dem unter [13]) zitierten Satze von S. Mazurkiewicz gibt es zu jeder Kurve $K \in \mathfrak{K}$ eine auf M oder auf N erklärte Homöomorphie h_K, derart dass $h_K(M) \subset K$ oder $h_K(N) \subset K$ besteht. Seien H_M und H_N die Klassen dieser Homöomorphien. Wären die beiden Klassen abzählbar, so gäbe es ein $h \in H_M$, bezw. $h \in H_N$, derart dass die Menge $h(M)$, bezw. $h(N)$, in unabzählbar vielen Kurven von $\mathfrak{K}$ enthalten sein müsste. Dann würden aber bereits in der aus diesen Kurven bestehenden Teilklasse $\mathfrak{C}$ von $\mathfrak{K}$, auf Grund des Hilfssatzes und nach (2), je zwei Kurven miteinander wesentlich verschlungen. Wir können also annehmen, dass die Menge H_M, bezw. H_N, eine unabzählbare ist. Der Raum G^M (bezw. G^N) sämtlicher stetiger Abbildungen von M (bezw. von N) auf Teilmengen der Kugel G [15]), in welchem H_M, bezw. H_N, als dessen Teilmenge liegt, ist bekanntlich separabel [16]). Daher enthält H_M, bezw. H_N, als Teilmenge dieses Raumes ein Verdichtungselement h'.

Dem Hilfssatze und der Behauptung (2) zufolge gibt es ein $\varepsilon > 0$ derart, dass je zwei ε-Deformationen der Kurve $h'(M)$, bezw. $h'(N)$, miteinander wesentlich verschlungene Bilder ergeben. Sei $\mathfrak{C}$ die aus denjenigen Kurven K bestehende Teilklasse von $\mathfrak{K}$, für welche $h_K \in H_M$ (bezw. H_N) und $\varrho(h_K, h') < \varepsilon$ gilt. Da h' ein Verdichtungselement von H_M (bezw. von H_N) ist, so ist $\mathfrak{C}$ sicher unabzählbar. Für je zwei Kurven K_i $(i = 1, 2)$ von $\mathfrak{C}$ haben wir $\varrho(h_{K_i}, h') < \varepsilon$, so dass die beiden Teilkurven $h_{K_i}(M)$, bezw. $h_{K_i}(N)$, von K_i, als ε-Deformationsbilder der Kurve $h'(M)$, bezw. $h'(N)$, miteinander wesentlich verschlungen sind. Umsomehr sind die ganzen Kurven K_i miteinander wesentlich verschlungen, w. z. b. w.

In gewissem Maasse kann der Satz II auch als ein räumlicher Analogon des Moore'schen Satzes über das Verhalten von sogenannten *Trioden* auf der Ebene angesehen werden [17]). Übrigens lässt sich der Satz von R. L. Moore über die Trioden direkt aus dem soeben bewiesenen Satz II erhalten, und zwar durch Projezierung der auf den Zylindermengen dieser Trioden gezeichneten Kurven M auf die Ebene, in der die Trioden liegen.

[15]) Für Bezeichnungsweise und Metrisierung dieses Raumes s. z. B. C. Kuratowski, *Topologie I*, S. 199.

[16]) Vgl. M. Fréchet, *Thèse*, Paris 1906, S. 36 und K. Borsuk, *Sur les rétractes*, Fund. Math. XVII (1931), S. 165.

[17]) Vgl. R. L. Moore, *Concerning triods in the plane*..., Proceed. Nat. Acad. of Sc., 14 (1928), S. 85—88, sowie den verallgemeinerten Satz in *Concerning triodic-continua in the plane*, Fund. Math. XIII (1929), S. 261—263.

On the differentiability of multiple integrals.

By

A. Zygmund (Wilno).

§ 1.

1. Let $f(x, y)$ be a function of two variables, defined in the square (S) $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and let

$$F(Q) = \int\int_Q f(x, y)\, dx\, dy,$$

where $Q \subset S$ in an arbitrary rectangle with sides parallel to the axis. Let $Q(x_0, x_0 + h; y_0, y_0 + k)$ denote the rectangle $x_0 \leqslant x \leqslant x_0 + h$, $y_0 \leqslant y \leqslant y_0 + k$. It is well-known that for almost every (x_0, y_0) in S we have $\lim F(Q)/|Q| = f(x_0, y_0)$ [1]), where $Q = Q(x_0, x_0 + h; y_0, y_0 + k)$ and h, k tend *regularly* to 0, that is the ratios h/k and k/h are bounded.

It has been recently proved by Saks that

(i) if f is bounded, the word „regularly" in the above theorem may be omitted [2])

(ii) there exists an integrable (summable) f such that at every point $(x_0, y_0) \in S$ we have $\limsup\limits_{h,k \to 0} F(Q)/|Q| = +\infty$, where $Q = Q(x_0, x_0 + h; y_0, y_0 + k)$ [3]).

The object of this paper is to generalize the first of these results and to apply this generalization to the theory of double Fourier

[1]) We denote by $|Q|$ the measure of Q.

[2]) Saks, *Théorie de l'intégrale*, Monogr. Mat. II, Warszawa, 1933, p. 1—290, esp. p. 231—232. See also: F. Riesz, Fund. Math. 22 (1934), 221—225; Busemann u. Feller, ibid., 226—256, esp. 242.

[3]) Saks, Fund. Math., 22 (1934), p. 257—261; also Busemann u. Feller, l. c., esp. 243—247.

series. The argument can easily be extended to spaces of higher dimensions.

2. ***Theorem I.*** *If $f(x, y) \in L^p$, $p > 1$, then at almost every point (x_0, y_0) we have $\lim F(Q)/|Q| = f(x_0, y_0)$, where $Q = Q(x_0, x_0 + h;\, y_0, y_0 + k)$ and h, k tend to 0 independently of each other* [4]).

From this theorem we see that there exists an essential, and rather unexpected, difference between the classes L and L^p, $p > 1$. We shall establish Theorem 1 using an argument which has proved very useful in the theory of trigonometrical series [5]). The idea is very simple, although some details make the proof a little long. It consists, roughly speaking, in showing that the function

$$f^*(x_0, y_0) = \operatorname*{Max}_{h,k} \frac{F(Q)}{|Q|} \tag{1}$$

is integrable over S. It follows in particular that $\limsup\limits_{h,k\to 0} F(Q)/|Q|$ is finite almost everywhere. It will be then a simple matter to prove Theorem 1.

It will be convenient to consider first rectangles Q with centre at (x_0, y_0): $Q = Q(x_0 - h, x_0 + h;\, y_0 - k, y_0 + k)$. It is also convenient to have $f(x, y)$ defined for all x, y by the condition of periodicity: $f(x + m, y + n) = f(x, y)$, $(m, n = 0, \pm 1, \pm 2, \ldots)$.

3. Let $L_h(x)$, $0 \leqslant h \leqslant 1/2$, be the function of x, of period 1 equal to $1/2h$ for $-h \leqslant x \leqslant h$, and to 0 elsewhere (mod 1). If $(x_0, y_0) \in S$, $Q = Q(x_0 - h, x_0 + h;\, y_0 - k, y_0 + k)$, then

$$F(Q)/|Q| = \int_S\!\!\int f(u, v)\, L_h(u - x_0)\, L_k(v - y_0)\, du\, dv. \tag{2}$$

We shall have to consider, in the sequel, the number $q = q(p)$ connected with p by the relation $1/p + 1/q = 1$. Since it is enough to prove Theorem 1 for a sequence of values of p, $p > 1$, tending

[4]) Once Theorem 1 has been established, it is not difficult to see that at almost every (x_0, y_0) we have $\lim \dfrac{F(Q)}{|Q|} = f(x_0, y_0)$, where $Q = Q(x_0 + h,\, x_0 + h_1;\, y_0 + k\; y_0 + k_1)$ provided that the ratios $h/(h_1 - h)$, $k/(k_1 - k)$ are bounded as $h, k \to 0$.

[5]) Kolmogoroff and Seliverstoff, Rendiconti della Reale Accademia dei Lincei, **3** (1926), 307—310. Plessner, Crelles Journal, **155** (1925), 15—25. Paley, Proceedings of the London Math. Society, **31** (1930), 289—300.

to 1, we may without any loss of generality consider only such p for which q is an integer: $q = 2, 3, \ldots$

Lemma. *Let $f(x, y) \in L^p$, $p > 1$, $f \geqslant 0$, in S. Let $Q = Q(x_0 - h, x_0 + h;\, y_0 - k,\, y_0 + k)$, where the numbers h, k are arbitrary measurable functions of x, y: $h = h(x, y)$, $k = k(x, y)$, satisfying the inequalites $0 < h \leqslant 2^{-q}$, $0 < k \leqslant 2^{-q}$. Then the ratio $F(Q)/|Q|$, which depends only on x, y, is integrable, and*

$$\int_S\!\!\int \frac{F(Q)}{|Q|}\, dx\, dy \leqslant C_p \left[\int_S\!\!\int f^p\, dx\, dy\right]^{1/p}, \tag{3}$$

where the constant C_p depends only on p.

Let us integrate the equation (2) over S. Inverting the order of integration and applying Hölder's inequality we find that

$$\int_S\!\!\int \frac{F(Q)}{|Q|}\, dx\, dy = \int_S\!\!\int f(u,v)\, du\, dv \left[\int_S\!\!\int L_h(u - x)\, L_k(v - y)\, dx\, dy\right] \leqslant$$
$$\leqslant I_p[f] \left[\int_S\!\!\int du\, dv \left\{\int_S\!\!\int L_h(u - x)\, L_k(v - y)\, dx\, dy\right\}^q\right]^{1/q},$$

where $I_p[f]$ denotes the second factor on the right in (3) and q is an integer, $1/p + 1/q = 1$. Replacing the q-th power of the integral in curly brackets by a product of q equal integrals, and changing the order of integration, we represent the integral in square brackets in the last inequality in the form

$$\int_{y_q}\!\!\cdots\!\int_{y_1}\int_{x_q}\!\!\cdots\!\int_{x_1} \left\{\left[\int_0^1 \prod_{i=1}^{q} L_{h_i}(u - x_i)\, du\right]\left[\int_0^1 \prod_{i=1}^{q} L_{k_i}(v - y_i)\, dv\right]\right\} dx_1 \ldots dy_q, \tag{4}$$

where $h_i = h(x_i, y_i)$, $k_i = k(x_i, y_i)$.

4. Given any numbers h_i, $0 \leqslant h_i \leqslant 1/4$, we consider the expression

$$R(x_1, \ldots x_n;\, h_1, \ldots h_n) = \int_0^1 L_{h_1}(u - x_1) \ldots L_{h_n}(u - x_n)\, du.$$

Assuming that h_1 is the smallest of the numbers $h_1, \ldots, h_n$, we will prove that

$$R(x_1, \ldots, x_n;\, h_1, \ldots, h_n) \leqslant 2^{n-1} L_{2h_2}(x_2 - x_1) \ldots L_{2h_n}(x_n - x_1). \tag{5}$$

First of all, $R(x_1, x_2, \ldots, x_n;\, h_1, \ldots, h_n) = R(0, x_2', \ldots, x_n';\, h_1, \ldots, h_n)$ where $x_i' = x_i - x_1$, $2 \leqslant i \leqslant n$. Now the integral defining $R(0, x_2', \ldots, x_n';\, h_1, \ldots, h_n)$ attains its maximum, equal to $(2h_1)\ (2h_1 \cdot 2h_2 \ldots 2h_n)^{-1} =$

$=(2h_2\dots 2h_n)^{-1}$ when $|x_2'|\leqslant h_2-h_1,\dots,|x_n'|\leqslant h_n-h_1$, and vanishes if at least one of the inequalities $|x_2'|\geqslant h_2+h_1,\dots |x_n'|\geqslant h_n+h_1$, or, à fortiori, of the inequalities $|x_2'|\geqslant 2h_2,\dots |x_n'|\geqslant 2h_n$, is satisfied. Thence we deduce, without difficulty, that $R(0,x_2',\dots x_n';h_1,\dots h_n)\leqslant$ $\leqslant 2^{n-1}L_{2h_2}(x_2'),\dots L_{2h_n}(x_n')$, i. e. the inequality (5).

Using this result we see that the expression in curly brackets in (4) does not exceed

$$2^{2(q-1)}\left[\sum_{k=1}^{q}\prod_{i=1}^{q}{}^{(k)}L_{2h_i}(x_k-x_i)\right]\left[\sum_{l=1}^{q}\prod_{j=1}^{q}{}^{(l)}L_{2k_j}(y_l-y_j)\right]=$$
$$=2^{2(q-1)}\sum_{k,l=1}^{q}\left\{\prod_{i=1}^{q}{}^{(k)}L_{2h_i}(x_k-x_i)\prod_{j=1}^{q}{}^{(l)}L_{2k_j}(y_l-y_j)\right\},$$

where the upper indices k and l indicate that the terms corresponding to $i=k$, $j=l$ are omitted. Let us integrate every term of the last sum with respect to $x_1,\dots x_q$, $y_1,\dots y_q$. Considering any particular term of this sum, that is fixing k and l, we may integrate first with respect to x_k and y_l. These variables now play the rôle of u,v, and we see that the integral (4) divided by $2^{2(q-1)}$ does not exceed a sum of q^2 integrals of the same form, except that now the number of factors in the products is not q but $q-1$, and that the quantities h, k are doubled. Repeating this argument we obtain finally that the integral (4) does not exceed a constant multiple of the sum

$$\sum_{r,s=1}^{q}\int\limits_{y_s}\int\limits_{x_r}\left[\int_0^1 L_{2^{q-1}h_r}(u-x_r)\,du\right]\left[\int_0^1 L_{2^{q-1}k_s}(v-y_s)\,dv\right]dx_r\,dy_s,$$

and every term of this sum is equal to 1. This proves the lemma.

5. Now we are in a position to prove Theorem I. Let us define $f^*(x_0,y_0)$ by the equation (1), where $f\geqslant 0$, $Q=Q(x_0-h,x_0+h;$ $y_0-k;\ y_0+k)$ and $0<h\leqslant 2^{-q}$, $0<k\leqslant 2^{-q}$. We may plainly substitute $f^*(x,y)$ for $F(Q)/|Q|$ in (3). From the new inequality we see that the upper limit of $F(Q)/|Q|$, as $h,k\to 0$, exceeds $\varepsilon/4$ in a set E of measure $<4\,C_p\,I_p[f]/\varepsilon$. If follows that, if for Q we take rectangles with (x,y) as their lower left-hand side corner, $Q=Q(x,x+h;y,y+k)$, the upper limit of $F(Q)/|Q|$ as $h,k\to 0$ does not exceed ε outside E.

Let φ be a continuous function such that $I_p[|f-\varphi|]\leqslant\varepsilon^2/4\,C_p$, $\psi=f-\varphi$, and let $\Phi(Q)$, $\Psi(Q)$ denote the integrals analogous to $F(Q)$, $Q=Q(x,x+h;y,y+k)$, with φ and ψ instead of f. At every point x,y we have $\Phi(Q)/|Q|\to\varphi(x,y)$ as $h,k\to 0$. The upper limit of $|\Psi(Q)|/|Q|$, as $h,k\to 0$, is $\leqslant\varepsilon$ except at a set E_1 of measure $<\varepsilon$. Moreover, the set E_2 of points where $|f-\varphi|>\varepsilon$ is less than $(\varepsilon/4\,C_p)^p$. Since $F(Q)=\Phi(Q)+\Psi(Q)$, we conclude that, except at the set E_1+E_2 of measure $<\varepsilon+(\varepsilon/4\,C_p)^p$, the upper limit of $|F(Q)/|Q|-f(x,y)|$, as $h,k\to 0$, does not exceed 2ε, and, ε being arbitrary, Theorem I follows.

§ 2.

6. Theorem I does not exhaust the problems concerning the differentiability of double integrals. We shall return to some of these problems in another paper. Here we will only make a few remarks on the summability of double Fourier series. Let

$$K_n(t)=\frac{1}{2(n+1)}\left(\frac{\sin\,(n+1)\frac{t}{2}}{\sin\frac{t}{2}}\right)^2$$

be the well-known Fejér kernel, and let $f(x,y)$ be a function integrable in the square $0\leqslant x\leqslant 2\pi$, $0\leqslant y\leqslant 2\pi$, and of period 2π: $f(x+2\pi,y)=f(x,y+2\pi)=f(x,y)$. If $\sigma_{m,n}(x,y)$ are the Fejér means of the Fourier series of $f(x,y)$, we have the formula [6])

$$\sigma_{m,n}(x,y)-f(x,y)=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi\varphi_{x,y}(u,v)K_m(u)K_n(v)\,du\,dv,$$

where $\varphi_{x,y}(u,v)=f(x+u,y+v)+f(x+u,y-v)+f(x-u,y+v)+$ $+f(x-u,y-v)-4f(x,y)$.

From the second of the Saks results mentionned on p. 143 it is easy to deduce that there exists an f such that, at every point (x,y), $\limsup\sigma_{m,n}(x,y)=+\infty$ as $m,n\to\infty$ independently of each other [7]). Conversely, Theorem I of this paper enables us to prove the following

[6]) For the theory of double Fourier series see e. g. Tonelli, *Serie Trigonometriche*, Bologna 1928, p. 1–526, esp p 435 sq.

[7]) Saks Fund Math. 22 (1934), p. 260 The same negative result holds for Abel's summability

Theorem II. *If* $f(x,y) \in L^p$, $p>1$, *then* $\sigma_{m,n}(x,y) \to f(x,y)$ *almost everywhere as* $m, n \to \infty$.

We begin by a few remarks

(i) In the lemma which we proved in § 1 we supposed that $0<h\leqslant 2^{-q}$, $0<k\leqslant 2^{-q}$. If, instead of the unit square, we considered an arbitrary square with sides of length d, we should obtain (by a transformation of the variables), an analogous result for $0\leqslant h\leqslant 2^{-q}d$, $0<k\leqslant 2^{-q}d$. Let $d=2^q\pi$.

If $f(x,y)$ is of period 2π, it is, à fortiori, of period $2^q\pi$, and so, if S now denotes the square $0\leqslant x\leqslant 2\pi$, $0\leqslant y\leqslant 2\pi$, we have the inequality (3), for $0<h\leqslant\pi$, $0<k\leqslant\pi$.

(ii) Considering instead of $f(u,v)$ the function $\varphi_{x,y}(u,v)$, introducing the function

$$\Phi_{x,y}(h,k)=\int_0^h\int_0^k|\varphi_{x,y}(u,v)|\,du\,dv$$

and repeating the well-known Lebesgue argument [8]), we deduce from (i) and from Theorem I that for almost every point x, y there exists a constant $C=C_{x,y}<\infty$ such that

$$\Phi_{x,y}(h,k)\leqslant Chk,\quad 0<h\leqslant\pi,\quad 0<k\leqslant\pi,$$

and that $\Phi_{x,y}(h,k)/hk\to 0$ as $h,k\to 0$.

(iii) $K_n(t)\leqslant An/(1+n^2t^2)$, $|t|\leqslant\pi$, where A is an absolute constant [9]).

7. Let $a_n(t)=An/(1+n^2t^2)$. Integrating by parts the right-hand side of the inequality

$$\pi^2|\sigma_{m,n}(x,y)-f(x,y)|\leqslant\int_0^\pi\int_0^\pi|\varphi_{x,y}(u,v)|\,a_m(u)\,a_n(v)\,du\,dv,$$

first with respect to u and then with respect to v, we find that $\pi^2|\sigma_{m,n}(x,y)-f(x,y)|$ does not exceed

$$(6)\quad\left|\begin{aligned}&a_m(\pi)\,a_n(\pi)\,\Phi(\pi,\pi)+\\ &-a_m(\pi)\int_0^\pi\Phi(\pi,v)\,a_n'(v)\,dv-a_n(\pi)\int_0^\pi\Phi(u,\pi)\,a_m'(u)\,du+\\ &+\int_0^\pi\int_0^\pi\Phi(u,v)\,a_m'(u)\,a_n'(v)\,du\,dv,\end{aligned}\right.$$

[8]) See Lebesgue, *Leçons sur les séries trigométriques*, p. 13.

[9]) See e. g. Tonelli, loc. cit., p. 175—176. This follows simply from the fact that $K_n(t)=O(n)$, if $0\leqslant t\leqslant\pi$, $K_n(t)=O(nt^{-2})$, if $1/n\leqslant t\leqslant\pi$.

where, for simplicity, the suffixes x, y are omitted. The first term tends manifestly to 0. The same is true for the second and third terms since we may replace $\Phi(u,\pi)$, $\Phi(\pi,v)$ by $C\pi u$, $C\pi v$ and the integral of $t\,a_n'(t)$ over $(0,\pi)$ is bounded. If we replaced $\Phi(u,v)$ by Cuv in the fourth integral, we should obtain that this integral is bounded. To show that it tends to 0 as $m,n\to\infty$, let $\varepsilon>0$ be a given number and let $\delta=\delta(\varepsilon)$ be such that $\Phi(u,v)\leqslant\varepsilon uv$ for $0\leqslant u\leqslant\delta$, $0<v\leqslant\delta$. Decomposing the integral considered into two, one extended over the square $0\leqslant u\leqslant\delta$, $0\leqslant v\leqslant\delta$ and the other over the rest of the square $0\leqslant u\leqslant\pi$, $0\leqslant v\leqslant\pi$, we easily find that the first of the integrals is $\leqslant A_1\,\varepsilon$, where A_1 is an absolute constant, and the second tends to 0 as $m,n\to\infty$. Since ε may be arbitrarily small, this completes the proof of Theorem 2.

8. Remark. The summability considered in Theorem II may be described as summability $(C,1,1)$. We may consider also the method (C,ε,δ), $\varepsilon>0$, $\delta>0$ and an argument similar to that given above permits to extend Theorem 2 to this more general case. We need only observe that, if $K_n^\delta(t)$ denotes the (C,δ) means of the series $\frac{1}{2}+\cos t+\cos 2t+\ldots$, then $|K_n^\delta(t)|\leqslant Bn$, $|K_n^\delta(t)|\leqslant B_1n^{-\delta}t^{-\delta-1}$ if $1/n\leqslant t\leqslant\pi$ [10]) and so $|K_n^\delta(t)|\leqslant B_2n/(1+n^{\delta+1}t^{\delta+1})$, where B, B_1, B_2 depend only on δ but not on n, t.

Finally, a theorem analogous to Theorem 2 holds also for Abel's summability.

[10]) M. Riesz, Acta Szeged, 1 (1922), 104—113.