A sufficient condition for a function of intervals to be monotone.

By

A. J. Ward (Cambridge, England).

In this note I prove the following theorem, solving a problem raised by C. Carathéodory:

If $F(I)$ is a function of intervals which is additive, continuous [1]*, and of bounded variation in some interval I_0, and if, at each point interior to I_0, we have*

$$\overline{\lim_{\delta\to 0}} \frac{F[R_\delta(x, y)]}{\delta^2} < 0,$$

where $R_\delta(x, y)$ is the square of centre (x, y) and side 2δ (with sides parallel to the axes), then $F(I)$ is a negative (non-increasing) function in I_0.

Let $P(I)$ and $N(I)$ be respectively the positive variation, and the absolute value of the negative variation, of F in I, so that $F(I)=P(I)-N(I)$. We shall show that $P(I_0)=0$.

Suppose if possible that $P(I_0)>0$. Choose any sequence of numbers $\delta_n>0$ such that $\delta_n\to 0$ as $n\to\infty$, and let E_n be the set of points (x, y) such that

$$F[R_\delta(x, y)]<0$$

whenever

$$\delta \leqslant \delta_n.$$

[1]) That is, $F(I)\to 0$ (uniformly) as the area of I tends to 0. The following example shows that it is not sufficient to assume only that $F(I)\to 0$ as the diameter of I tends to 0. Let I be the interval $x_1\leqslant x\leqslant x_2$, $y_1\leqslant y\leqslant y_2$. If $x_1=0<x_2$, write $F(I)=y_2-y_1$. If $x_2=0>x_1$, write $F(I)=-(y_2-y_1)$. In all other cases write $F(I)=0$. This function is obviously additive and of bounded variation, and its symmetric derivative is 0 everywhere.

We can find a finite set of non-overlapping intervals J_i in I_0 such that

$$\sum_i F(J_i) > \left(1-\frac{1}{100}\right) P(I_0) \geqslant \left(1-\frac{1}{100}\right) \sum_i P(J_i).$$

At least one of these intervals, say J_1, satisfies

$$F(J_1) > \left(1-\frac{1}{100}\right) P(J_1) > 0,$$

and therefore

$$N(J_1) < \frac{1}{100} P(J_1) < \frac{1}{50} F(J_1). \tag{1}$$

Choose ε_1 such that

$$|F(I)| < \frac{1}{8} F(J_1) \tag{2}$$

whenever I lies in I_0 and has area less than ε_1. Let l_1 be the length of the longer side of J_1. Take any h_1 such that $h_1\leqslant\delta_1$ and

$$h_1 < \frac{1}{4}\varepsilon_1/l_1, \tag{3}$$

and consider the squares of the form

$$r h_1 \leqslant x \leqslant (r+1) h_1, \qquad s h_1 \leqslant y \leqslant (s+1) h_1,$$

r, s being integers. Let $A_1, A_2, \ldots, A_m$ (say) be those squares of this form which lie inside J_1 and are distant more than h_1 from the boundary of J_1. (Such squares must exist, for otherwise the shorter side of J_1 would be of length at most $4h_1$; that is, by (3), the area of J_1 would be less than ε_1, which, by (2), is obviously impossible.) Let A be the rectangle $\sum_{i=1}^{m} A_i$; then J_1-A can be divided into four rectangles of length at most l_1 and breadth at most $2h_1$. Using (2) and (3), we see that, since the function F is additive,

$$F(A) > \frac{1}{2} F(J_1). \tag{4}$$

We are now going to pick out a certain selection of the squares A_i, which we shall call S_i, $i=1, 2, \ldots, p$, say. When we have defined any S_i, T_i will denote the concentric square of five times the side. S_1 shall be that square A_i for which $F(A_i)$ is greatest. If there are two or more such squares, we choose one of them arbitrarily. Then it is easy to see that

$$F(S_1) \geqslant \frac{1}{25} F(AT_1).$$

Among those squares A_i which do not lie in T_1, we choose S_2 as that square for which $F(A_i)$ is greatest. Then

$$F(S_2) \geqslant \frac{1}{25} F(AT_2 - T_1) \text{ [2]}.$$

Among those squares A_i which do not lie in T_1 or in T_2, we choose S_3 as that square for which $F(A_i)$ is greatest. By proceeding in this way until the squares A_i are exhausted, we obtain a finite set of squares $S_1, S_2, \ldots, S_p$, say, such that, for $i=1, 2, \ldots$,

$$F(S_i) \geqslant F(S_{i+1}),$$

(5) $$F(S_i) \geqslant \frac{1}{25} F\{AT_i - \sum_{j=1}^{i-1} T_j\},$$

$$S_{i+1} \cdot \sum_{j=1}^{i} T_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{p} T_i \supset A.$$

Then, since the figures $(AT_i - \sum_{j=1}^{i-1} T_j)$ do not overlap, we have, by (5),

$$F(A) = \sum_{i=1}^{p} F(AT_i - \sum_{j=1}^{i-1} T_j) \leqslant 25 \sum_{i=1}^{p} F(S_i).$$

Hence, from (4),

$$\sum_{i=1}^{p} F(S_i) > \frac{1}{50} F(J_1).$$

If $S_1, S_2, \ldots, S_q$ are those S_i such that $F(S_i) > 0$, it follows at once that

$$\sum_{i=1}^{q} F(S_i) > \frac{1}{50} F(J_1).$$

Suppose if possible that each S_i, $i \leqslant q$, contains a point of E_1, say (x_i, y_i). Consider the squares

$$R_i = R_{h_1}(x_i, y_i).$$

[2]) If E is a figure composed of a finite number of non-overlapping intervals $I_1, I_2, \ldots, I_N$, $F(E)$ denotes $\sum_{i=1}^{N} F(I_i)$.

Clearly $R_i \supset S_i$, and so $P(R_i) \geqslant P(S_i)$. On the other hand, since $h_1 \leqslant \delta_1$ and (x_i, y_i) is in E_1, we see that $F(R_i) < 0$. That is,

$$N(R_i) \geqslant P(R_i) \geqslant P(S_i) \geqslant F(S_i),$$

and so

$$\sum_{i=1}^{q} N(R_i) \geqslant \sum_{i=1}^{q} F(S_i) > \frac{1}{50} F(J_1).$$

On the other hand, since S_i is distant more than h_1 from the boundary of J_1, R_i lies in J_1. Also, if $i > j$, S_i lies outside T_j, that is, $d(S_i, S_j) \geqslant 2h_1$, and so R_i, R_j do not overlap. It follows that

$$N(J_1) \geqslant \sum_{i=1}^{q} N(R_i) > \frac{1}{50} F(J_1),$$

contradicting (1). Hence one at least of $S_1, S_2, \ldots, S_q$ contains no point of E_1.

That is, we have found an interval, which we call I_1, contained in the interior of J_1, such that $F(I_1) > 0$, and therefore certainly $P(I_1) > 0$, while $I_1 E_1 = 0$. We can now repeat the whole argument with I_1 for J_1, E_2 for E_1, and δ_2 for δ_1, and so on as many times as we like. We obtain a decreasing sequence of intervals I_n, interior to J_1, such that $I_n E_n = 0$. These intervals must have a common point (x, y) which belongs to no set E_n. Then clearly

$$\overline{\lim_{\delta \to 0}} \frac{F[R_\delta(x, y)]}{\delta^2} \geqslant 0.$$

Thus the theorem is proved.

It is easy to see that we can sharpen the theorem in either of two ways. If we leave the hypotheses unaltered, we can conclude that $F(I)$ is strictly negative (not zero) for intervals lying in I_0. On the other hand, to conclude that $F(I)$ is negative in the wide sense, it is necessary to know only that

$$\overline{\lim_{\delta \to 0}} \frac{F[R_\delta(x, y)]}{\delta^2} \leqslant 0.$$

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