Vu la définition de la suite transfinie (3), on ne peut pas avoir $y_\xi \prec x_0$ pour $\xi<\vartheta$. Il existe donc un nombre ordinal $\alpha<\vartheta$ tel que y_α non $\prec x_0$. On a donc à plus forte raison y_ξ *non* $\prec x_0$ pour $\alpha<\xi<\vartheta$ (puisque $y_\alpha \prec y_\xi$ pour $\alpha<\xi<\vartheta$).

Soit ξ un nombre ordinal tel que $\alpha\leqslant\xi<\vartheta$. On a donc y_ξ *non* $\prec x_0$ et, vu la définition de la relation $\prec$, il existe un undice i tel que $a_i(y_\xi)\geqslant a_i(x_0)$, d'où $y_\xi \epsilon U$ en vertu du lemme précité. On a donc $y_\xi \epsilon U$ pour $\alpha\leqslant\xi<\vartheta$, c. à d. $\underset{y_\xi}{\mathrm{E}}[\alpha\leqslant\xi]\subset U$. Nous avons ainsi démontré qu'il existe pour tout ensemble ouvert $U\subset\mathcal{R}$ un nombre ordinal $\alpha<\vartheta$ tel que $\underset{y_\xi}{\mathrm{E}}[\alpha\leqslant\xi]\subset U$.

Soit maintenant Γ un G_δ linéaire quelconque contenant $\mathcal{R}$. Il existe donc une suite infinie $U_1, U_2, \dots$ d'ensembles ouverts telle que

$$\Gamma = U_1 U_2 \dots \tag{6}$$

D'après $\Gamma\subset\mathcal{R}$ et (6), on a $U_n\subset\mathcal{R}$ pour $n=1,2,\dots$ et, comme nous venons de démontrer, il existe pour tout n naturel un nombre ordinal $\alpha_n<\vartheta$ tel que

$$\underset{y_\xi}{\mathrm{E}}[\alpha_n\leqslant\xi]\subset U_n. \tag{7}$$

Comme nous savons, il existe un nombre ordinal $\alpha<\vartheta$, tel que $\alpha>\alpha_n$ pour $n=1,2,\dots$ Nous avons donc

$$\underset{y_\xi}{\mathrm{E}}[\alpha<\xi]\subset\underset{y_\xi}{\mathrm{E}}[\alpha_n\leqslant\xi] \qquad \text{pour} \quad n=1,2,\dots$$

ce qui donne d'après (7) et (6) $\underset{y_\xi}{\mathrm{E}}[\alpha<\xi]\subset\Gamma$.

Or, le nombre ordinal ϑ étant, comme nous savons, non confinal avec ω, l'ensemble $\underset{y_\xi}{\mathrm{E}}[\alpha<\xi]$ est indénombrable. L'ensemble $E\Gamma$ est donc indénombrable. Ceci étant vrai pour tout ensemble Γ qui est un G_δ contenant $\mathcal{R}$, nous concluons que l'ensemble $\mathcal{R}$ n'est pas un G_δ relativement à $E+\mathcal{R}$.

Ainsi l'ensemble $E+\mathcal{R}$ ne jouit pas de la propriété λ.

Some Methods of Proving Measurability.

By

H. D. Ursell (Leeds).

1. The present paper is part of an attempt to systematise as far as possible the cases in which we can say that a function is measurable (B) or measurable (L). The limit-processes used in elementary analysis can be divided into *sequence* limiting processes, e. g. $\underset{(n)}{\text{u. bd.}}$, $\lim\limits_{n\to\infty}$ and *continuum* limiting processes, e. g. $\underset{(x)}{\text{u. bd.}}$, $\lim\limits_{x\to\infty}$.

In elementary analysis there is an exact parallelism between the two kinds of limit process. But in the more advanced parts of analysis, in which not every function is continuous, this parallelism breaks down.

A sequence limiting process preserves measurability (B) but a continuum limiting process may destroy it. For instance, Neubauer has shown[1]) that the partial derivatives of a function measurable (B) are not themselves necessarily measurable (B).

A sequence limiting process preserves measurability (L) but a continuum limiting process may destroy it. For instance the formula

$$F(x) = \underset{(y)}{\text{upper bound}}\, f(x,y) \tag{1}$$

may produce a non-measurable function $F(x)$ from a measurable function $f(x,y)$: or from a null-function $f(x,y)$, i. e. from a function $f(x,y)=0$ almost everywhere, it may produce a function $F(x)$ which even if measurable need not be a null function. In fact we can form

[1]) M. Neubauer, Monatshefte für Math. und Phys. **38** (1931), p. 139.

a function $\varphi(x)$ $(0 \leqslant x \leqslant 1)$ such that $y=\varphi(x)$ maps the segment $\langle 0,1\rangle$ of the x-axis on a set Y of measure zero on the y-axis. If now E be any set on the segment $0\leqslant x\leqslant 1$ and if we define

$$f(x,y)=\begin{cases} 1 \text{ when } x\in E,\ y=\varphi(x),\\ 0 \text{ otherwise,}\end{cases}$$

then $f(x,y)$ is certainly a null function. But (1) gives

$$F(x)=\begin{cases} 1 \text{ when } x\in E,\\ 0 \text{ otherwise,}\end{cases}$$

so that $F(x)$ need not be measurable (L) and, if it is, need not be a null-function.

§§ 2-6 are closely connected with one another. We take first the formula

$$\varphi(y)=\underset{x\in E(y)}{\text{upper bound}}\, f(x)$$

where $E(y)$ is a variable set in x-space depending on the point y in y-space. This formula produces a function of y from a function of x. [The upper bound is taken, as a limit can always be reduced to a repeated bound.] *Whatever* $f(x)$ *may be*, $\varphi(y)$ will be lower semi-continuous provided the set of y — we denote it by $E'(x)$ — for which $x\in E(y)$ is open.

This result is generalised in various ways. In § 2 we aim at measurability (B) and are able to work throughout in abstract spaces. § 3 shows that some generalisations which § 2 might suggest are in fact false. In § 4 we aim only at measurability (L) and accordingly the y-space becomes a coordinate space. Here we introduce the phrase "regular set" for a set such that if a point y belongs to the set then there is a small angular region of vertex y contained entirely in the set. These sets have the property we have enunciated in Lemma 3 — namely that the sum of any aggregate of regular sets is a regular set — in common with open sets. This property plays an essential rôle in our work. § 5 is devoted to the corresponding results for a slightly different formula which includes the formula

$$M(x)=\overline{\lim_{\xi\to x}} f(\xi)$$

as a particular case. It is included for the sake of completeness but the theory of this formula does not seem as interesting as the theory of the upper bound. § 6 gives a final analysis of what assumptions in the foregoing work are necessary.

§§ 7, 8 are devoted to results of a type in which, given a function f or a set E in (x,y)-space we argue from hypotheses in the separate spaces $x=$const., $y=$const. to the measurability of f or E. The notion of a regular set finds another application here.

2. Let $f(x)$ be any real function of a point x in an abstract space. Let $E(y)$ be a variable set in this space depending on a point y in a topological space. The function

$$\varphi(y)=\underset{x\in E(y)}{\text{upper bound}}\, f(x) \tag{1}$$

is defined for all y except those for which $E(y)$ is empty- but it may take $+\infty$ as a value. We may if we wish take $\varphi(y)=-\infty$ if $E(y)$ is empty.

Instead of considering variable sets $E(y)$ in the x-space we consider, in the cross-product of the two spaces, the single set E which consists of all points (x,y) such that $x\in E(y)$. This set E defines the correspondence between x and y and we could call $\varphi(y)$ *the upper transform of* $f(x)$ by this correspondence.

For any given x we denote by $E'(x)$ the set of points y such that $(x,y)\in E$. We shall speak of $E(y)$, $E'(x)$ as *sections* of E (although this term is not quite precise, $E(y)$ for instance being a set of points in x-space, not in (x,y)-space.) And when we speak of $E'(x)$ being, say, open, we shall mean open in y-space, not in (x,y)-space.

It is convenient at once to generalise (1) by considering functions $f(x,y)$ depending on y as well as x. Thus we write

$$\varphi(y)=\underset{x\in E(y)}{\text{upper bound}}\, f(x,y). \tag{2}$$

Theorem 1. *If all the sections $E'(x)$ of E are open and if $f(x,y)$ is continuous — or at least lower semi-continuous — in y for each fixed x, then $\varphi(x)$ is lower semi-continuous.*

The proof is immediate. We have

$$\underset{y}{\mathrm{E}}[\varphi(y)>K]=\underset{y}{\mathrm{E}}[f(x,y)>K \text{ for some } x\in E(y)]$$
$$=\sum_{\text{all } x}\underset{y}{\mathrm{E}}[y\in E'(x),\, f(x,y)>K]$$
$$=\text{sum of open sets}=\text{open set.}$$

As an example of Theorem 1 suppose that $F(I)$ is a function of interval, not necessarily additive, in a Cartesian space of coordinates $(y_1, y_2, \ldots)$ and suppose that we calculate an upper derivative of it by the formula

$$(3) \qquad \overline{\mathfrak{D}}F(y) = \overline{\lim_{\delta(I)\to 0}} \frac{F(I)}{|I|}$$

for intervals I containing y *in their interiors*. Here $|I|$ denotes the volume of I and $\delta(I)$ its diameter. Then

$$(4) \qquad \overline{\mathfrak{D}}F(y) = \lim_{n\to\infty} \varphi\left(y; \frac{1}{n}\right)$$

where

$$(5) \qquad \varphi(y;h) = \underset{\delta(I)<h}{\text{upper bound}} \frac{F(I)}{|I|}$$

for intervals I containing y in their interiors. By Theorem 1, $\varphi(y;h)$ is l.s.c. as a function of y for any fixed h: the place of x in the theorem is here taken by the variable interval I. Hence $\overline{\mathfrak{D}}F(y)$ is the limit of a decreasing sequence of l.s.c. functions and is of class 2 at most.

An upper derivative of a function of intervals can be defined in many ways, according to what intervals are allowed. E.g. we may admit only cubes, or only intervals whose vertices are irrational, or the term interval may include any parallelopiped. The above argument will always apply provided the points y for which any given interval I is relevant form an open set.

Theorem 2. *Let $z = (z_1, z_2, \ldots, z_t)$ be a point of a Cartesian t-dimensional space and let $z_1, z_2, \ldots, z_t$ be functions of a point $y = (y_1, y_2, \ldots, y_s)$ of a Cartesian s-dimensional space which are measurable (B) and of Baire class α at most. Let $f(x,y) = F(x,z)$ where F is continuous — or at least lower semi-continuous — in z for each fixed x. Let*

$$\varphi(y) = \underset{x\in E(y)}{\text{upper bound}}\, f(x,y).$$

If the sections $E'(x)$ of E are open then $\varphi(y)$ is measurable (B) and of Baire class $\alpha+1$ at most.

The relations expressing $z_1, z_2, \ldots, z_t$ as functions of $y_1, y_2, \ldots, y_s$ determine a set S in the $(s+t)$-dimensional (y,z)-space. Taking S as the y-space of Theorem 1 and defining the functions $F^*(x,t)$ and $\psi(t)$ for $t = (y,z) \in S$ as follows: $F^*(x,t) = F(y,z)$, $\psi(t) = \sup F^*(x,t)$, $x \in E^*(t)$ where $E^*(t) = E(y)$ we see that the hypotheses of that theorem are satisfied. Hence $\psi(t)$, considered as a function of position on S, is l.s.c. and the set $\underset{t}{\mathrm{E}}[t\in S, \psi(t) > K]$ is open relative to S. For each point $t \in S$ we have $F(x,z) = f(x,y)$ and hence

$$F^*(x,t) = f(x,y), \qquad \psi(t) = \sup_{x\in E(y)} f(x,y) = \varphi(y).$$

Consequently $\underset{t}{\mathrm{E}}[t\in S, \psi(t)>K] = \underset{t}{\mathrm{E}}[t\in S, \varphi(y)>K]$ and the set $\underset{t}{\mathrm{E}}[t\in S, \varphi(y)>K]$ is open relative to S. The set $\underset{t}{\mathrm{E}}[\varphi(y)>K]$ is therefore the sum of a sequence of sets

$$(6) \qquad E(a,a',b,b') = \underset{y}{\mathrm{E}}[a_i < y_i < a'_i,\ b_j < z_j < b'_j]$$

each of which is measurable (B) and of type O_α. It follows that $\underset{y}{\mathrm{E}}[\varphi(y)>K]$ is measurable (B) and of type O_α. Hence $\varphi(y)$ is measurable (B) and of Baire class $(\alpha+1)$ at most.

As an example of Theorem 2 let x, y be single variables and let

$$(7) \qquad f(x,y) = \frac{f(x)-f(y)}{x-y} = \frac{f(x)-z_1}{x-z_2}$$

where $z_1 = f(y)$, $z_2 = y$. If $f(y)$ is measurable (B) of class α then

$$(8) \qquad \varphi(y;h) = \underset{y<x<y+h}{\text{upper bound}} \frac{f(x)-f(y)}{x-y}$$

is measurable (B) of class $\alpha+1$ at most and hence

$$(9) \qquad \overline{D}^+ f(y) = \overline{\lim_{x\to y+}} \frac{f(x)-f(y)}{x-y} = \lim_{n\to\infty} \varphi\left(y; \frac{1}{n}\right)$$

is measurable (B) and of class $\alpha+2$ at most [2]).

Note that in Theorem 2 we require that $F(x,z)$ shall be continuous — or at least l.s.c. — in z at each (x,z) which arises from a point (x,y) of E. But it is not necessary that F should even be defined for other values of x and z — e.g. for $x = z_2$ in our example — nor that the continuity over the relevant values of (x,z) should be uniform.

---

[2]) W. Sierpiński, Fund. Math. **3**, p. 123.

3. If X-space is topological and if we consider the (x,y)-space as a topological space with the ordinary topology[3]) the condition concerning E will be satisfied when E is an open set. This might suggest that if E is not open but constructed from open sets in some simple way — for instance if E is a G_δ — then $\varphi(y)$ would be measurable-B or even belong to some fixed Baire class. Generalisations of this kind seem however to be impossible. The definition

$$(1) \qquad F(x) = \underset{(y)}{\text{upper bound}}\, f(x,y)$$

of § 1 corresponds in fact to a set E in ordinary space given by

$$(2) \qquad 0 \leqslant x \leqslant 1, \qquad 0 \leqslant y \leqslant 1, \qquad z = x$$

where x,y,z are ordinary Cartesian coordinates. In fact

$$(3) \qquad F(z) = \underset{(x,y)\subset E(z)}{\text{upper bound}}\, f(x,y),$$

which is of our form (1). E is in this case a G_δ and as special as one could desire. But as we have seen F need not be measurable.

The argument, in fact, requires whole regions.

We might still hope to prove $\varphi(y)$ measurable when E, instead of being open, is the closed envelope of an open set. It is possible to prove something of this nature (§ 4) but only with further restrictions on E. In fact we shall now construct a set E which is a closed region but can produce a non-measurable $\varphi(y)$.

We take the x-space and y-space to be Cartesian spaces of the same dimensions and we fix the coordinate system in each. Let B be a nowhere dense closed set of positive measure in x-space. Let S_0 be a division of the space into cubes and let S_{n+1} be the division obtained from S_n by bisecting its cubes each way. For each closed cube Q_n, of the division S_n, which contains at least one point of B we examine its successive subdivisions until we find in it cubes Q_{n+k} which do not meet B. We then form the sum σ_{Q_n} of these cubes Q_{n+k}: the integer k is the same for all of them and depends only on Q_n. We then form the cross-product

$$(4) \qquad (Q_n)_x \times (\sigma_{Q_n})_y$$

i. e. the set of points (x,y) such that

$$(5) \qquad x \subset Q_n, \qquad y \subset \sigma_{Q_n}.$$

[3]) See C. Kuratowski, *Topologie I*, Monografie Matematyczne **3**, p. 135.

the second condition meaning that the point in x-space with the same coordinates as the point y belongs to σ_{Q_n}. We define E to be the sum of all these cross-products plus the set given by

$$(6) \qquad y = x \subset B.$$

It is easily seen that E is a closed region in the (x,y)-space: moreover each section $E'(x)$ is a closed region.

If $x \not\subset B$ then $E'(x)$ contains no point $y \subset B$. If $x \subset B$ then $E'(x)$ contains the point x itself but no other point $y \subset B$.

Now let B^* be a non-measurable subset[4]) of B and consider

$$(7) \qquad C = \sum_{x \subset B^*} E'(x).$$

It constains B^* but no other points of B. Thus its intersection with the measurable set B is non-measurable, and so it cannot be measurable itself. But if

$$(8) \qquad f(x) = \begin{cases} 1 & \text{for } x \subset B^*, \\ 0 & \text{elsewhere,} \end{cases}$$

then

$$(9) \qquad \varphi(y) = \begin{cases} 1 & \text{for } y \subset C, \\ 0 & \text{elsewhere,} \end{cases}$$

so that $\varphi(y)$ is non-measurable.

This example, it is true, requires a non-measurable $f(x)$ to make $\varphi(y)$ non-measurable. But by a topological transformation $x \to \xi$ which converts B into a null-set, $f(x)$ becomes a measurable function, in fact a null-function, of ξ. There is no difficulty in defining such a transformation, especially in particular cases, e. g. when the x-space is 1-dimensional. If we leave the y-space untransformed we then get a non-measurable $\varphi(y)$ as the upper transform of a null-function $f(x(\xi))$ by a closed region.

4. In this § we suppose that the y-space is a Cartesian space of dimensions s, with coordinates $y_1, y_2, \ldots, y_s$.

Lemma 1. *A set Y in y-space has the property that each point $y \subset Y$ is the vertex of an n-dimensional cube $Q(y)$:*

$$(1) \qquad y_k \leqslant \eta_k \leqslant y_k + a \qquad (k = 1, 2, \ldots, s;\ a = a(y))$$

which is contained entirely in Y. Then Y is measurable.

[4]) On the existence of B^* cf. W. Wilkosz, Fund. Math. **1**, p. 82, § 3.

Observe first that for each $y \subset Y$ we can take cubes $Q(y)$ arbitrarily small. These cubes Q then cover Y in the sense of Vitali. We can therefore choose a sequence Q_n of such cubes in such a way that

(2) $$Y - \sum Q_n \quad \text{is of measure zero.}$$

But each $Q_n \subset Y$ and hence

(3) $$Y = \sum Q_n + N,$$

where N is a null-set. Thus Y is measurable.

Notice that Y is the sum of an open set and of a null-set on its boundary. We might speak of Y as being *open in the positive directions*.

Lemma 1 may be proved [5]) without Vitali's Theorem as follows. Suppose Y bounded and write

(4) $$Y = G + B$$

where G is the open kernel of Y and B the rest. Let G_ε be the part of G within a distance ε of the boundary of G. Then G_ε is measurable and its measure tends to 0 with ε. Let B_δ be the part of B for which $a(y) > \delta$. Then a small positive translation transforms B_δ into a subset of G_ε. Hence

(5) $$m_e B_\delta \leqslant m_e G_\varepsilon$$

for any positive δ, ε. It follows that B_δ is a null-set for each δ: and as B_δ increases to B as δ decreases to 0, B itself is a null-set.

Lemma 2. *A set Y has the property that each point $y \subset Y$ is the vertex of an n-dimensional angular region $\Omega(y) \subset Y$. Then Y is measurable.*

The phrase *angular region* means an open set composed entirely of open segments all having the vertex y of the region as a common endpoint. By a suitable linear transformation

(6) $$y_k = \sum_{(h)} a_{hk} y'_k$$

with rational coefficients of non-vanishing determinant we can secure that the region in y'-space corresponding to a given $\Omega(y)$ contains a cube $Q(y')$ as in Lemma 1, except for y' itself.

[5]) S. Banach, Fund. Math. **6**, p. 173, has the same result and gives still another proof.

The linear transformations with rational coefficients can be arranged in a single sequence $\{T_m\}$ $(m=1,2,\ldots)$. Let Y_m be the set of points of Y for which T_m is suitable in the above sense. The sets Y_m may of course overlap. By Lemma 1, Y_m is measurable: so also therefore is $Y = \sum Y_m$.

Definition. Let us say that a set Y with the property of Lemma 2 is *regular*.

Lemma 3. *The sum of any aggregate (finite, enumerable or more than enumerable) of regular sets is regular.*

Theorem 3. *If the sections $E'(x)$ of E are all regular and if*

(7) $$\varphi(y) = \underset{x \subset E(y)}{\text{upper bound}}\, f(x)$$

then $\varphi(y)$ is measurable.

In fact the set $\underset{y}{\mathrm{E}}[\varphi(y) > K]$ is the sum of the sets $E'(x)$ for those values of x for which $f(x) > K$. By Lemma 3, it is regular: by Lemma 2, therefore, measurable.

As an example of Theorem 3 suppose that $F(I)$ is a function of interval, not necessarily additive, and that we define its upper derivative by the usual formula

(8) $$\overline{D}\, F(y) = \overline{\lim_{\delta(Q)\to 0}} \frac{F(Q)}{|Q|}$$

for cubes Q containing y either inside or on the boundary. Then

(9) $$\overline{D}\, F(y) = \lim_{n\to\infty} \varphi\left(y; \frac{1}{n}\right)$$

where

(10) $$\varphi(y;h) = \underset{\delta(Q)<h}{\text{upper bound}} \frac{F(Q)}{|Q|}$$

for cubes Q containing y. By Theorem 3, $\varphi(y;h)$ is measurable for any fixed h: hence $\overline{D}F(y)$ is also measurable [6]).

The property expressed by Lemma 3 is precisely what is needed for the kind of Theorem we want. Any sub-class of the class

[6]) This result is due to S. Banach, l.c. Our method applies of course to many variations in the nature of the intervals used in defining the upper derivative.

of measurable sets which possesses this property will give us a theorem of the type of Theorem 3. The only such sub-classes we know are:

(i) open sets,

(ii) sums of subsequences of a given sequence of measurable sets,

(iii) regular sets,

(iv) regular sets in which restrictions are placed on the angle $\Omega(y)$, for instance the sets open in the positive directions.

Sets such that any boundary point of the set which belongs to it is linearly accessible from the interior of the set have the property of Lemma 3: but such a set need not be measurable [7]).

A regular set need not be measurable (B), nor even a Suslin set. For instance if E consists of the square

$$0<x<1, \qquad 0<y<1 \tag{11}$$

together with a non-measurable linear set on the segment

$$0<x<1, \qquad y=0 \tag{12}$$

then it is regular but its section by $y=0$ (a Suslin set) is not measurable, therefore not a Suslin set.

The hypothesis of Theorem 3 cannot be replaced by the hypothesis that E is regular: this last is neither necessary nor sufficient for the sets $E'(x)$ to be regular. For instance if a set E of a 3-dimensional space is given by

$$y\leqslant x_1\leqslant y+1, \qquad y\leqslant x_2\leqslant y+1 \tag{13}$$

it is regular but the section $E'(x_1,x_2)$ is not regular when $|x_1-x_2|=1$.

The sets $E'(x)$ will certainly be regular if E is regular and if in addition *the angle* $\Omega(x,y)$ at a point $(x,y)\subset E$ *meets the space* $x=$ const. *through its vertex.*

[7]) O. Nikodym, Fund. Math. **10**, p. 116, has constructed in the plane a closed set F of positive measure, every point of which is linearly accessible from the complementary set G. The set $G+$ a non-measurable subset of F (cf. W. Wilkosz, l. c.) would give us our example.

Theorem 4. *Let $z=(z_1,z_2,...,z_t)$ be a point of a Cartesian t-dimensional space and let $z_1,z_2,...,z_t$ be measurable functions of a point $y=(y_1,y_2,...,y_s)$ of a s-dimensional space and let $f(x,y)=F(x,z)$ where $F(x,z)$ is continuous — or at least lower semi-continuous — in z for each fixed x. If E is an open set — or if at least its sections $E'(x)$ are all regular — then*

$$\varphi(y)=\underset{x\subset E(y)}{\text{upper bound}}\, f(x,y) \tag{14}$$

is measurable.

To prove this theorem we use

Lemma 4 (Lusin's Theorem). *If $z(y)$ is a measurable function of y in a closed interval I then given $\varepsilon>0$ we can find a closed set F such that $G=I-F$ is of measure less than ε and that $z(y)$ is continuous on F.*

We return now to Theorem 4. We determine sets F_j such that z_j is continuous on F_j and write $F=\Pi F_j$, $G=I-F$. Then the z_j are simultaneously continuous on F.

If E is open then $F\cdot \underset{y}{\mathrm{E}}[\varphi(y)>K]$ is open relative to F: in particular it is measurable. Hence

$$m_e \underset{y}{\mathrm{E}}[\varphi(y)>K]-m_i \underset{y}{\mathrm{E}}[\varphi(y)>K]\leqslant mG \tag{18}$$

and since G is of arbitrarily small measure it follows that $\underset{y}{\mathrm{E}}[\varphi(y)>K]$ is measurable.

If we assume only that the sections $E'(x)$ are regular then we find that $F\cdot\underset{y}{\mathrm{E}}[\varphi(y)>K]$ is regular relative to F, i. e. if a point y_0 belongs to F and if $\varphi(y_0)>K$ then there is an angular region $\Omega(y_0)$ of vertex y_0 such that $\Phi(y)>K$ for every y belonging to F in this region. If to the set $F\cdot\underset{y}{\mathrm{E}}[\varphi(y)>K]$ we add all these angular regions we get a regular set R and $F\cdot\underset{y}{\mathrm{E}}[\varphi(y)>K]=FR$. Hence $F\cdot\underset{y}{\mathrm{E}}[\varphi(y)>K]$ is measurable and the proof of the theorem is completed as above.

As an example of Theorem 4 let x,y be single variables and let

$$f(x,y)=\frac{f(x)-f(y)}{x-y}=\frac{f(x)-z_1}{x-z_2}. \tag{19}$$

If f is a measurable function then we find that $\overline{D}^+f(y)$ is also measurable [8]).

[8]) S. Banach, Fund. Math. **3**, p. 128, H. Auerbach, Fund. Math. **7**, p. 263.

5. The methods of § 2 would tell us that the function

(1) $$f_\varrho(x) = \underset{\delta(x,\xi)<\varrho}{\text{upper bound}}\, f(\xi)$$

is l.s.c. and hence that

(2) $$M(x) = \lim_{\varrho\to 0} f_\varrho(x)$$

is of class 2 at most. In actual fact it is u.s.c. and so of class 1 at most. The upper semi-continuity depends upon a different argument: it arises rather in the process of taking the limit as $\varrho\to 0$ than in the process of taking the upper bound.

Suppose that we have a decreasing sequence E_n of sets in (x,y)-space and that we define

(3) $$\varphi_n(y) = \underset{x\subset E_n(y)}{\text{upper bound}}\, f(x),$$

(4) $$\varphi(y) = \lim_{n\to\infty} \varphi_n(y).$$

Here $E_n(y)$ denotes a section of E_n given by $y=\text{const}$. The limit certainly exists as $\varphi_n(y)$ is monotonic in n. We may write simply

(5) $$\varphi(y) = \overline{\lim_{x\subset E_n(y)}}\, f(x).$$

Theorem 5. *If $E_N(y)\subset E_n(y_0)$ for all N, y satisfying*

(6) $$\delta(y,y_0)<\delta_0=\delta_0(y_0,n),$$
$$N>N_0=N_0(y_0,n,y),$$

then $\varphi(y)$ is upper semi-continuous.

The proof is immediate. If $\varphi(y_0)<K$ then there is an n so great that $\varphi_n(y_0)<K$, say $\varphi_n(y_0)=K-\varepsilon$ where $\varepsilon>0$. Then $f(x)\leqslant K-\varepsilon$ for all $x\subset E_n(y_0)$ in particular for all $x\subset E_N(y)$ if y, N satisfy (6). Hence

$$\varphi_N(y)\leqslant K-\varepsilon \text{ for } N>N_0$$
$$\varphi(y)\leqslant K-\varepsilon, \qquad \varphi(y)<K$$

for all y in $\delta(y,y_0)<\delta_0$. Thus the set $\underset{y}{\mathrm{E}}[\varphi(y)<K]$ is open and so $\varphi(y)$ is u. s. c.

The hypothesis concerning the sets E_n is quite complicated compared with the condition in Theorem 1 that E be open: and it does not seem possible to simplify it substantially even by sacrificing some generality. Another difficulty enters when we replace $f(x)$ by $f(x,y)$. We now write:

(7) $$\varphi_n(y) = \underset{x\subset E_n(y)}{\text{upper bound}}\, f(x,y),$$

(8) $$\varphi(y) = \lim_{n\to\infty}\varphi_n(y)$$

Theorem 6. *Let $z=(z_1,z_2,\dots,z_t)$ be a point of a Cartesian t-dimensional space, let $z_1,z_2,\dots,z_t$ be functions of a point $y=(y_1,y_2,\dots,y_s)$ of a Cartesian s-dimensional space which are measurable (B) and of Baire class α at most. Let $f(x,y)=F(x,z)$ where F is continuous — or at least upper semi-continuous — in z uniformly in x. If $E_N(y)\subset E_n(y_0)$ for all N, y satisfying (6) then $\varphi(y)$ is measurable (B) and of Baire class $\alpha+1$ at most.*

The relations connecting y and z determine a set S in the $(s+t)$-dimensional (y,z)-space. We easily find that $\varphi(y)$ is u.s.c. as a function of position on S. The result stated then follows as in Theorem 2.

The new difficulty is that we need to assume F not only continuous in z but uniformly so.

We shall disregard Theorem 3 and proceed at once to the analogue of Theorem 4.

Theorem 7. *Let $z=(z_1,z_2,\dots,z_t)$ be a point of a Cartesian t-dimensional space, let $z_1,z_2,\dots,z_t$ be measurable functions of a point $y=(y_1,y_2,\dots,y_s)$ of a Cartesian s-dimensional space and let $f(x,y)=F(x,z)$ where F is continuous — or at least upper semi-continuous — in z uniformly in x. If $E_N(y)\subset E_n(y_0)$ for all N, y satisfying*

(9) $$y\subset\Omega_n(y_0), \qquad N>N_0=N_0(y_0,n,y)$$

then $\varphi(y)$ is measurable.

As in Theorem 4 we first construct a closed set F in y-space on which the z are simultaneously continuous. We then show that the set $F\underset{y}{\mathrm{E}}[\varphi(y)<K]$ is regular relative to F and hence measurable. And since the complement of F is of arbitrarily small measure it follows that $\underset{y}{\mathrm{E}}[\varphi(y)<K]$ is measurable.

6. In this § we make certain general observations on Theorems 1–7, in particular on how far the hypotheses made are necessary.

(i) We have supposed the x-space, like the y-space, metrical. But x may vary in any aggregate, even one in which no topology is defined, and the theorems in question will remain true with the single exception that it will not be possible, in Theorems 1–4, to speak of E being open. In applications the x-space is invariably metrical and usually the same as the y-space.

(ii) In Theorems 2 and 6 we supposed the y-space Cartesian and introduced a z-space which we also supposed Cartesian. Now Borel sets and Baire classes can be defined in any metrical space and if we keep the z-space Cartesian we need not restrict the y-space. The proofs require only a slight change if the y-space is separable: but if it is not separable we must proceed as follows. To fix the ideas take the case of Theorem 2. If the point (y_0, z_0) of S gives $\varphi(y_0) > K$ then, since φ is l.s.c. on S, there is a neighbourhood

$$\delta(y, y_0) < \delta_0, \qquad b_j < z_j < b'_j \qquad (j = 1, 2, \ldots, t)$$

which includes (y_0, z_0) and is such that $\varphi(y) > K$ for any point (y, z) of S which lies in this neighbourhood. We denote by $E(b, b', \delta_0)$ the set of all points y_0 such that the numbers b_j, b'_j and δ_0 are appropriate in this sense to (y_0, z_0). Then $E(b, b', \delta_0)$ increases as δ_0 decreases. Let $E(b, b')$ be its limit as $\delta_0 \to 0$. It is easily seen that $E(b, b')$ is a relatively open subset of

$$\underset{y}{\mathrm{E}}[b_j < z_j < b'_j]$$

and hence of type O_α. The set $\underset{y}{\mathrm{E}}[\varphi(y) > K]$ is the sum of the sets $E(b, b')$ for all rational values of b_j, b'_j. It is thus itself of type O_α and hence $\varphi(y)$ of class $(\alpha+1)$ at most.

(iii) It is also possible to generalise the z-space in Theorems 2, 4, 6, and 7. We can replace it by any separable space, e.g. by a Hilbert space or by Jessen's Q_α [9]). If with any point y a definite point z is associated we say that the point z is a *function* of the point y and write

$$z = \Psi(y).$$

[9]) B. Jessen, Acta Math. **63**, p. 249.

Given a set Z in z-space we obtain from it a set

$$\Psi^{-1}(Z) = \underset{y}{\mathrm{E}}[\Psi(y) \subset Z]$$

in y-space, the *original* of Z. If for every open set V in z-space $\Psi^{-1}(V)$ is open then we say that Ψ is a continuous function. If $\Psi^{-1}(V)$ is of type O_α then we say that Ψ is of Borel [10]) class α. And if (in Theorems 4 and 7 where in the nature of the case the y-space is Cartesian) $\Psi^{-1}(V)$ is measurable we say that Ψ is measurable.

The z-space being separable, we can find in it a sequence $\{V_n\}$ of open sets which define the topology of the space: i.e. given any point z we can find in the sequence arbitrarily small neighbourhoods of z. In the case of Theorem 2 and 6 we simply replace the sets

$$b_j < z_j < b'_j,$$

where b_j, b'_j are rational, by the sets V_n. For Theorems 4 and 7 we start with the generalisation of Lusin's Theorem:

If the point z of a separable space is a measurable function of the point $y = (y_1, y_2, \ldots, y_s)$ in an interval I then we can find a closed set F such that z is continuous on F while $G = I - F$ is of arbitrarily small measure.

From this generalisation of Lusin's Theorem the proofs proceed as before.

In conclusion we observe that the z-space, supposed separable and metrical, is homeomorphic to a subset of Hilbert space. Let $z_1, z_2, \ldots$ be the coordinates in this space. Then

(a) if z is of Borel class α each z_n is of Baire class α at most, if each z_n is of Baire class α at most then z is of Borel class α.

(b) if z is measurable, each z_n is measurable: and conversely.

Thus the generalisations we have just made amount roughly to making t infinite.

7. In the sequel x-space will be a measurable subset of a Cartesian space, y-space will be Cartesian and the word *function* will mean a numerical function. We shall consider the Borel classes relative to x- and y-spaces given and to their Cartesian-product.

[10]) The identity between Borel classes so defined and Baire classes does not hold without conditions on the z-space: cf. F. Hausdorff, *Mengenlehre*, (1927), p. 268, and C. Kuratowski, *Topologie I*, Monografie Matematyczne **3**, p. 187–188.

Theorem 8. *If $f(x,y)$ is measurable as a function of x for each fixed y and continuous as a function of y for each fixed x then it is measurable as a function of (x,y).*

We shall suppose the y-space 1-dimensional. For each positive integer n we divide up the range of y by points $\frac{m}{2^n}$ (m integral) and define

$$f_n(x,y)=f\left(x,\frac{m}{2^n}\right) \quad \text{if} \quad \frac{m}{2^n}\leqslant y<\frac{m+1}{2^n}.$$

Then f_n is measurable in (x,y) and $f_n\to f$ as $n\to\infty$. Hence f is measurable.

If now the y-space is s-dimensional, say

$$y=(y_1,y_2,\dots,y_s),$$

then from the hypotheses of the theorem we find successively that f is

measurable in (x,y_1) for fixed $(y_2,\dots,y_s)$
measurable in (x,y_1,y_2) for fixed $(y_3,\dots,y_s)$
measurable in $(x,y_1,y_2,\dots,y_s)$.

We may remark in conclusion that it is sufficient if f is continuous in y for almost all x and measurable in x for an everywhere dense set of points y.

Theorem 8 illustrates a type of result in which, given a function f or a set E in (x,y)-space we argue from hypotheses in the separate spaces $x=$const., $y=$const. to the measurability of f or E.

The theorem on sets corresponding to Theorem 8 would be: *if the sections $E(y)$ are all measurable and the sections $E'(x)$ all open (or all closed) then E is measurable.* This is false: Sierpiński[11]) has constructed a set E for which the sections $E'(x)$ and $E(y)$ are all closed but which nevertheless is not measurable.

In view of the examples afforded by this set and its complement the following theorem is perhaps somewhat surprising.

Theorem 9. *If the sections $E'(x)$ are all open, the sections $E(y)$ all closed, then E is of type O_1 (i.e. it is an F_σ) and hence measurable.*

[11]) W. Sierpiński, Fund. Math. **1**, p. 112.

Let $y_1,y_2,\dots,y_s$ be the coordinates in y-space and let I be an open rational interval in that space:

$$\mathrm{I:}\qquad r_k<y_k<r'_k \qquad (k=1,2,\dots,s)$$

where r_k, r'_k are rational numbers. The set

$$\underset{x}{\mathrm{E}}\,[I\subset E'(x)]$$

is closed. Its cross-product with I, namely

$$\underset{x,y}{\mathrm{E}}\,[y\subset I\subset E'(x)]$$

is an F_σ. And E is the sum of these sets for all rational intervals I in y-space.

Corollary. *The set E is also*[12]) *of type F_1 (i.e. it is a G_δ).*

For the complement of E satisfies the hypotheses of the theorem (with x and y interchanged) and so is an F_σ.

Theorem 10. *If the sections $E'(x)$ are all regular and the sections $E(y)$ all closed then E is measurable.*

As in Theorem 9 the set

$$\underset{x}{\mathrm{E}}\,[I\subset E'(x)]$$

is closed and

$$\underset{x,y}{\mathrm{E}}\,[y\subset I\subset E'(x)]$$

is an F_σ. Let G be the sum of these sets for all rational intervals I in y-space. Then G is an F_σ.

Clearly $G'(x)$ is the open kernel of $E'(x)$.

We shall show that $E-G$ is of measure zero. Let B consist of the points of $E-G$ at which $E'(x)$ is open in the positive directions and let B_δ consist of the points of B for which $a(y)>\delta$ in the notation of § 4. Denote for each open interval I: $r_k<y<r'_k$ $(k=1,2,\dots,s)$ by I_ε the open interval

$$r_k+\varepsilon<y_k<r'_k \qquad (k=1,2,\dots,s)$$

if such exists, i.e. if the numbers r'_k-r_k all exceed ε, and otherweise let I_ε denote the empty set. Let H_ε be the sum of all the sets

$$\underset{x,y}{\mathrm{E}}\,[y\subset I_\varepsilon,\ I\subset E'(x)].$$

[12]) It is thus an "ensemble ambigu" of class 1.

It is an F_σ and increases to G as ε decreases to 0. Write

$$G_\varepsilon = G - H_\varepsilon.$$

Then G_ε is measurable and (if it is confined to a finite interval)

$$mG_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

By a translation of length less than $\operatorname{Min}(\varepsilon, \delta)$ in the positive direction B_δ can be transformed into a subset of G_ε. Hence

$$m_e B_\delta \leqslant m G_\varepsilon$$

and thus B_δ is a null-set. Since B_δ increases to B as δ decreases to zero it follows that B is also a null-set.

The result for the set B can be extended to the whole of $E-G$ as in § 4, Lemma 2. Thus E is the sum of the Borel set G and a null-set and so is measurable.

In conclusion we may mention that just as the regular sets introduced in § 4 imitate open sets in that they have the property expressed by Lemma 3 so the complementary sets — let us call them CR sets — imitate closed sets in that the product of any aggregate of CR sets is a CR set. We can therefore replace the condition that $E(y)$ be closed in Theorem 10 by the condition that $E(y)$ by CR. The same proof applies save that "closed", "an F_σ" must be replaced by "CR", "measurable" respectively.

8. Theorem 11. *If $f(x,y)$ is measurable in x for each fixed y and increases with y for each fixed x then it is measurable in (x,y).*

Theorem 12. *If the sections $E(y)$ of a set E are all measurable and increase with y then the set E is measurable.*

In these theorems "increase" is understood in the wide sense. We shall also suppose y a simple variable. The theorems remain true when $y=(y_1, y_2, \dots, y_s)$ is a multiple variable provided the phrase "increase with y" is defined as "increase with each y_i $(i=1,2,\dots,s)$". This extension follows at once from the simple case by induction.

Theorem 11 will follow from Theorem 12 applied to the sets

$$E = \mathop{\mathrm{E}}_{x,y}[f(x,y) > K].$$

To prove Theorem 12 first confine the set E to a finite interval I:

$$\alpha_j \leqslant x_j \leqslant \beta_j, \qquad a \leqslant y \leqslant b.$$

Divide up the range of y by points $y_1, y_2, \dots, y_{n-1}$,

$$a = y_0 < y_1 < \dots < y_n = b,$$

in such a way that

$$y_r - y_{r-1} < \varepsilon \qquad (r=1,2,\dots,n).$$

We obtain an inner approximation F to E by taking

$$F(y) = E(y_{r-1}) \quad \text{in} \quad y_{r-1} \leqslant y < y_r.$$

and an outer approximation G by taking

$$G(y) = G(y_{r-1}) \quad \text{in} \quad y_{r-1} < y \leqslant y_r.$$

Then F, G are both measurable and $F \subset E \subset G$. But writing m_r for $mE(y_r)$ we find that

$$m(G-F) = \sum (y_r - y_{r-1})(m_r - m_{r-1})$$
$$\leqslant \varepsilon \sum (m_r - m_{r-1}) = \varepsilon (m_n - m_0) = K\varepsilon.$$

K depending only on E and the interval I. Hence E, or rather EI, included between two measurable sets whose measures differ by arbitrarily little, is measurable. This being true for any interval I, the whole set E is measurable.

The phrase "increases with y" in the hypothesis of Theorem 11 cannot be replaced by "is of bounded variation in y". For the characteristic function $\psi(x,y)$ of the Sierpiński set E in mentioned § 7 is b. v. in y for each fixed x and b. v., a fortiori measurable, in x for each fixed y. But we can make an extension in that direction as follows.

Theorem 13. *If $f(x,y)$ is measurable in x for each fixed y and monotonic in y for each fixed x then it is measuralbe in (x,y).*

Here again we suppose y a simple variable: the extension to the case of a multiple variable y is obtained by induction. Let X_1 be the set of values of x for which $f(x,y)$ is non-decreasing in y and let X_2 be the set of values of x for which $f(x,y)$ is non-increasing. Every x belongs by hypothesis either to X_1 or to X_2 — or to both, in which case $f(x,y)$ would be a constant for that particular x.

Let η denote generically a rational value of y. Then

$$X_1 = \prod_{\eta_1 > \eta_2} \mathop{\mathrm{E}}_{x} [f(x,\eta_1) \geqslant f(x,\eta_2)],$$

$$X_2 = \prod_{\eta_1 > \eta_2} \mathop{\mathrm{E}}_{x} [f(x,\eta_1) \leqslant f(x,\eta_2)].$$

Thus X_1, X_2, as products of sequences of measurable sets, are themselves measurable.

On the set S_1 in (x,y)-space given by S_1:

$$x \subset X_1$$

$f(x,y)$ is measurable by[13]) Theorem 11. Similarly it is measurable on the set S_2 in (x,y)-space given by S_2:

$$x \subset X_2.$$

These sets being measurable and filling the (x,y)-space it follows that f is measurable.

In conclusion we may observe that in Theorems 11 and 13 the condition "for each fixed y" may be replaced by the same condition for an everywhere dense set of values of y and the condition "for each fixed x" by the same condition "for almost all x".

---

[13]) The relativised form of Theorem 11 required here can be proved in the same way as Theorem 11 itself. It can also be *deduced* from Theorem 11. E. g. if we define $f_1 = f$ in S_1, $f_1 = 0$ elsewhere then f_1 is measurable by Theorem 11.