The strong summability of Fourier series.

By

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§ 1. Introduction.

1.1. A series

$$A_0 + A_1 + A_2 + \ldots$$

may be said to be *strongly summable*, with index k and sum s, if $k > 0$,

$$s_n = A_0 + A_1 + \ldots + A_n,$$

and

$$\frac{1}{n+1} \sum_0^n |s_\nu - s|^k \to 0 \tag{1.1.1}$$

when $n \to \infty$. It follows from Hölder's inequality that (1.1.1) says the more the larger k.

Suppose now that $f(t)$ is a periodic function of the class L^r, where $r > 1$, that

$$A_0 + \sum_1^\infty A_n = \frac{1}{2} a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt) \tag{1.1.2}$$

is the Fourier series of $f(t)$, that

$$\varphi(x, t) = \frac{1}{2} \{f(x+t) + f(x-t) - 2s\}, \tag{1.1.3}$$

$$\Phi(x, t) = \int_0^t \varphi(x, u)\, du, \tag{1.1.4}$$

and

$$\Phi_r^*(x, t) = \int_0^t |\varphi(x, u)|^r\, du. \tag{1.1.5}$$

If

$$\Phi(x, t) = o(t) \tag{1.1.6}$$

and

$$\Phi_r^*(x, t) = O(t) \tag{1.1.7}$$

for small t, then the series (1.1.2) *is strongly summable, to sum s, for every k* [1]. The conditions (1.1.6) and (1.1.7) are satisfied for almost all x when $s = f(x)$. In particular

$$\Phi_r^*(x, t) = o(t), \tag{1.1.8}$$

which includes both (1.1.6) and (1.1.7), is a sufficient condition for strong summability.

The proof fails when $r = 1$, and this case of the problem has remained unsolved [2].

1.2. Our main purpose here is to settle this unsolved problem. Our solution is (as was to be expected) negative; (1.1.8) is not sufficient, when $r = 1$, for strong summability with any index k [3]. We prove, however, a good deal more. If $k > 0$, and

$$\chi = \chi(n) = o(\sqrt{\log n}), \tag{1.2.1}$$

then (1.1.8), with $r = 1$, does not imply

$$\sum_0^n |s_\nu - s|^k = o(n \chi^k). \tag{1.2.2}$$

We state the proof primarily for the case $k = 1$.

[1]) Hardy and Littlewood (**4**, Theorem 1). The first theorem of this character appeared in **3**, and the theorem stated here is the result of successive generalisations by Carleman (**1**), Sutton (**8**), and ourselves.

Still further generalisations were made by Paley (**7**). Thus (1.1.6) may be replaced by

$$\varphi(x, t) \to 0 \qquad (C, l)$$

with any l, and (1.1.7) by

$$\int_0^t \psi(|\varphi(x, u)|)\, du = O(t),$$

where $\psi(w)$ may be, for example, any of

$$w\,(\log^+ w)^{1+\delta}, \quad w \log^+ w\,(\log^+ \log^+ w)^{1+\delta}, \ldots \qquad (\delta > 0).$$

[2]) See, for example, Zygmund (**9**, 240).

[3]) However small.

This negative result raises a further problem. The proof suggests that the function $\sqrt{\log n}$ should be, in a sense, a „best possible“ function; and in § 3 we prove that, substantially, this is so, since (1.1.8) implies

(1.2.3) $$\sum_{0}^{n} |s_\nu - s|^k = o\left(n (\log n)^{\frac{1}{2}k}\right)$$

at any rate when $k \leqq 2$.

This result completes the main purpose of the paper. In § 4 we discuss, more cursorily, some further problems left open by our work.

§ 2. Negative Theorems.

2.1. In § 2 we suppose that $f(t)$ is even and

$$x = 0, \qquad s = 0,$$

so that

$$\varphi(x, t) = f(t).$$

The numbers

$$A, \; B, \; C, \ldots$$

are positive „world-constants“, which preserve their identity throughout the argument, unless the contrary is stated expressly.

2.2. ***Theorem 1.*** *Suppose that* $\chi = \chi(n)$ *is an increasing function of* n, *and that*

(2.2.1) $$\chi(n) = o(\sqrt{\log n}).$$

Then there is an integrable function $f(t)$ *for which*

(2.2.2) $$\int_{0}^{t} |f(u)|\, du = o(t)$$

and

(2.2.3) $$\sum_{0}^{n} |s_\nu| \neq o(n\chi).$$

We begin by transforming the theorem. We may suppose that

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(t)\, dt = 0,$$

and then (2.2.3) is equivalent to

$$\sum_{1}^{n} |t_\nu| \neq o(n\chi),$$

where

$$t_n = a_1 + a_2 + \ldots + a_{n-1} + \frac{1}{2} a_n .$$

Now

$$t_n = n\, b_n,$$

where b_n is the Fourier constant

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} \cos nt\, g(t)\, dt$$

of the even and integrable function [4])

$$g(t) = \frac{1}{2}\int_{t}^{\pi} \cot \frac{1}{2} u\, f(u)\, du.$$

Since

$$g'(t) = -\frac{1}{2} \cot \frac{1}{2} t\, f(t),$$

(2.2.2) is equivalent to

(2.2.4) $$\int_{0}^{t} u\, |g'(u)|\, du = o(t).$$

Hence Theorem 1 is equivalent to

Theorem 2. *There is a function*

$$g(b) \sim \frac{1}{2} b_0 + \sum_{1}^{\infty} b_n \cos nt,$$

satisfying (2.2.4), *for which*

(2.2.5) $$\sum_{1}^{n} \nu\, |b_\nu| \neq o(n\chi).$$

It is in this form that we shall consider the theorem.

[4]) See Hardy (2).

Lemmas for Theorems 1 and 2.

2.3. ***Lemma 1.*** *Suppose that*

(2.3.1) $$n = 10^k, \qquad k \geqq 3, \qquad l = [\log k].$$

Then there are positive constants A, B, and an even function

$$f(t) = f_n(t)$$

possessing the following properties.

(a) $f(t) = \varepsilon_r$, *where* $\varepsilon_r = \pm 1$, *in the intervals*

$$\delta_r = \left(\frac{\pi}{10^{r+1}}, \frac{\pi}{10^r}\right) \qquad (l \leqq r < k),$$

and $f(t) = 0$ *if* $t > 10^{-l}\pi$ *or* $0 \leqq t < 10^{-k}\pi$.

(b) $f(t)$ *satisfies*

(2.3.2) $$\int_0^t u\,|df(u)| \leqq At, \qquad (0 \leqq t \leqq \pi),$$

(2.3.3) $$\int_0^t u\,|df(u)| = 0 \qquad \left(0 \leqq t \leqq \frac{\pi}{10^k}\right).$$

(c) *The Fourier constants* $c_\nu = c_\nu(f)$ *of* f *satisfy*

(2.3.4) $$\sum_1^n \nu\,|c_\nu(f)| \geqq Bn\sqrt{\log n}.$$

This is the critical lemma; when it is proved, the remainder of the proof of Theorems 1 and 2 will be a matter of routine.

In the first place, (2.3.3) is obvious.

Next, $f(t)$ is a step function with discontinuities, of magnitude 2, at some of the ends of the intervals δ_r. If

$$10^{-s-1}\pi \leqq t \leqq 10^{-s}\pi \qquad (l \leqq s < k)$$

then

$$\int_0^t u\,|df(u)| \leqq 2\sum_s^{k-1}\frac{\pi}{10^r} \leqq \frac{2\pi}{10^s}\cdot\frac{10}{9} \leqq \frac{200\,\pi}{9}t,$$

which proves (2.3.2).

It remains to prove that, if the ε_r are selected properly, c_ν satisfies (2.3.4).

2.4. Now

$$c_\nu(f) = \sum_{r=l}^{k-1}\varepsilon_r\int_{10^{-r-1}\pi}^{10^{-r}\pi}\cos\nu t\,dt,$$

(2.4.1) $$\nu\,c_\nu(f) = \sum_{r=l}^{k-1}\varepsilon_r\,\Delta_{r,\nu},$$

where

(2.4.2) $$\Delta_{r,\nu} = \sin\frac{\nu\pi}{10^r} - \sin\frac{\nu\pi}{10^{r+1}}.$$

Suppose, if possible, that

(2.4.3) $$\sum_1^n \nu\,|c_\nu| \leqq Cn\sqrt{\log n}$$

for every set of ε_r; and let

'Av'

denote an average taken over the 2^{k-l} sets of ε_r. Then, after (2.4.3),

(2.4.4) $$\sum_1^n Av\,(\nu\,|c_\nu|) \leqq Cn\sqrt{\log n}.$$

But [5]) there is a D such that

(2.4.5) $$Av\,(\nu\,|c_\nu|) = Av\left(\left|\sum_{r=l}^{k-1}\varepsilon_r\,\Delta_{r,\nu}\right|\right) \geqq D\sqrt{\sum_{r=l}^{k-1}\Delta_{r,\nu}^2};$$

and therefore, after (2.4.4),

(2.4.6) $$\sum_{\nu=1}^n\sqrt{\sum_{r=l}^{k-1}\Delta_{r,\nu}^2} \leqq \frac{C}{D}n\sqrt{\log n}.$$

If we can prove that (2.4.6) is false, for some C, we shall have proved that (2.4.3) is false for at any rate one selection of the ε_r; and this will complete the proof of the lemma.

[5]) Littlewood (6, Lemma 4).

2.5. We suppose now that

$$0 < \nu < n = 10^k, \quad \nu = i_1 i_2 \dots i_k$$

(in the decimal scale), and consider the conditions under which it is possible that

$$|\Delta| = |\Delta_{r,\nu}| < \frac{1}{100}, \quad r < k.$$

If $r < k$ then

$$\begin{aligned} \Delta &= \sin\frac{\nu\pi}{10^r} - \sin\frac{\nu\pi}{10^{r+1}} \\ &= (-1)^{i_{k-r}} \sin\left(\frac{i_{k-r+1}}{10} + \dots + \frac{i_k}{10^r}\right)\pi \\ &\quad - (-1)^{i_{k-r-1}} \sin\left(\frac{i_{k-r}}{10} + \dots + \frac{i_k}{10^{r+1}}\right)\pi \\ &= \pm \sin\theta \pm \sin\varphi, \end{aligned}$$

say; and $|\sin\theta|$ and $|\sin\varphi|$ differ by more than $\frac{1}{100}$ unless i_{k-r+1} has one of the 6 values

$$i_{k-r}, \quad i_{k-r}-1, \quad i_{k-r}+1, \quad 10-i_{k-r}, \quad 10-i_{k-r}+1, \quad 10-i_{k-r}-1$$ [6].

Hence

$$|\Delta| > \frac{1}{100}$$

unless i_{k-r+1} satisfies this condition.

We may regard each of the $10^k - 1$ values of ν as determined by successive choice of the digits

$$i_1, i_2, \dots, i_{k-r}, i_{k-r+1}, \dots, i_k.$$

In general there are 10 possible choices of i_{k-r+1}; but if i_{k-r+1} is to be chosen so that $|\Delta| < \frac{1}{100}$, then there are at most 6.

[6]) Divide the angle π into ten equal sectors. Then φ lies in a sector which is not the same as or adjacent to that containing θ, nor supplementary to such a sector; and

$$||\sin\theta| - |\sin\varphi|| > \int_{\frac{1}{2}\pi - \frac{1}{10}\pi}^{\frac{1}{2}\pi} \cos u\, du = 1 - \cos\frac{\pi}{10} > \frac{1}{100}.$$

Suppose now that, for a particular ν,

(2.5.1) $$\sum_{r=l}^{k-1} \Delta_{r,\nu}^2 \leqq h^2,$$

where $h = h(n)$ is a function of n (or of k) to be chosen later. Then

$$\Delta > \frac{1}{100}$$

for at most

$$\lambda = [10^4 h^2]$$

values of r. At most λ of the choices for ν are unrestricted, and the remainder, at least $k - l - \lambda$, restricted. The λ unrestricted choices can correspond to at most

$$\binom{k}{\lambda}$$

sets of positions, and the total number of values of ν which satisfy (2.5.1) is at most

(2.5.2) $$m = 10^l \binom{k}{\lambda} 10^\lambda 6^{k-l-\lambda}.$$

We take

$$h = E\sqrt{\log n}$$

and find an upper bound for m for large n. We have

$$\lambda = [10^4 E^2 \log n] = \gamma k,$$

where

$$\gamma \leqq \frac{10^4 E^2}{\log 10}$$

is small with E. Also

$$\begin{aligned} \binom{k}{\lambda} &= \frac{k(k-1)\dots(k-\lambda+1)}{\lambda!} \\ &< \frac{k^\lambda}{\lambda!} \leqq \frac{k^{\gamma k} e^{\gamma k}}{(\gamma k)^{\gamma k}} = 10^{\delta k} = n^\delta, \end{aligned}$$

where

$$\delta = \frac{\gamma - \gamma\log\gamma}{\log 10}$$

is small with E and γ. We can therefore choose E so that

(2.5.3) $$\binom{k}{\lambda} < n^{\frac{1}{30}}$$

and

(2.5.4) $$10^{\lambda} = 10^{\gamma k} = n^{\gamma} < n^{\frac{1}{80}}$$

for sufficiently large n. Finally

(2.5.5) $$10^{l} = 10^{[\log k]} < n^{\frac{1}{80}}$$

for large n, and

(2.5.6) $$6^{k-l-\lambda} < 6^{k} = n^{\frac{\log 6}{\log 10}} = n^{\cdot 78\cdots} < n^{\frac{8}{10}};$$

and these four inequalities and (2.5.2) show that

$$m < n^{\frac{3}{80}+\frac{8}{10}} = n^{\frac{9}{10}}$$

for large n.

It follows that

$$\sum_{r=l}^{k-1} \Delta_{r,\nu}^2 \geqq h^2 = E^2 \log n$$

for at least

$$n - n^{\frac{9}{10}} > \frac{1}{2} n$$

values of n, and therefore that

$$\sum_{\nu=1}^{n} \sqrt{\sum_{r=l}^{k-1} \Delta_{r,\nu}^2} \geqq \frac{1}{2} E n \sqrt{\log n}.$$

This contradicts (2.4.6) when

$$C < \frac{1}{2} D E;$$

and the contradiction proves the lemma.

2.6. **Lemma 2.** *Suppose that n, k, l satisfy* (2.3.1). *Then there are positive constants H, K, and an even function*

$$g(t) = g_n(t),$$

with the following properties.

(b′) $g(t)$ *satisfies*

(2.6.1) $$\int_0^t u\,|g'(u)|\,du \leqq Ht \qquad (0 \leqq t \leqq \pi),$$

(2.6.2) $$\int_0^t u\,|g'(u)|\,du = 0 \qquad \left(0 \leqq t \leqq \frac{\pi}{10^k}\right).$$

(c′) *The Fourier constants $c_\nu = c_\nu(g)$ of g satisfy*

(2.6.3) $$\sum_1^n \nu\,|c_\nu(g)| \geqq K n \sqrt{\log n}.$$

The f of Lemma 1 has discontinuities at some of the points $10^{-r}\pi$; we construct g by smoothing away these discontinuities appropriately. If, for example, $l < r < k-1$, and f jumps from -1 to $+1$ at $10^{-r}\pi$, then we join the points

$$(10^{-r}\pi - n^{-3}, -1), \qquad (10^{-r}\pi + n^{-3}, 1)$$

by a straight line, and take this as part of the definition of g. A jump from $+1$ to -1, or a jump of half the height when r is l or $k-1$, is dealt with similarly. Since g is stationary except along these lines, and

$$\int_{10^{-r}\pi-n^{-3}}^{10^{-r}\pi+n^{-3}} u\,|g'(u)|\,du = \int_{10^{-r}\pi-n^{-3}}^{10^{-r}\pi+n^{-3}} u\,|df(u)|,$$

g has the properties (2.6.1) and (2.6.2). Also f and g differ by at most 1, and that in a set of measure at most

$$\frac{2k}{n^3} < L\frac{\log n}{n^3}.$$

Hence

$$|c_\nu(f) - c_\nu(g)| < L\frac{\log n}{n^3};$$

and

$$\sum_1^n \nu\,|c_\nu(f)|, \qquad \sum_1^n \nu\,|c_\nu(g)|$$

differ by less than

$$L\frac{\log n}{n},$$

so that g has the property (2.6.3).

2.7. In this paragraph we vary our notation a little. M is a world-constant wherever it occurs, but the constants in different places are not the same.

Lemma 3. *The function g of Lemma 2 has also the properties*

(2.7.1) $$\sum_{1}^{m} \nu\,|c_\nu(g)| \leqq M m \log n,$$

(2.7.2) $$\sum_{1}^{m} \nu\,|c_\nu(g)| \leqq M m \log m,$$

for $m > 1$.

(i) Since

$$\nu\,|c_\nu(g)| \leqq M\,V(g),$$

where $V(g)$ is the total variation of g, and $V(g)$ is less than Mk or $M\log n$, g has the property (2.7.1).

(ii) We have

$$|g| \leqq 1, \quad G(t) = \int_0^t u\,|g'(u)|\,du \leqq Ht.$$

Now

(2.7.3) $$c_\nu(g) = \frac{2}{\pi}\left(\int_0^{\pi/\nu} + \int_{\pi/\nu}^{\pi}\right) \cos \nu t\, g(t)\,dt = J_1 + J_2,$$

say. Here

(2.7.4) $$|J_1| \leqq \frac{2}{\pi}\int_0^{\pi/\nu} |g(t)|\,dt \leqq \frac{M}{\nu},$$

since $|g| \leqq 1$. Also

$$J_2 = \frac{2}{\pi}\int_{\pi/\nu}^{\pi} \cos \nu t\, g(t)\,dt = -\frac{2}{\pi\nu}\int_{\pi/\nu}^{\pi} \sin \nu t\, g'(t)\,dt,$$

(2.7.5) $$|J_2| \leqq \frac{M}{\nu}\int_{\pi/\nu}^{\pi} |g'(t)|\,dt = \frac{M}{\nu}\int_{\pi/\nu}^{\pi} \frac{G'(t)}{t}\,dt$$

$$= \frac{M}{\nu}\left(\frac{G(\pi)}{\pi} - \frac{G(\pi/\nu)}{\pi/\nu}\right) + \frac{M}{\nu}\int_{\pi/\nu}^{\pi} \frac{G(t)}{t^2}\,dt$$

$$\leqq \frac{M}{\nu} + \frac{MH}{\nu}\int_{\pi/\nu}^{\pi} \frac{dt}{t} \leqq M\frac{\log \nu}{\nu}.$$

From (2.7.3), (2.7.4), and (2.7.5) we deduce

$$|c_\nu(g)| \leqq M\frac{\log \nu}{\nu},$$

and so (2.7.2).

Proof of Theorem 2.

2.8. We can now prove Theorem 2 (and so Theorem 1). We define $g(t)$ by

$$g(t) = \sum_{s=1}^{\infty} \zeta_s\, g_{n_s}(t) = \sum_{s=1}^{\infty} \frac{g_{n_s}(t)}{\eta_s},$$

where $\eta_s > 0$ and $n_s = 10^{k_s}$ are sequences, to be chosen later, which tend to infinity, when $s \to \infty$, with great rapidity. It is plain that, if the increase of n_s is sufficiently rapid, at most one g_{n_s} differs from 0 for any t. Also $g_{n_s} = 0$ if $t < 10^{-k_s}\pi = \pi/n_s$, and

$$\int_0^t u\,|g'_{n_s}(u)|\,du = O(t)$$

uniformly in s. Hence

$$\int_0^t u\,|g'(u)|\,du \leqq \sum \zeta_s \int_0^t u\,|g'_{n_s}(u)|\,du = O\left(t \sum_{10^{-k_s} \leqq t} \zeta_s\right) = o(t),$$

which is (2.2.4). It remains to verify (2.2.5).

We have

$$c_\nu(g) = \sum_s \zeta_s\, c_\nu(g_{n_s}),$$

$$|c_\nu(g)| \geqq \zeta_\sigma\,|c_\nu(g_{n_\sigma})| - \sum_{s<\sigma} \zeta_s\,|c_\nu(g_{n_s})| - \sum_{s>\sigma} \zeta_s\,|c_\nu(g_{n_s})|,$$

(2.8.1) $$S = \sum_{\nu=1}^{n_\sigma} \nu\,|c_\nu(g)| \geqq \zeta_\sigma \sum_{\nu=1}^{n_\sigma} \nu\,|c_\nu(g_{n_\sigma})|$$

$$- \sum_{s<\sigma} \zeta_s \sum_{\nu=1}^{n_\sigma} \nu\,|c_\nu(g_{n_s})| - \sum_{s>\sigma} \zeta_s \sum_{\nu=1}^{n_\sigma} \nu\,|c_\nu(g_{n_s})|$$

$$= S^* - S_1 - S_2,$$

say.

In the first place, by (2.6.3),

(2.8.2) $$S^* > K\zeta_\sigma n_\sigma \sqrt{\log n_\sigma} \succ n_\sigma \chi(n_\sigma)$$ [7]

if

(2.8.3) $$\frac{\sqrt{\log n_\sigma}}{\chi(n_\sigma)} \succ \eta_\sigma.$$

Next, in S_1, $s < \sigma$ and, by (2.7.1),

$$\sum_{\nu=1}^{n_\sigma} \nu\,|c_\nu(g_{n_s})| \leqq M n_\sigma \log n_s \leqq M n_\sigma \log n_{\sigma-1},$$

so that

$$S_1 \leqq M n_\sigma \log n_{\sigma-1} \sum_s \zeta_s \leqq M n_\sigma \log n_{\sigma-1},$$

and

(2.8.4) $$S_1 \prec n_\sigma \chi(n_\sigma)$$

if

(2.8.5) $$\log n_{\sigma-1} \prec \chi(n_\sigma).$$

For this it is only necessary that n_σ should tend to infinity with sufficient rapidity.

Finally, in S_2, $s > \sigma$ and, by (2.7.2),

$$\sum_{\nu=1}^{n_\sigma} \nu\,|c_\nu(g_{n_s})| \leqq M n_\sigma \log n_\sigma,$$

Hence

$$S_2 \leqq M n_\sigma \log n_\sigma (\zeta_{\sigma+1} + \zeta_{\sigma+2} + \ldots) \leqq M n_\sigma \log n_\sigma\, \zeta_{\sigma+1},$$

if ζ_s decreases with sufficient rapidity. It follows that

(2.8.6) $$S_2 \prec n_\sigma \chi(n_\sigma)$$

if

(2.8.7) $$\frac{\log n_\sigma}{\chi(n_\sigma)} \prec \eta_{\sigma+1}.$$

[7]) We write $\varphi \succ \psi$ if $\varphi/\psi \to \infty$ when the variable involved in φ and ψ tends to infinity, and $\varphi \prec \psi$ if $\varphi/\psi \to 0$.

Collecting our results from (2.8.1), (2.8.2), (2.8.4) and (2.8.6), we see that

$$S = \sum_1^{n_\sigma} \nu\,|c_\nu(g)| \succ n_\sigma \chi(n_\sigma)$$

if n_σ and η_σ tend to infinity both with sufficient rapidity and in such a manner as to satisfy (2.8.3) and (2.8.7). Since we can satisfy these conditions by successive choice of

$$\eta_1,\ n_1,\ \eta_2,\ n_2,\ \eta_3, \ldots,$$

the theorem follows.

We have supposed that the k of § 1 is 1. It is plain that the same argument proves that

$$\sum_0^n |s_\nu|^k \neq o(n\chi^k)$$

for any positive k.

§ 3. Positive theorems.

3.1. The notation of this section differs from that of § 2.

We suppose that $u(\theta)$ is periodic and integrable; that

$$u(\theta) \sim \frac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta) = \frac{1}{2}A_0 + \sum_1^\infty A_n(\theta);$$

that

$$\varphi(\theta, t) = \frac{1}{2}\{u(\theta+t) + u(\theta-t) - 2s\};$$

and that

$$s_n = s_n(\theta) = \frac{1}{2}A_0 + \sum_1^n A_\nu(\theta).$$

We also suppose that

$$u(r, \theta) = \frac{1}{2}A_0 + \sum_1^\infty A_n(\theta)\, r^n$$

is the harmonic function associated with $u(\theta)$, or Poisson integral of $u(\theta)$; and that

$$f(z) = f(re^{i\theta}) = \sum c_n z^n = \sum c_n r^n e^{ni\theta},$$

where

$$c_0 = \frac{1}{2} a_0, \qquad c_n = a_n - i\, b_n \qquad (n > 0),$$

is the analytic function, regular for $r < 1$, whose real part is $u(r, \theta)$ and whose imaginary part vanishes at the origin.

The A are positive world-constants, different on different occasions, while C is a positive number which remains the same throughout a proposition and its proof.

Theorem 3. *If*

$$\int_0^t |\varphi(\theta, w)|\, dw = o(t), \tag{3.1.1}$$

then

$$\sum_0^n (s_\nu - s)^2 = o(n \log n) \tag{3.1.2}$$

and

$$\sum_0^n |s_\nu - s| = o(n \sqrt{\log n}). \tag{3.1.3}$$

We need only prove (3.1.2), since (3.1.3) then follows by Cauchy's inequality. The theorem shows that the condition (2.2.1), imposed on $\chi(n)$ in Theorems 1 and 2, is the „best possible" condition of its kind.

The series

$$\frac{1}{2} A_0 - s + \sum_1^\infty A_n(\theta) \cos nt$$

is the Fourier series in t of the even function $\varphi(\theta, t)$, and converges to zero when s_n converges to s. We may therefore suppose, without real loss of generality, that $\theta = 0$, $s = 0$, and that $u(t)$ is an even function of t. In these circumstances $\varphi(\theta, t) = u(t)$. The letters θ and φ then disappear from the theorem, and there will be no inconvenience in using them in other senses.

Lemmas for Theorem 3.

3.2. ***Lemma 4.*** *If δ is positive, and θ real, then*

$$\int_{-\infty}^{\infty} \frac{\delta \varphi^2}{(\delta^2 + \varphi^2)(\delta^2 + (\varphi - \theta)^2)}\, d\varphi < A, \tag{3.2.1}$$

$$\int_{-\infty}^{\infty} \frac{\delta |\theta| |\varphi|}{(\delta^2 + \varphi^2)(\delta^2 + (\varphi - \theta)^2)}\, d\varphi < A. \tag{3.2.2}$$

(i) Since $\varphi^2 < \delta^2 + \varphi^2$, we have

$$\int_{-\infty}^{\infty} \frac{\delta \varphi^2}{(\delta^2 + \varphi^2)(\delta^2 + (\varphi - \theta)^2)}\, d\varphi < \int_{-\infty}^{\infty} \frac{\delta\, d\varphi}{\delta^2 + (\varphi - \theta)^2} = \pi.$$

(ii) We may suppose θ positive. It is then sufficient to consider the integral over $(0, \infty)$, the other part being smaller.

We write

$$\int_0^\infty \frac{\delta \theta \varphi}{(\delta^2 + \varphi^2)(\delta^2 + (\varphi - \theta)^2)}\, d\varphi = \int_0^{\frac{1}{2}\theta} + \int_{\frac{1}{2}\theta}^{2\theta} + \int_{2\theta}^{\infty} = J_1 + J_2 + J_3.$$

In J_1,

$$\delta^2 + (\varphi - \theta)^2 > (\varphi - \theta)^2 \geqq \frac{1}{4} \theta^2 \geqq \frac{1}{2} \theta \varphi,$$

so that

$$J_1 < 2 \int_0^\infty \frac{\delta\, d\varphi}{\delta^2 + \varphi^2} = \pi.$$

In J_2,

$$\delta^2 + \varphi^2 > \varphi^2 \geqq \frac{1}{2} \theta \varphi,$$

so that

$$J_2 < 2 \int_{-\infty}^{\infty} \frac{\delta\, d\varphi}{\delta^2 + (\varphi - \theta)^2} = 2\pi.$$

Finally, in J_3, $\theta \leqq \frac{1}{2} \varphi$ and so

$$\delta^2 + (\varphi - \theta)^2 > \frac{1}{4}(\delta^2 + \varphi^2),$$

$$J_3 < 2 \int_0^\infty \frac{\delta \varphi^2\, d\varphi}{(\delta^2 + \varphi^2)^2} = \frac{1}{2} \pi.$$

3.3. ***Lemma 5.*** *If*

(3.3.1) $$\left|\int_0^\theta |u(t)|\,dt\right| \leqq C|\theta|,$$

for all θ, *then*

(3.3.2) $$\left|\int_0^\theta |u(r,t)|\,dt\right| \leqq AC|\theta|$$

for $r \leqq 1$.

We may suppose that

$$0 < \theta \leqq \frac{1}{2}\pi.$$

We write

$$\delta = 1 - r,$$

so that $0 \leqq \delta \leqq 1$. Then

$$u(r,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\,u(\varphi)}{1-2r\cos(\varphi-t)+r^2}\,d\varphi,$$

$$|u(r,t)| \leqq A\int_{-\pi}^{\pi} \frac{\delta\,|u(\varphi)|}{\delta^2+(\varphi-t)^2}\,d\varphi,$$

(3.3.3) $$\int_0^\theta |u(r,t)|\,dt \leqq A\int_{-\pi}^{\pi} |u(\varphi)|\,\chi(\varphi,\theta,\delta)\,d\varphi,$$

where

(3.3.4) $$\chi(\varphi,\theta,\delta) = \int_0^\theta \frac{\delta\,dt}{\delta^2+(\varphi-t)^2}.$$

It will be enough to prove that

(3.3.5) $$J = J(\theta,\delta) = \int_0^\pi |u(\varphi)|\,\chi(\varphi,\theta,\delta)\,d\varphi \leqq AC\theta.$$

If

$$U(\varphi) = \int_0^\varphi |u(t)|\,dt \leqq C\varphi,$$

then

(3.3.6) $$J = \int_0^\pi U'(\varphi)\,\chi(\varphi,\theta,\delta)\,d\varphi = U(\pi)\,\chi(\pi,\theta,\delta) - \int_0^\pi U(\varphi)\,\frac{\partial\chi}{\partial\varphi}\,d\varphi.$$

The first term is plainly less than $AC\theta$. In the second, we have

$$\frac{\partial\chi}{\partial\varphi} = -\int_0^\theta \frac{d}{dt}\left(\frac{\delta}{\delta^2+(\varphi-t)^2}\right)dt = \frac{\delta}{\delta^2+\varphi^2} - \frac{\delta}{\delta^2+(\varphi-\theta)^2},$$

$$\left|\frac{\partial\chi}{\partial\varphi}\right| \leqq A\,\frac{\delta(\theta\varphi+\theta^2)}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)},$$

$$\left|\int_0^\pi U(\varphi)\,\frac{\partial\chi}{\partial\varphi}\,d\varphi\right| \leqq AC\int_0^\pi \frac{\delta\theta\varphi^2+\delta\theta^2\varphi}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)}\,d\varphi \leqq AC\theta,$$

by Lemma 4.

3.4. ***Lemma 6.*** *If*

(3.4.1) $$\int_0^\theta |u(t)|\,dt = o(|\theta|)$$

then

(3.4.2) $$\int_0^\theta |u(r,t)|\,dt = o(|\theta|),$$

uniformly for $1 - r \leqq |\theta|$.

We may suppose θ positive. The conclusion then asserts that, given a positive ε, we have

$$\int_0^\theta |u(r,t)|\,dt \leqq \varepsilon\theta$$

for

$$\delta = 1 - r \leqq \theta \leqq \zeta(\varepsilon)\ [8].$$

[8]) We state and prove what we shall actually require. The restrictions on δ and θ are no doubt stronger than is necessary.

It is of course not true that

$$\int_0^\theta |u(r,t)|\,dt = o(|\theta|)$$

uniformly for $r \leqq 1$; in fact

$$\int_0^\theta |u(r,t)|\,dt \sim \theta\,u(r,0),$$

for a fixed r, and $u(r,0)$ is not generally 0.

It is enough to show that the integral $J(\theta, \delta)$ of (3.3.5) satisfies such an inequality.

We transform $J(\theta, \delta)$ as in (3.3.6). Then

$$U(\pi)\,\chi(\pi, \theta, \delta) = U(\pi) \int_0^\theta \frac{\delta\, dt}{\delta^2 + (\pi - t)^2}$$

is less than a multiple of $\delta\theta$, and is $o(\theta)$ in the sense required. It is therefore sufficient to prove that

$$\int_0^\pi \frac{\delta\,\theta\,\varphi\, U(\varphi)}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)}\, d\varphi = o(\theta) \tag{3.4.3}$$

and

$$\int_0^\pi \frac{\delta\,\theta^2\, U(\varphi)}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)}\, d\varphi = o(\theta). \tag{3.4.4}$$

The argument is, after Lemma 4, the same for either integral. We choose $\eta = \eta(\varepsilon)$ so that

$$U(\varphi) \leqq \varepsilon\varphi \qquad (0 < \varphi \leqq \eta).$$

Then (taking the first integral for example) we have

$$\int_0^\eta \frac{\delta\,\theta\,\varphi\, U(\varphi)}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)}\, d\varphi \leqq \varepsilon \int_0^\infty \frac{\delta\,\theta\,\varphi^2}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)}\, d\varphi \leqq A\varepsilon\theta. \tag{3.4.5}$$

But, if $\theta < \frac{1}{2}\eta$, we have also

$$\int_\eta^\pi \frac{\delta\,\theta\,\varphi\, U(\varphi)}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)}\, d\varphi < C\delta\theta \int_\eta^\infty \frac{\varphi^2\, d\varphi}{\varphi^2\cdot\frac{1}{4}\varphi^2} = 4C\delta\theta \int_\eta^\pi \frac{d\varphi}{\varphi^2} = \tag{3.4.6}$$

$$= \frac{4\,C\,\delta\,\theta}{\eta} \leqq \frac{4\,C\,\theta^2}{\eta} = o(\theta);$$

and (3.4.3) follows from (3.4.5) and (3.4.6).

3.5. ***Lemma* 7.** *If $u(t)$ satisfies the condition of Lemma 5, then*

$$\int_0^\theta |f'(r\,e^{it})|\, dt \leqq \frac{A\,C\,|\theta|}{1-r}. \tag{3.5.1}$$

It is enough to prove that

$$\int_0^\theta |u_r(r, t)|\, dt \leqq \frac{AC\theta}{\delta}, \tag{3.5.2}$$

$$\int_0^\theta |u_t(r, t)|\, dt \leqq \frac{AC\theta}{\delta}, \tag{3.5.3}$$

for $\theta > 0$, $\delta < \frac{1}{2}$. Here the suffixes denote partial differentiations with respect to r and t.

These inequalities are corollaries of those occurring in the proof of Lemma 5. For

$$u_r(r, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\varphi) \frac{\partial}{\partial r}\left(\frac{1-r^2}{1 - 2r\cos(\varphi - t) + r^2}\right) d\varphi,$$

and

$$\left|\frac{\partial}{\partial r}\left(\frac{1-r^2}{1-2r\cos(\varphi-t)+r^2}\right)\right| = \frac{\left|2(1-r)^2 - 4(1+r^2)\sin^2\frac{1}{2}(\varphi-t)\right|}{\left((1-r)^2 + 4r\sin^2\frac{1}{2}(\varphi-t)\right)^2}$$

is less than

$$A\frac{\delta^2+(\varphi-t)^2}{(\delta^2+(\varphi-t)^2)^2} = \frac{A}{\delta^2+(\varphi-t)^2}.$$

Hence

$$\int_0^\theta |u_r(r, t)|\, dt$$

is majorised by an integral which is, apart from a factor δ, that occurring in the proof of Lemma 5. And similarly

$$\left|\frac{\partial}{\partial t}\left(\frac{1-r^2}{1-2r\cos(\varphi-t)+r^2}\right)\right| = \frac{2r(1-r^2)\,|\sin(\varphi-t)|}{(1-2r\cos(\varphi-t)+r^2)^2}$$

is less than

$$A\frac{\delta\,|\varphi - t|}{(\delta^2+(\varphi-t)^2)^2} \leqq \frac{A}{\delta^2+(\varphi-t)^2}.$$

***Lemma* 8.** *If $u(t)$ satisfies the conditions of Lemma 6, then*

$$\int_0^\theta |f'(r\,e^{it})|\, dt = o\left(\frac{|\theta|}{1-r}\right), \tag{3.5.4}$$

uniformly for $1 - r \leqq |\theta|$.

This follows in the same way from the proof of Lemma 6.

3.6. ***Lemma 9.*** *If*

(3.6.1) $$\left|\int_0^\theta u(t)\,dt\right| \leqq C|\theta|,$$

for all θ, *and* a fortiori *if* $u(t)$ *satisfies* (3.3.1) *then*

(3.6.2) $$\left|\frac{u(r,\theta)}{1-z}\right| \leqq \frac{AC}{1-r};$$

and if

(3.6.3) $$\int_0^\theta u(t)\,dt = o(|\theta|),$$

and a fortiori *if* $u(t)$ *satisfies* (3.4.1), *then*

(3.6.4) $$\frac{u(r,\theta)}{1-z} = o\left(\frac{1}{1-r}\right)$$

when z *tends to* 1 *from inside the unit circle.*

We shall need only one clause of the lemma, viz. that (3.3.1) implies (3.6.2); but it is as easy to prove the stronger form of this proposition, which has some independent interest.

It is familiar that

$$|u(r,\theta)| \leqq \frac{AC}{1-r},$$

and we may therefore suppose, in the first part of the lemma, that $0 < \theta \leqq \frac{1}{2}\pi$. Then, if

$$U_1(\varphi) = \int_0^\varphi u(t)\,dt,$$

we have

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\,u(\varphi)}{1-2r\cos(\varphi-\theta)+r^2}\,d\varphi$$

$$= \frac{1}{2\pi}\left[\frac{(1-r^2)\,U_1(\varphi)}{1-2r\cos(\varphi-\theta)+r^2}\right]_{-\pi}^{\pi} +$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2r(1-r^2)\sin(\varphi-\theta)\,U_1(\varphi)}{(1-2r\cos(\varphi-\theta)+r^2)^2}\,d\varphi.$$

The first term does not exceed AC. The second has a majorant

$$J = AC\int_{-\pi}^{\pi} \frac{\delta\,|\varphi|\,|\varphi-\theta|}{(\delta^2+(\varphi-\theta)^2)^2}\,d\varphi \leqq AC\int_{-\pi}^{\pi} \frac{\delta(\varphi-\theta)^2}{(\delta^2+(\varphi-\theta)^2)^2}\,d\varphi +$$

$$+ AC\int_{-\pi}^{\pi} \frac{\delta\,\theta\,|\varphi-\theta|}{(\delta^2+(\varphi-\theta)^2)^2}\,d\varphi = J_1 + J_2,$$

say. Here

$$J_1 \leqq AC\int_{-\infty}^{\infty} \frac{\delta\,d\varphi}{\delta^2+(\varphi-\theta)^2} \leqq AC$$

and

$$J_2 \leqq AC\theta\int_{-\infty}^{\infty} \frac{d\varphi}{\delta^2+(\varphi-\theta)^2} \leqq \frac{AC\theta}{\delta}.$$

Hence

$$J \leqq AC\left(1+\frac{\theta}{\delta}\right),$$

and so

$$\left|\frac{u(r,t)}{1-z}\right| \leqq \frac{AC}{|1-z|} + \frac{AC}{\delta} \leqq \frac{AC}{\delta}.$$

This proves the first half of the lemma. The second (with o) is not wanted here, but we sketch the proof for the sake of completeness. It is enough to show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2r(1-r^2)\sin(\varphi-\theta)\,U_1(\varphi)}{(1-2r\cos(\varphi-\theta)+r^2)^2}\,d\varphi = o(1) + o\left(\frac{|\theta|}{\delta}\right).$$

We choose $\zeta = \zeta(\varepsilon)$ so that $|U_1(\varphi)| \leqq \varepsilon|\varphi|$ for $|\varphi| \leqq \zeta$. The argument which precedes then shows that

$$\left|\frac{1}{2\pi}\int_{-\zeta}^{\zeta}\right| \leqq A\varepsilon\int_{-\zeta}^{\zeta} \frac{\delta\,|\varphi|\,|\varphi-\theta|}{(\delta^2+(\varphi-\theta)^2)^2}\,d\varphi \leqq A\varepsilon\left(1+\frac{|\theta|}{\delta}\right),$$

and the rest of the integral tends to zero with δ and θ, for any fixed ζ.

It will be observed that here we need not make any reservation about the relative magnitudes of θ and δ.

3.7. ***Lemma 10.*** *If* $u(t)$ *satisfies* (3.3.1) *then*

(3.7.1)
$$\left|\frac{f'(z)}{1-z}\right| \leqq \frac{AC}{(1-r)^2};$$

and if $u(t)$ *satisfies* (3.4.1) *then*

(3.7.2)
$$\frac{f'(z)}{1-z} = o\left(\frac{1}{(1-r)^2}\right).$$

We prove the first clause only; the second is not required, and the reader will have no difficulty in making the appropriate modifications in the proof.

It is enough to prove that

$$\left|\frac{u_r(r,\theta)}{1-z}\right| \leqq \frac{AC}{\delta^2}, \qquad \left|\frac{u_\theta(r,\theta)}{1-z}\right| \leqq \frac{AC}{\delta^2}.$$

But [9]) neither $|u_r|$ nor $|u_\theta|$ exceeds the sum of

$$A\int_0^\pi \frac{|u(\varphi)|}{\delta^2+(\varphi-\theta)^2}\,d\varphi = A\frac{U(\pi)}{\delta^2+(\pi-t)^2} + 2A\int_0^\pi \frac{U(\varphi)\,|\varphi-\theta|}{(\delta^2+(\varphi-\theta)^2)^2}\,d\varphi$$

and a similar contribution arising from negative values of φ. The integral here has a majorant which differs from that of Lemma 9 only in the absence of a factor δ, and the conclusion follows.

Proof of Theorem 3.

3.8. We may suppose (as we explained in § 3.1) that

$$u(t) \sim \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nt$$

is even, and that $\theta = 0$, $s = 0$.

We write

$$f(z) = \sum c_n z^n, \qquad c_0 = \frac{1}{2}a_0, \qquad c_n = a_n \qquad (n > 0),$$

$$s_n = s_n(f) = c_0 + c_1 + \dots + c_n, \qquad \sigma_n = \sigma_n(f) = \frac{s_0+s_1+\dots+s_n}{n+1};$$

so that

$$s_n - \sigma_n = \frac{c_1+2c_2+\dots+nc_n}{n+1} = \frac{s_n(f')}{n+1}.$$

[9]) See § 3.5.

Then

$$\sum_1^n \sigma_\nu^2 = o(n)$$

by the theorem of Fejér and Lebesgue, and it is sufficient to prove that

$$\sum_0^n \left(\frac{s_\nu(f')}{\nu+1}\right)^2 = o(n\log n)$$

or, what is the same thing, that

(3.8.1)
$$\sum_0^n (s_\nu(f'))^2 = o(n^3\log n).$$

Suppose now that

$$\delta = 1 - r = \frac{1}{n}.$$

Then

$$\sum_0^n (s_\nu(f'))^2 \leqq \left(1-\frac{1}{n}\right)^{-2n} \sum_0^\infty (s_\nu(f'))^2 r^{2\nu}.$$

The first factor is less than A for $n \geqq 2$, and

(3.8.2)
$$\sum_0^\infty (s_\nu(f'))^2 r^{2\nu} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{f'(z)}{1-z}\right|^2 d\theta.$$

Hence (3.8.1) will follow from

(3.8.3)
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{f'(z)}{1-z}\right|^2 d\theta = o\left(\frac{1}{\delta^3}\log\frac{1}{\delta}\right).$$

It will be sufficient to consider the part of the integral in which θ is positive. We write

(3.8.4)
$$J = \int_0^\pi \left|\frac{f'(z)}{1-z}\right|^2 d\theta = \int_0^\delta + \int_\delta^\pi = J_1 + J_2,$$

and show that J_1 and J_2 are of the form required.

If $F(\theta) = \int_0^\theta |f'(re^{it})|\,dt$ then $F(\theta) = O\left(\frac{\theta}{\delta}\right)$, by Lemma 7;

and $\frac{f'(z)}{1-z} = O\left(\frac{1}{\delta^2}\right)$ by Lemma 10. Hence

$$(3.8.5)\quad J_1 = O\left(\frac{1}{\delta^2}\right)\int_0^\delta \frac{|f'(z)|}{|1-z|}\,d\theta = O\left(\frac{1}{\delta^2}\right)\int_0^\delta \frac{|f'(re^{i\theta})|}{(\delta^2+\theta^2)^{\frac12}}\,d\theta$$

$$= O\left(\frac{1}{\delta^2}\right)\left\{\frac{F(\delta)}{(\delta^2+\delta^2)^{\frac12}} + \int_0^\delta \frac{\theta F(\theta)}{(\delta^2+\theta^2)^{\frac32}}\,d\theta\right\}$$

$$= O\left(\frac{1}{\delta^2}\cdot\frac{\delta}{\delta\cdot\delta}\right) + O\left(\frac{1}{\delta^3}\int_0^\delta \frac{\theta^2\,d\theta}{(\delta^2+\theta^2)^{\frac32}}\right)$$

$$= O\left(\frac{1}{\delta^3}\right) + O\left(\frac{1}{\delta^3}\int_0^1 \frac{u^2\,du}{(1+u^2)^{\frac32}}\right) = O\left(\frac{1}{\delta^3}\right) = o\left(\frac{1}{\delta^3}\log\frac{1}{\delta}\right).$$

On the other hand

$$(3.8.6)\qquad J_2 = O\left(\frac{1}{\delta^2}\right)\int_\delta^\pi \frac{|f'(re^{i\theta})|}{(\delta^2+\theta^2)^{\frac12}}\,d\theta.$$

The integral here is

$$(3.8.7)\qquad \frac{F(\pi)}{(\delta^2+\pi^2)^{\frac12}} - \frac{F(\delta)}{(2\delta^2)^{\frac12}} + \int_\delta^\pi \frac{\theta F(\theta)}{(\delta^2+\theta^2)^{\frac32}}\,d\theta.$$

The first two terms give $O(1)$ and $O(1/\delta)$. In the last, since $\delta \leqq \theta$, we may use (3.5.4), so that

$$\int_\delta^\pi \frac{\theta F(\theta)}{(\delta^2+\theta^2)^{\frac32}}\,d\theta = o\left(\int_\delta^\pi \frac{\theta^2\,d\theta}{\delta(\delta^2+\theta^2)^{\frac32}}\right) = o\left(\frac{1}{\delta}\int_1^{\pi/\delta}\frac{u^2\,du}{(1+u^2)^{\frac32}}\right) = o\left(\frac{1}{\delta}\log\frac{1}{\delta}\right).$$

Hence, after (3.8.6) and (3.8.7), we obtain

$$(3.8.8)\qquad J_2 = o\left(\frac{1}{\delta^3}\log\frac{1}{\delta}\right);$$

and the theorem follows from (3.8.5) and (3.8.8).

§ 4. Conclusion.

4. 1. We conclude with a few miscellaneous remarks, in part concerning problems left open by our analysis.

(1) There is no difficult in proving that, under the conditions of Theorem 3,

$$S_n = \sum_0^n |s_\nu - s|^k = o\left(n(\log n)^{k-1}\right)$$

for $k \geqq 2$. We replace (3.8.2) by

$$\sum_0^\infty |s_n(f')|^k r^{kn} \leqq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{f'(z)}{1-z}\right|^{k'} d\theta\right)^{k-1} \text{[10]},$$

where $k' = \frac{k}{k-1}$, and repeat the argument with the obvious changes. The argument of § 2 shows that S_n may be nearly as large as

$$n(\log n)^{\frac12 k};$$

but the gap between $(\log n)^{\frac12 k}$ and $(\log n)^{k-1}$ remains open.

On the other hand Theorem 3, together with Hölder's inequality, shows that

$$S_n = o\left(n(\log n)^{\frac12 k}\right)$$

when $k \leqq 2$; and this, after § 2, is the best possible result.

(2) It is natural to ask whether the Fourier series of an integrable $f(\theta)$ is strongly summable for almost all θ. Our theorems do not settle this question, though they may suggest that the answer is negative. The series is not necessarily strongly summable in the „Lebesgue set" of $f(\theta)$, but it may conceivably be so in some other „full" set of θ. Thus, after Kolmogoroff, Seliverstoff, and Plessner,

$$s_n = o\left(\sqrt{\log n}\right),$$

when $f(\theta)$ is L^2, for almost all θ; but this is not necessarily true in the Lebesgue set or even at a point of continuity.

This is no doubt the most interesting question still left open.

(3) We have asked ourselves whether

$$\int_0^t |\varphi(x,u)|\,(1+\log^+|\varphi(x,u)|)\,du = o(t)$$

is a sufficient condition for strong summability [11], but without result.

(4) We might define a weaker type of „strong summability"; we might define it, for example,

$$(4.1.1)\qquad \sum_1^n \frac{|s_\nu - s|}{\nu} = o(\log n)$$

[10]) This is effectively one of the Hausdorff inequalities.

[11]) Compare the results of Paley referred to in [1]).

or

$$\sum_{2}^{n} \frac{|s_\nu - s|}{\nu \log \nu} = o(\log\log n),$$

or by similar equations with $|s_\nu - s|^k$. The most interesting of these notions is (4.1.1), which may be called „strong logarithmic summability"; and we may ask whether this property is a consequence of the conditions of Theorem 3. The answer is again negative, as may be shown by an appropriate modification of the argument of § 2.

There is therefore some interest in our final theorem, which follows.

Theorem 4. If

$$f(z) = \sum c_n z^n$$

is a power series of the complex class L, and

$$\int_0^\theta |f(e^{it})|\, dt = o(|\theta|), \tag{4.1.2}$$

then

$$\sum_{1}^{n} \frac{|s_\nu|}{\nu} = o(\log n). \tag{4.1.3}$$

We say that

$$g(z) = \sum b_n z^n$$

belongs to L if

$$\mu(r, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{i\theta})|\, d\theta$$

is bounded for $r < 1$. It is known [12]) that then

$$\sum_{0}^{\infty} \frac{|b_n|}{n+1} r^n \leqq A\,\mu(r, g). \tag{4.1.4}$$

Suppose now that $f(z)$ satisfies the conditions of Theorem 4, and apply (4.1.4) to

$$g(z) = \frac{f(z)}{1-z} = \sum s_n z^n$$

[12]) Hardy and Littlewood (5, 208, Theorem 16).

on the circle $r = 1 - \delta = 1 - \frac{1}{n}$. Then

$$\sum_{0}^{\infty} \frac{|s_n|}{n+1} r^n \leqq A \int_{-\pi}^{\pi} \left|\frac{f(z)}{1-z}\right| d\theta.$$

An argument like that of § 3.8 shows that the last integral is $o\left(\log \frac{1}{\delta}\right)$, and (4.1.3) follows.

Here then there is a difference between Fourier series and power series: it is the only one which we have found connected with this particular problem.

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