ganze Zahl $>k_0$. Diese m Punkte nenne ich $(u_1^{(\mu)}, \ldots, u_n^{(\mu)})$ $(\mu=1,2,\ldots,m)$; ich wähle dabei $(u_1', \ldots, u_n')$ so, dass für jedes ganze $\mu \geqq 2$ und $\leqq m$ die erste nicht verschwindende der Zahlen

$$v_1^{(\mu)}=\frac{u_1^{(\mu)}-u_1'}{l}, \ldots, v_n^{(\mu)}=\frac{u_n^{(\mu)}-u_n'}{l}$$

positiv ist. Für $\mu=2,\ldots,m$ ist dann $(v_1^{(\mu)}, \ldots, v_n^{(\mu)})$ ein nicht mit dem Koordinatenursprung zusammenfallender Gitterpunkt, der, da $\left(\frac{k_1 u_1^{(\mu)}}{l}, \ldots, \frac{k_n u_n^{(\mu)}}{l}\right)$ und $\left(\frac{k_1 u_1'}{l}, \ldots, \frac{k_n u_n'}{l}\right)$ zu M gehören, der Menge N angehört. Hiermit ist Satz 7 bewiesen.

Dieser Satz liefert u. a. Satz 1 der ersten Mitteilung (man setze $k_0=1$) und die Blichfeldtsche Behauptung [2]): ein im n-dimensionalen Raum liegender konvexer Körper M mit Volumen $V>2^n k$ und mit einem Mittelpunkt im Koordinatenursprung enthält ausser diesem Mittelpunkt mehr als $k-1$ Paare Gitterpunkte (man wähle $k_1=\ldots=k_n=2$; $k_0=k$ und $N=M$).

Ist die Menge N beschränkt und abgeschlossen, so darf in Satz 7 die Voraussetzung $V>k_0 k_1 \ldots k_n$ durch $V \geqq k_0 k_1 \ldots k_n$ ersetzt werden.

(Eingegangen am 16. November 1935.)

[2]) H. F. Blichfeldt, Notes on geometry of numbers, Bull. Amer. Math. Soc 27 (1921), S. 150, 152—153.

On sequences of positive integers.

By

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1. Let $a_1, a_2, \ldots$ be any sequence of (different) positive integers, and let $b_1, b_2, \ldots$ be the sequence consisting of all positive integers which are divisible by at least one a. We define

$$A_1=\frac{1}{a_1},$$

$$A_2=\frac{1}{a_2}-\frac{1}{[a_1, a_2]},$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$A_\nu=\frac{1}{a_\nu}-\sum_{\mu<\nu}\frac{1}{[a_\mu, a_\nu]}+\sum_{\lambda<\mu<\nu}\frac{1}{[a_\lambda, a_\mu, a_\nu]}-\ldots,$$

where $[a,b,c,\ldots]$ denotes the least common multiple of $a,b,c,\ldots$ Then A_ν is easily seen to be the density of those integers which are divisible by a_ν but not by any one of $a_1, \ldots, a_{\nu-1}$. Hence $A_\nu \geqq 0$, and $\sum_1^m A_\nu$, being the density of those integers which are divisible by at least one of $a_1, \ldots, a_m$, is less than 1. If we define

$$A=\sum_1^\infty A_\nu$$

then $0<A\leqq 1$, and it is reasonable to expect that A is the density

in some sense of the sequence $\{b_i\}$. It was proved by Besicovitch[1]) that the sequence $\{b_i\}$ may have different upper and lower densities. We shall prove (§ 2) that the "logarithmic density" of $\{b_i\}$ exists and has the value A, and also that the *lower* density of $\{b_i\}$ has the value A.

In § 3 we use the former of these results to prove that if a sequence $a_1, a_2, \ldots$ of positive integers has the property

$$\overline{\lim_{x=\infty}} (\log x)^{-1} \sum_{a_m \leqq x} a_m^{-1} > 0,$$

then it a has subsequence $a_{i_1}, a_{i_2}, \ldots$ in which $a_{i_k} \mid a_{i_{k+1}}$ $(k = 1, 2, \ldots)$. Naturally every sequence of positive lower density satisfies the condition.

2. Let $\theta(n)$ be 1 if n is a b_i (i. e. if there is an $a_j \mid n$) and 0 otherwise. Let

$$F(s) = \sum_1^\infty \theta(n) n^{-s} \qquad (s > 1).$$

Let

$$A_\nu(s) = \frac{1}{a_\nu^s} - \sum_{\mu<\nu} \frac{1}{[a_\mu, a_\nu]^s} + \sum_{\lambda<\mu<\nu} \frac{1}{[a_\lambda, a_\mu, a_\nu]^s} - \ldots,$$

so that $A_\nu(1) = A_\nu$, and

$$A(s) = \sum_1^\infty A_\nu(s).$$

Then it is easily seen that

$$F(s) = \zeta(s) A(s)$$

for $s > 1$.

Lemma 1: *If $1 < s_1 < s_2$, then for any m,*

$$\sum_1^m A_\nu(s_2) \leqq \sum_1^m A_\nu(s_1).$$

Proof: Let $\theta_m(n)$ be 1 if n is divisible by any one of $a_1, \ldots, a_m$ and 0 otherwise, and let $F_m(s) = \sum_{n=1}^\infty \theta_m(n) n^{-s}$. As before

$$F_m(s) = \zeta(s) \sum_1^m A_\nu(s).$$

We have the inequality

(1)
$$\theta_m(n) \log n \geqq \sum_{d|n} \theta_m(d) \Lambda\left(\frac{n}{d}\right)$$

for all n. For if $\theta_m(n) = 0$ then $\theta_m(d) = 0$ for all $d \mid n$, and if $\theta_m(n) = 1$ then

$$\log n = \sum_{d|n} \Lambda\left(\frac{n}{d}\right) \geqq \sum_{d|n} \theta_m(d) \Lambda\left(\frac{n}{d}\right).$$

From (1):

$$\sum_{n=1}^\infty \theta_m(n) \log n \, n^{-s} \geqq \left(\sum_{n=1}^\infty \theta_m(n) n^{-s}\right)\left(\sum_{n=1}^\infty \Lambda(n) n^{-s}\right)$$

for $s > 1$, i. e.

$$-F_m'(s) \geqq F_m(s)\left(-\frac{\zeta'(s)}{\zeta(s)}\right),$$

hence

$$\frac{d}{ds}\left(\sum_1^m A_\nu(s)\right) \leqq 0$$

for $s > 1$, which proves the Lemma.

Lemma 2: *$A(s) \to A$ as $s \to 1$ $(s > 1)$.*

Proof: By Lemma 1, we have for $s > 1$ and any m,

$$\sum_1^m A_\nu(s) \leqq \lim_{s=1} \sum_1^m A_\nu(s) = \sum_1^m A_\nu \leqq A,$$

hence $A(s) \leqq A$. But

$$\underline{\lim_{s=1}} A(s) \geqq \lim_{s=1} \sum_1^m A_\nu(s) = \sum_1^m A_\nu,$$

and so

$$\underline{\lim_{s=1}} A(s) \geqq A,$$

which proves the Lemma.

Theorem 1: (*a*) $\lim_{x=\infty} (\log x)^{-1} \sum_{n=1}^x \theta(n) n^{-1}$ *exists and has the value A.*

(*b*) $\underline{\lim_{x=\infty}} x^{-1} \sum_{n=1}^x \theta(n) = A.$

Proof: By lemma 2,

(2)
$$F(s) = \sum_1^\infty \theta(n) n^{-s} \sim \frac{A}{s-1}$$

[1]) Math. Annalen 110 (1934), 336 — 341.

as $s \to 1$ $(s > 1)$. Part (a) of the Theorem follows from this by a Tauberian theorem due to Hardy and Littlewood.[2])

As regards (b), it is obvious from the meaning of $\sum_{1}^{m} A_\nu$ as a density that the lower limit in (b) is $\geqq A$, and if equality did not hold we should have

$$s_n = \sum_{l=1}^{n} \theta(l) > (A+\delta)\, n$$

for some $\delta > 0$ and all $n \geqq N$, and so

$$F(s) = \sum_{1}^{\infty} s_n \left(n^{-s} - (n+1)^{-s} \right) > (A+\delta) \sum_{N+1}^{\infty} n^{-s},$$

which on making $s \to 1$ contradicts (2).

3. Theorem 2: *If $a_1, a_2, \ldots$ is a sequence of (different) positive integers, and*

$$\alpha = \overline{\lim_{x=\infty}}\, (\log x)^{-1} \sum_{a_n \leqq x} a_n^{-1} > 0,$$

then there exists a subsequence $a_{i_1}, a_{i_2}, \ldots$ such that $a_{i_k} \mid a_{i_{k+1}}$ $(k = 1, 2, \ldots)$.

Proof: It suffices to prove that there exists an a_i such that

$$\overline{\lim_{x=\infty}}\, (\log x)^{-1} \sum_{\substack{a_n \leqq x \\ a_i \mid a_n}} a_n^{-1} > 0. \tag{3}$$

We take r so large that

$$\sum_{\nu > r} A_\nu < \alpha, \tag{4}$$

and we shall prove that there exists an a_i with $i \leqq r$ satisfying (3). If the left side of (3) were zero for $i \leqq r$, we should have

$$\alpha = \overline{\lim_{x=\infty}}\, (\log x)^{-1} \sum_{\substack{a_n \leqq x \\ a_1 \nmid a_n, \ldots, a_r \nmid a_n}} a_n^{-1}$$

[2]) Proc. London Math. Soc. (2) 13 (1914), 174—191, Theorem 16.

$$\leqq \overline{\lim_{x=\infty}}\, (\log x)^{-1} \sum_{\substack{n=1 \\ a_1 \nmid n, \ldots, a_r \nmid n}}^{x} \theta(n)\, n^{-1}.$$

By Theorem 1 (a) the last expression has the value

$$A - \sum_{\nu=1}^{r} A_\nu.$$

From (4) we have a contradiction.

The condition in Theorem 2 is easily seen to be best possible of its kind, i. e. one can construct sequences $\{a_i\}$ for which

$$(\log x)^{-1} \sum_{a_n \leqq x} a_n^{-1}$$

tends to zero arbitrarily slowly, but in which no subsequence with the desired property exists.

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