On trigonometric series conjugate to Fourier series of two variables [1].

By

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1. Let us consider a trigonometric series [2])

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{1.1}$$

Conjugate to the series (1.1) is called the series

$$\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx). \tag{1.2}$$

The series (1.1) and (1.2) are respectively the real and the imaginary part of the power series $\sum_{n=0}^{\infty}c_n e^{inx}$, where $c_0=\frac{1}{2}a_0$, and $c_n=a_n-ib_n$ for $n>0$.

The series (1.1) is the Fourier series of a function $f(x)$, if a_n and b_n are of the form

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\,dt,\qquad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt.$$

[1]) This paper is a translation of a Ph. D. thesis presented at the University of Wilno in 1939. The Polish original („O szeregach trygonometrycznych sprzężonych do szeregów Fouriera dwu zmiennych“) was to appear in „Wiadomości Matematyczne“, and was already available in reprint form. The outbreak of the war stopped the publication of the journal and the paper did not appear in print.

The author, an instructor of the University of Wilno and officer in the Polish Army, was killed during the war.

Editors.

[2]) For the theory of trigonometric series of a single variable, see A. Zygmund, *Trigonometrical Series*, Monografie Matematyczne t. 5, Warszawa-Lwów, 1935. The book will be quoted TS.

In this paper we consider exclusively integrals in the Lebesgue sense.

It follows from the theorem of Riesz-Fischer that if (1.1) is the Fourier series of a function $f(x)\in L^2$, then (1.2) is also the Fourier series of a function $g(x)\in L^2$. This theorem was extended by M. Riesz to the Fourier series of functions of the class L^p, $p>1$. On the other hand, it is well known that if (1.1) is the Fourier series of a function $f(x)\in L$, the conjugate series need not be a Fourier series.

We shall say that a function $f(x)$ of period 2π belongs to the class $L^{p,k}$, if the function $|f(x)|^p\{\log^+|f(x)|\}^k$ is integrable.

It has been proved [3]) that if $f(x)$ belongs to the class $L^{1,k}$, $k\geqslant 1$, then the series conjugate to the Fourier series of $f(x)$ is the Fourier series of a function $\bar{f}(x)$ belonging to the class $L^{1,k-1}$. In particular, if $f(x)\in L^{1,1}$, then $\bar{f}(x)\in L$.

In what follows, by $\bar{f}(x)$ we shall mean the (generalized) sum of the series conjugate to the Fourier series of the function $f(x)$. It is well known that

$$\bar{f}(x)=-\frac{1}{\pi}\int_0^{\pi}\frac{f(x+t)-f(x-t)}{2\tan\frac{1}{2}t}\,dt,\qquad\text{where}\qquad \int_0^{\pi}=\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi}. \tag{1.3}$$

A function $f(x,y)$, of period 2π with respect to each variable, will be said to belong to the class L^p, if the integral $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^p\,dx\,dy$ exists. We define the class $L^{p,k}$ similarly.

The analogue for two variables of the series (1.1) is the series

$$\sum_{m,n=0}^{\infty}A_{m,n}(x,y), \tag{1.4}$$

where

$$A_{0,0}=\tfrac{1}{4}a_{0,0},\qquad A_{0,n}=\tfrac{1}{2}(a_{0,n}\cos ny+b_{0,n}\sin ny),$$
$$A_{m,0}=\tfrac{1}{2}(a_{m,0}\cos mx+c_{m,0}\sin mx),$$
$$A_{m,n}=a_{m,n}\cos mx\cos ny+b_{m,n}\cos mx\sin ny+c_{m,n}\sin mx\cos ny+d_{m,n}\sin mx\sin ny,$$

[3]) See TS, p. 165.

m and n being positive. The series (1.4) is the Fourier series of a function $f(x,y)$, if

$$a_{m,n}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos mx \cos ny\, dx\, dy$$

for $m,n=0,1,2,\ldots$, and if analogue formulas hold for $b_{m,n}$, $c_{m,n}$ and $d_{m,n}$.

The purpose of this paper is to obtain certain results for the series conjugate to Fourier series of two variables [4].

The notion of conjugacy for double series has a slightly different character than in the case of single series. This follows from the fact that every series (1.1) is the real part of a power series; the series conjugate to (1.1) is defined as the imaginary part of that power series. On the other hand, not every series (1.4) is the real part of a power series of two variables. If this is to be the case, certain relations between the coefficients must be satisfied, namely

$$a_{m,n}=-d_{m,n}, \qquad b_{m,n}=c_{m,n}.$$

Let us introduce the following definitions

$$(1.5)\qquad {}^*\mathfrak{S}(f)=\sum_{m,n}{}^*A_{m,n}(x,y), \qquad \mathfrak{S}^*(f)=\sum_{m,n}A^*_{m,n}(x,y)$$

$$(1.6)\qquad \bar{\mathfrak{S}}(f)=\sum_{m,n}\bar{A}_{m,n}(x,y).$$

Here, for every trigonometric polynomial A of two variables, ${}^*A$ means the polynomial conjugate to A, the latter being treated as a function of a single variable x; A^* is defined similarly, the roles of x and y being interchanged. Finally, $\bar{A}=\{{}^*A\}^*$. Correspondingly the (generalized) sums of the series (1.5) and (1.6) will be denoted by ${}^*f(x,y)$, $f^*(x,y)$ and $\bar{f}(x,y)$. The series (1.6) will be called *conjugate* to the series (1.4). On account of the formula (1.3), the functions ${}^*f(x,y)$ and $f^*(x,y)$ may be written as follows

$$(1.7)\qquad \begin{aligned} {}^*f(x,y)&=-\frac{1}{\pi}\int_0^{\pi}\frac{f(x+u,y)-f(x-u,y)}{2\tan\frac{1}{2}u}\,du,\\ f^*(x,y)&=-\frac{1}{\pi}\int_0^{\pi}\frac{f(x,y+v)-f(x,y-v)}{2\tan\frac{1}{2}v}\,dv,\end{aligned}$$

[4]) For the general information concerning the trigonometric series of two variables see L. Tonelli, *Serie Trigonometriche*, Bologna, 1928.

the integrals having the same meaning as in (1.3). Purely formally, $\bar{f}(x,y)$ may be written

$$\bar{f}(x,y)=-\frac{1}{\pi}\int_0^{\pi}\frac{{}^*f(x,y+v)-{}^*f(x,y-v)}{2\tan\frac{1}{2}v}\,dv.$$

Substituting for ${}^*f(x,y)$ its value from (1.7) we get (again arguing formally)

$$\bar{f}(x,y)=\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\frac{f(x+u,y+v)-f(x-u,y+v)-f(x+u,y-v)+f(x-u,y-v)}{2\tan\frac{1}{2}u\cdot 2\tan\frac{1}{2}v}\,du\,dv.$$

We shall show that, if $f(x,y)$ satisfies certain conditions, this integral exists at almost every point (x,y), if we understand it in the following sense

$$(1.9)\qquad \int_0^{\pi}\int_0^{\pi}=\lim_{\varepsilon,\eta\to 0}\int_{\varepsilon}^{\pi}\int_{\eta}^{\pi}.$$

Here ε and η tend to 0 independently of each other. We shall then say that the integral (1.9) converges *strongly*, to distinguish this case from the one in which ε and η tend to 0 in such a way that the ratios ε/η and η/ε remain bounded.

In what follows, we shall use the symbol $\int_{-\pi}^{(\xi)\pi} g(x)\,dx$ to denote the integral extended over the sum of the intervals $(-\pi,-\xi)$ and (ξ,π). Similarly,

$$\int_{-\pi}^{(\varepsilon)\pi}\int_{-\pi}^{(\eta)\pi} g(x,y)\,dx\,dy$$

will mean the integral extended over the domain $\varepsilon\leqslant|x|\leqslant\pi, \eta\leqslant|y|\leqslant\pi$.

The main object of this paper is the proof of the following two theorems.

Theorem 1. *If the function $f(x,y)$ is of period 2π with respect to x and y, and belongs to the class L^p, $p>1$, then the function*

$$\bar{f}(x,y)=\lim_{\varepsilon,\eta\to 0}\frac{1}{\pi^2}\int_{-\pi}^{(\varepsilon)\pi}\int_{-\pi}^{(\eta)\pi}\frac{f(x+u,y+v)}{2\tan\frac{1}{2}u\cdot 2\tan\frac{1}{2}v}\,du\,dv \tag{1.10}$$

exists for almost every point (x,y) and satisfies the inequality [5])

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\bar{f}|^p\,dx\,dy\leqslant A_p\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f|^p\,dx\,dy.$$

Theorem 2. *If the function $f(x,y)$ belongs to the class $L^{1,k}$ where $k\geqslant 3$, then the function $\bar{f}(x,y)$ defined by the formula* (1.10) *exists almost everywhere and belongs to the class $L^{1,k-2}$.*

Theorem 1 is an analogue of the result quoted above of M. Riesz.

2. The proof of Theorem 1 is based on a few lemmas.

Lemma 1. *If the function $f(x)$ of period 2π belongs to the class L^p, $p>1$, then the conjugate function $\bar{f}(x)$, the n-th partial sum $s_n(x)$ of the Fourier series of $f(x)$, and the n-th partial sum $\bar{s}_n(x)$ of the conjugate series satisfy the inequalities*

$$\begin{gathered}\int_{-\pi}^{\pi}|\bar{f}(x)|^p dx\leqslant A_p\int_{-\pi}^{\pi}|f(x)|\,dx,\qquad \int_{-\pi}^{\pi}|s_n(x)|^p dx\leqslant A_p\int_{-\pi}^{\pi}|f(x)|^p dx,\\ \int_{-\pi}^{\pi}|\bar{s}_n(x)|^p dx\leqslant A_p\int_{-\pi}^{\pi}|f(x)|^p dx.\end{gathered}\tag{2.1}$$

This result is very well known and is due to M. Riesz [6]).

In what follows we shall use the following notations

$$\begin{aligned} M(x,y;f)&=\operatorname*{Sup}_{h,k}\frac{1}{hk}\int_{-h}^{h}\int_{-k}^{k}|f(x+u,y+v)|\,du\,dv,\\ M_1(x,y;f)&=\operatorname*{Sup}_{h}\frac{1}{h}\int_{-h}^{h}|f(x+u,y)|\,du,\\ M_2(x,y;f)&=\operatorname*{Sup}_{k}\frac{1}{k}\int_{-k}^{k}|f(x,y+v)|\,dv,\end{aligned}\tag{2.2}$$

with $0<h\leqslant\pi$, $0<k\leqslant\pi$.

[5]) In this paper, A_p and B_p will denote constants (not always the same) depending exclusively on the parameter p.

[6]) TS, p. 147.

Lemma 2. *Let $f(x,y)$ be a function of period 2π with respect to x and y, and of the class L^p, $p>1$. Then*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\{M(x,y;f)\}^p\,dx\,dy\leqslant A_p\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^p\,dx\,dy.$$

This result is also known [7])

Lemma 3. *The Fourier series of a function $f(x,y)$ of the class L^p, $p>1$, is almost everywhere summable by the method of the first arithmetic mean* [8]).

3. ***Lemma 4.*** *Let $f(x)$ be a function defined in the interval $(-\pi,\pi)$, and let $\mu=\operatorname*{Sup}_h h^{-1}\int_{-\pi}^{h}|f(x)|\,dx$. Then*

$$h\int_{-\pi}^{(h)\pi}\frac{|f(x)|}{x_2}\,dx\leqslant 2\mu. \tag{3.1}$$

Proof. Putting $F(u)=\int_{-u}^{u}|f(t)|\,dt$, we may transcribe the left-hand side in the form

$$h\int_{h}^{\pi}\frac{F'(u)}{u^2}\,du=h\left[\frac{F(u)}{u}\right]_h^{\pi}+2h\int_{h}^{\pi}\frac{F(u)}{u^3}\,du$$

$$\leqslant\frac{h}{\pi}\mu+2\mu h\int_{h}^{\pi}\frac{du}{u^2}=\frac{h}{\pi}\mu+2\mu h\left\{\frac{1}{h}-\frac{1}{\pi}\right\}\leqslant 2\mu.$$

Lemma 5. *If $f(x)$ is of period 2π and of the class L^p, $p>1$, the function*

$$\tilde{f}(x)=\operatorname*{Sup}_{h}\left|\int_{h}^{\pi}\frac{f(x+u)-f(x-u)}{2\tan\frac{1}{2}u}\,du\right|$$

satisfies the inequality

$$\int_{-\pi}^{\pi}\tilde{f}^p(x)\,dx\leqslant A_p\int_{-\pi}^{\pi}|f(x)|^p dx.$$

[7]) See Jessen, Marcinkiewicz and Zygmund, *Note on the Differentiability of Multiple Integrals*, Fund. Math. **25** (1935), pp. 217-234, esp. 219.

[8]) See A. Zygmund, *On the differentiability of multiple integrals*, Fund. Math. **23** (1934), pp. 143-149, or the paper quoted in footnote 4).

This lemma is known[9]). It implies at once

Lemma 6. *If the function $f(x,y)$ of period 2π with respect to x and y is of the class L^p, $p>1$, the functions*

$$\tilde{f}_1(x,y)=\operatorname*{Sup}_h\left|\int_{-\pi}^{(h)\pi}\frac{f(x+u,y)}{2\tan\frac{1}{2}u}du\right|,\qquad \tilde{f}_2(x,y)=\operatorname*{Sup}_h\left|\int_{-\pi}^{(h)\pi}\frac{f(x,y+v)}{2\tan\frac{1}{2}v}dv\right|$$

satisfy the inequalities

$$\int_{-\pi}^{\pi}\tilde{f}_1^p(x,y)\,dx\leqslant A_p\int_{-\pi}^{\pi}|f(x,y)|^p\,dx,\qquad \int_{-\pi}^{\pi}\tilde{f}_2^p(x,y)\,dy\leqslant A_p\int_{-\pi}^{\pi}|f(x,y)|^p\,dy.$$

Lemma 7. *If (1.4) is the Fourier series of a function $f(x,y)$ of the class L^p, $p>1$, then the series (1.4) and the series $\sum_{m,n}{}^*A_{m,n}(x,y)$, $\sum_{m,n}A^*_{m,n}(x,y)$ all converge in the mean of order p to the functions $f(x,y)$, ${}^*f(x,y)$, $f^*(x,y)$ respectively.*

Proof. Let us consider the partial sums

$$R=\sum_{m=0}^{m_1}\sum_{n=0}^{n_1}A_{m,n}(x,y),\quad {}^*R=\sum_{m=0}^{m_1}\sum_{n=0}^{n_1}{}^*A_{m,n}(x,y),\quad R^*=\sum_{m=0}^{m_1}\sum_{n=0}^{n_1}A^*_{m,n}(x,y).$$

A repeated application of the second inequality (2.1) gives

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|R(x,y)|^p\,dx\,dy\leqslant A_p\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^p\,dx\,dy.$$

Applying the first inequality (2.1), we also get

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|{}^*R(x,y)|^p\,dx\,dy\leqslant A_p\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|R(x,y)|^p\,dx\,dy\leqslant A_p\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^p\,dx\,dy.$$

A similar inequality is satisfied by the function R^*, analogous to ${}^*R$. From these formulas we easily deduce

Lemma 8. *If $f(x,y)\in L^p$, $p>1$, and if*

$$f(x,y)\sim\sum_{m,n}A_{m,n}(x,y),$$

the series $\sum_{m,n}\bar{A}_{m,n}(x,y)$ converges in the mean of order p.

This result follows if we apply Lemma 7 twice.

[9]) TS, p. 260.

4. We may now pass to the proof of Theorem 1. Let $f(x,y)$ satisfy the hypotheses of Theorem 1, and let $\bar{f}(x,y)$ be the (generalized) sum of the series $\bar{\mathfrak{S}}(f)$ (cf. (1.6)). On account of Lemma 8, $\bar{f}(x,y)\in L^p$ and $\bar{\mathfrak{S}}(f)=\mathfrak{S}(\bar{f})$. In virtue of Lemma 3, $\bar{\mathfrak{S}}(f)$ is almost everywhere summable by the method of the first arithmetic mean to sum $\bar{f}(x,y)$.

The first arithmetic means of the series (1.4) and (1.6) will be denoted by $\sigma_{m,n}(x,y)$ and $\bar{\sigma}_{m,n}(x,y)$ respectively. Thus

$$\bar{\sigma}_{m,n}(x,y)=\sum_{k=0}^{m}\sum_{l=0}^{n}\left(1-\frac{k}{m+1}\right)\left(1-\frac{l}{n+1}\right)\bar{A}_{k,l}(x,y),$$

$$\pi^2\bar{\sigma}_{m,n}(x,y)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x+u,y+v)\,\bar{K}_m(u)\,\bar{K}_n(v)\,du\,dv$$

$$=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x+u,y+v)\left[\frac{1}{2\tan\frac{1}{2}u}-\frac{\sin(m+1)u}{(m+1)(2\sin\frac{1}{2}u)^2}\right]\left[\frac{1}{2\tan\frac{1}{2}v}-\frac{\sin(n+1)v}{(n+1)(2\sin\frac{1}{2}v)^2}\right]du\,dv.$$

If we set

$$\bar{K}_n(u)=\frac{1}{2\tan\frac{1}{2}u}-R_n(u),$$

we get the following inequalities

$$|\bar{K}_n(u)|<n,\qquad\text{for all } u \tag{4.1}$$

$$|R_n(u)|<\pi^2/4u,\qquad\text{for } |u|\leqslant 1/n \tag{4.2}$$

$$|R_n(u)|<\pi^2/4nu^2\qquad\text{for } 1/n<|u|\leqslant\pi. \tag{4.3}$$

Let ε and η be two positive numbers less than π, and let m and n denote the positive integers defined by the inequalities

$$1/(m+1)<\varepsilon\leqslant 1/m,\qquad 1/(n+1)<\eta\leqslant 1/n. \tag{4.4}$$

We split the fundamental square $-\pi\leqslant u\leqslant\pi$, $-\pi\leqslant v\leqslant\pi$ into four domains

I $(-\varepsilon\leqslant u\leqslant\varepsilon,\ -\eta\leqslant v\leqslant\eta)$, II $(\varepsilon\leqslant|u|\leqslant\pi,\ -\eta\leqslant v\leqslant\eta)$

III $(-\varepsilon\leqslant u\leqslant\varepsilon,\ \eta\leqslant|v|\leqslant\pi)$, IV $(\varepsilon\leqslant|u|\leqslant\pi,\ \eta\leqslant|v|\leqslant\pi)$.

Then

$$\pi^2\bar{\sigma}_{m,n}(x,y)=\iint\limits_{\mathrm{I}}+\iint\limits_{\mathrm{II}}+\iint\limits_{\mathrm{III}}+\iint\limits_{\mathrm{IV}} \tag{4.5}$$

$$=A_{\varepsilon,\eta}+B_{\varepsilon,\eta}+C_{\varepsilon,\eta}+D_{\varepsilon,\eta},$$

say. We may write

$$D_{\varepsilon,\eta}=\int\limits_{-\pi}^{(\varepsilon)\,\pi}\int\limits_{-\pi}^{(\eta)\,\pi} f(x+u,y+v)\,\bar{K}_m(u)\,\bar{K}_n(v)\,du\,dv=\int\limits_{-\pi}^{(\varepsilon)\,\pi}\int\limits_{-\pi}^{(\eta)\,\pi}\frac{f(x+u,y+v)}{2\tan\frac{1}{2}u\tan\frac{1}{2}v}\,du\,dv$$

$$-\int\limits_{-\pi}^{(\varepsilon)\,\pi}\int\limits_{-\pi}^{(\eta)\,\pi}\frac{f(x+u,y+v)\cdot\sin(n+1)v}{2\tan\frac{1}{2}u\cdot(n+1)\cdot(2\sin\frac{1}{2}v)^2}\,du\,dv-$$

$$-\int\limits_{-\pi}^{(\varepsilon)\,\pi}\int\limits_{-\pi}^{(\eta)\,\pi}\frac{f(x+u,y+v)\cdot\sin(m+1)u}{2\tan\frac{1}{2}v\cdot(m+1)\cdot(2\sin\frac{1}{2}u)^2}\,du\,dv$$

$$+\int\limits_{-\pi}^{(\varepsilon)\,\pi}\int\limits_{-\pi}^{(\eta)\,\pi}\frac{f(x+u,y+v)\cdot\sin(m+1)u\cdot\sin\frac{1}{2}(n+1)v}{(m+1)(n+1)(2\sin\frac{1}{2}u)^2\cdot(2\sin\frac{1}{2}v)^2}\,du\,dv=$$

$$=D'_{\varepsilon,\eta}-D''_{\varepsilon,\eta}-D'''_{\varepsilon,\eta}+D''''_{\varepsilon,\eta}$$

say. Hence

$$|\pi^2\bar{\sigma}_{m,n}(x,y)-D'_{\varepsilon,\eta}|\leqslant|A_{\varepsilon,\eta}|+|B_{\varepsilon,\eta}|+|C_{\varepsilon,\eta}|+|D''_{\varepsilon,\eta}|+|D'''_{\varepsilon,\eta}|+|D''''_{\varepsilon,\eta}|. \tag{4.6}$$

We shall give an estimate for the right-hand side of the inequality (4.6). From (4.5) and (4.1) we get

$$|A_{\varepsilon,\eta}|\leqslant mn\int\limits_{-1/m}^{1/n}\int\limits_{-1/n}^{1/n}|f(x+u,y+v)|\,du\,dv\leqslant M(x,y;f)$$

where $M(x,y;f)$ is defined by the formula (2.2). Furthermore (cf. (4.5))

$$B_{\varepsilon,\eta}=\int\limits_{-\eta}^{\eta}\bar{K}_n(v)\,dv\int\limits_{-\pi}^{(\varepsilon)\,\pi}\frac{f(x+u,y+v)}{2\tan\frac{1}{2}u}\,du-\int\limits_{-\eta}^{\eta}\bar{K}_n(v)\,dv\int\limits_{-\pi}^{(\varepsilon)\,\eta}\frac{f(x+u,y+v)}{(m+1)(2\sin\frac{1}{2}u)^2}\,du$$

$$=\beta'_{\varepsilon,\eta}-\beta''_{\varepsilon,\eta},$$

say, and

$$|B_{\varepsilon,\eta}|\leqslant|B'_{\varepsilon,\eta}|+|B''_{\varepsilon,\eta}|. \tag{4.8}$$

Using the notation of Lemma 6, and in virtue of the inequality (4.1) we get

$$|B'_{\varepsilon,\eta}|\leqslant n\int\limits_{-\eta}^{\eta}\Bigg|\int\limits_{-\pi}^{(\varepsilon)\,\pi}\frac{f(x+u,y+v)}{2\tan\frac{1}{2}u}\,du\Bigg|\,dv\leqslant n\int\limits_{-1/n}^{1/n}\tilde{f}_1(x,y+v)\,dv\leqslant M_2(x,y;\tilde{f}_1).$$

The inequality (4.3) and Lemma 4 give

$$|B''_{\varepsilon,p}|\leqslant\frac{\pi^2n}{4}\int\limits_{-1/n}^{1/n}\frac{2}{m+1}\Bigg\{\int\limits_{-\pi}^{(1/m+1)\,\pi}\frac{|f(x+u,y+v)|}{u^2}\,du\Bigg\}\,dv\leqslant$$

$$\leqslant\frac{\pi^2n}{4}\int\limits_{-1/n}^{1/n}4M_1(x,y+v;f)\,dv.$$

Finally,

$$|B''_{\varepsilon,\eta}|\leqslant\pi^2M_2(x,y;M_1).$$

On account of the inequality (4.8),

$$|B_{\varepsilon,\eta}|\leqslant M_2(x,y;\tilde{f}_1)+\pi^2M_2(x,y;M_1). \tag{4.9}$$

Similarly we get

$$|C_{\varepsilon,\eta}|\leqslant M_1(x,y;\tilde{f}_2)+\pi^2M_1(x,y;M_2). \tag{4.10}$$

Using the notation of Lemma 6 we deduce from (4.3)

$$|D''_{\varepsilon,\eta}|\leqslant\frac{\pi^2}{4n}\int\limits_{-\pi}^{\pi}\frac{dv}{v^2}\Bigg\{\sup_{\varepsilon}\Bigg|\int\limits_{-\pi}^{(\varepsilon)\,\pi}\frac{f(x+u,y+v)}{2\tan\frac{1}{2}u}\,du\Bigg|\Bigg\}$$

$$\leqslant\frac{\pi^2}{2(n+1)}\Bigg\{\int\limits_{-\pi}^{(1/n+1)\,\pi}\frac{\tilde{f}_1(x,y+v)}{v^2}\,dv\Bigg\}.$$

On account of Lemma 4, this gives

$$|D''_{\varepsilon,\eta}|\leqslant\frac{\pi^2}{2}\,2M_2(x,y;\tilde{f}_1)=\pi^2M_2(x,y;\tilde{f}_1). \tag{4.11}$$

Similarly,

$$|D'''_{\varepsilon,\eta}|\leqslant\pi^2M_1(x,y;\tilde{f}_2). \tag{4.12}$$

Finally, we easily estimate $D''''_{\varepsilon,\eta}$ if we use Lemma 4 and the inequality (4.3)

$$|D''''_{\varepsilon,\eta}|\leqslant\pi^4M_1(x,y;M_2). \tag{4.13}$$

From the relation (4.6) and taking into account the inequalities from (4.7) up to (4.3) we get

$$|\pi^2\bar{\sigma}_{m,n}(x,y)-D'_{\varepsilon,\eta}|\leqslant M(x,y;f)+(1+\pi^2)M_2(x,y;\tilde{f}_1)+ \tag{4.14}$$
$$+(1+\pi^2)M_1(x,y;\tilde{f}_2)+\pi^2M_2(x,y;M_1)+\pi^2(1+\pi^2)M_1(x,y;M_2).$$

From Lemmas 2 and 6 we see that each term on the right of (4.14) is a function of the class L^p, $p>1$. Denoting the sum of these terms by $h(x,y)$, we have

$$|\pi^2\bar{\sigma}_{m,n}(x,y)-D'_{\varepsilon,\eta}|\leqslant h(x,y) \tag{4.15}$$

and furthermore

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h^p(x,y)\,dx\,dy\leqslant A_p\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^p\,dx\,dy. \tag{4.16}$$

From (4.15) and (4.6) we see that

$$\limsup|\pi^2\bar{\sigma}_{m,n}(x,y)-D'_{\varepsilon,\eta}|$$

is finite for almost every point (x,y). We shall prove that this limit is equal to 0 almost everywhere. For this purpose we set $f(x,y)=f_1(x,y)+f_2(x,y)$, where $f_1(x,y)$ is a trigonometric polynomial of two variables and $f_2(x,y)$ satisfies the condition

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f_2(x,y)|^p\,dx\,dy\leqslant\delta^p, \tag{4.17}$$

δ denoting a sufficiently small number. Obviously

$$\bar{\sigma}_{m,n}(x,y;f)=\bar{\sigma}_{m,n}(x,y;f_1)+\bar{\sigma}_{m,n}(x,y;f_2)$$

and

$$|\pi^2\bar{\sigma}_{m,n}(f)-D'_{\varepsilon,\eta}(f)|\leqslant|\pi^2\bar{\sigma}_{m,n}(f_1)-D'_{\varepsilon,\eta}(f_1)|+|\pi^2\bar{\sigma}_{m,n}(f_2)-D'_{\varepsilon,\eta}(f_2)|=K_1+K_2$$

say. Clearly, $\pi^2\sigma_{m,n}(f_1)$ tends uniformly to the integral we obtain from (1.8) by replacing there $f(x,y)$ by $f_1(x,y)$. Let us denote this integral by I. Since

$$|f(x+u,y+v)-f(x-u,y+v)-f(x+u,y-v)+f(x-u,y-v)|\leqslant\text{Const}\,|uv|$$

the integral I converges absolutely, and (even uniformly)

$$D'_{\varepsilon,\eta}(f_1)-\pi^2I\to 0$$

so that $K_1\to 0$. By the inequality (4.15), $K_2\leqslant h_2(x,y)$, where $h_2(x,y)$ is a function derived from $f_2(x,y)$ in the same way as $h(x,y)$ was derived from $f(x,y)$. Using the formulas (4.15) and (4.17) we see that the measure of the set where $h_2(x,y)>\delta^{1/2}$ is less than $B_p\delta^{p/2}$, where B_p is a constant depending only on p. Hence the measure of the set in which $|\pi^2\bar{\sigma}_{m,n}(f_2)-D'_{\varepsilon,\eta}(f_2)|>\delta^{1/2}$ for all m and n sufficiently large is less than $2B_p\delta^{p/2}$. Since δ may be arbitrarily small, we infer that $\lim|\pi^2\bar{\sigma}_{m,n}(f)-D'_{\varepsilon,\eta}(f)|=0$ at almost every point (x,y). Lemma 3 and the relations (4.4) thus give the required result

$$\lim_{\varepsilon,\eta\to 0}\int_{-\pi}^{(\varepsilon)\pi}\int_{-\pi}^{(\eta)\pi}\frac{f(x+u,y+v)}{2\tan\frac{1}{2}u\cdot 2\tan\frac{1}{2}v}\,du\,dv=\tilde{f}(x,y)$$

at almost every point (x,y).

From (4.14) and from the fact that the function $\sigma(x,y)=\sup_{m,n}|\sigma_{m,n}(x,y)|$ satisfies the inequality

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sigma^p(x,y)\,dx\,dy\leqslant A_p\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^p\,dx\,dy$$ [10])

easily follows the following

Theorem 3. *If $f(x,y)\in L^p$, $p>1$, then the function*

$$\gamma(x,y)=\sup_{0<\varepsilon,\eta\leqslant\pi}\left|\int_{-\pi}^{(\varepsilon)\pi}\int_{-\pi}^{(\eta)\pi}\frac{f(x+u,y+v)}{2\tan\frac{1}{2}u\cdot 2\tan\frac{1}{2}v}\,du\,dv\right|$$

satisfies the inequality

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\gamma^p(x,y)\,dx\,dy\leqslant A_p\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^p\,dx\,dy.$$

This theorem also proves the inequality of Theorem 1.

5. The proof of Theorem 2 is analogous to that of Theorem 1, and we may, therefore, be more concise now. Instead of Lemmas 1, 2, 3 we shall use the following lemmas:

[10]) See the paper quoted in footnote 7, p. 234.

Lemma 9 [11]. *If $f(x)$ is of the class $L^{1,k}$, $k \geqslant 1$, then $\bar{f}(x)$ belongs to the class $L^{1,k-1}$, and*

$$\int_{-\pi}^{\pi} |\bar{f}(x)| \{\log^+ |\bar{f}(x)|\}^{k-1} dx \leqslant A_k \int_{-\pi}^{\pi} |f(x)| \{\log^+ |f(x)|\}^k dx + B_k.$$

Lemma 10 [12]. *If $f(x,y) \in L^{1,k}$, where $k \geqslant 2$, then $M(x,y;f) \in L^{1,k-2}$ and*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} M(x,y) \{\log^+ |M(x,y)|\}^{k-2} dx\,dy \leqslant$$
$$\leqslant A_k \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x,y)| \{\log^+ |f(x,y)|\}^k dx\,dy + B_k.$$

Lemma 11 [13]. *The Fourier series of a function $f(x,y)$ of the class $L^{1,1}$ is summable almost everywhere by the method of the first arithmetic mean.*

The following lemma, in which we use the notation of Lemma 6, is an analogue of the latter.

Lemma 12. *If $f(x,y) \in L^{1,k}$, $k \geqslant 1$, then* [14]

$$\int_{-\pi}^{\pi} \tilde{f}_1(x,y) \{\log^+ \tilde{f}_1(x,y)\}^{k-1} dx \leqslant A_k \int_{-\pi}^{\pi} |f(x,y)| \{\log^+ |f(x,y)|\}^k dx + B_k,$$

$$\int_{-\pi}^{\pi} \tilde{f}_2(x,y) \{\log^+ \tilde{f}_2(x,y)\}^{k-1} dy \leqslant A_k \int_{-\pi}^{\pi} |f(x,y)| \{\log^+ |f(x,y)|\}^k dy + B_k.$$

Lemma 13 [15]. *If $f(x,y) \in L^{1,k}$, $k \geqslant 1$ then the functions $M_1(x,y;f)$ and $M_2(x,y;f)$ of § 2 belong to the class $L^{1,k-1}$.*

Passing to the proof of Theorem 2, we show that under the hypotheses of that theorem the function $\bar{f}(x,y)$ belongs to the class $L^{1,k-2}$. In fact, if we treat $f(x,y)$ as a function of a single variable x, and form the conjugate function (we denoted the latter by $^*f(x,y)$), then, on account of Lemma 9, $^*f(x,y)$ belongs for almost every y to the class $L^{1,k-1}$ with respect to x. Integrating the inequality of

[11] TS, p. 150.
[12] See Jessen, Marcinkiewicz and Zygmund, loc. cit.
[13] See Jessen, Marcinkiewicz and Zygmund, loc. cit., p. 230.
[14] TS, p. 250.
[15] See Hardy, Littlewood and Pólya, *Inequalities*, Cambridge, 1934, p. 291, and Jessen, Marcinkiewicz and Zygmund, loc. cit.

Lemma 9 with respect to y, we come to the conclusion that $^*f(x,y) \in L^{1,k-1}$. We now treat $^*f(x,y)$ as a function of the sole variable y; the function conjugate to $^*f(x,y)$ with respect to y will be $\bar{f}(x,y)$. On account of Lemma 9, $\bar{f}(x,y)$ is of the class $L^{1,k-2}$. Since $k \geqslant 3$, and in view of Lemma 11, the Fourier series of $\bar{f}(x,y)$ will be summable by the method of the first arithmetic mean.

The estimation of the terms of the right-hand side of the inequality (4.6) leads to the inequality (4.14). By Lemmas 10, 12, 13, all the functions on the right of (4.14) are of the class $L^{1,k-2}$. We shall therefore have the inequalities

$$|\pi^2 \bar{\sigma}_{m,n}(x,y) - D'_{\varepsilon,\eta}| \leqslant H(x,y)$$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} H(x,y) \{\log^+ H(x,y)\}^{k-2} dx\,dy \leqslant$$
$$\leqslant A_k \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x,y)| \{\log^+ |f(x,y)|\}^k dx\,dy + B_k.$$

Let us apply the last inequality to the function $\lambda \cdot f(x,y)$, where λ is a constant greater than 1 and such that $B/\lambda < \delta$. Then

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} H(x,y) \{\log^+ H(x,y)\}^{k-2} dx\,dy \leqslant$$
$$\leqslant A_k \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x,y)| \{\log^+ \lambda |f(x,y)|\}^k dx\,dy + \delta.$$

Repeating the previous argument we find that

$$\limsup |\pi^2 \sigma_{m,n}(x,y) - D'_{\varepsilon,\eta}| < \delta$$

almost everywhere. Since δ is arbitrary, Theorem 2 is proved.

Remark. It seems very likely that the integral (1.8) exists (in the strong sense), if $f(x,y) \in L^{1,1}$. This theorem I am unable to prove. However, an argument analogous to the proof of Theorem 2 gives the following

Theorem 4. *If $f(x,y) \in L^{1,1}$, then*

$$\bar{\sigma}_{m,n}(x,y) - \frac{1}{\pi^2} \overset{(\varepsilon)}{\int_{-\pi}^{\pi}} \overset{(\eta)}{\int_{-\pi}^{\pi}} \frac{f(x+u,y+v)}{2\tan\frac{1}{2}u \cdot 2\tan\frac{1}{2}v}\, du\,dv \to 0$$

as $\varepsilon, \eta \to 0$, and m, n satisfy the condition (4.4).

The proof of this theorem is based on the following lemmas:

Lemma 14. *If $g(x,y)$ is integrable, the inequalities*

$$\Big[\int_{-\pi}^{\pi}\{M_1(x,y;g)\}^{\alpha}\,dx\Big]^{1/\alpha}\leqslant A_\alpha\int_{-\pi}^{\pi}|g(x,y)|\,dx$$

$$\Big[\int_{-\pi}^{\pi}\{M_2(x,y;g)\}^{\alpha}\,dy\Big]^{1/\alpha}\leqslant A_\alpha\int_{-\pi}^{\pi}|g(x,y)|\,dy$$

hold for every $0<\alpha<1$ [16]).

Lemma 15. *If $f(x,y)$ is of the class $L^{1,1}$, then*

$$\Big[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\{M(x,y;f)\}^{\alpha}\,dx\,dy\Big]^{1/\alpha}\leqslant A_\alpha\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|\log^+|f(x,y)|\,dx\,dy+B_\alpha$$

for every $0<\alpha<1$ [17]).

6. In the previous arguments we confined our attention for simplicity to functions of two variables, but the foregoing results can be extended without difficulty to the case of n variables. For example, we have the following theorems:

Theorem 5. *If $f(x_1,x_2,\ldots,x_n)$ is of period 2π with respect to each of the variables, and is of the class L^p, $p>1$, the integral*

$$\bar{f}(x_1x_2,\ldots,x_n)=$$

$$(6.1)\qquad =\lim_{h_1,\ldots,h_n\to 0}\Big\{\Big(-\frac{1}{\pi}\Big)^n\int_{-\pi}^{(h_1)\ \pi}\cdots\int_{-\pi}^{(h_n)\ \pi}\frac{f(x_1+u_1,\ldots,x_n+u_n)}{2\tan\frac12u_1\ldots 2\tan\frac12u_n}\,du_1\ldots du_n$$

exists almost everywhere and is a function of the class L^p.

Theorem 6. *If $f(x_1,x_2,\ldots,x_n)$, of period 2π with respect to each variable, is of the class $L^{1,k}$, $k\geqslant n+1$, then the limit* (6.1) *exists almost everywhere and belongs to the class $L^{1,k-n}$.*

[16]) See Jessen, Marcinkiewicz and Zygmund, loc. cit., p. 222.
[17]) Loc. cit., p. 223.

7. Recently, Marcinkiewicz and Zygmund proved the following result which will be stated as

Lemma 16 [18]). *If $f(x,y)\in L$, then at almost every point (x,y) the first arithmetic means $\sigma_{m,n}(x,y)$ of the Fourier series of $f(x,y)$ satisfy the condition*

$$\sigma_{m,n}(x,y)\to f(x,y)$$

for $m,n\to\infty$, provided the ratios m/n and n/m are bounded.

The methods used in the proof of that result give the following theorem analogous to Theorem 4.

Theorem 7. *If $f(x,y)\in L$, then almost everywhere*

$$\bar{\sigma}_{m,n}(x,y)-\int_{-\pi}^{(\varepsilon)\ \pi}\int_{-\pi}^{(\eta)\ \pi}\frac{f(x+u,y+v)}{2\tan\frac12u\cdot 2\tan\frac12v}\,du\,dv\to 0$$

provided that ε and η tend to 0 in such a way that the ratios ε/η and η/ε are bounded, and m, n are defined by the conditions (4.4).

We shall not give the proof of the theorem because it would be too long. We shall however establish the following result:

Theorem 8. *Let $f(x,y)$ belong to the class $L^{1,2}$. Then*

$$\lim_{\varepsilon,\eta\to 0}\frac{1}{\pi^2}\int_{-\pi}^{(\varepsilon)\ \pi}\int_{-\pi}^{(\eta)\ \pi}\frac{f(x+u,y+v)}{2\tan\frac12u\cdot 2\tan\frac12v}\,du\,dv=\bar{f}(x,y)$$

almost everywhere, provided ε and η tend to 0 in such a way that the ratios ε/η and η/ε are bonnded.

This result becomes a direct consequence of the argument of § 5, provided we use Lemma 15. In fact, under the assumptions of Theorem 8, the function $f(x,y)$ will be of the class L and, in virtue of Lemma 15,

$$(7.1)\qquad \bar{\sigma}_{m,n}(x,y)\to\bar{f}(x,y)$$

[18]) Marcinkiewicz and Zygmund, *On the summability of double Fourier series*, Fund. Math. **32** (1939), pp. 122-132.

almost everywhere, provided the ratios m/n and n/m are bounded. Estimating the right-hand side of the inequality (4.6) and arguing as in §§ 4, 5, we come to the conclusion that

$$\lim_{m,n\to\infty}\left\{\pi^2\bar{\sigma}_{m,n}(x,y)-\int_{-\pi}^{(\varepsilon)\;\pi}\int_{-\pi}^{(\eta)\;\pi}\frac{f(x+u,y+v)}{2\tan\frac{1}{2}u\cdot 2\tan\frac{1}{2}v}\,du\,dv\right\}=0.$$

The boundedness of the ratios ε/η and η/ε, and (4.4) imply the boundedness of the ratios m/n and n/m. Comparing (7.1) and (7.2) we get the required result.

Sur l'ensemble des points singuliers d'une fonction d'une variable réelle admettant les dérivées de tous les ordres.

Par

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Introduction.

1. Le but de ce travail est de caractériser d'une manière topologique l'ensemble des points singuliers des fonctions d'une variable réelle admettant les dérivées de tous les ordres.

Etant donné un ensemble Z de nombres réels dense en soi, j'appelle *fonction de classe D_∞ sur Z* toute fonction qui admet en chaque point de Z les dérivées de tous les ordres par rapport à Z. Dans le cas où ces dernières sont finies (donc continues), la fonction sera dite *de classe C_∞ sur Z*.

En particulier, lorsque l'ensemble Z est ouvert, les classes D_∞ et C_∞ coïncident.

Enfin, j'appelle *fonctions de classe C_∞*, tout court, les fonctions qui sont de classe C_∞ sur l'axe des x tout entier.

Si $f(x)$ est une fonction de classe C_∞, on peut former, pour tout x, la série de Taylor:

$$T_f(x,h)=f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\dots$$

dont le rayon de convergence $r_f(x)$ (en tant que celui de la série en h) est défini par la formule de Cauchy-Hadamard:

$$r_f(x)=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{\dfrac{|f^{(n)}(x)|}{n!}}},$$

en posant $r_f(x)=0$ ou $r_f(x)=+\infty$, suivant que le dénominateur est infini ou s'annule.