6. It should be clear that we have already proved this final

Theorem: In order that a T-space possess a monotonic, complete, finite-covering-by-open-sets-system, it is necessary and sufficient that it be completely separable and compact.

We shall conclude with the following remarks. It is well-known that a completely separable and compact T-space need not be, metrizable, but *if it is not* then it must also fail to be *regular*. *Further*, in a completely separable, compact T-space, *regularity* is implied by the weaker separation property (Hausdorff) that to each pair of points x and y there exist *mutually exclusive* open sets U_x and U_y, containing x and y respectively. Therefore, if a T-space posseses an m.c.-system and is *not* metrizable, it must contain at least one pair of points which cannot belong to mutually exclusive open sets. Therefore, the theorem with we opened this paper is a special case of our final one.

For completeness sake, it is perhaps worth while to give the argument upon which our last remarks are based. Suppose that X is a closed subset of a T-space, and y a point of that space. Suppose, further, to to each point x of X we may associate a pair of mutually exclusive neighborhoods U_x and U_y^x, $U_x \supset x$, $U_y^x \supset y$. Now, if our T-space is completely separable and compact, then, first of all, we may suppose that the open sets are drawn from a countable fundamental set and, secondly, we may apply the Heine-Borel theorem which we proved for these spaces, in **3.** It will be clear that our proof applies to closed subsets of the space, also. Then we conclude that there exists a finite set of open sets $U_{x_i}, i=1, 2, ..., N$, whose *sum* covers X. The *product* of the open sets $U_y^{x_i}$, $i=1, ..., N$, is an open set containing y, which has no point in common with that *sum*. This is regularity.

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Symmetrical Cut Sets.

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I. Introduction.

After a survey of characterizations of the simple closed curve and simple closed surface [1]) it has occurred to us that the principle of symmetry, which has been used to only a limited extent in analysis situs [2]), might be advantageous in forming such characterizations.

To this end we shall say that a set S is a *symmetrical cut set* of a set M if $M-S$ can be expressed as the sum of two mutually separated sets M_1 and M_2 which are such that there exists a continuous (1—1) correspondance Δ having the properties that, $\Delta(M_1+S)=M_2+S$, and $\Delta(S)=S$. If S consists of a finite number of points it will be called a *permutable symmetrical cut set* provided that $\Delta(P_i)=P_{i+1}$ $(i=1, 2, ..., n-1)$, and $\Delta(P_n)=P_1$. The set S will be called a *strong symmetrical cut set* of M if, in addition to being a symmetrical cut set of M as defined above, $\Delta(P)=P$, for every point P of S. Hereafter the sets M_1 and M_2, defined above, will be referred to as *symmetric separates* of M with respect to S.

It is easy to see that every pair of distinct points of a simple closed curve is a symmetrical cut set of the curve; likewise it has been shown that every simple closed curve of a simple closed surface is a strong symmetrical cut set of the latter [3]). On the other hand,

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[1]) A *simple closed surface* is the homeomorph of the unit sphere $x^2+y^2+z^2=1$ in cartesian 3-space.

[2]) H. M. Gehman, *Centers of symmetry in analysis situs*, Amer. Jour. Math. 52 (1930), pp. 543—547.

[3]) A. Schoenflies, *Beiträge zur Theorie der Punktmengen*, III, Math. Ann. 62 (1906), p. 324, and J. R. Kline, *A new proof of a theorem due to Schoenflies*, Proc. Nat. Acad. Sc., 6 (1920), pp. 529—531.

even though an equatorial band of a simple closed surface satisfies the definition of symmetrical cut set, it fails to be a strong symmetrical cut set.

We are led to inquire, then, if we assume that every pair of distinct points of a connected [4]) set M in a locally compact, metric space is a symmetrical cut set of M, what additional conditions must be imposed in order to characterize M as a simple closed curve? We shall find that if M is further restricted to be either 1) closed, or 2) locally connected [5]), M is a simple closed curve. If, however, we assume that every pair of points is a permutable symmetrical cut set, we find that we need impose only the further restriction that M be locally connected at a single point to insure that M is a simple closed curve.

It will be shown that the property of being a strong symmetrical cut set of a simple closed surface is sufficient to characterize a simple closed curve. Hence we can state that „A necessary and sufficient condition that C be a strong symmetrical cut set of a simple closed surface M is that C be a simple closed curve of M„. A continuous curve *) such that no pair of points disconnects it, can be identified as simple closed surface by the additional requirement that every simple closed curve be a strong symmetrical cut set. It is interesting to note in this connection that a continuum consisting of two simple closed curves interecting in two distinct points is a continuous curve every simple closed curve of which is a strong symmetrical cut set. However it fails to meet the conditions required of a simple closed surface in that it may be disconnected by the omission of a pair of points.

The author wishes to acknowledge his deep obligation to Professor R. L. Wilder for his many suggestions and criticism in the development of this paper.

[4]) In the sense of Lennes-Hausdorff.

[5]) In the sense in which we use the term, a set M is *locally connected* at a point P if, for every neighborhood U of P, there exists a neighborhood V of P, contained in U, such that all points of M in V lie in a connected subset of M which itself lies in U.

*) By *continuous curve* (= Jordan continuum = Peano continuum) is meant a compact metric continuum which is locally connected.

2. Preliminary Lemmas.

Before proceeding to the main problems of this paper, some general lemmas regarding symmetrical cuts will be established [6]).

Lemma 1. *Every strong symmetrical cut set of a set M is closed relative to M.*

Let S be a strong symmetrical cut set of M such that $M-S=M_1+M_2$, where M_1 and M_2 are symmetric separates of M with respect to S, and let P be a limit point of S in M. Suppose that P is a point of M_1. Since M_2+S is homeomorphic with M_1+S, there exists a point P' in M_2 corresponding to P. There exist neighborhoods U and V of P and P' respectively such that U and V have no point in common. Since P is a limit point of S, there are points of S within the neighborhood U. Let this set of points be denoted by S_1, and let $S-S_1=S_2$. Being a limit point of S, P must be a limit point of either S_1 or S_2. But since there are no points of S_2 within the neighborhood U, it follows that P must be a limit point of S_1. From the definition of strong symmetrical cut set it is clear that P' must likewise be a limit point of S_1. But this is impossible since the neighborhood V of P' contains no points of S_1. Similarly P cannot be a point of M_2, hence P belongs to S.

Lemma 2. *Every strong symmetrical cut set of a simple closed surface is connected.*

Let S be a strong symmetrical cut set of a simple closed surface M such that $M-S=M_1+M_2$, where M_1 and M_2 are symmetric separates with respect to S. We will assume that $S=S_1+S_2$, where S_1 and S_2 are mutually separated. Let a and b be points of S_1 and S_2 respectively. There exists a continuum K in $M-S$ separating a and b [7]). The set of points S_1' of S in the domain [8]) of $M-S$ that contains a is a closed set of points, by virtue of Lemma 1, and so also is the set $S_2'=S-S_1'$. There exists then an arc t of M

[6]) It is obvious form the proofs that these lemmas hold in very general spaces.

[7]) B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fund. Math. 2 (1921), pp. 206—255.

[8]) If M is a continuous curve, and J is any closed subset of M, any component of $M-J$ is a *domain*.

whose endpoints a' and b', and only these, lie in S, and such that a' and b' lie in S_1' and S_2' respectively. Obviously the arc t intersects K. Let the connected set $K+<t>$ [9]) be contained in M_1, since it, cannot have points in both M_1 and M_2. On account of the homeomorphism between M_1 and M_2, there exists an arc t' form a' to b' such that $<t'>$ is contained in M_2. But the arc t' likewise intersects K, violating the hypothesis that M_1 and M_2 are separated.

Lemma 3. *If M is a set of more than two points containing a non-vacuous subset S which is closed relative to M, and a point Q not contained in S, such that 1) $M-S$ can be expressed as the sum of two mutually separated sets M_1 and M_2 which are homeomorphic, 2) $M-Q$ is connected, then M is connected* [10]).

The set $M-S$ contains at least two points, Q in M_1, say, and R, the correspondent of Q in M_2 under the homeomorphism between M_1 and M_2. If Q is not a limit point of $M-Q$, Q is not a limit point of M_1-Q. Hence R is not a limit point of M_2-R. Then $M-Q=[(M_1-Q)+S+(M_2-R)]+R$, which is a separation of $M-Q$, contrary to the hypothesis. Therefore Q is a limit point of $M-Q$, and M is connected.

Corollary. *If S is a non-vacuous strong symmetrical cut set of M, and M contains a non-cut point Q, which is not contained in S, then M is connected.*

By Lemma 1, S is closed relative to M, and the definition of strong symmetrical cut set requires that M_1 and M_2 be homeomorphic.

Lemma 4. *If M_1 and M_2 are symmetric separates of M with respect to a symmetrical cut set S, then M_1 and M_2 have equal numbers of components.*

This is obvious from the homeomorphism between M_1 and M_2.

Lemma 5. *No pair of distinct points of an arc is a symmetrical cut set of the arc.*

Let A and B be two distinct points of an arc $M(=PQ)$ where in the order from P to Q, $A<B$. By Lemma 4, A and B are not

[9]) By $<t>$ will be meant the arc t without its end-points.
[10]) See H. M. Gehman, loc. cit., Theorem 4.

both interior points of PQ. Hence two cases arise, 1) where one of the points A, B is an interior point of M, and 2) where both A and B are end-points of M. Clearly we can assign notation so that $M-(A+B)=M_1+M_2$, where $M_1=<AB>$ and $M_2=PA>+<BQ$. In the first case there exists no homeomorphism Δ such that $\Delta(M_1+A+B)=M_2+A+B$ since M_1+A+B is connected and M_2+A+B is not connected. In the second case $M_2=0$, and M is not separated by the omission of $(A+B)$.

Lemma 6. *If M is a connected set, and A and B are any two connected subsets of M such that $M-(A+B)$ is the sum of two mutually separated sets M_1 and M_2, then at least one of the sets M_1+A+B, M_2+A+B is connected.*

Let us assume the contrary and show that we are led to a contradiction of the hypothesis that M is connected. Let $M_1+A+B=X+Y$, where the sets X and Y are mutually separated. Similarly let $M_2+A+B=W+Z$, where W and Z are mutually separated. We will consider three possibilities: 1) A and B are contained in different sects*) in each of the above separations, that is, A is contained in X and W, and B is contained in Y and Z; 2) A and B, are contained in the same sect in one separation, and different sects in the other, that is, that both A and B are contained in X, A is contained in W, and B in Z; 3) A and B are contained in the same sect in each separation, that is, that both A and B are contained in each of the sets X, W. In 1), $X+W$ is separated from $Y+Z$, in 2), Y is separated from $(X+W+Z)$, and in 3), Z is separated from $(X+Y+W)$, in each case separating M. Hence the theorem follows.

Corollary. *Let M be a connected set of which every pair of points is a symmetrical cut set. If A and B are two distinct points of M, and M_1 and M_2 are symmetric separates of M with respect to $(A+B)$, then both the sets M_1+A+B, M_2+A+B are connected.*

*) If K is a subset of M such that K and $(M-K)$ are mutually separated, K will be called a *sect* of M.

3. Characterization of the simple closed curve by means of symmetrical cut sets [11]).

Theorem 1. If M is a connected set of points of which every pair of distinct points is a symmetrical cut set, no point of M is a cut-point.

Let us assume that there exists a point P of M such that $M-P=R_1+R_2$, where R_1 and R_2 are mutually separated. Let A be a point of R_1, and B be a point of R_2. Then $M-(A+B)=M_1+M_2$, where M_1 and M_2 are symmetric separates of M with respect to $(A+B)$. By the corrollary to Lemma 6, the sets M_1+A+B and M_2+A+B are connected. Suppose that P is contained in M_2+A+B. Then the set M_1+A+B is contained in (R_1+R_2), and, being connected, must be contained in either R_1 or R_2, say the former. Thus B, a point of R_2, is contained in R_1, giving a contradiction. Hence M has no cut-point.

Theorem 2. If M is a connected set of which every pair of distinct points is a symmetrical cut set, then M is a quasi-closed curve [12]).

This is a consequence of Theorem 1, and a theorem of R. L. Wilder [13]).

Corollary 1. Let M be a connected set of which every pair of distinct points is a symmetrical cut set. If a pair of points A, B separates a pair of points P, Q then conversely P and Q separate A and B.

Let $M-(A+B)=M_1+M_2$, where M_1 and M_2 are symmetric separates of M with respect to $(A+B)$, with P in M_1, and Q in M_2. Also set $M-(P+Q)=N_1+N_2$, where N_1 and N_2 are symmetric separates of M with respect to $(P+Q)$, with A in N_1. We will assume that B is likewise contained in N_1. By the corollary to Lemma 6, N_2+P+Q is connected. This set is a component of (M_1+M_2), since both A and B were assumed to be in N_1, and hence is contained in either M_1 or M_2, violating the hypothesis regarding the separation of P and Q.

[11]) The theorems in this section relating especially to simple closed curves may be considered as holding in any locally compact metric space.

[12]) *A quasi-closed curve* is a set of points which remains connected on the omission of any connected subset.

[13]) R. L. Wilder, *Concerning simple continuous curves and related point sets*, Amer. Jour. Math. 53 (1931), pp. 39—55; see Theorem 4.

Corollary 2. Let M be a connected set of with every pair of distinct points is a symmetrical cut set. If A and B are two points of M, and M_1 and M_2 are symmetric separates of M with respect to $(A+B)$, then the sets M_1+A+B and M_2+A+B are irreducibly connected from A to B.

This follows from the definition of quasi-closed curve given by Wilder [14]).

The following theorem may now be stated: *Let M be a connected set of points such that every pair of distinct points is a symmetrical cut set of M. If M is either* 1) *closed, or* 2) *locally connected, M is a simple closed curve* [15]).

Theorem 3. Let M be a connected set of points such that 1) *every pair of distinct points of M is a permutable symmetrical cut set of M,* 2) *M is locally connected at one point. Then M is locally connected.*

Let Q be a point at which M is locally connected, and suppose that there exists a point P at which M does not have this property. Then $M-(P+Q)=M_1+M_2$, where M_1 and M_2 are symmetric separates of M with respect to $(P+Q)$. By Corollary 2 above, the sets M_1+P+Q and M_2+P+Q are irreducibly connected from P to Q. There exists a neighborhood U of P such that, if V is any neighborhood of P contained in U, there exists within V a point Z having the property that no connected subset of M containing P and Z lies within U. Let Z be contained in M_1. The set M_1+P+Q is not locally connected at P since M is not locally connected at P. But this is impossible because the set M_2+P+Q is locally connected at Q, and P corresponds to Q under the homeomorphism defined by the permutable symmetrical cut set $(P+Q)$.

Theorem 4. If a set M is connected, locally connected at one point, and such that every pair of distinct points is a permutable symmetrical cut set, M is a simple closed curve.

By Theorem 2, M is a quasi-closed curve, and by Theorem 3, M is locally connected. It follows from a theorem of R. L. Wilder [16]) that M is a simple closed curve.

[14]) Loc. cit., page 45.

[15]) R. L. Wilder, loc. cit., see Theorems 6 and 7.

[16]) Loc. cit., Theorem 6.

Theorem 5. *If M is a closed compact set of more than four points such that for every pair of distinct points, A, B, $M-(A+B)=M_1+M_2$, where M_1 and M_2 are mutually separated, connected, and homeomorphic, then M is a simple closed curve.*

Let P and Q be points of M_1 and M_2 respectively. Then $M-(P+Q)=(M_1-P)+(M_2-Q)+(A+B)$. Since $M-(P+Q)$ is the sum of two connected, but mutually separated sets, it follows that A and B are limit points of the set (M_1+M_2). Either A and B are limit points of the same sect, or A is a limit point of the one, and B of the other.

First, let both A and B be limit points of M_1. Corresponding, under the above homeomorphism, to a sequence of points $\{m_i\}$ of M_1, which have A as a sequential limit point, is a sequence of points $\{m_i''\}$ of M_2 which have a sequential limit point X, since M is compact. Since N is closed, X is a point of M, but is contained in neither M_1 nor M_2. Hence X is identical with either A or B, and M is connected, since it is the sum of the two connected sets (M_1+A+B) and M_2 having a common limit point.

Secondly, suppose that A is a limit point of M_1, and B is a limit point of M_2. We need consider only the possibility that (M_1+A) and (M_2+B) are mutually separated. As M contains more than four points, the sets M_1 and M_2 each contain at least two points. Let x and y be two points of M_1, and suppose that $M-(x+y)=U+V$, where U and V are mutually separated, connected, and homeomorphic, according to the hypothesis. Let the connected set (M_2+B) be contained in V. If $N=(M_1+A)-(x+y)$ is separated, $M-(x+y)$ is the sum of at least three mutually separated sets, contrary to the hypothesis. Then N is connected and is contained in U, for otherwise U is vacuous. Hence both x and y are limit points of U, and M is connected as in the first case.

In any case, then, M is a connected set, and it folows from a theorem of R. L. Moore that M is a simple closed curve[17]).

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[17]) R. L. Moore, *Concerning simple continuous curves*, Trans. Amer. Math. Soc. 21 (1920), pp. 313—320, Theorem 4.

4. Characterization of the simple closed curve as a strong symmetrical cut set of a simple closed surface.

Lemma 7. *Let C be a strong symmetrical cut set of a continuous curve M. Then there exists a subset B of C, and domains X_1 and X_2 of M such that 1) B is the common boundary[18]) of the domains X_1 and X_2, and 2) B is strong symmetrical cut set of the set $N=X_1+X_2+B$.*

Let $M-C=M_1+M_2$, where M_1 and M_2 are symmetric separates of M with respect to C, and suppose that $M_1=X_1+Y_1$, where X_1 is a component of M_1, and X_1 and Y_1 are mutually separated, if Y_1 is non-vacuous. Let Δ be the homeomorphism between (M_1+C) and (M_2+C) as given in the definition of strong symmetrical cut set. Let $\Delta(X_1)=X_2$. Then $M_2=X_2+Y_2$, where X_2 and Y_2 are mutually separated, if Y_2 is non-vacuous, and X_2 is connected. Since, by Lemma 1, C is closed, X_1 and X_2 are domains of M. Let B be the boundary of X_1. Clearly B is a subset of C. On account of the homeomorphism Δ between (M_1+C) and (M_2+C), and the invariance of C under this homeomorphism, B is likewise the boundary of X_2. It follows from the definition of strong symmetrical cut set, and the nature of the homeomorphism Δ, that B is a strong symmetrical cut set of the connected set $N=X_1+X_2+B$.

Theorem 6. *Let B be a strong symmetrical cut set of a set N on a simple closed surface M such that $N-B=X_1+X_2$, where X_1 and X_2, the symmetric separates of N, are domains of M having B as their common boundary. Then prime ends of X_1 and X_2 are of the first kind*[19]).

Let ε be a prime end of X_1, defined by a set of crosscuts $\{t_i\}$ of X_1 that are open arcs of concentric circles coverging to a point P[20]). Let $\{t_i'\}$ be a set of open arcs in X_2 homeomorphic with the arcs $\{t_i\}$. The cross-cuts $\{t_i'\}$ likewise converge to P on account of the homeomorphism between X_1+B and X_2+B. The arcs $\{t_i\}$ and $\{t_i'\}$, together with their end-points on B, form a set of simple closed curves J_i, since the end-points of the arcs $\{t_i\}$ lie in B and hence are invariant under the homeomorphism between X_1+B and

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[18]) If X is a domain of a continuous curve M, the *boundary* of X consists of those points of $M-X$ which are limit points of X.

[19]) C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, Math. Ann. 73 (1913), pp. 323—370.

[20]) C. Carathéodory, loc. cit.; see Satz VIII.

X_2+B. Let the complementary domains of J_1 be I_1 and E_1, the former containing P. In the sequence of simple closed curves, let J_2 be the first that does not meet J_1, and let the complementary domains with respect to J_2 be I_2 and E_2, the former containing P, and so on. We will show that I_2 is contained in I_1.

Let $X_1-t_1=G_1+H_1$, P being a boundary point of the former. Similarly, let $X_1-t_2=G_2+H_2$, etc. Each G_i is contained in I_i. We will suppose that J_1 is contained in I_2. Then I_2 will contain the domains G_2, $G_1-\overline{G}_2$ [21]), and H_1. Consequently I_2 contains $B-(x+y)$ where $x+y=J_2\cdot B$. But this is impossible, for there exist in E_2 points of both X_1 and X_2, and consequently points of B, since an arc in E_2 joining points of X_1 and X_2 must meet B. Since J_1 is not contained in I_2, it is contained in E_2. Then, since both I_1, and I_2 contain P, it follows that $I_1\cdot I_2\neq 0$, and I_1 contains I_2. There exist then infinitely many simple closed curves J_i, such that for all i, I_i contains J_{i+1}.

It will now follows that ε contains the single point P. For the set of points K contained in ε is the set of points of B to which the domains G_i converge. Since I_i contains G_i, and P is the only point common to the sets I_i, the domains G_i must converge to P, and P is the only point in ε. Therefore ε is a prime and of the first kind [22]).

Theorem 7. *Let B the common boundary of two domains X_1, and X_2 on a simple closed surface M. Then if the prime ends of X_1 and X_2 are all of the first kind, B is a simple closed curve.*

Since B is the common boundary of two domains on a simple closed surface, B is connected, and has no cut-points. We will prove that B is disconnected by the omission of any pair of points α and β. There exist arcs s and s' from α to β which, except for their end-points, lie in X_1 and X_2 respectively. The existence of these arcs follows from the fact the prime ends of both domains are all of the first kind. The sum of the arcs s and s' is a simple closed curve K which has only the points α and β in common with B. By the Jordan Curve Theorem, $M-K=N_1+N_2$, where N_1 and N_2 are mutually separated. The set $B-(\alpha+\beta)=B\cdot N_1+B\cdot N_2$, neither of which is a null set for, if x and y are points of s and s' respectively, there exists in N_1+x+y an arc xy which must contain a point

[21]) $\overline{G}$ denotes the domain G plus its boundary.

[22]) C. Carathéodory, loc. cit., page 362.

of B, and a similar arc in N_2+x+y. Hence B is disconnected by the omission of any two of its points. It follows from a theorem of Wilder [23]) that B is a simple closed curve [24]).

Theorem 8. *Let C be a strong symmetrical cut set of a simple closed surface. Then C is a simple closed curve.*

By Lemma 7, and Theorems 6 and 7, C has a subset B which is a simple closed curve, hence separating M into precisely two domains. As in Lemma 7, we have $M-C=M_1+M_2$, where M_1 and M_2 are symmetric separates of M with respect to C. Since B is contained in C, we have $M-C$ contained in $M-B$. But, $M-B=X_1+X_2$, where X_1 and X_2 are connected symmetric separates with respect to B, and (X_1+X_2) contains the set (M_1+M_2). In view of the fact that X_1 is contained in M_1 [25]), and X_2 is contained in M_2, we can conclude that $X_1=M_1$, and $X_2=M_2$. Hence $B=C$, and the theorem follows.

This theorem may be stated as follows: Let C be a subset of a simple closed surface M such that $M-C=M_1+M_2$, separate, where there exists a homeomorphism between M_1+C and M_2+C, which on C is the identity. Then C is a simple closed curve.

5. Characterization of the simple closed surface by means of symmetrical cut sets.

In this section M will denote a continuous curve having the following properties [26]), (H), no pair of distinct points of M disconnects M, (I), every simple closed curve of M is a strong symmetrical cut set of M.

We note that from property (H) it follows that M has no cut-point, and hence is cyclicly connected [27]); in particular, then, M contains at least one simple closed curve.

[23]) Loc. cit., Corollary, p. 48.

[24]) This theorem may also be demonstrated on the basis of certain results of M. Torhorst and R. L. Moore concerning the relations of continuous curves to their complements in the plane. We believe that the above elementary proof is preferable, however.

[25]) See proof of Lemma 7.

[26]) L. Zippin, *On continuous curves and the Jordan curve theorem*, Amer. Jour. Math. 52 (1930), pp. 331—350.

[27]) See, for instance, C. Kuratowski and G. T. Whyburn, *Sur les éléments cycliques et leur applications*, Fund. Math. 16 (1930), pp. 305—331, and references to earlier papers of Ayres and Whyburn contained therein.

Lemma 8. *If J is any simple closed curve of M, and a and b are distinct points of J separating J into two open arcs s and t, having no points in common, then $M-J$ contains a component with limit points in both s and t.*

Suppose the contrary. We will define a set C_t as follows: C_t contains t, and all components of $M-J$ having limit points on t. On account of property (H), $M-J$ is non-vacuous, and every component of $M-J$ has limit points in either s or t, for no such component can have its boundary exclusively in the two points a and b. Let $M-(a+b+C_t)=R$. The set R is non-vacuous since it contains the open arc s. Then $M-(a+b)=C_t+R$, where, because of the local connectedness of M, C_t and R are mutually separated, contrary to property (H). Hence there exists a component of $M-J$ which has limit points in both s and t.

Theorem 9. *If t is an arc of a simple closed curve of M, then $M-t$ is connected.*

Let J be a simple closed curve of M, t an arc of J, and t_1 the complementary arc. We will suppose that t separates M, that is, that $M-t=A_1+A_2$, where A_1 and A_2 are mutually separated. The open arc $<t_1>$ is contained in either A_1 or A_2, say the latter. Let X be a component of A_1. Then X is likewise a component of $M-J$. Let $M-J=M_1+M_2$, where M_1 and M_2 are symmetric separates with respect to J, and let X be contained in M_1. Because of property (H), t must contain at least three boundary points of X. Let α and β be the first and last of these boundary points in a given order on t.

By Lemma 8, there exists a component C of $M-J$ having limit points in both components of $J-(\alpha+\beta)$. Then $C\cdot X=0$. Either α and β are accessible from X or they are limit points of accessible points[28]). Hence points a and b of t, which are accessible from X, may be taken arbitrarily near to α and β respectively. We will choose the accessible points a and b so that the arc $<ab>$ of t contains a boundary point P of X as well as a limit point of C, and let s be an arc from a to b which, except for its end-points, lies in X. There exists an arc s' from a to b which, except for its end-points, lies in X', the symmetric homeomorph of X in M_2. Clearly $C\cdot X'=0$. Let $s+s'=K$, a simple closed curve, and consequently a symmetrical cut set of M.

In $\overline{X}$ let $P'Q$ be an arc lying, except for P', wholly in X, meeting s only at Q, and such that P' is a point of $<ab>$ (either P or an accessible point near P). Let $P'Q'$ be the symmetric homeomorph of $P'Q$ in $\overline{X}'$. We will define D, a connected subset of $M-K$ as follows: $D=C+P'P>+P'Q'>+[J-(a+b)]$. Both D and D', the symmetric homeomorph of D with respect to K, have limit points on each of the open arcs $<s>$ and $<s'>$. Since the simple closed curve J separates $<s>$ and $<s'>$, D' intersects J in some point γ of one of the open arcs $<ab>$ of J. But D and D' would then have the point γ in common since D contains $J-(a+b)$ which set in turn contains γ. But this is impossible, for symmetric separates with respect to K can have no points in common. Hence t does not separate M.

Theorem 10. *Let M be a continuous curve such that* 1) *no pair of distinct points disconnects M, and* 2) *every simple closed curve of M is a strong symmetrical cut set of M. Then M is a simple closed surface.*

We have observed that M satisfies non-vacuously the condition that every simple closed curve of M disconnects M. The preceding theorem has established the fact that no arc of a simple closed curve disconnects M. It follows by a theorem of Zippin that M is a simple closed surface[29]).

[28]) R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Coll. Pub. 13, Theorem 2, p. 89.

[29]) L. Zippin, loc. cit., Theorem 3.