Note on homology theory for locally bicompact spaces.

By

Morris Kline (Princeton).

Kolmogoroff has defined [1]) two types of homology groups for the various dimensions of a locally bicompact Hausdorff space. The first of these is the isomorph of the Vietoris group of the same dimension, in the case of compact metric spaces. The second of the two, namely his $B_0^r(R,J)$, is the character group of the group of the same dimension of the first type when both are applied to any locally bicompact space. Alexander has given [2]) a definition of homology groups for general spaces in terms of a symbolic space of symbolic complexes. It is the purpose of this note to show that *in locally bicompact spaces the r-dimensional Kolmogoroff group of the second type is the isomorph of the Alexander group of the same dimension.* As a consequence, the relations of Alexander's groups to the Čech, Alexandroff, and Vietoris groups are established, the relations of the Čech and Alexandroff approaches to Vietoris's being well known.

1. The definition of *Kolmogoroff's groups* $B_0^r(R,J)$, $r=0,1,2,\ldots$, is essentially as follows. He denotes by $\bar{f}^r$ any function of $r+1$ points of the space R which satisfies the following conditions:

(a) $\bar{f}^r$ is defined and single-valued for all sets of $r+1$ points of R;

(b) $\bar{f}^r$ takes values belonging to a discrete Abelian group J;

(c) $\bar{f}^r$ is skew-symmetric in its arguments;

(d) There exists for each $\bar{f}^r$, a finite system $S_{\bar{f}^r}$ of disjoined subsets bicompact in R such that $\bar{f}^r(p_0,\ldots,p_r)=\bar{f}^r(p_0',\ldots,p_r')$ if, whatever be i, the points p_i and p_i' belong to the same element of the system $S_{\bar{f}^r}$.

[1]) Kolmogoroff A.: C. R. Paris **200** (1936), pp. 1144-47, pp. 1325-27, and pp. 1558-1560.

[2]) Alexander J. W.: *On the Connectivity Ring of an Abstract Space*, Annals of Math. **37** (1936), pp. 698-708.

Two functions $\bar{f}_1^r$ and $\bar{f}_2^r$ are called *equivalent* if there exists, for each point p of R, a neighborhood $V(p)$ such that

$$\bar{f}_1^r(p_0,\ldots,p_r)=\bar{f}_2^r(p_0,\ldots,p_r)$$

whenever all the points p_i belong to a $V(p)$. A class of equivalent functions $\bar{f}^r$ is called a *complex* of dimension r and is indicated by f^r. The complexes f^r form a group F^r.

The boundary of $\bar{f}^r$ is defined as the function $\bar{f}^{r+1}$, where

$$\bar{f}^{r+1}(p_0,\ldots,p_{r+1})=\sum_{i=0}^{r+1}(-1)^i\bar{f}^r(p_0,\ldots,p_{i-1},p_{i+1},\ldots,p_{r+1}). \tag{1}$$

Because the boundaries of equivalent functions are equivalent the boundary of a complex can be defined as the complex of the boundaries. A complex is called a *cycle* if its boundary is the zero complex. The *bounding cycles* are defined as usual and the group of residue classes determined by the cycles modulo the bounding cycles in the *Betti group* $B_0^r(R,J)$.

2. The definition of *Alexander's groups* amounts to the following. An r-simplex is any set of $r+1$ points (vertices). In a bicompact Hausdorff space [3]), each covering of the space by a finite number of open sets determines a complex which consists of all the simplexes which can be formed with the points of the space provided only that the vertices of any one simplex must lie within at least one element of the covering. In a locally bicompact space, a complex is determined by a covering of any bicompact subset of the space by a finite number of sets open in the space. The complex consists only of simplexes which lie on that subset and whose vertices lie entirely within at least one element of the covering. (Since the subset is bicompact, we can close the sets of the covering and the intersection of these closed sets with the bicompact subset are therefore also bicompact subsets of the space. Hence we may say, for the case of a locally bicompact space, that a complex consists of all simplexes lying on a finite collection of bicompact subsets of the space, provided each simplex is entirely within at least one of these subsets).

[3]) The term *Hausdorff space* includes the separation axiom for two points.

In the case of a bicompact space an *r-function* is a skew-symmetric function Φ defined over all the r-simplexes which can be formed with the points of the space, the values of Φ belonging to a discrete Abelian group J. The *derived* of Φ is an $(r+1)$-function Φ' such that

$$\Phi'(p_0,\dots,p_{r+1})=\sum_{i=0}^{r+1}(-1)^i\,\Phi(p_0,\dots,p_{i-1},p_{i+1},\dots,p_{r+1}). \tag{2}$$

Φ is *exact* if Φ' vanishes on *any one* complex. Φ is derived if it is identical on some complex with the derived of some $(r-1)$-function. The exact (derived) r-functions form a group because the sum of two exact (derived) functions Φ_1 and Φ_2 is exact (derived) on the complex determined by the common refinement of the coverings determining the complexes on which Φ_1 and Φ_2 are exact (derived). The group of exact r-functions modulo the subgroup of derived r-functions is the r-dimensional *Betti group* of the space R.

In the case of a locally bicompact space an r-function Φ is a skew-symmetric function defined on all the r-dimensional simplexes of a bicompact subset of a space R. The values of Φ belong to J. The derived function Φ' of Φ is defined by (2). Φ is called *exact* if Φ' vanishes on some complex determined by a covering of the bicompact subset on which it is defined. Φ is called *derived* if it is identical on such a complex with the derived function of some $(r-1)$-function. If Φ_1 and Φ_2 are defined on the bicompact subsets B_1 and B_2, we define $\Phi_1+\Phi_2$ on B_1+B_2 as the function whose value on any r-simplex of B_1+B_2 is the sum of the values of Φ_1 and Φ_2 if both are defined on this simplex, or is Φ_1 or Φ_2, if only Φ_1 or Φ_2 is defined. Then the exact functions form a group. For the sum of two exact functions Φ_1 and Φ_2 is exact, since if C_1 (C_2) is a covering determining a complex on $B_1(B_2)$ on which $\Phi_1(\Phi_2)$ is exact, the sets of C_1 and the set $R-B_1$, on the one hand, and the sets of C_2 and the set $R-B_2$, on the other, form two coverings of the space R. These coverings have a common refinement which furnishes a covering of B_1+B_2 on the complex of which, $\Phi_1+\Phi_2$ is exact. Similarly, the derived functions form a group. Then the r-dimensional *Betti group* of R is the group of exact r-functions modulo the subgroup or derived r-functions.

3. The isomorphism between the r-dimensional Alexander and Kolmogoroff groups in bicompact Hausdorff spaces is immediately established. We remark only that, because of Kolmogoroff's notion of equivalence and the bicompactness of the space, in each [4]) K-complex [5]) there is at least one K-function which takes values on a complex in Alexander's sense.

4. We proceed to the case of a locally bicompact space R which we can suppose not to be bicompact. Then we can imbed R in a bicompact space $\bar{R}$ [6]) by adding the point ξ and by making the necessary adjustments in the topology of this extension of R.

Now consider any A-function Φ defined on some bicompact subset B of R. Let Φ correspond to the function which has the same values that Φ has on B but which is 0 on $R-B$. This latter function is a K-function, condition (d) of **1** being satisfied for the following reason. Because every locally bicompact space is regular [7]) we can find a neighborhood $U(p)$ of a point p in R such that $U(p)$ does not intersect B. $U(p)$ is bicompact in R and satisfies condition (d). If Φ is exact (derived), then, in view of the Kolmogoroff concept of equivalence, the corresponding K-function belongs to a cycle (bounding cycle). Hence the correspondence of A- to K-functions sets up a single-valued mapping of the A-homology classes into the K-homology classes. It remains to be shown that to each K-homology class there corresponds exactly one A-class. We need concern ourselves then only with K-functions which belong to complexes that are cycles.

We make use of Kolmogoroff's notion of a fundamental system [8]). Such a system can be constructed for R in the following

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[4]) We shall use the letters K and A to stand for Kolmogoroff and Alexander.

[5]) The word *complex* is used in two different senses. In the Kolmogoroff theory it is a class of equivalent functions. In Alexander's theory it is a geometrical structure.

[6]) Alexandroff P. and Hopf H.: *Topologie I*, Berlin 1935, p. 93.

[7]) Alexandroff P. and Urysohn P.: *Mémoire sur les Espaces Topologiques*, Proceed. Acad. Amsterdam **14** (1929), p. 71.

[8]) Kolmogoroff, loc. cit., pp. 1326-1327. A decomposition of R into a finite or infinite number of disjoined sets B_α is called *locally finite* if each bicompact set in R has points in common with only a finite number of B_α. A system S of locally finite decompositions Σ is called a *fundamental system* if it possesses the following properties:

1) If Σ' and Σ'' are two decompositions of S the decomposition $\Sigma'\cdot\Sigma''$ (i. e., the family of sets obtained by intersecting each set of Σ' with each set of Σ'') also belongs to S.

2) Whatever be the finite system of open sets covering a subset A bicompact in R, there exists among the decompositions of R belonging to S at least one for which the elements which have points in common with A are contained each in at least one of the open sets.

way. Choose any neighborhood U of ξ in $\bar{R}$. In $\bar{R}-U$, which, as a closed subset of $\bar{R}$, is bicompact as a space, select any covering by a finite number of disjoined sets. Then U, together with this finite number of sets covering $\bar{R}-U$, is a locally finite decomposition [8]) of $\bar{R}$, and by omitting ξ, a locally finite decomposition of R. We form all possible decompositions obtainable by using all choices of $U(\xi)$ and, with each, all ways of decomposing $\bar{R}-U$ into a finite number of disjoined sets. Such a collection of decompositions forms a fundamental system for R.

Now let f be any cycle. According to a theorem of Kolmogoroff [9]) there exists a cycle f_1 homologous to f and constant in relation to each of its arguments on the elements of some decomposition Σ of any fundamental system. Let Σ_1 be the decomposition of the fundamental system just constructed on which f_1 is constant in relation to each of its arguments. In the $U(\xi)$ belonging to Σ_1, f_1 is 0 because it is skew-symmetric. As a bicompact space $\bar{R}$ is certainly regular. Hence there must be a neighborhood $U_1(\xi)$ whose closure is contained in $U(\xi)$. To each point of $\bar{R}-U_1$ let us select a neighborhood, requiring only that if a point lie in $U-U_1$, the neighborhood be interior to U, whereas if the point lie in $\bar{R}-U$, the neighborhood should not intersect $\bar{U}_1$. These neighborhoods, together with U_1, are a covering of $\bar{R}$. Hence we can select a finite number of them to cover $\bar{R}$. We define a function $\bar{f}_2$ to have the same value on any simplex lying within one of these neighborhoods that some definite function belonging to f_1 has. For other simplexes $\bar{f}_2$ is to have the value 0. $\bar{f}_2$ is equivalent in Kolmogoroff's sense to the functions belonging to f_1. Because f_1 is a cycle, the function Φ_2 which has the same values as $\bar{f}_2$ on the simplexes of $\bar{R}-U_1$ is an exact A-function on the bicompact subset $\bar{R}-U_1$ of R. Hence to each K-homology class there is at least one A-homology class. Moreover, if the K-class in the class of bounding cycles, the corresponding A-class is derived. Hence the isomorphism between the two groups is established.

[9]) loc. cit., p. 1326.

The Institute For Advanced Study
Princeton, N. J.

Boolesche Ringe mit geordneter Basis.

Von

Andrzej Mostowski und Alfred Tarski (Warszawa).

Die allgemeine Boolesche Algebra stellt bekanntlich ein formales Schema dar, das verschiedenartiger Realisierungen fähig ist. Im vorliegenden Aufsatz betrachten wir eine neue Art der Booleschen Algebra, die bisher nicht untersucht wurde und die eine ziemlich natürliche Verallgemeinerung der abzählbaren Realisierungen der Booleschen Algebra darstellt.

Die Boolesche Algebra kann entweder als eine selbständige mathematische Theorie oder als ein Kapitel der abstrakten Algebra oder auch der Mengenlehre betrachtet werden. Wir haben uns hier für den algebraischen Weg entschieden und kleiden alle unsere Theoreme in die Form von Sätzen über gewisse algebraische Ringe (nämlich sog. Boolesche Ringe). Nichtsdestoweniger lassen diese Ergebnisse eine einfache topologische und abstrakt-mengentheoretische Deutung zu.

In § 1 bringen wir die Definitionen der Booleschen Ringe, mit denen wir uns weiter befassen, und leiten einen Satz ab, der die Struktur der Elemente von solchen Ringen beschreibt. In § 2 befassen wir uns mit den Begriffen der Isomorphie und Homomorphie. In § 3 untersuchen wir näher eine spezielle Art der uns hier interessierenden Ringe, nämlich solche, die eine zerstreute Basis haben. In § 4 geben wir schließlich eine Charakterisierung der Primideale in den hier betrachteten Booleschen Ringen an.

Präzisere Ergebnisse lassen sich erreichen, wenn man sich auf Ringe mit einer wohlgeordneten Basis beschränkt. Es ist eine Fortsetzung der vorliegenden Arbeit beabsichtigt, wo diese Probleme nebst einigen Anwendungen auf die Theorie der abzählbaren Booleschen Ringe besprochen werden sollen [1]).

[1]) Zur Auffassung der Booleschen Algebra als einer selbständigen Disziplin vgl. L. Couturat: *l'Algèbre de la Logique* (2. Aufl., Paris, 1914); das Buch be-