Let $b=3$. By (31), $F=P-5y+11$, $y\geqq-6$. For $y\geqq3$, 31 is not a sum of five values. The least positive integer not a sum of four is 27 if $y\geqq2$ or $y\geqq1$, 53 if $y\geqq0$ or $y\geqq-1$, 696 if $y\geqq-2$, 1631 if $y\geqq-3$, 1652 if $y\geqq-4$ or $y\geqq-5$ or $y\geqq-6$. For $y\geqq0$, 53, 85, 217, 351, 391, 472 are the only integers $\leqq501$ which are not sums of four values of F. We readily conclude that all $\leqq2700$ are sums of five values.

Let $b=5$. By (31), $F=P-14y+50$, $y\geqq-10$. The least integer not a sum of five values of F is 37 if $y\geqq4$, and 63 if $y\geqq3$. Also 19 is not a sum of four values with $y\geqq-10$. Using the twenty-four integers $\leqq500$ which are not sums of four values of F for $y\geqq2$, we find that all $\leqq3000$ are sums of five.

Let $b=6$. Then $F=P-20y+85$, $y\geqq-12$. Then 13 is not a sum of four values. For $y\geqq4$, 122 is not a sum of five. All integers $\leqq3775$ are sums of five values of F for $y\geqq3$.

Finally, let $b=7$. Then $F=P-27y+133$, $y\geqq-14$. Then 5 is not a sum of four values. For $y\geqq5$, 43 is not a sum of five. Every integer $\leqq10000$ is a sum of five values of F for $y\geqq4$.

On the arithmetical density of the sum of two sequences one of which forms a basis for the integers.

By

P. Erdös (Manchester).

Let a_1, a_2, ... be any given sequence of positive steadily increasing integers and suppose there are $x=f(n)$ of them not exceeding a number n, so that

$$a_x \leqq n < a_{x+1}.$$

The density $\eth$ of the sequence is defined by Schnirelmann as the lower bound of the numbers $f(n)/n$, $n=1, 2, \ldots$. Thus if $a_1 \neq 1$, $\eth=0$.

Clearly $f(n) \geqq \eth n$.

Suppose also that the steadily increasing set

$$A_0 = 0, A_1, A_2, \ldots$$

forms a basis of order l of the positive integers. This means that every positive integer can be expressed as the sum of at most l of the A's. I prove the following

Theorem: *If $\eth'$ is the density of the sequence $a+A$, i. e. of the integers which can be expressed as the sum of an a and an A, then*

$$\eth' \geqq \eth + \frac{\eth(1-\eth)}{2l}.$$

Particular cases of this theorem have been proved by Khintchine and Buchstab in an entirely different and more complicated way.

I prove my theorem as a particular case of a more general one. Let the positive integers $\leq n$ not included among the a's be denoted by $b_1, b_2, \ldots,$ and let

$$b_y \leq n < b_{y+1}.$$

Put

$$E = b_1 + b_2 + \ldots + b_y - \frac{1}{2} y (y + 1),$$

Clearly $E \geq 0$, since $b_1 \geq 1$, $b_2 \geq 2$ etc. Then there exist at least $x + \dfrac{E}{l\,n}$ integers $\leq n$ of the form $a + A$, where in fact we need only use $A = 0$ and a single other A. This theorem is deduced from the one that there are at least $\dfrac{E}{l\,n}$ of the $b's \leq n$ which can be represented in the form $a + A$, and in fact only a single A is used.

We require the

Lemma: *An integer $J > 0$ exists such that there are at least $\dfrac{E}{n}$ of the $b's$ among the integers $\leq n$ in the set $a_1 + J$, $a_2 + J, \ldots$.*

For the number of solutions of the equation

$$a + v = b$$

in positive integers v and $a's$, $b's \leq n$ is E. Thus for given $b = b_r$, there are $b_r - r$ solutions since the number of $a's < b_r$ is clearly $b_r - r$ and every such a gives a solution v. Hence summing for $r = 1, 2, \ldots, y$, the total number of solutions is

$$E = \sum_{r=1}^{y} (b_r - r).$$

But there are at most n possible values of v, namely $1, 2, \ldots, n$ and so for at least one value of v, say J, there are not less than $\dfrac{E}{n}$ solutions of $a + J = b$ in $a's$ and $b's$ not greater than n.

Now express J as a sum of exactly l $A's$, say

$$J = A_1 + A_2 + \ldots + A_l,$$

by including a sufficient number of $A_0's$ among the $A's$ if need be and where A_1 need not denote the first, A_2 the second etc.

Denote by μ_s the number of $b's$ in the set $a + A_s$, $s = 1, 2, \ldots, l$. I prove now that

$$\mu_1 + \mu_2 + \ldots + \mu_l \geq \frac{E}{n}.$$

For in the set of integers given by

$$a + A_1 + A_2,$$

there are at most $\mu_1 + \mu_2$ of the $b's$. Thus the set $a + A_1$ contains μ_1 of the $b's$ together with some $a's$. When we add A_2 to the numbers of the set $a + A_1$, the μ_1 $b's$ give at most μ_1 $b's$, while the $a's$ give at most μ_2 $b's$. Now take the set $a + A_1 + A_2 + A_3$. This contains at most $\mu_1 + \mu_2 + \mu_3$ of the $b's$ by precisely the same argument applied to the sum of $a + A_1 + A_2$ and A_3. Similarly the set $a + A_1 + A_2 + \ldots + A_l$, i. e. $a + J$ will contain at most $\mu_1 + \mu_2 + \ldots + \mu_l$ of the $b's$. But since the set $a + J$ contained at least $\dfrac{E}{n}$ of the $b's$, clearly one of the μ's say $\mu_k \geq \dfrac{E}{l\,n}$, and so the set $a + A_k$ contains at least $\dfrac{E}{l\,n}$ of the $b's \leq n$. Now the set $a + A_0$, since $A_0 = 0$, consists of exactly the x $a's$. Hence the set $a + A$ including $A_k = 0$ contains at least $x + \dfrac{E}{l\,n}$ different integers $\leq n$.

Suppose now the $a's$ have a density δ with $\delta < 1$ which is no loss of generality. We have $f(b_r) \geq \delta b_r$ hence $b_r - r = f(b_r) \geq \delta b_r$, $b_r \geq \dfrac{r}{1 - \delta}$, and therefore

$$E \geq \frac{1 + 2 + \ldots + y}{1 - \delta} - \frac{y(y+1)}{2} \geq \frac{\delta}{2(1 - \delta)} y (y + 1).$$

Hence for the number N of integers $\leq n$ in the set $a + A$

$$N \geq x + \frac{\delta}{2(1 - \delta)} \frac{y^2}{l\,n} = \varphi(x) \qquad (y = n - x),$$

say. For $x \geq \delta n$

$$\varphi'(x) = 1 - \frac{\delta}{2(1 - \delta)} \frac{2(n - x)}{l\,n} \geq 1 - \frac{\delta}{l} > 0,$$

i. e.

$$N \geq \varphi(x) \geq \varphi(\delta n) = \delta n + \frac{\delta}{2(1 - \delta)} \frac{(1 - \delta)^2 n^2}{l\,n}.$$

hence
$$N \geqq n\left(\delta + \frac{\delta(1-\delta)}{2l}\right),$$

and this is the theorem.

I can prove in the same way that if a sequence $a_1, a_2, \ldots$ is given and there are $f(n)$ of the a's not exceeding n, then in the set $|a \pm A|$, there are at least

$$f(n) + \frac{f(n)(n-f(n))}{2l}$$

numbers not exceeding n.

Before closing my paper I would express my sincere gratitude to Prof. L. J. Mordell for having so kindly helped me with my ms.

A note on the distribution of primes.

By

A. E. Ingham (Cambridge).

1. If $\pi(x)$ denotes as usual the number of primes not exceeding x, then, by the prime number theorem, $\pi(x) \sim \mathrm{li}\, x$ (as $x \to \infty$), and, by a well known theorem of Littlewood [1]),

$$(1) \qquad \pi(x) - \mathrm{li}\, x = \Omega \pm \left(\frac{x^{\frac{1}{2}}}{\log x} \log \log \log x\right).$$

The first aim of this paper is to give a proof of (1) without the use of the Phragmén-Lindelöf theorem which was an essential feature of Littlewood's original proof; and the second is to adapt the method to the proof of the following result, in which Θ denotes the upper bound of the real parts of the zeros of the Riemann zeta-function $\zeta(s) = \zeta(\sigma + it)$.

Theorem A. *If Θ is attained, i. e. if $\zeta(s)$ has a zero on the line $\sigma = \Theta$, then there exists an absolute constant $A > 1$ such that, for all $x > 1$, the interval $(x, A x)$ contains an integer n and an integer n' satisfying*

$$\pi(n) < \mathrm{li}\, n, \quad \pi(n') > \mathrm{li}\, n'.$$

The possibility of dispensing with the Phragmén-Lindelöf theorem and the resulting advantages for the detailed study of the difference

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[1]) See E. Landau, *Vorlesungen über Zahlentheorie* (Leipzig, 1927), II, 123—150; or A. E. Ingham, *The distribution of prime numbers* (Cambridge, 1932), Chapter V. [These books will be quoted as „Vorlesungen" and „Prime numbers" respectively]. The notation is that of *Prime numbers*.