable in the sense of Denjoy-Perron, and $\underline{R}(x)$ is finite, then (ii) is a Fourier-Denjoy series.

The method of proving these theorems presents a distinct analogy to some investigations qiven in II (458—472). In that paper, an essential, if inconspicuous, part was played by functions possessing the property of being upper semi-continuous (u. s. c.) on some portion of any perfect set on which they are defined. We did not then draw explicit attention to this class of functions. This was first done by Saks (loc. cit.). In the present paper we say that such functions possess the property $R^*$. This property is a natural extension of the property $R$. A function possesses the property $R$ if it is continuous on some portion of any perfect set on which it is defined.

We next consider the problem of replacing the condition $a_n =$

[1]) Saks: *Journal Lond. Math. Soc.* 7 (1932) 247—251.

$= o(n)$, $b_n = o(n)$ by a more general one. We are able to replace this by a condition which is much more general than might be anticipated, namely, by the condition

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{-n}^{n} (a_m - i b_m) e^{imx} = 0 \tag{iv}$$

for all $x$. To prove this requires a delicate piece of analysis. In the first place, we must prove generalisations of Theorems I and II. In Theorem I for example, the condition „$F(x)$ is continuous“, is to be replaced by „$F(x)$ is approximately continuous and has the property $R^*$“, while the condition „$F'(x)$ is unique“ is to be replaced by „the approximate derivative $F_a(x)$ of $F(x)$ satisfies $F_a(x) < \infty$, and possesses the property $R^*$“. It is a fact that under the conditions of Theorem I we must have „$F'(x) < \infty$ and possesses the property $R^*$“, so that we are indeed concerned with an effective generalisation of that theorem. If $\overline{D}^3 F(x)$, $\underline{D}^3 F(x)$ satisfy the same conditions as in Theorem I, we can infer the same consequences as in that theorem. This generalisation constitutes Theorem V, and Theorem VI is a corresponding generalisation of Theorem II. As for the new uniqueness theorem it is easy to show that when (iv) holds,

$$F(x) = \sum \frac{B_n(x)}{n^3}$$

converges for all $x$. We can therefore talk of the sums $(R, 3)$ of

$$\sum_{1}^{\infty} A_n(x). \tag{v}$$

A somewhat elaborate analysis shows that if the sums $(R, 3)$ of (v) subject to (iv), satisfy the conditions of Theorem III, then $F(x)$ satisfies all the conditions which enable us to assert, by Theorem VI, that it is of the form (iii).

We further consider summability $(R, 4)$. The upper and lower sums $(R, 4)$ of $\sum_{1}^{\infty} a_n$ are the upper and lower limits as $\theta \to 0$ of

$$\sum_{1}^{\infty} a_n \left(\frac{\sin n\theta}{n\theta}\right)^4.$$

Results for trigonometric series, similar to the two previously mentioned hold when the upper and lower sums $(R, 4)$ satisfy the conditions previously satisfied by the sums $(R, 3)$. To prove this, requires an examination of the fourth symmetric derivatives of continuous functions, with varying hypotheses concerning its derivatives. Our investigation, is, in part, somewhat more general than is required for the application to trigonometric series; and is by no means an immediate analogy of the investigation of the third symmetric derivative. We have arranged the proofs so as to employ two interesting mean value theorems (Theorems IV, IX).



## II. The third symmetric derivative.

**2.** We write

$$\Delta F(x, h) = F(x + h) - F(x - h),$$

$$\Delta^r F(x, h) = \Delta \cdot \Delta^{r-1} F(x, h) = \Delta^{r-1} F(x + h, h) - \Delta^{r-1} F(x - h, h),$$
$$(r = 2, 3, 4)$$

so that, in particular [1],

$$\Delta^2 F(x, h) = F(x + 2h) + F(x - 2h) - 2F(x);$$

further,

$$\overline{D}^r F(x) = \overline{\lim_{h\to 0}}\, \Delta^r F(x, h) \,|\, (2h)^r, \quad \underline{D}^r F(x) = \underline{\lim_{h\to 0}}\, \Delta^r F(x, h) \,|\, (2h)^r,$$

and

$$D^r F(x) = \lim \Delta^r F(x, h) \,|\, (2h)^r$$

when the limit exists.

In this section we give a number of known results which are required in the proof of the first theorem.

**Lemma 1.** (de la Vallée Poussin). *Let $F(x)$ be continuous for $a \leqslant x \leqslant b$, and $\overline{D}^2 F(x) \geqslant 0$ for $a < x < b$. Then for $a \leqslant x <$*

[1]) When we are only considering the second symmetric derivative, it is convenient to write

$$\Delta^2 F(x, h) = F(x + h) + F(x - h) - 2F(x),$$

as was done in II. When, as in the present paper, we are considering the symmetric derivatives of various orders, the definition given in the text is the more convenient.

$< y < z \leqslant b$, *we have*

$$(1) \qquad \frac{F(x)-F(y)}{x-y} \leqslant \frac{F(y)-F(z)}{y-z}.$$

**Lemma 2.** (*Steinhaus*). *If $F(x)$ satisfies the conditions of lemma 1, then $F(x)$ has increasing derivatives, $D^2F(x)$ exists p. p. (presque partout) in $(a,b)$, and is integrable L in $(a+\epsilon,\ b-\epsilon)$ for $0<\epsilon<(b-a)/2$.*

By lemma 1, we have for $a \leqslant x < y < z < w \leqslant b$,

$$\frac{F(x)-F(y)}{x-y} \leqslant \frac{F(z)-F(w)}{z-w}.$$

Letting $y \to x$, $w \to z$ in a suitable manner, we see that $D^+F(x)$ is non-diminishing. Hence

$$H(x) = \lim_{h\to 0} \frac{D^+F(x+h) - D^+F(x-h)}{2h}$$

exists p. p. and is integrable $L$ in $(a+\epsilon,\ b-\epsilon)$. Further, at a point $x$ at which $H(x)$ exists,

$$\frac{\Delta^2 F(x, h/2)}{h^2} = \frac{1}{h^2}\int_0^h \{D^+F(x+t) - D^+F(x-t)\}\,dt \to H(x)$$

as $h \to 0$; so that $D^2F(x) = H(x)$.

**Lemma 3.** *Let $\varphi(x)$ be a function which, for each $x$ satisfying $a \leqslant x \leqslant b$ has a unique value which is finite or $-\infty$. Let $\varphi(x)$ be u. s. c. in $(a,b)$ and $\overline{D}^2\varphi(x) \geqslant 0$ for $a<x<b$. If $\alpha$ and $\beta$ be two points satisfying $a<\alpha<\beta<b$, and such that $\varphi(\alpha), \varphi(\beta)$ are finite, then*

$$\varphi(x) \leqslant \varphi(\alpha) + \frac{x-\alpha}{\beta-\alpha}(\varphi(\beta)-\varphi(\alpha)) \qquad (\alpha<x<\beta)$$

For a proof, see II lemma 14.

**Definition.** A finite function $f(x)$, defined in the closed interval $(a,b)$ is convex in that interval if

$$2f\left(\frac{x_1+x_2}{2}\right) \leqslant f(x_1) + f(x_2). \qquad a \leqslant x_1 \leqslant b,\ a \leqslant x_2 \leqslant b.$$



A finite function $f(x)$, defined in an open interval, is convex in that interval, if it is convex in every closed interval interior to the open interval.

**Lemma 4.** (Sierpiński)[1]. *If the finite function $f(x)$ is convex in $a<x<b$, and measurable L, then it is continuous in $a<x<b$.*

If the function $f(x)$ is continuous in $A<x<B$ and convex in that interval, then $f(B-h)$ tends as $h>0$ tends to 0, to a limit which is finite or $+\infty$. For the conditions of lemma 1 are satisfied in every interval $(a,b)$ interior to $(A,B)$. By lemma 2, $D^+f$ is non-diminishing in $A<x<B$. It therefore tends, as $x \to B$, to a limit which is finite or $+\infty$. In the first case, $\lim f(B-h)$ is finite; and in the second case, for $0<h<h_1$, $f(B-h)$ increases as $h$ diminishes, so that $\lim f(B-h)$ is finite or $+\infty$. Further, $f(A+h)$ tends as $h>0$ tends to 0, to a limit which is finite or $+\infty$. For $D^+f$ tends, as $x \to A$, to a limit which is finite or $-\infty$. In the first case, $\lim f(A+h)$ is finite; and in the second case, for $0<h<h_1$, $f(A+h)$ increases as $h$ diminishes, and as $\lim f(A+h)$ is either finite or $+\infty$.

**Lemma 5.** *Let $\{f_\alpha(x)\}$ be a family of continuous functions defined on a perfect set $\Pi$ for $\beta \geqslant \alpha > \alpha_0$. Suppose that for every $\epsilon$ satisfying $0<\epsilon<\beta-\alpha_0$, $f_\alpha(x)$ is continuous as a function of $(\alpha, x)$ for $\beta \geqslant \alpha \geqslant \alpha_0+\epsilon, x \subset \Pi$. Let $\underline{\lim}_{\alpha\to\alpha_0} f_\alpha(x) > -\infty$ for $x \subset \Pi$. Then there is a portion $\bar{\omega}$ of $\Pi$ and a constant $K = K(\bar{\omega})$, such that for $x \subset \bar{\omega}. f_\alpha(x) > K$ for $\beta \geqslant \alpha > \alpha_0$.*

For the method of proof, see II lemma 8.

**3. Definition.** A function $f(x)$ which has a single value, finite or infinite, at each point of an interval $(a,b)$, will be said to possess the property $R^*$ in that interval, if given any perfect set $\Pi$ in $(a,b)$, there is a portion of $\Pi$ on which $f(x)$ is u. s. c.

**Theorem I.** *If in an open interval $I$, $F(x)$ is continuous, $F'(x)$ has a unique value, $\overline{D}^2F(x) > 0$, $\underline{D}^2F(x) > -\infty$, then $F'(x)$ is continuous and $\overline{D}^2F'(x) \geqslant 0$.*

We require five lemmas.

[1]) Sierpiński *Fund. Math.* 1 (1920) 125—129.

**Lemma 6.** *Under the conditions of the theorem, $\bar{D}^2 F'(x) \geqslant \underline{D}^3 F(x)$.*

It is to be understood that in forming the upper limit of

$$\frac{F'(x+2h)+F'(x-2h)-2F'(x)}{(2h)^2} \tag{2}$$

the values of $h$ for which (2) has no meaning, are to be neglected. At an assigned point $\xi$, we have for some $K = K(\xi)$, $\underline{D}^3 F(\xi) > K$. Hence, if $G(x) = F(x) - Kx^3/6$, then $\underline{D}^3 G(\xi) > 0$. For all sufficiently small $h$ we therefore have

$$G(x+3h) - 3G(x+h) + 3G(x-h) - G(x-3h) > 0,$$

and so

$$\Delta G(\xi, 3h)/6h > \Delta G(\xi, h)/2h.$$

Thus $\Delta G(\xi, h/3^n)/(2h/3^n)$ forms a diminishing sequence which tends to $G'(\xi)$. Hence $G'(\xi) < \infty$, and so $F'(\xi) < \infty$. Thus $F'(x)$ is either finite or $-\infty$. The set of points at which $F'(x) = -\infty$ forms a set of measure zero, so that the quotient (2) has a meaning for almost all $h$.

To prove the lemma, we may suppose that at the given point $x$, $\underline{D}^3 F(x) > 0$, and it is then sufficient to show that $\bar{D}^2 F'(x) \geqslant 0$. For sufficiently small $h$, we have by the preceding argument,

$$\frac{F(x+h)-F(x-h)}{2h} > F'(x). \tag{3}$$

Write $\varphi(t) = F(x+t) - F(x-t)$. Since $\dfrac{d}{dt} F(x+t)$, $\dfrac{d}{dt}\{-F(x-t)\}$ are finite or $-\infty$, $\varphi'(t)$ has a unique value, and we can apply the mean value theorem. We have

$$\begin{aligned} F(x+h) - F(x-h) &= \varphi(h) - \varphi(0) \\ &= h\varphi'(\theta h) \qquad (0<\theta<1) \\ &= h[F'(x+\theta h) + F'(x-\theta h)]. \end{aligned}$$

Hence (3) gives $\Delta^2 F'(x, \theta h/2) > 0$. Since $h$ is arbitrarily small, this implies $\bar{D}^2 F'(x) \geqslant 0$.

**Lemma 7.** *Under the conditions of the theorem, $F'(x)$ has the property $R^*$.*

It is sufficient to show that $F'(x)$ has the property $R^*$ in every interval $(a, b)$ completely interior to $I = (A, B)$. Let $3h_0 = \min(B-b, a-A)$, and consider the family of functions

$$f_h(x) = \frac{\Delta^3 F(x,h)}{(2h)^3}. \qquad (x \subset (a,b),\ 0 < h \leqslant h_0.$$

By lemma 5, given a perfect set $\Pi \subset (a,b)$ there is a portion $\tilde{\omega}$ of $\Pi$, and a $K = K(\tilde{\omega})$ such that

$$\Delta^3 F(x,h)/(2h)^3 > K. \qquad (x \subset \tilde{\omega},\ 0 < h \leqslant h_0).$$

Let $G(x) = F(x) - Kx^3/6$. Then

$$\Delta^3 G(x,h) > 0. \qquad (x \subset \tilde{\omega},\ 0 < h \leqslant h_0)$$

Hence $\Delta G(x, h_0/3^n)/(2h_0/3^n)$ is a diminishing sequence of functions, each of which is continuous on $\tilde{\omega}$. Their limit, $G'(x)$ is therefore u. s. c. on $\tilde{\omega}$; so then is $F'(x)$.

**Lemma 8.** *If $F'(x)$ is u. s. c. in $(a,b)$, and $\underline{D}^3 F(x) > K$ in that interval, then $D^3 F(x)$ exists p. p.*

We may without loss of generality suppose that $K = 0$. Then $\bar{D}^2 F'(x) > 0$ by lemma 6. Also, by an argument used in the proof of that lemma, $F'(x) < \infty$. We first show that $F'(x)$ is finite in $d = (a,b)$. If not, there are two possibilities.

(i) There exist two points $\alpha$ and $\beta > \alpha$ such that $F'(\alpha) = -\infty$, $F'(\beta) = -\infty$. We can then find $\gamma$ and $\delta$ such that $\gamma < \alpha < \beta < \delta$ and such that $F'(\gamma)$, $F'(\delta)$ are finite while $F'(\gamma) < -N$, $F'(\delta) < -N$, where $N$ is arbitrarily assigned. For, as regards $\delta$,

$$\frac{F(\beta+h)-F(\beta)}{h} = F'(\beta+\theta h), \qquad (0<\theta<1)$$

and the first member tends to $-\infty$ as $h \to 0$; and similarly for $\gamma$. By lemma 3, $F'(x) < -N$ in $(\alpha, \beta)$ and since $N$ is arbitrary, we have $F'(x) = -\infty$ in an interval, which is impossible.

(ii) There is only one point $\alpha$ such that $F'(\alpha) = -\infty$. Then $F'(x)$ being finite and u. s. c. in the open interval $(\alpha, b)$, it is convex in that interval by lemma 3, and continuous by lemma 4. Hence $F'(\alpha+0)$ is finite or $+\infty$. But

$$F'(\alpha) = \lim \frac{F(\alpha+h)-F(\alpha)}{h} \qquad (h>0)$$

$$\begin{aligned} &= \lim F'(\alpha + \theta h) \qquad (0 < \theta < 1) \\ &= F'(\alpha + 0). \end{aligned}$$

This contradicts $F'(\alpha) = -\infty$. Thus $F'(x)$ is finite in d. By lemma 3 it is convex, and by lemma 4, continuous. By lemma 2, $D^+F'(x)$ is increasing. Hence, p. p. in $(a,b)$,

$$f(x) = \lim_{h\to 0} \frac{D^+F'(x+h) - D^+F'(x)}{h}$$

exists. Let $x$ be a point at which $f(x)$ exists. Then

$$\begin{aligned} \frac{\Delta^3 F(x,h)}{(2h)^3} &= \frac{\Delta^2 F(x+h,h) - \Delta^2 F(x-h,h)}{(2h)^3} \\ &= \frac{\Delta^2 F'(\xi,h)}{(2h)^2} \qquad (x-h<\xi<x+h) \end{aligned}$$

by the mean value theorem. Further,

$$\frac{\Delta^2 F'(\xi,h)}{(2h)^2} = \frac{1}{(2h)^2}\int_0^{2h} \{D^+F'(\xi+t) - D^+F'(\xi-t)\}\, dt. \tag{4}$$

By the definition of $f(x)$ we have

$$\begin{aligned} D^+F'(\xi+t) - D^+F'(x) &= (\xi+t-x)[f(x)+\epsilon(t,\xi)], \\ D^+F'(\xi-t) - D^+F'(x) &= (\xi-t-x)[f(x)+\eta(t,\xi)], \end{aligned}$$

where

$$\epsilon^2 + \eta^2 < \zeta^2. \qquad (o < h < h(\zeta))$$

Thus (4) becomes

$$\begin{aligned} \Delta^2 F'(\xi,h)/(2h)^2 &= \frac{1}{(2h)^2}\int_0^{2h} [2t\, f(x) + o(h)]\, dt \\ &\to f(x) \end{aligned}$$

as $h \to 0$. Hence $D^3F(x) = f(x)$.

**Lemma 9.** *If $F'(x)$ possesses the property $R^*$ in $(a,b)$ and $F'(x) < \infty$, $\overline{D}^2 F'(x) > 0$ p. p., $\overline{D}^2 F'(x) > -\infty$, then $F'(x)$ is continuous and convex.*

If $E$ be the set of zero measure at which $\overline{D}^2 F'(x) \leqslant 0$, then we can find a continuous increasing function $g(x)$ such that $g'(x) = +\infty$ for $x \subset E$ while $0 < g(x) < \epsilon$ for an assigned $\epsilon > 0$. Writing



$$G(x) = \int_a^x g(t)\, dt,$$

and $H(x) = F'(x) + G(x)$, we have $D^2 H(x) > 0$ for all $x$, while $H(x)$ possesses the other properties of $F'(x)$. If then we show that $H(x)$ is continuous and convex, the lemma will follow since $H(x) \to F'(x)$ uniformly as $\epsilon \to 0$. We may therefore suppose that $D^2 F'(x) > 0$ everywhere.

By the definition of the property $R^*$, there is a non-dense closed set $q_1$ such that if $d = (\alpha,\beta)$ be an interval contiguous to $q_1$, then $F'(x)$ is u. s. c. in the open interval. As in the proof of lemma 8, we can show that $F'(x)$ is finite in this open interval. By lemma 3 it so convex, and hence continuous. Hence $F'(\beta - 0)$ is finite or $+\infty$. But $F'(\beta) = \lim F'(\beta - h)$, $h > 0$. Since $F'(\beta) < \infty$, we have $F'(\beta) = F'(\beta - 0)$, both being finite. Similarly for $\alpha$. Thus $F'(x)$ is continuous and convex in the closed contiguous intervals of $q_1$. Hence it is continuous and convex in the open contiguous intervals of $q_1'$, the derived set of $q_1$. By the above argument, it is therefore continuous and convex in the closed contiguous intervals of $q_1'$. Proceeding in this way, we see that $F'(x)$ is continuous and convex in the closed contiguous intervals of $p_1$, the perfect kernel of $q_1$.

By the definition of the property $R^*$, there is a closed set $q_2 \subset p_1$ non-dense in $p_1$, such that if $d$ be *a* contiguous interval (open) of $q_2$, then $F'(x)$ is u. s. c. on $p_1 d$. Since $F'(x)$ is continuous and convex in a closed contiguous interval of $p_1$, it attains its maximum in such an interval at an end point, i. e. a point of $p_1$. Hence $F'(x)$ is u. s. c. in d. We thus arrive at the same situation as before, with $q_1$, replaced by $q_2 \subset q_1$ and non-dense in $q_1$. By transfinite induction we infer that $F'(x)$ is u. s. c. in $(a,b)$. By an argument already used in the proof of this lemma, it follows that $F'(x)$ is continuous and convex.

**Lemma 10.** *If $F'(x)$ is continuous and convex in $(a,b)$, then $\underline{D}^3 F(x) \geqslant 0$ in that interval.*

For

$$\begin{aligned} \Delta^3 F(x,h) &= \Delta^2 F(x+h,h) - \Delta^2 F(x-h,h) \\ &= 2h\,\Delta^2 F'(\xi,h) \qquad (x-h<\xi<x+h) \\ &\geqslant 0. \end{aligned}$$

We can now prove the main theorem. By lemma 7, $F'(x)$ possesses the property $R^*$. Hence, and by lemma 5, there is a non-dense closed set $F_1$ with the following property. If $d$ be a contiguous interval of $F_1$, and $\delta$ be completely interior to $d$, there is a $K=K(\delta)$ such that $D^3F(x)>K$ for $x\subset\delta$, while $F'(x)$ is u. s. c. in $\delta$. By lemma 8, $\bar{D}^3F(x)$ exists p. p. in $\delta$. Hence $\bar{D}^3F(x)>0$ p. p. in $\delta$, and so by lemma 6, $\bar{D}^2F'(x)>0$ p. p. in $\delta$ and $\bar{D}^2F'(x)>-\infty$ everywhere in $\delta$. By lemma 9, $F'(x)$ is continuous and convex in $\delta$.

Since $\delta$ is an arbitrary interval within $d$, $F'(x)$ is continuous and convex in the open interval $d$, and therefore, since $F'(x)<\infty$, it follows by an argument we have already used, that $F'(x)$ is continuous and convex in the closed interval $d$. By repeating the argument, we infer that $F'(x)$ is continuous and convex in the closed intervals contiguous to $\pi_1$ the perfect kernel of $F_1$.

By lemma 7 and lemma 5, there is a closed set $F_2\subset\Pi_1$, non-dense in $\Pi_1$, with the following property. If $d$ be a contiguous interval of $F_2$ and $\delta$ be completely interior to $d$, there is a $K=K(\delta)$ such that $D^3F(x)>K$ for $x\subset\Pi_1\delta$, while $F'(x)$ is u. s. c. on $\Pi_1\delta$. Since $F'(x)$ is continuous and convex in the closed contiguous intervals of $\Pi_1\delta$, $F'(x)$ is u. s. c. in $\delta$. Further, $D^3F(x)\geqslant \mathrm{Min}\,(K,0)$ for $x\subset\Pi_1\delta$, while by lemma 10, this holds also for $x\subset\delta.\,C\Pi_1\delta$. By lemma 8, $\bar{D}^3F(x)$ exists p. p. in $\delta$, and so $\bar{D}^3F(x)>0$ p. p. in $\delta$. By lemma 6, $\bar{D}^2F'(x)>0$ p. p. in $\delta$ and $\bar{D}^2F'(x)>-\infty$ everywhere in $\delta$. By the above argument it follows that $F'(x)$ is continuous and convex in the closed contiguous intervals of $\Pi_2$ the perfect kernel of $\Pi_1$. By transfinite induction, we manifestly arrive at the desired result.

4. **Theorem II.** *If $F(x)$ is continuous in $(a,b)$, $F'(x)$ has a unique value, $\bar{D}^3F(x)\geqslant f(x)$ where $f(x)$ is integrable $D$, (i. e. in the sense of Denjoy-Perron), and $\underline{D}^3F(x)$ is finite, then $\underline{D}^3F(x)$ is integrable $D$ in $(a+\epsilon,\,b-\epsilon)$ for $0<\epsilon<(b-a)/2$, and for $a<\alpha<\beta<b$,*

$$F'(x)=\int_{\alpha}^{x}dy\int_{\alpha}^{y}\underline{D}^3F(t)\,dt+px+q. \qquad (\alpha\leqslant x\leqslant\beta)$$

**Lemma 11.** *Let $g(t)$ be defined in the neighbourhood of $t=x$. Let $g'(t)$ be continuous and $D^+g'(t)$ be bounded in that neighbourhood. Then*

$$\bar{D}\,D^+g'(x)\geqslant\bar{D}^3g(x)\geqslant\underline{D}^3g(x)\geqslant\underline{D}\,D^+g'(x).$$



It can be verified that

$$\frac{\Delta^3g(x,h)}{(2h)^3}=\frac{3}{(2h)^3}\int_0^h dy\int_y^{3y}\{D^+g'(x+z)-D^+g'(x-z)\}\,dz,$$

so that the result follows by the mean value theorem.

Turning to the proof of the theorem, we may suppose that $f(x)$ is finite. Let $m(x)$ be a minor function of $f(x)$ in $(a,b)$. Write

$$\mu(x)=\int_a^x m(t)\,dt,\quad M(x)=\int_a^x \mu(t)\,dt.$$

At every point,

$$\overline{\lim_{h\to}}\ \frac{m(x+h)-m(x-h)}{2h}\leqslant f(x).$$

By lemma 11,

$$\bar{D}^3M(x)\leqslant f(x).$$

Write $G(x)=F(x)-M(x)$. Then,

$$\bar{D}^3G(x)\geqslant\bar{D}^3F(x)-\bar{D}^3M(x)\geqslant0,$$

and

$$\underline{D}^3G(x)\geqslant\underline{D}^3F(x)-\bar{D}^3M(x)>-\infty$$

Further $M'(x)$ and therefore $G'(x)$ has a unique value at every point. By Theorem I, $G'(x)$ is continuous and convex in the open interval $(a,b)$. Let $m(x)$ tend (uniformly) to

$$\int_a^x f(t)\,dt.$$

Then

$$G'(x)=F'(x)-\int_a^x m(t)\,dt$$

tends (uniformly) to

$$H(x)=F'(x)-\int_a^x dy\int_a^y f(t)\,dt,$$

which is accordingly continuous and convex. Consequently, $D^+H(x)$

is non-diminshing in $(a,b)$. Now $D^+H(x)$ must be continuous in the open interval. For if not, there is a point $\xi$ such that

$$\lim D^+H(\xi+h)-D^+H(\xi-h)>0.$$

Then

$$\lim D^+F'(\xi+h)-D^+F'(\xi-h)>0. \tag{5}$$

But

$$D^+F'(x)=D^+H(x)-\int_a^x f(t)\,dt$$

is bounded in the neigh bourhood of $\xi$, and so by an adaptation of the proof of lemma 11, we see that (5) implies $D^3F(\xi)=+\infty$, a contradiction. Thus $H'(x)$ is continuous and non-diminishing. We can therefore write

$$F''(x)=\int_a^x f(t)\,dt+\Lambda(x) \qquad (a<x<b) \tag{6}$$

where $\Lambda(x)$ is continuous and non-diminishing. This equation tells us that $F'''(x)$ exists p. p. in $(a,b)$ and is integrable in $(a+\epsilon, b-\epsilon)$. By lemma 11, $D^3F(x)=F'''(x)$ p. p. Hence $\underline{D}^3F(x)$ is integrable $D$ in $(a+\epsilon, b-\epsilon)$.

Let $a<\alpha<\beta<b$. Consider the interval $(\alpha,\beta)$. We can apply to this interval the above reasoning which lead to (6), replacing $f(x)$ by $\underline{D}^3F(x)$, since the latter is now known to be integrable in $(\alpha,\beta)$. We accordingly arrive at the equation

$$F''(x)=\int_\alpha^x \underline{D}^3F(t)\,dt+\lambda(x), \qquad (\alpha\leqslant x\leqslant\beta)$$

where $\lambda(x)$ is continuous and non-diminishing.

Let now $K(x)$ denote a major function of $\underline{D}^3F(x)$ for the interval $(\alpha,\beta)$. We can write

$$K(x)=\int_\alpha^x \underline{D}^3F(t)\,dt+\sigma(x), \qquad (\alpha\leqslant x\leqslant\beta)$$

where $\sigma(x)$ is continuous and non-diminishing. By lemma 11, $\underline{D}F'''(x)<\overline{D}^3F(x)$, and by the definition of $K(x)$, $\underline{D}K(x)\geqslant \overline{D}^3F(x)$. Hence, since $\underline{D}^3F(x)$ is finite, $\underline{D}[F''(x)-K(x)]<0$. This implies that the function

$$F''(x)-K(x)=\lambda(x)-\sigma(x),$$



which is continuous and of bounded variation, is non-increasing. Let $K(x)$ tend (uniformly) to

$$\int_\alpha^x \underline{D}^3F(t)\,dt.$$

Then $\sigma(x)\to 0$, from which it follows that $\lambda(x)$ is non-increasing. But $\lambda(x)$ is non-diminishing. Hence $\lambda(x)=p$, a constant. Thus

$$F''(x)=\int_\alpha^x \underline{D}^3F(t)\,dt+p, \qquad (\alpha\leqslant x\leqslant\beta)$$

which is the desired result.

5. **Theorem III.** *Let*

$$\sum_1^\infty A_n(x) \qquad a_n=o(n),\ b_n=o(n)$$

*be a trigonometric series. Let $\overline{R}(x)$, $\underline{R}(x)$ denote its upper and lower sums $(R,3)$. Let $\underline{R}(x)$ be finite everywhere, and $\overline{R}(x)\geqslant f(x)$, where $f(x)$ is integrable $D$. Then the series is a* F o u r i e r *series.*

Write

$$F(x)=\Sigma B_n(x)/n^3.$$

Then $\overline{D}^3F(x)\geqslant f(x)$ and $\underline{D}^3F(x)$ is finite. We proceed to show that $F'(x)$ has a unique value. Let

$$\varphi(x,\theta)=\frac{\Delta^3F(x,\theta)}{(2\theta)^3}.$$

Then

$$\varphi(x,\theta)=\sum_1^\infty A_n(x)\left(\frac{\sin n\theta}{n\theta}\right)^3. \tag{7}$$

Since

$$\sum\left|A_n(x)\left(\frac{\sin n\theta}{n\theta}\right)^3\right|<\frac{C}{\theta^3}$$

where $C$ is an absolute constant, we have the equation

$$\int_\xi^\infty \varphi(x,\theta)\,\theta\,d\theta = \sum A_n(x) \int_\xi^\infty \left(\frac{\sin n\theta}{n\theta}\right)^3 \theta\,d\theta. \qquad (\xi > 0)$$

Now

$$\int_\xi^\infty \left(\frac{\sin n\theta}{n\theta}\right)^3 \theta\,d\theta = \frac{1}{n^2}\int_{n\xi}^\infty \left(\frac{\sin \theta}{\theta}\right)^3 \theta\,d\theta$$
$$= \frac{1}{n^2}\psi(n\xi)$$

where

$$\psi(t) = \int_t^\infty \left(\frac{\sin\theta}{\theta}\right)^3 \theta\,d\theta.$$

We have

$$|\psi(t)| < 4/t^2, \qquad (t > 1)$$
$$|\psi(t) - B| < t, \qquad (0 < t \leqslant 1)$$

where

$$B = \int_0^\infty \left(\frac{\sin\theta}{\theta}\right)^3 \theta\,d\theta.$$

Since we are considering a fixed value of $x$, we write $A_n(x)/n^2 = \alpha_n$, and then $\alpha_n = o(^1/n)$. Then

(8) $$\int_\xi^\infty \varphi(x,\theta)\theta\,d\theta = \sum \alpha_n \psi(n\xi).$$

By hypothesis,

$$\varliminf_{\theta\to 0} \varphi(x,\theta) > -\infty.$$

Hence the first number of (8) tends, as $\xi \to 0$, to a finite limit or to $+\infty$. Let $N = N(\xi)$ denote the integer which satisfies $N\xi \leqslant 1 < (N+1)\xi$. Then

$$\varlimsup_{\xi\to 0}\left| B\sum_1^N \alpha_n - \sum_1^\infty \alpha_n\psi(n\xi)\right| \leqslant \varlimsup_{\xi\to 0}\sum_{M+1}^N |\alpha_n(B-\psi(n\xi))| +$$
$$+ \varlimsup_{\xi\to 0}\sum_{N+1}^\infty |\alpha_n\psi(n\xi)|$$
$$< \epsilon N\xi + 4\epsilon\sum_{N+1}^\infty \frac{1}{n^3\xi^2},$$



where $M = M(\epsilon)$ is chosen so that $|n\alpha_n| < \epsilon$ for $n > M$. Thus

$$\sum_1^m \alpha_n = \sum_1^m A_n(x)/n^2$$

tends to a finite limit or to $+\infty$ as $m \to \infty$. Hence, since $A_n(x) = o(n)$,

$$\sum_1^\infty \frac{A_n(x)}{n^2}\left(\frac{\sin n\theta}{n\theta}\right)$$

tends as $\theta \to 0$ to a finite limit or to $+\infty$; i. e. $DF(x)$ has a unique value. Since $a_n = 0(n), b_n = 0(n)$,

$$\lim \Delta^2 F(x,\theta)/\theta = 0.$$

Hence

$$F'(x) = -\sum A_n(x)/n^2$$

has a unique value. Theorem II now gives

$$F'(x) = \int_0^x dy \int_0^y \underline{D}^3 F(t)\,dt + Ax + B$$

for all $x > 0$. Hence the given series is the Fourier series of $\underline{D}^3 F(x)$.

6. In Theorem III, the condition $a_n = o(n), b_n = o(n)$, may be replaced by the more general condition

$$\sum_1^n |a_m| + |b_m| = o(n^2).$$

This, in its turn, may replaced by a still more general condition. Writing $c_n = \frac{1}{2}(a_n - i\,b_n)$, the condition in question is

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{-n}^n c_m e^{imx} = 0$$

for all $x$, where $c_{-m}$ is the conjugate of $c_m$. In order to prove this we require some preliminary theorems.

Given a finite function $F(x)$, if $\{F(x+h) - F(x)\}/h$ tends to a unique limit, finite or infinite, as $h \to 0$ on a set of unit metric density at the origin, we denote the limit by $F_a(x)$.

**Theorem IV.** *If $F(x)$ is defined in the open interval $I$, and (i) $F(x)$ is approximately continuous, (ii) $F(x)$ has the property $R^*$, (iii) $F_a(x)$ has a unique value, then given two points of $I$, $\alpha$ and $\beta > \alpha$, there is a $\xi$ satisfying $\alpha < \xi < \beta$ such that*

$$F(\beta) - F(\alpha) = (\beta - \alpha) F_a(\xi).$$

The point of the theorem lies in the fact that $F_a(x)$ is not restricted to be finite. When $F_a(x)$ is finite, the condition (ii) is superfluous. This has been proved by Khintchine[1].

To prove the theorem, we require three lemmas.

**Lemma 12.** (Denjoy[2]). *If $F(x)$ is approximately continuous $a \leqslant x \leqslant b$, then $F(x)$ takes every value between $F(a)$ and $F(b)$.*

**Lemma 13.** *If $F(x)$ is finite in $a \leqslant x \leqslant b$, and $F_a(x)$ is unique and positive, then $F(x)$ is non-diminishing.*

To each point $x$ of $a \leqslant x < b$ we make correspond a point $x' > x$, $x' \leqslant b$ such that $F(x') > F(x)$ and such that the mean density of points $\zeta$ in $(x, x')$ such that $F(\zeta) > F(x)$, exceeds $1/2$. We construct a Lebesgue chain starting from $a$. If we obtain a sequence of points $(x_n)$ of the chain tending to $\xi \leqslant b$, then since $F(x_{n+1}) > F(x_n)$, $\lim F(x_n)$ has a unique value. We must have $F(\xi) \geqslant \lim F(x_n)$. For if $F(\xi) < \lim F(x_n) - 2\eta$, $\eta > 0$, then for $n \geqslant N$, $F(\xi) < F(x_n) - \eta$, and on a set of $h$ whose metric density on the right at the origin exceeds $1/2$, we would have

$$\frac{F(\xi - h) - F(\xi)}{-h} \to -\infty,$$

contrary to $F_a(\xi) > 0$. If $\xi < b$, the construction of the chain can be continued. If the points $x_\alpha$ have been defined for all ordinals $\alpha < \beta$, and $x_\alpha \to \xi \leqslant b$, we see by a similar argument, that $F(\xi) \geqslant \lim F(x_\alpha)$. Hence $F(b) \geqslant F(a)$. Applying the argument to any interval within $(a, b)$, we obtain the desired result.

**Lemma 14.** *If $F(x)$ is defined in $(a, b)$, and (i) $F(x)$ is approximately continuous, (ii) $F(x)$ is u. s. c., (iii) $F_a(x)$ is unique, then $F_a(x)$ takes every value between any two values which it takes.*

[1]) Khintchine *Fund. Math.* 9 (1927) 212—279 (243).
[2]) Denjoy *Bull. Soc. Math. de France* 43 (1915) 161—248 (179).



We may suppose that $F_a(x)$ takes a positive value, and a negative value and we must then show that it takes the value zero. We can find a pair of points $x_1, x_2$, such that $x_2 > x_1$, $F(x_2) - F(x_1) > 0$; and a pair of points $y_1, y_2$, distinct from $x_1, x_2$, such that $y_2 > y_1$, $F(y_2) - F(y_1) < 0$. By lemma 12, we can find a pair of points $z_1$ and $z_2 > z_1$, such that $F(z_2) - F(z_1) = 0$. In $(z_1, z_2)$, either $F(x)$ is constant, in which case the lemma is proved, or else, since $F(x)$ is u. s. c., there is a $\xi$, $z_1 < \xi < z_2$, at which $F(x)$ attains its maximum in $(z_1, z_2)$. Since

$$\frac{F(\xi + h) - F(\xi)}{h} \leqslant 0, \quad \frac{F(\xi - h) - F(\xi)}{-h} \geqslant 0,$$

for $0 < h \leqslant h(\xi)$, and $F_a(\xi)$ is unique, we must have $F_a(\xi) = F'(\xi) = 0$ This proves the lemma.

We can now prove Theorem IV. It is no loss of generality to suppose that $F(\beta) - F(\alpha) = 0$. We must prove that there is a $\xi$ in $(\alpha, \beta)$ such that $F_a(\xi) = 0$. Suppose on the contrary that $F_a(x) \neq 0$. for $\alpha < x < \beta$.

Since $F(x)$ has the property $R^*$, there is a non-dense closed set $F_1$ in $(\alpha\beta)$ such that if $\delta$ denote a contiguous interval of $F_1$, then $F(x)$ is u. s. c. in $\delta$. By lemma 14. $F_a(x)$ is of one sign in $\delta$, and hence $F(x)$ is monotone in $\delta$ by lemma 13. In virtue of the approximate continuity of $F(x)$, $F(x)$ is monotone and continuous in $\delta$ taken closed. Let $\xi$ be an isolated point of $F_1$. Then $F(x)$ is monotone in each of the intervals $(\xi - \epsilon \leqslant x \leqslant \xi)$, $(\xi \leqslant x \leqslant \xi + \epsilon)$, where $\epsilon$ is chosen so that $\xi$ is the only point of $F_1$ in $(\xi - \epsilon, \xi + \epsilon)$. If now $F(x)$ were not monotone in the same sense in each of these intervals. we would necessarily have $F_a(\xi) = F'(\xi) = 0$, contrary to hypothesis. We thus see that $F(x)$ is monotone and continuous in each of the contiguous intervals. taken closed, of $P_1$ the perfect kernel of $F_1$.

There is a closed set $F_2 \subset P_1$, non-dense in $P_1$, such that if $\delta$ be a contiguous interval of $F_2$, and $d$ be a closed interval interior to $\delta$, then $F(x)$ is u. s. c. on $dP_1$. Since $F(x)$ is monotone and continuous on the closed contiguous intervals of $dP_1$, it attains its maximum in each such contiguous interval at an end point. i. e. a point of $dP_1$. Hence $F(x)$ is u. s. c. on $d$. Hence as before, $F(x)$ is monotone and continuous in $d$. We thus see that $F(x)$ is monotone and continuous in each of the closed contiguous interval of $P_2$, the perfect kernel of $F_2$. Proceeding in this manner, we infer that $F(x)$ is monotone in $(\alpha,\beta)$. Since $F(\beta)=F(\alpha)$, $F(x)$ is constant in $(\alpha,\beta)$. This contradicts $F_a(x)\neq 0$, and proves the theorem.

7. **Theorem V.** *If $F(x)$ is defined in an open interval $I$, and*

*(i) $F(x)$ is approximately continuous and has the property $R^*$,*

*(ii) $F_a(x)<\infty$ is unique and has the property $R^*$.*

*(iii) $\overline{D}^3F(x)>0, \underline{D}^3F(x)>-\infty$,*

*then $F'(x)$ is continuous and $\overline{D}^2F'(x)\geqslant 0$.*

**Lemma 15.** *Under the conditions of the theorem, $\overline{D}^2F_a(x)\geqslant \underline{D}^3F(x)$.*

It is necessary to make a preliminary remark concerning the meaning of $\overline{D}^2F_a(x)$. If $F_a(x)$ is finite, then $\Delta^2F_a(x,h)$ has a unique value for all sufficiently small $h$. If $F_a(x)$ is not finite, then $F_a(x)=-\infty$. There must be arbitrarily small values of $h$ for which $F_a(x+h)+F_a(x-h)$ is finite, and it is for such $h$ that $\overline{\lim}\, \Delta^2F_a(x,h/2)/h^2$ is to calculated. We then obtain $\overline{D}^2F_a(x)=+\infty$. To see that there are arbitrarily small $h$ with the stated property, we observe that $F_a(x+h)+F_a(x-h)$ is the unique approximate derivative of $\varphi(h)=F(x+h)-F(x-h)$. If for every $h$ satisfying $0<h\leqslant h_0, \varphi_a(h)$ were not finite, we would have $\varphi_a(h)=-\infty$ in $0<h\leqslant h_0$. By lemma 13, this would imply $\varphi(h_2)-\varphi(h_1)<K(h_2-h_1)$ for $0<h_1<h_2<h_0$ and all $K$, which is impossible.

To prove the lemma, we may suppose that at the given point $x$, $\underline{D}^3F(x)>0$, and it is then sufficient to show that $\overline{D}^2F_a(x)\geqslant 0$. We have, for all sufficiently small $h$,

$$\frac{F(x+h)-F(x-h)}{2h}>\frac{F(x+h/3)-F(x-h/3)}{(2h/3)}. \tag{9}$$

If $F_a(x)=-\infty$, then in virtue of the above remark, $\overline{D}^2F_a(x)=+\infty$ We may therefore suppose that $F_a(x)$ is finite. If now $\overline{D}^2F_a(x)<0$, there is an $\eta>0$, such that for $0<h\leqslant h(\eta)$,

$$F_a(x+h)+F_a(x-h)-2F_a(x)<-\eta h^2. \tag{10}$$

By Theorem IV, there is a $\theta=\theta(h)$ satisfying $0<\theta<1$, such that

$$F(x+h)-F(x-h)=h[F_a(x+\theta h)+F_a(x-\theta h)].$$



Hence by (10),

$$\frac{F(x+h)-F(x-h)}{2h}<F_a(x)-\frac{\eta}{2}h^2, \qquad (0<h\leqslant h(\eta)$$

and so

$$\frac{F(x+h)-F(x-h)}{2h}<F_a(x)-\frac{\eta}{2}\left(\frac{h(\eta)}{3}\right)^2. \qquad \left(\frac{h(\eta)}{3}\leqslant h\leqslant h(\eta)\right)$$

By (9), the preceding inequality holds also for $0<h\leqslant h(\eta)/3$. Hence

$$F_a(x)\leqslant F_a(x)-\frac{\eta}{2}\left(\frac{h(\eta)}{3}\right)^2$$

which is impossible since $F_a(x)$ is finite.

**Lemma 16.** *Under the conditions of the theorem, if $F_a(x)$ is u. s. c. in $(a,b)$, and $\underline{D}^3F(x)>K$ in that interval, then $D^3F(x)$ exists p. p.*

We may suppose that $K=0$. Then $D^2F_a(x)>0$ by lemma 15. We first show that $F_a(x)$ is finite in $d=(a,b)$. If not, there are two possibilities.

(i) There exist two points $\alpha$ and $\beta>\alpha$, such that $F_a(\alpha)=-\infty$, $F_a(\beta)=-\infty$. We can then find $\gamma$ and $\delta$ such that $\gamma<\alpha<\beta<\delta$, and such that $F_a(\gamma), F_a(\delta)$ are finite, while $F_a(\gamma)<-N, F_a(\delta)<-N$, where $N$ is arbitrarily assigned. For as regards $\delta$, we can find an arbitrarily small $h>0$ such that

$$\frac{F(\beta+h)-F(\beta)}{h}<-N,$$

and by Theorem IV there is a $\theta$, $0<\theta<1$, such that

$$\frac{F(\beta+h)-F(\beta)}{h}=F_a(\beta+\theta h).$$

Similarly for $\gamma$. By lemma 3, $F_a(x)<-N$ in $(\alpha,\beta)$, and so by lemma 13, $F(\beta)-F(\alpha)<-N(\beta-\alpha)$ for all $N$, which is impossible.

(ii) There is only one point $\alpha$ such that $F_a(x)=-\infty$. Then $F_a(x)$ being finite and u. s. c. in $(\alpha,b)$, it is convex in that interval by lemma 3, and continuous by lemma 4. Manifestly, $F_a(x)=F'(x)$ in $\alpha<x<b$, and $F'(\alpha+0)$ is finite or $+\infty$. For a set

of $h>0$ of unit metric density on the right at the origin,

$$F_a(\alpha)=\lim \frac{F(\alpha+h)-F(\alpha)}{h};$$

and by Theorem IV, to each such $h$ there corresponds a $\theta, 0<\theta<1$, such that

$$\frac{F(\alpha+h)-F(\alpha)}{h}=F_a(\alpha+\theta h)=F'(\alpha+\theta h).$$

Hence $F_a(\alpha)=F'(\alpha+0)$, which contradicts $F_a(\alpha)=-\infty$. Thus $F_a(x)$ is finite in $d$. By lemmas 3 and 4, it is convex and continuous, so that $F_a(x)=F'(x)$. The remainder of the proof is the same as for lemma 8.

**Lemma 17.** *If under the conditions of the theorem, $F_a(x)$ has the property $R^*$ in $(a,b)$, $F_a(x)<\infty$. $\overline{D}^2F_a(x)>0$ p.p., $\underline{D}^2F_a(x)>-\infty$, then $F_a(x)$ is continuous and convex.*

In virtue of the preceding argument, the proof is an obvious adaptation of the proof of lemma 9.

We can now prove Theorem V by adapting the proof of Theorem I and using the preceding three lemmas instead of lemmas 6, 8, 9 respectively. We do not require an analogue of lemma 7 since we have supposed explicitly that $F_a(x)$ has the property $R^*$.

**Theorem VI.** *If $F(x)$ is defined in $a<x<b$, and*

(i) *$F(x)$ is approximately continuous and has the property $R^*$,*

(ii) *$F_a(x)<\infty$ is unique and has the property $R^*$,*

(iii) *$\overline{D}^3F(x)\geqslant f(x)$, where $f(x)$ is integrable $D$, and $\underline{D}^3F(x)$ is finite,*

*then $\underline{D}^3F(x)$ is integrable in $(a+\epsilon, b-\epsilon)$ for $0<\epsilon<(b-a)/2$, and for $a<\alpha<\beta<b$,*

$$F_a(x)=F'(x)=\int_\alpha^x dy\int_\alpha^y \underline{D}^3F(t)\,dt+px+q. \qquad (\alpha\leqslant x\leqslant\beta)$$

The proof is similar to that of Theorem II; we use Theorem V instead of Theorem I.

8. **Lemma 18.** *Let $\Sigma A_n(x)$ be a trigonometric series such that*

$$\lim \frac{1}{n^2}\sum_1^n B_m(x)=0$$



*for all $x$. Then*

$$F(x)=\sum B_n(x)/n^3$$

*is convergent.*

Let

$$\sigma_n(x)=\sum_1^n B_m(x),$$

so that for any assigned $x$, $\sigma_n(x)=o(n^2)$. Then

$$\sum_p^q B_n(x)/n^3=\sum_p^q \frac{\sigma_n(x)-\sigma_{n-1}(x)}{n^3}$$

$$=-\frac{\sigma_{p-1}(x)}{p^3}+\sum_{n=p}^{q-1}\sigma_n(x)\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right)+\frac{\sigma_q(x)}{q^3}$$

$$=o\left(\frac{1}{p}\right)+o\left(\sum_p^{q-1}\frac{1}{n^2}\right)+o\left(\frac{1}{q}\right),$$

whence the result.

Thus, given a trigonometric series which satisfies the conditions of lemma 18, we can form the expression

$$\frac{\Delta^3F(x,h)}{(2h)^3}=\sum_1^\infty A_n(x)\left(\frac{\sin nh}{nh}\right)^3.$$

We now have.

**Theorem VII.** *Let $\Sigma A_n(x)$ be a trigonometric sories such that*

(11) $$\lim \frac{1}{n^2}\sum_{-n}^n c_m e^{imx}=0$$

*for all $x$. Let $\overline{R}(x)$, $\underline{R}(x)$ denote its upper and lower sums $(R,3)$. Let $\underline{R}(x)$ be finite everywhere, and $\overline{R}(x)\geqslant f(x)$, where $f(x)$ is integrable $D$. Then the series is a* F o u r i e r *series.*

The condition (11) implies that

(12) $$\lim \frac{1}{n^2}\sum_1^n A_m(x)=0,$$

(13) $$\lim \frac{1}{n^2}\sum_1^n B_m(x)=0.$$

By (13) and lemna 18, we can write

$$F(x) = \sum \frac{B_n(x)}{n^3}. \tag{14}$$

**Lemma 19.** *If* $\alpha_n = o(1), \beta_n = o(1)$, *then*

$$\Phi(x) \sim -\sum (\beta_n \cos nx - \alpha_n \sin nx)/n$$

*is approximately continuous at any point at which the series converges.*

In this lemma, $\Phi(x)$ is defined to be equal to the sum of the series wherever the latter converges, i. e. almost everywhere. We may suppose that the point of convergence under consideration is $x=0$. It is no loss of generality to suppose that

$$\sum \frac{\beta_n}{n} = 0, \tag{15}$$

and we must prove that $\Phi(x) \to 0$ approximately as $h \to 0$, for which it is sufficient to show that

$$\varphi_1(h) = \sum \beta_n \frac{\cos nh}{n},$$

$$\varphi_2(h) = \sum \alpha_n \frac{\sin nh}{n},$$

each tend to 0 approximately. Write

$$s_n = \sum_1^n \frac{\beta_m}{m}.$$

Let $[x]$ denote the greatest integer which does not exceed $x$. Write

$$\nu = \nu(h) = [h^{-1}].$$

Then

$$\varphi_1(h) = \sum_1^{\nu} + \sum_{\nu+1}^{\infty}$$
$$= \tau_1(h) + \tau_2(h).$$

Let $\epsilon$ satisfy

$$1 > \epsilon > 0, \qquad \epsilon\left(\log\frac{1}{\epsilon}\right)^2 < 1.$$



There is an $N = N(\epsilon)$, such that for $n \geqslant N, |s_n| < \epsilon$. Then

$$\tau_1(h) = \sum_1^N \frac{\beta_n}{n} \cos nh + \sum_{N+}^{\nu} (s_n - s_{n-1}) \cos nh$$
$$= \sum_1^N \frac{\beta_n}{n} \cos nh - s_N \cos Nh + \sum_{N+1}^{\nu} s_n(\cos nh - \cos(n+1)h) +$$
$$+ s_\nu \cos \nu h.$$

In $(0,1)$, $\cos x$ diminishes. Hence

$$|\tau_1(h)| < \left|\sum_1^N \frac{\beta_n}{n} \cos nh\right| + 3\epsilon$$
$$< 4\epsilon$$

for $0 < h \leqslant h(\epsilon)$. We can choose $h(\epsilon)$ so that

$$|\beta_n| < \epsilon^2 \qquad n \geqslant [h^{-1}(\epsilon)].$$

Write

$$\mu = [h^{-1}(\epsilon)], \quad \lambda = [(\epsilon h(\epsilon))^{-1}],$$

and

$$G(h) = G(\epsilon, h) = \sum_{\mu}^{\infty} \frac{\beta_n}{n} \cos nh.$$

Let $E$ denote the set of points in $\epsilon h(\epsilon) \leqslant h \leqslant h(\epsilon)$ at which $|\tau_2(h)| > \epsilon$. For $h$ in this interval, we have

$$|\tau_2(h) - G(h)| < \sum_{\mu}^{\lambda} \frac{|\beta_n|}{n} < 2\epsilon^2 \log\frac{1}{\epsilon},$$

and so

$$\epsilon^2 m E < \int_{\epsilon h(\epsilon)}^{h(\epsilon)} \tau_2^2(h)\,dh < 2\int_{\epsilon h(\epsilon)}^{h(\epsilon)} [\tau_2(h) - G(h)]^2 dh + 2\int_0^{\pi} G^2(h)\,dh$$
$$< 8h(\epsilon)\epsilon^4\left(\log\frac{1}{\epsilon}\right)^2 + 2\pi \sum_{\mu}^{\infty} \frac{\beta_n^2}{n^2}$$
$$< 8h(\epsilon)\epsilon^3 + 2\pi h(\epsilon)\epsilon^4$$
$$< 10h(\epsilon)\epsilon^3$$

for relevant $\epsilon$. Hence the set of points in $(-h(\epsilon), h(\epsilon))$ at which $|\tau_2(h)| > \epsilon$ is of measure less than $2\epsilon h(\epsilon) + 20\epsilon h(\epsilon)$. Hence the set of points in $(-h(\epsilon), h(\epsilon))$ at which $|\varphi_1(h)| > 5\epsilon$ is of measure less than $22\epsilon h(\epsilon)$, so that $\varphi_1(h) \to 0$ approximately.

For $\varphi_2(h)$ we write

$$\varphi_2(h) = \sum_1^{\nu} + \sum_{\nu+1}^{\infty}$$
$$= L_1(h) + L_2(h).$$

The function $L_2(h)$ is treated in precisely the same fashion as $\tau_2(h)$. Further,

$$L_1(h) = \sum_1^N + \sum_{N+1}^{\nu},$$

where $N$ is chosen so that $|a_n| < \epsilon$ for $n \geqslant N$. Then

$$|L_1(h)| \leqslant \left|\sum_1^N\right| + h\epsilon \sum_{N+1}^{\nu} 1$$
$$\leqslant 2\epsilon$$

for $0 < h \leqslant h(\epsilon)$, so that the conclusion follows as before.

**Lemma 20.** *The function $F(x)$ defined by* (14) *is approximately continuous, and has the property $R$, and, a fortiori, the property $R^*$.*

Since the coefficients in (14) are $o(1/n)$, and (14) is everywhere convergent, it follows by lemma 19, that $F(x)$ is approximately continuous. To prove that $F(x)$ has the property $R$, we observe that there is a finite positive function $M(x)$ such that for each $x$,

$$\left|\sum_1^n B_m(x)\right| < n^2 M(x). \qquad (n = 1, 2, \dots)$$

The proof of lemma 18 gives

$$\left|\sum_{p+1}^{\infty} \frac{B_n(x)}{n^3}\right| < M(x)\frac{2}{p}.$$

Write

$$F_p(x) = \sum_1^p \frac{B_n(x)}{n^3}.$$

Then $F_p(x)$ is continuous, $F_p(x) \to F(x)$ as $p \to \infty$, and

$$|F_p(x) - F_q(x)| < M(x)\left(\frac{2}{p} + \frac{2}{q}\right).$$



Let $E_m$, where $m$ is a positive integer, denote the closed set of points in $(0, 2\pi)$ at which

$$|F_p(x) - F_q(x)| \leqslant \frac{4m}{p}. \quad (q = p+1, p+2, \dots)$$

Since $M(x)$ is finite, $\Sigma E_m = (0, 2\pi)$, and there is uniform convergence in each $E_m$. Given any perfect set $P \subset (0, 2\pi)$, there is an $m$ such that $E_m$ contains a portion $\tilde{\omega}$ of $P$. On this portion there is uniform convergence. Hence $F(x)$ is continuous on $\tilde{\omega}$, and thus has the property $R$.

9. **Lemma 21.** *The series $-\Sigma A_n(x)/n^2$ is either convergent, or properly divergent to $-\infty$; i. e.*

$$\lim_{m\to\infty} \sum_1^m \frac{A_n(x)}{n^2}$$

*is finite or $+\infty$.*

Since for a given $x$, $A_n(x) = o(n^2)$, the series

$$f(\theta) \sim \sum_1^{\infty} \frac{A_n(x)}{n^3} \sin 3n\theta$$

is a Fourier series. It may be multiplied by $1/\theta^2$ and integrated term over $(\xi, \infty)$, where $\xi > 0$. The same is true of

$$\varphi(\theta) \sim \sum_1^{\infty} \sin n\theta,$$

and therefore of

$$\frac{3\varphi(\theta) - f(\theta)}{4} = \sum_1^{\infty} \frac{A_n(x)}{n^3} \sin^3 n\theta.$$

We have, however, by the definition of $F(x)$,

$$\frac{\Delta^3 F(x, \theta)}{8} = \sum_1^{\infty} \frac{A_n(x)}{n^3} \sin^3 n\theta.$$

Hence

$$\int_{\xi}^{\infty} \frac{\Delta^3 F(x, \theta)}{(2\theta)^3} \theta\, d\theta = \sum_1^{\infty} \frac{A_n(x)}{n^2} \int_{\xi}^{\infty} \frac{\sin^3 n\theta}{\theta^2}\, d\theta$$
$$= \sum_1^{\infty} \frac{A_n(x)}{n^3} \int_{n\xi}^{\infty} \left(\frac{\sin\theta}{\theta}\right)^3 \theta\, d\theta.$$

By hypothesis,

$$\lim_{\theta\to 0}\frac{\Delta^3 F(x,\theta)}{(2\theta)^3}>-\infty.$$

Hence

$$\sum_1^\infty\frac{A_n(x)}{n^2}\int_{n\xi}^{\infty}\left(\frac{\sin\theta}{\theta}\right)^3\theta\,d\theta$$

tends as $\xi\to 0$ to a finite limit, or to $+\infty$.

We now use the notation of the proof of Theorem III. Let $\psi(t)$ and $B$ and $N=N(\xi)$ have the meanings there given. We have

$$\overline{\lim_{\xi\to 0}}\left|B\sum_1^N\frac{A_n(x)}{n^2}-\sum_1^\infty\frac{A_n(x)}{n^2}\psi(n\xi)\right|\leqslant\overline{\lim}\left|\sum_{M+1}^N\frac{A_n(x)}{n^2}(B-\psi(n\xi))\right|+$$
$$+\overline{\lim}\left|\sum_{N+1}^\infty\frac{A_n(x)}{n^2}\psi(n\xi)\right|$$
$$\leqslant\tau_1+\tau_2,$$

where $M<N$ is an assigned integer. Choose $M=M(\epsilon,x)$ so that

$$\left|\frac{1}{n^2}\sum_1^n A_m(x)\right|<\epsilon, \qquad (n>M)$$

and write

$$s_n=s_n(x)=\sum_1^n A_m(x).$$

Then

$$\tau_1=\overline{\lim}\left|\sum_{M+1}^N(s_n-s_{n-1})\frac{B-\psi(n\xi)}{n^2}\right|$$
$$\leqslant\overline{\lim}\left|s_M\frac{B-\psi(M+1)\xi}{(M+1)^2}\right|+\overline{\lim}\left|\sum_{M+1}^{N-1}s_k\left\{\frac{B-\psi(n\xi)}{n^2}-\right.\right.$$
$$\left.\left.-\frac{B-\psi(n+1)\xi}{(n+1)^3}\right\}\right|+\overline{\lim}\left|s_N\frac{B-\psi(N\xi)}{N^2}\right|$$
$$\leqslant H_1+H_2+H_3.$$

Since $|\psi(t)-B|<t$, we have

$$|H_1|<\epsilon(M+1)\xi<\epsilon;\quad |H_3|<\epsilon N\xi<\epsilon;$$
$$|H_2|<\overline{\lim}\,\epsilon\sum_{M+1}^{N-1}n^2\left|\left\{\frac{B-\psi(n\xi)}{n^2}-\frac{B-\psi(n+1)\xi}{(n+1)^2}\right\}\right|.$$



Now

$$\left|\frac{B-\psi(n\xi)}{n^2}-\frac{B-\psi(n+1)\xi}{(n+1)^2}\right|=\left|\frac{\{B-\psi(n\xi)\}-\{B-\psi(n+1)\xi\}}{n^2}+\right.$$
$$\left.+\{B-\psi(n+1)\xi\}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)\right|$$
$$\leqslant\left|\frac{\psi(n+1)\xi-\psi(n\xi)}{n^2}\right|+\frac{2|B-\psi(n+1)\xi|}{n^3}.$$

Further,

$$|\psi(n+1)\xi-\psi(n\xi)|=\left|\int_{n\xi}^{(n+1)\xi}\left(\frac{\sin\theta}{\theta}\right)^3\theta\,d\theta\right|$$
$$=\left|\xi\left(\frac{\sin\zeta}{\zeta}\right)^3\zeta\right| \qquad n\xi<\zeta<(n+1)\xi$$
$$<2n\xi^2.$$

Hence

$$|H_2|<\overline{\lim}\,\epsilon\sum_{M+1}^{N-1}\left[2n\xi^2+\frac{2(n+1)\xi}{n}\right]$$
$$<\overline{\lim}\,\epsilon\sum_{M+1}^{N-1}(2n\xi^2+4\xi)<6\epsilon.$$

Finally,

$$\tau_2=\overline{\lim}\left|\sum_{N+1}^\infty(s_n-s_{n-1})\frac{\psi(n\xi)}{n^2}\right|$$
$$\leqslant\overline{\lim}\left|\frac{s_N\psi(n\xi)}{N^2}\right|+\overline{\lim}\left|\sum_{N+1}^\infty s_n\left(\frac{\psi(n\xi)}{n^2}-\frac{\psi(n+1)\xi}{(n+1)^2}\right)\right|.$$

The first term is zero. Is evaluate the second, we observe that

$$\left|\frac{\psi(n\xi)}{n^2}-\frac{\psi(n+1)\xi}{(n+1)^3}\right|\leqslant\left|\frac{\psi(n\xi)-\psi(n+1)\xi}{n^2}\right|+\frac{2|\psi(n+1)\xi|}{n^3}$$
$$\leqslant\frac{1}{n^2}\int_{n\xi}^{(n+1)\xi}\frac{d\theta}{\theta^2}+\frac{8}{n^5\xi}$$
$$\leqslant\frac{1}{n^4\xi}+\frac{8}{n^5\xi}<\frac{9}{n^4\xi},$$

so that

$$\overline{\lim}\sum_{N+1}^\infty s_n\left(\frac{\psi(n\xi)}{n^2}-\frac{\psi(n+1)\xi}{(n+1)^2}\right)<\epsilon\,\overline{\lim}\sum_{N+1}^\infty\frac{9}{n^2\xi}<9\epsilon.$$

We have thus proved that

$$\lim_{\xi\to 0}\left[B\sum_{1}^{N}\frac{A_n(x)}{n^2}-\sum_{1}^{\infty}\frac{A_n(x)}{n^2}\psi(n\xi)\right]=0;$$

and since we have shown that the sum of the infinite series tends to a finite limit or to $+\infty$ as $\xi\to 0$, it follows that

$$\lim_{N\to\infty}\sum_{1}^{N}\frac{A_n(x)}{n^2}$$

is finite or $+\infty$. This proves the lemma.

We can now write

$$G(x)=-\sum_{1}^{\infty}\frac{A_n(x)}{n^2} \tag{16}$$

where $G(x)$ is finite or $-\infty$.

**Lemma 22.** *The function* $G(x)$ *defined by* (16) *possesses the property* $R^*$.

We have from the preceding proof

$$G(x)=\lim_{\xi\to 0}G(x,\xi)$$

where

$$G(x,\xi)=-\int_{\xi}^{\infty}\frac{\Delta^3F(x,\theta)}{(2\theta)^3}=-\sum_{1}^{\infty}\frac{A_n(x)}{n^2}\int_{n\xi}^{\infty}\left(\frac{\sin\theta}{\theta}\right)^3\theta\,d\theta.$$

Now

$$\left|\int_{n\xi}^{\infty}\left(\frac{\sin\theta}{\theta}\right)\theta\,d\theta\right|<\frac{K}{n^2\xi^2}.$$

Since $a_n=o(n^2)$, $b_n=o(n^2)$, it follows that $G(x,\xi)$ is a continuous function of $x$ for each $\xi>0$. We have

$$G(x,\xi')-G(x,\xi)=-\int_{\xi'}^{\xi}\frac{\Delta^3F(x,\theta)}{(2\theta)^3}\theta\,d\theta. \qquad (0<\xi'<\xi)$$

We are given that

$$\lim_{\theta\to 0}\frac{\Delta^3F(x,\theta)}{(2\theta)^3}>-\infty.$$



Hence, to each $x$ there corresponds a $\theta(x)>0$ and a $K(x)$, $0<K(x)<\infty$, such that

$$-\frac{\Delta^3F(x.\theta)}{(2\theta)^3}<K(x). \qquad (0<\theta\leqslant\theta(x).$$

Thus

$$G(x,\xi')-G(x,\xi)\leqslant\xi K(x), \qquad (0<\xi'<\xi\leqslant\xi(x)$$

or

$$G\left(x,\frac{1}{n+p}\right)-G\left(x,\frac{1}{n}\right)\geqslant\frac{1}{n}K(x). \qquad (n\geqslant n(x), p=1,2,\ldots)$$

We suppose that $n(x)$ is the least positive integer for which the preceding relation holds. Let $H_m$ be the set of points $x$ in $(0,\ 2\pi)$ at whiah $n(x)\leqslant m$, $K(x)\leqslant m$, where $m$ is a positive integer. Then at every point of $H_m$

$$G\left(x,\frac{1}{n+p}\right)-G\left(x,\frac{1}{n}\right)\leqslant\frac{m}{n}, \qquad (n\geqslant m,\ p=1,2,\ldots) \tag{17}$$

and, since $K(x)$ is finite, $\Sigma H_m=(0,2\pi)$. Let $E_m=H_m+H'_m$. Since $G(x,\xi)$ is a continuous function of $x$, (14) holds for $x\subset E_m$. Now $E_m$ is closed and $\Sigma E_m=(0,2\pi)$. Hence, given a perfect set $\Pi$ in $(0,2\pi)$, there is a portion $\tilde{\omega}$ of $\Pi$ such that for some $m$, $E_m$ contains $\tilde{\omega}$. Then (14) holds for $x\subset\tilde{\omega}$. Letting $p\to\infty$, we get

$$G(x)-G\left(x,\frac{1}{n}\right)\leqslant\frac{m}{n} \qquad (n\geqslant m)$$

for $x\subset\tilde{\omega}$; or

$$-G(x)\geqslant-G\left(x,\frac{1}{n}\right)-\frac{m}{n} \qquad (n\geqslant m)$$

for $x\subset\tilde{\omega}$. By II lemma 11, $-G(x)$ is lower semi-continuous on $\tilde{\omega}$. This proves the lemma.

**10. Lemma 23** (Rajchman and Zygmund)[1]. *If* $\lim a_n=0(C,1)$, *then*

$$\varphi(\theta)\sim\sum_{1}^{\infty}a_n\frac{\sin^2 n\theta}{n^2\theta}$$

*tends to* 0 *approximately as* $\theta\to 0$.

[1] Rajchman and Zygmund *Bull. Polonaise* (1925) 69—80.

**Lemma 24** (Rajchman and Zygmund)[1]. *If $\Sigma\alpha_n$ convergest hen*

$$\sum \alpha_n \frac{\sin n\theta}{n\theta}$$

*tends to $\Sigma\alpha_n$ approximately as $\theta\to 0$.*

**Lemma 25.** *The function $F(x)$ has a unique approximate derivative $F_a(x)=G(x)$.*

We are given that

$$\lim \frac{1}{n^3}\sum_1^n B_m(x)=0$$

for each $x$. For a particular $x$, write $\sigma_n=\sum_1^n B_m(x)$. Then

$$\sum_1^n \frac{B_m(x)}{m}=\sum_1^n(\sigma_m-\sigma_{m-1})\frac{1}{m}$$
$$=\sum_1^{n-1}\sigma_m\left(\frac{1}{m}-\frac{1}{m+1}\right)+\frac{\sigma_n}{n}$$
$$=o(n)$$

Thus

$$\lim \frac{B_n(x)}{n}=0. \qquad (C,1)$$

Hence by lemma 23,

$$\sum \frac{B_n(x)\sin^2 n\theta}{n^3\quad\theta}$$

tends to 0 approximately as $\theta\to 0$; i. e.

$$\frac{F(x+\theta)+F(x-\theta)-2F(x)}{\theta}\to 0$$

approximately. To prove the lemma, it is therefore sufficient to show that

$$\frac{F(x+\theta)-F(x-\theta)}{2\theta}\to G(x)$$

approximately; i. e. that

$$\sum_1^\infty \frac{A_n(x)}{n^2}\left(\frac{\sin n\theta}{n\theta}\right)\to\sum_1^\infty\frac{A_n(x)}{n^2}$$

[1]) Rajchman and Zygmund, *loc. cit.*



approximately. Now if $\Sigma A_n(x)/n^2$ converges, the result follows from lemma 24. We may therefore suppose that

$$\lim_{N\to\infty}\sum_1^N\frac{A_n(x)}{n^2}=+\infty.$$

Let

$$s_n=s_n(x)=\sum_1^n A_m(x),$$

$$S_n=S_n(x)=\sum_1^n\frac{A_m(x)}{m^2},$$

so that

$$S_n\to+\infty.$$

Write

$$\lambda=[\theta^{-1}].$$

Then,

$$H(\theta)=\sum_1^\infty\frac{A_n(x)}{n^2}\left(\frac{\sin n\theta}{n\theta}\right)$$
$$=\sum_1^\lambda+\sum_{\lambda+1}^\infty=H_1(\theta)+H_2(\theta).$$

Now

$$H_1(\theta)=\sum_1^N+\sum_{N+1}^\lambda=\tau_1(\theta)+\tau_2(\theta),$$

where $N$ is an assigned integer. We have

$$\tau_2(\theta)=\sum_{N+1}^\lambda(S_n-S_{n-1})\frac{\sin n\theta}{n\theta}$$

$$=-S_N\frac{\sin N\theta}{N\theta}+\sum_{N+1}^{\lambda-1}S_n\left(\frac{\sin n\theta}{n\theta}-\frac{\sin(n+1)\theta}{(n+1)\theta}\right)+S_n\frac{\sin\lambda\theta}{\lambda\theta}.$$

Chose $N=N(\epsilon)$ so that $S_n>1/\epsilon$ for $n\geqslant N$. Then

$$H_1(\theta)>\tau_1(\theta)-S_N\frac{\sin N\theta}{n\theta}+\frac{1}{\epsilon}\sum_{N+1}^{\lambda-1}\frac{\sin n\theta}{n\theta}-\frac{\sin(n+1)\theta}{(n+1)\theta}\Big)+\frac{1}{2\epsilon}.$$

Further, for $0 < \theta \leqslant \theta(\epsilon)$, $\tau_1(\theta) > S_N - \epsilon$, so that $H_1(\theta) > \frac{1}{2\epsilon}$ for $0 < \theta \leqslant \theta(\epsilon)$. It remains then to prove that $H_2(\theta) \to 0$ approximately. We have

$$H_2(\theta) = \sum_{\lambda+1}^{\infty} (s_n - s_{n-1}) \frac{\sin n\theta}{n^3 \theta}$$
$$= -s_\lambda \frac{\sin \lambda\theta}{\lambda^3 \theta} + \sum_{\lambda+1}^{\infty} s_n \left( \frac{\sin n\theta}{n^3 \theta} - \frac{\sin (n+1)\theta}{(n+1)^3 \theta} \right).$$

Since, $s_\lambda = o(\lambda^2)$, the first term tends to zero as $\theta \to 0$. Further

$$\frac{\sin n\theta}{n^3} - \frac{\sin (n+1)\theta}{(u+1} = \sin n\theta \left( \frac{1}{n^3} - \frac{1}{(n+1)} \right) +$$
$$+ \frac{\sin n\theta (1 - \cos \theta)}{n+1)^3} - \frac{\cos n\theta \sin \theta}{(n+1)^3}.$$

Hence

$$\sum_{\lambda+1}^{\infty} s_n \left( \frac{\sin n\theta}{n^3 \theta} - \frac{\sin (n+1)\theta}{(n+1)^3} \right) = \sum s_n \frac{\sin n\theta}{\theta} \left( \frac{1}{n^3} - \frac{1}{(n+1)^3} \right) +$$
$$+ \frac{1 - \cos \theta}{\theta} \sum s_n \frac{\sin n\theta}{(n+1)^3} + \frac{\sin \theta}{\theta} \sum s_n \frac{\cos n\theta}{(n+1)^3}$$
$$= K_1(\theta) + \frac{1 - \cos \theta}{\theta} K_2(\theta) + \frac{\sin \theta}{\theta} K^3(\theta).$$

We have

$$|K_1(\theta)| = o\left( \theta^{-1} \sum_{\lambda+1}^{\infty} \frac{1}{n^2} \right) = o(1),$$

and it remains to prove that $K_2(\theta) \to 0$ approximately, and that $K_3(\theta) \to 0$ approximately. Observing that $s_n/(n+1)^3$ can be written $\epsilon_n/n$, where $\epsilon_n \to 0$, we see that the treatment in lemma 19 of $\tau_2(h), L_2(h)$, establishes the rrsult.

We have now proved that the function $F(x)$ defined by (14) possesses in every interval all the properties required by Theorem IV. By that theorem, we accordingly have for every $x > 0$,

$$F'(x) = \int_0^x dy \int_0^y \underline{D}^3 F(t)\, dt + 2Ax + B.$$



Thus

$$\sum B_n(x)/n^3$$

is the Fourier series of

$$F(x) = \int_0^x dy \int_0^y dz \int_0^z \underline{D}^3 F(t)\, dt + Ax^2 + Bx + C,$$

which is equivalent io the desired result.

## III. The fourth symmetric derivative.

**11.** A measurable function $F(x)$ defined in $a < x < b$ has a unique second derivative $F''(x)$ in that interval if (*i*) $F'(x)$ is unique in that interval, i. e.

$$\lim_{h \to} \frac{F(x+h) - F(x)}{h}$$

is unique, and (*ii*), if

$$(18) \qquad \lim_{h \to 0} \frac{F'(x+h) - F'(x)}{h} = F''(x)$$

is unique. The second condition requires $F'(x)$ to be finite. If for example $F'(x) = -\infty$, there are positive values of $h$, arbitrarily small, such that $F'(x+h)$ is finite. Hence there is a sequence of $h > 0$, which tends to zero such that the limit (18) is $+\infty$ There are also positive values of $h$, arbitrarily small, such that $F'(x+h)$ is finite; and therefore a sequence of $h > 0$ which tends to zero, such that

$$\lim \frac{F'(x-h) - F'(x)}{-h} = -\infty.$$

This proves our assertion. Consequently, $F(x)$ is continuous in $a < x < b$.

**Theorem VIII.** *If $F(x)$ is continuous in the open interval $I$, and $F'(x)$ is finite in that interval and $F''(x)$ unique; if further $\overline{D}^4 F(x) > 0$, $\underline{D}^4 F(x) > -\infty$, then $F''(x)$ is continuous and $\overline{D}^2 F''(x) \geqslant 0$.*

The proof is not altogether analogous to that of Theorem I. We first prove the analogue of lemma 7.

**Lemma 7a.** *Under the conditions of the theorem, $F''(x)$ has the property $R^*$.*

It is sufficient to show that $F''(x)$ has the property $R^*$ in every interval $(a,b)$ completely interior to $I=(A,B)$. Let

$$4h_0=\operatorname{Min}(B-b,\,A-a),$$

and consider the family of functions

$$f_h(x)=\frac{\Delta^4 F(x,h)}{(2h)^4}. \qquad (x\subset(a,b)\quad 0<h\leqslant h_0)$$

By lemma 5, given a perfect set $\Pi\subset(a,b)$, there is a portion $\tilde\omega$ of $\Pi$ and a $K=K(\tilde\omega)$, such that

$$\frac{\Delta^4 F(x,h)}{(2h)^4}>K. \qquad (x\subset\tilde\omega,\quad 0<h\leqslant h_0)$$

Let $G(x)=F(x)-Kx^4/24$. Then

$$\Delta^4 G(x,h)>0. \qquad (x\subset\tilde\omega,\quad 0<h\leqslant h_0)$$

Hence $\Delta^2 G(x,h)/(2h)^2$ is a diminishing sequence of functions each of which is continuous on $\tilde\omega$. Their limit is therefore u. s. c. on $\tilde\omega$. Since $G'(x)$ is finite for all $x$, we have by a classical theorem of Cauchy

$$\frac{\Delta^2 G(x,h)}{(2h)^2}=\frac{G'(x+2\theta h)-G'(x-2\theta h)}{4\theta h}; \qquad (0<\theta<1)$$

and since $G''(x)$ has a unique value, the limit of the last expression is $G''(x)$. Thus $G''(x)$, and therefore $F''(x)$ is u. s. c. on $\tilde\omega$.

We shall deduce the analogue of lemma 6 from the following theorem.

**Theorem IX.** *Let $F(x)$ be continuous in an interval $I$ and let* (i) *$F'(x)$ be finite;* (ii) *$F''(x)<\infty$ be unique and possess the property $R^*$. If $\alpha$ and $\beta>\alpha$ be two points of $I$, then there is a $\xi$, $\alpha<\xi<\beta$, such that*

$$F'(\beta)-F'(\alpha)=(\beta-\alpha)\,F''(\xi).$$

We may suppose that $F'(\beta)=F'(\alpha)$, and we must then show that there is a $\xi$ such that $F''\xi=0$. Suppose on the contrary that $F''(x)\neq 0$ for $\alpha<\xi<\beta$. We can easily prove, by the construction of a Lebesgue chain, the following lemma.



**Lemma 26.** *If $f'(x)$ is finite for $a\leqslant x\leqslant b$, and $f''(x)>0$, then $f'(x)$ is non-diminishing.*

We now proceed with the proof of Theorem IX. Since $F''(x)$ has the property $R^*$, there is a non-dense closed set $F_1$ in $(\alpha,\beta)$, such that if $\delta$ be a contiguous interval of $F_1$, and $d$ be a closed interval interior to $\delta$, then $F''(x)$ is u. s. c. in $d$. Hence $F''(x)\,(<\infty)$, has a finite upper bound in $d$. By lemma 26, $F'(x)$ is of bounded variation in $d$, and therefore, as the differential coefficient of a continuous function, must be continuous. Now $F''(x)$ is in $d$, the differential coefficient of a continuous function, and therefore takes every value between any two values which it takes. Since $F''(x)\neq 0$, it follows that $F'(x)$ is continuous and monotone in $d$. Hence it possesses this property in the open interval $\delta$; and so, as the differential coefficient of a continuous function, in the closed interval $\delta$.

If $\xi$ be an isolated point of $F_1$, we see that unless $F'(x)$ is monotone in the same sense on the right and left of $\xi$, then $F''(x)=0$, contrary to hypothesis. It follows that $F'(x)$ is monotone and continuous in the closed contiguous intervals of $P_1$, the perfect kernel of $F_1$.

There is a closed set $F_2^{(1)}\subset P_1$, non-dense in $P_1$, such that if $\delta$ be a contiguous interval of $F_2^{(1)}$, and $d$ be a closed interval in $\delta$, then $F''(x)$ is u. s. c. on $P_1 d$, and therefore has a finite upper bound on $P_1 d$. Let $(\alpha_n,\beta_n)$ denote a contiguous interval of $P_1 d$. We have proved that $F'(x)$ is continuous and monotone in $\alpha_n\leqslant x\leqslant\beta_n$. The set $G$ of points of $P_1 d$ in the neighbourhood of which the aggregate $\left\{\dfrac{F'(\beta_n)-F'(\alpha_n)}{\beta_n-\alpha_n}\right\}$ is not bounded above, is closed. We shall prove that it is non-dense on $P_1 d$. If not, there is a portion $P_1 d\Delta$ of $P_1 d$, where $\Delta$ denotes a closed interval such that if the contiguous intervals of $P_1 d\Delta$ be denoted by $(\alpha_\nu,\beta_\nu)$, then to each point $\xi$ of $P_1 d\Delta$ there corresponds a sub-sequence $(\alpha_{\nu'},\beta_{\nu'})$ of the $(\alpha_\nu,\beta_\nu)$ for which $(\alpha_{\nu'},\beta_{\nu'})\to\xi$, and

$$F'(\beta_{\nu'})-F'(\alpha_{\nu'})>0,\qquad \frac{F'(\beta_{\nu'})-F'(\alpha_{\nu'})}{\beta_{\nu'}-\alpha_{\nu'}}\to\infty.$$

Since $F'(x)$ is monotone in $\alpha_\nu\leqslant x\leqslant\beta_\nu$, it is non diminishing in any such interval if $F'(\beta_\nu)-F'(\alpha_\nu)>0$. Accordingly $F''(\alpha_\nu)$, $F''(\beta_\nu)$ are both non negative, and hence by (ii), finite. Hence $F'(x)$ is continuous at $z=\alpha_n$, $z=\beta_n$. We can select a sequence $(\alpha_r,\beta_r)$

of the contiguous intervals of $P_1 d\Delta$ such that the end points are everywhere dense in $P_1 d\Delta$, and such that

$$\frac{F'(\beta_r) - F'(\alpha_r)}{\beta_r - \alpha_r} > r.$$

In virtue of the continuity of $F'(x)$ at $\alpha_r$, there is an $\epsilon_r > 0$, $\epsilon_r \to 0$, such that for $\alpha_r - \epsilon_r \leqslant x \leqslant \alpha_r$,

$$\frac{F'(\beta_r) - F'(x)}{\beta_r - x} > r.$$

If $u_n = (\alpha_n - \epsilon_n, \alpha_n)$, then

$$R = \prod_{r=1}^{\infty} \sum_{n=r}^{\infty} u_n P_1 d\Delta$$

is a residual of $P_1 d\Delta$. Thus $R$ is not null, and if $\xi \subset R$, these is a sequence $\xi_n > \xi$, $\xi_n \to \xi$, such that

$$\frac{F'(\xi_n) - F'(\xi)}{\xi_n - \xi} \to \infty,$$

which contradicts $F''(x) < \infty$. This proves that $G$ is non-dense in $P_1 d$. Hence there is a closed interval $\Delta$ and a constant $K$ such that $P_1 d\Delta$ exists, and such that, (a) $F''(x) < K$ for $x \subset P_1 d\Delta$, (b) $\dfrac{F'(\beta) - F'(\alpha)}{\beta - \alpha} < K$ for every contiguous interval $(\alpha, \beta)$ of $P_1 d\Delta$. Manifesty, by the construction of a Lebesgue chain, we can show that if $\xi_1$ and $\xi_2 > \xi_1$ be any two points of $P_1 d\Delta$, then $F'(\xi_2) - F'(\xi_1) < K(\xi_2 - \xi_1)$. Hence the function $G(x)$ defined by,

(A) $G(x) = F'(x) \qquad x \subset P_1 d\Delta$

(B) $G(x)$ is linear in each closed contiguous interval of $P_1 d\Delta$

is of bounded variation in $d\Delta$. Since the (total) variation of $F'(x)$ in $(\alpha, \beta)$ equals the (total) variation of $G(x)$ in $(\alpha, \beta)$, namely $|F'(\beta) - F'(\alpha)|$, it follows that $F'(x)$ is of bounded variation in $d\Delta$. Hence as before, it is continuous and monotone in $d\Delta$. The portions of $P_1$ such as $P_1 d\Delta$ form an everywhere dense set in $P_1$, and their complement with respect to $P_1$ is a closed set $F_2 \supset F_2^{(1)}$, non-dense in $P_1$.



As at the beginning of the proof, we infer that $F'(x)$ is continuous and monotone in each of the closed contiguous intervals of $P_2$, the perfect kernel of $F_2$. Proceeding in this manner, we infer eventually, that $F'(x)$ is continuous and monotone in $(\alpha, \beta)$. Since $F''(x) \neq 0$ for $\alpha < x < \beta$, we arrive at a contradiction with $F'(\beta) = F'(\alpha)$. This proves the theorem.

**12.** We can now continue the proof of Theorem VIII.

**Lemma 6 a.** *Under the conditions of Theorem VIII,* $\overline{D}^2 F''(x) \geqslant \underline{D}^4 F(x)$.

We may suppose that at the point under consideration, $\underline{D}^4 F(x) > 0$, and we must then show that $\overline{D}^2 F''(x) \geqslant 0$. For all sufficiently small $h$, we have by the argument of lemma 7 a,

$$\frac{F(x+h) + F(x-h) - 2F(x)}{h^2} \geqslant F''(x).$$

By a classical theorem of Cauchy, this is the same as

$$\frac{F'(x+h) - F'(x-h)}{2h} \geqslant F''(x) \tag{19}$$

for all sufficiently small $h$.

The condition $\underline{D}^4 F(x) > -\infty$ implies that $F'' < \infty$ in I. Thus $\dfrac{d}{dh}\{F'(x+h) - F'(x-h)\} < \infty$, and by lemma 7 a, possesses the property $R^*$. Writing $\varphi(h) = F'(x+h) - F'(x-h)$, we can apply Theorem IX. We have

$$\begin{aligned} F'(x+h) - F'(x-h) &= \varphi(h) - \varphi(0) \\ &= h\,\varphi(\theta h) \qquad (0 < \theta < 1) \\ &= h[F''(x+\theta h) + F'(x - \theta h)]. \end{aligned}$$

Hence (19) gives $\Delta^2 F''(x, \theta h/2) > 0$. Since $h$ is arbitrary, this implies $\overline{D}^2 F''(x) \geqslant 0$.

**Lemma 8 a.** *If* $F''(x)$ *is u. s. c. in* $(a, b)$, $\underline{D}^4 F(x) > K$ *in that interval, then* $D^4 F(x)$ *exists p. p.*

We may suppose that $K = 0$. Then $\overline{D}^2 F''(x) > 0$ by lemma 6a. Further $F''(x) < \infty$. We can show that $F''(x)$ is finite in $d = (a, b)$

by an adaptation of the method of lemma 8; $F''(x)$ takes the place of $F'(x)$ in that method, and $F'(x)$ takes the place of $F(x)$. It then follows that $F''(x)$ is convex and continuous. Then $D^+F''(x)$ is increasing, and p. p. in $d$,

$$f(x)=\lim_{h\to 0}\frac{D^+F''(x+h)-D^+F''(x)}{h}$$

exists. Let $x$ be a point at which $f(x)$ exists. Then

$$\frac{\Delta^4 F(x,h)}{(2h)^4}=\frac{\Delta^3 F(x+h,h)-\Delta^3 F(x-h,h)}{(2h)^4}$$
$$=\frac{\Delta^3 F'(\xi,h)}{(2h)^3} \qquad (x-h<\xi<x+h)$$

We now use a relation given in the proof of lemma 11. We have

$$\frac{\Delta^3 F'(\xi,h)}{(2h)^3}=\frac{3}{(2h)^3}\int_0^h dy\int_y^{3y}\{D^+F''(\xi+z)-D^+F''(\xi-z)\}\,dz. \tag{20}$$

Further,

$$D^+F''(\xi+z)-D^+F''(x)=(\xi+z-z)\,[f(x)+\epsilon(\xi,z)],$$
$$D^+F''(\xi-z)-D^+F''(x)=(\xi-z-z)\,[f(x)+\eta(\xi,z)],$$

where

$$\epsilon^2+\eta^2<\zeta^2 \qquad 0<h\leqslant h(\zeta).$$

Thus (20) becomes

$$\frac{\Delta^3 F'(\xi,h)}{(2h)^3}=\frac{3}{(2h)^3}\int_0^h dy\int_y^{3y}\{2zf(x)+o(h)\}\,dz$$
$$\to f(x)$$

as $h\to 0$. Hence $D^4F(x)=f(x)$.

**Lemma 9 a.** *If $F''(x)$ posseses the property $R^*$ in $(a,b)$ and $F''(x)<\infty$, $\overline{D}^2F''(x)$ p. p., $\overline{D}^2F''(x)>-\infty$, then $F''(x)$ is continuous and convex.*

There is no difficulty in adapting the proof of lemma 9.

**Lemma 10 a.** *If $F''(x)$ is continuous and convex in $(a,b)$, then $\underline{D}^4F(x)\geqslant 0$ in that interval.*



For we can apply the mean value theorem twice, and obtain,

$$\begin{aligned}\Delta^4F(x,h)&=2h\,\Delta^3F'(\xi,h) &&(x-h<\xi<x+h)\\ &=(2h)^2\Delta^2F''(\zeta,h) &&(\xi-h<\zeta<\xi+h)\\ &\geqslant 0.\end{aligned}$$

The proof of Theorem VIII can now be completed as the proof of Theorem I was at the end of § 3, the analogous lemmas being used.

**Theorem X.** *If $F(x)$ is continuous in $(a,b)$, $F'(x)$ is finite, $F''(x)$ is unique, $\overline{D}^4F(x)\geqslant f(x)$, where $f(x)$ is integrable $D$, and $\underline{D}^3F(x)$ is finite, then $\underline{D}^3F(x)$ is integrable $D$ in $(a+\epsilon, b-\epsilon)$ for $0<\epsilon<(b-a)/2$, and for $a<\alpha<\beta<b$,*

$$F''(x)=\int_\alpha^x dy\int_\alpha^y \underline{D}^3F(t)\,dt+px+q \qquad (\alpha\leqslant x\leqslant\beta).$$

The proof is analogous to that of Theorem II, in virtue of the following analogue of lemma 11.

**Lemma 11 a.** *If $g(t)$ be defined in the neighbourhood of $t=x$, and $g'(t)$, $g''(t)$ are continuous and $D^+g''(t)$ is bounded in that neighbourhood, then*

$$\overline{D}\,D^+g''(x)\geqslant\overline{D}^4g(x)\geqslant\underline{D}^4g(x)\geqslant\underline{D}\,D^+g''(x).$$

It can be verified that

$$\frac{\Delta^4 g(x,h)}{(2h)^4}=\frac{1}{h^4}\int_0^h dt\int_0^t dy\int_{2y}^{4y}\{D^+g''(x+z)-D^+g''(x-z)\}\,dz,$$

so that the result follows by the mean value theorem.

**Theorem XI.** *Let*

$$\sum A_n(x) \qquad a_n=o(n),\ b_n=o(n)$$

*be a trigonometric series. Let $\overline{R}(x)$, $\underline{R}(x)$ denote its upper and lower sums $(R,4)$. Let $\underline{R}(x)$ be finite everywhere, and $\overline{R}(x)\geqslant f(x)$, where $f(x)$ is integrable $D$. Then the series is a Fourier series.*

The proof is analogous to that of Theorem III.

13. We consider finally the analyse of Theorem VII.

**Theorem VII a.** *Let* $\Sigma A_n(x)$ *be a trigonometric series such that*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{-n}^{n} c_m e^{imx} = 0$$

*for all* $x$. *Let* $\overline{R}(x)$, $\underline{R}(x)$ *denote its upper and lower sums* $(R, 4)$. *Let* $\underline{R}(x)$ *be finite,* $\overline{R}(x) \geqslant f(x)$, *where* $f(x)$ *is integrable* $D$. *Then the series is a Fourier series.*

We can define $F(x)$ as in (14). We write

$$H(x) = \sum \frac{A_n(x)}{n^4}.$$

Then $H(x)$ is continuous, and $H'(x) = F(x)$. The proof of lemma 20 holds in the present case. That the result of lemma 21 is true under the conditions of the present theorem, is seen by an easy adaptation of the proof of that lemma. We define $G(x)$ by (16). That $G(x)$ possesses the property $R^*$, follows by easy adaptations of the proof of lemma 22. Finally, lemma 25 remains true. Thus Theorem VII a will be proved, if we prove the following analogue of Theorem VI.

**Theorem VI a.** *If* $H(x)$ *is continuous in* $a < x < b$, *and*

(*i*) $F(x) = H'(x)$ *is approximately continuous and has the property* $R^*$,

(*ii*) $F_a(x) < \infty$ *is unique and has the property* $R^*$,

(*iii*) $\overline{D}^4 H(x) \geqslant f(x)$ *where* $f(x)$ *is integrable* $D$, *and* $\underline{D}^4 H(x)$ *is finite,*

*then* $\underline{D}^4 H(x)$ *is integrable* $D$ *in* $(a+\epsilon, b-\epsilon)$ *for* $0 < \epsilon < (b-a)/2$, *and for* $a < \alpha < \beta < b$,

$$F_a(x) = H''(x) = \int_\alpha^x dy \int_\alpha^y \underline{D}^4 H(t)\,dt + px + q. \quad (\alpha \leqslant x \leqslant \beta)$$

This theorem, in its turn, will be a consequence of the following analogue of Theorem V.

**Theorem V a.** *If* $H(x)$ *is defined and continuous in an open interval* $I$, *and,*



(*i*) $F(x) = H'(x)$ *is approximately continuous and has the property* $R^*$,

(*ii*) $F_a(x) < \infty$ *is unique and has the property* $R^*$,

(*iii*) $\overline{D}^4 H(x) > 0$, $\underline{D}^4 H(x) > -\infty$,

*then* $H''(x)$ *is continuous, and* $\underline{D}^2 H''(x) \geqslant 0$.

On examining the proof of Theorem V, (and of Theorem I), we see that lemma 15 is not altogether essential to the proof, but may be replaced by the following specialised proposition: Under the conditions of the theorem, if $F_a(x)$ is u. s. c. at $x$, then $\overline{D}^2 F_a(x) \geqslant \underline{D}^3 F(x)$. It is the analogue of this propositions which will correspond to lemma 15 in the proof of Theorem V a.

**Lemma 15 a.** *Under the conditions of the theorem, if* $F_a(x)$ *is u. s. c. at* $\xi$, *then* $\overline{D}^2 F_a(\xi) \geqslant \underline{D}^4 H(\xi)$.

We may suppose that at the point $\xi$. $\underline{D}^4 H(\xi) > 0$, and it is then sufficient to show that $\underline{D}^2 F_a(\xi) \geqslant 0$. We have for all sufficiently small $h$,

$$\text{(21)} \quad \frac{H(\xi+h)+H(\xi-h)-2H(\xi)}{h^2} > \frac{H(\xi+h/4)+H(\xi-h/4)-2H(\xi)}{(h/4)^2}.$$

As in the proof of lemma 15, we may suppose that $F_a(\xi)$ is finite. If now $\overline{D}^2 F_a(\xi) < 0$, there is an $\eta > 0$, such that

$$F_a(\xi+h) + F_a(\xi-h) - 2F_a(\xi) < -\eta h^2$$

for $0 < h \leqslant h(\eta)$. We now observe that by the theorem of Cauchy, there is a $\theta$, $0 < \theta < 1$, such that

$$\text{(22)} \quad \frac{H(\xi+h)+H(\xi-h)-2H(\xi)}{h^2} = \frac{F(\xi+\theta h) - F(\xi-\theta h)}{2\theta h}.$$

By Theorem IV, there is a $\theta'$, $0 < \theta' < 1$, such that

$$\frac{F(\xi+\theta h) - F(\xi-\theta h)}{2\theta h} = \frac{F_a(\xi+\theta\theta' h) + F_a(\xi-\theta\theta' h)}{2}.$$

We thus infer by the argument of lemma 15, that

$$\text{(23)} \quad \frac{H(\xi+h)+H(\xi-h)-2H(\xi)}{h^2} < F_a(\xi) - \frac{\eta}{2}\left(\frac{h(\eta)}{4}\right)^2. \quad (0<h\leqslant h(\eta))$$

We now use the property that $F_a(x)$ is u. s. c. at $\xi$. Then given $\epsilon > 0$, there is a $\delta > 0$ such that $F_a(x) < F_a(\xi) + \epsilon$ for $\xi - \delta \leqslant \leqslant x \leqslant \xi + \delta$. Since $F_a(\xi)$ is finite, $F_a(\xi) - (F_a(\xi) + \epsilon) < 0$ in $(\xi - \delta, \xi + \delta)$, and so by lemma 13, $\Phi(x) = F(x) - x\,(F_a(\xi) + \epsilon)$, is monotone. Further, at $\xi$, $\Phi(x)$ has a finite approximate derivative, $\Phi_a(\xi) = -\epsilon$. We now use the following lemma.

**Lemma 27.** (Khintchine[1])). *At a point $\xi$ at which the monotone function $f(x)$ has a finite approximate derivative $f_a(\xi)$, the differential coefficient of $f(x)$ exists, and $f'(\xi) = f_a(\xi)$.*

Hence $\Phi'(\xi) = \Phi_a(\xi)$. It follows that $F_a(\xi) = F'(\xi)$. The relation (22) then shows that

$$\lim \frac{H(\xi + h) + H(\xi - h) - 2H(\xi)}{h^2} = F_a(\xi).$$

Thus (23) gives

$$F_a(\xi) < F_a(\xi) - \frac{\eta}{2}\left(\frac{h(\eta)}{4}\right)^2.$$

This contradiction establishes the lemma.

**Lemma 16a.** *If $F_a(x)$ is u. s. c. in $(a, b)$, $\underline{D}^4 H(x) > K$ in that interval, then $D^4 H(x)$ exists p. p.*

We may suppose that $k = 0$. Then $\overline{D}^2 F_a(x) > 0$ by lemma 15a. The remainder of the proof is the same as for lemma 16.

Finally, lemma 17 remains unaltered in enunciation. The deduction of Theorem Va, now presents no new difficulties.

[1]) Khintchine, *Fund. Math.* 9 (1927) 212—279 (242).

# Absolut-additive abstrakte Mengenfunktionen.

Von

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Diese Note schliesst sich unmittelbar an eine andere Note des Verfassers an[1]), in welcher die Lebesguesche Integrations- und Differentiationstheorie für den Fall erörtert wurde, dass die Werte der (in kartesischen Räumen definierten) Funktionen nicht Zahlen sondern, allgemeiner, Elemente eines vollständigen, komplex-linearen (Vector-) Raumes sind. Es wurde nachgewiesen, dass sich die Begriffe und Sätze der Lebesgueschen Theorie in weitem Umfange aufrechterhalten lassen, mit Ausnahme der Besselschen Ungleichung für Fourierreihen, deren Versagen an einem Beispiel aufgewiesen wurde[2]), und mit Ausnahme des Satzes, dass jede absolut-additive Mengenfunktion fast überall differenzierbar ist, dessen Gültigkeit unentschieden blieb. — Ziel der vorliegenden Note ist es, an einem Gegenbeispiel nachzuweisen, dass auch dieser Differentiationssatz zu bestehen aufhört.

Diese Entscheidung dürfte, obwohl sie eine negative ist, von Interesse sein. Denn in der neueren Theorie der Potenzreihen

$$\sum_{n=0}^{\infty} a_n z^n$$

(mit komplexen Werten der Variablen $z$) spielt der Lebesguesche Differentiationssatz im wesentlichen nur an einer Stelle eine Rolle, nämlich beim Beweis des sogenannten Satzes von Fatou. In einem

[1]) *Fundamenta Math.* XX (1933), p. 262—276; vgl. auch daselbst.

[2]) Wegen eines prägnanteren Beispiels vgl. *Göttinger Nachrichten, Mathem.-Phys. Klasse* 1933, 178—180.