On the cartesian product of metric spaces.

By

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In this paper I shall prove three theorems on topological properties of subsets of the cartesian product [1]) $\mathcal{X}\times\mathcal{Y}$ of two metric spaces $\mathcal{X}$ and $\mathcal{Y}$ connected with the notion of first category and with the property of Baire. This subject has already been considered by Kuratowski and Ulam [2]) in the case where the space $\mathcal{Y}$ is separable. In this paper it is assumed merely that $\mathcal{Y}$ is a *locally separable* [3]), metric space and $\mathcal{X}$ is an arbitrary metric space.

The first theorem gives a formula for the set of all points at which the set $X\times Y\subset\mathcal{X}\times\mathcal{Y}$ is of second category [4]) in $\mathcal{X}\times\mathcal{Y}$. The second theorem gives a necessary and sufficient condition for the set $X\times Y$ to possess the property of Baire in $\mathcal{X}\times\mathcal{Y}$. The third is a generalization of Kuratowski's theorem on the geometric image of an arbitrary function $y=f(x)$ (which is an immediate consequence) [5]).

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[1]) The cartesian product $\mathcal{X}\times\mathcal{Y}$ of two metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is the set of all ordered pairs of elements (x,y) for which $x\in\mathcal{X}$ and $y\in\mathcal{Y}$. The distance between two pairs (x_1,y_1) and (x_2,y_2) is by definition: $\sqrt{|x_1-x_2|^2+|y_1-y_2|^2}$ where $|x_1-x_2|$ denotes the distance between x_1 and x_2 and $|y_1-y_2|$ the distance between y_1 and y_2.

[2]) C. Kuratowski et St. Ulam, [1].

[3]) A space $\mathcal{Z}$ is *locally separable* if every point $z\in\mathcal{Z}$ is contained in an open set U which considered as a space is separable.

[4]) A subset Z of a metric space $\mathcal{Z}$ is *of first category at a point* $z\in\mathcal{Z}$ if there exists an open set $U\subset\mathcal{Z}$ such that $z\in U$ and that the set UZ is of first category in $\mathcal{Z}$, i. e. that UZ is the sum of an enumerable sequence of nowhere dense sets. (When $\mathcal{Z}=\mathcal{X}\times\mathcal{Y}$ the set U may always be supposed of the form $U=G\times H$ where G and H are two open subsets of $\mathcal{X}$ and $\mathcal{Y}$ respectively). The set Z is *of second category at a point* $z\in\mathcal{Z}$ if it is not of first category at z.

[5]) *Corollary*, p. 292.

If $\mathcal{Z}$ is a metric space and $Z\subset\mathcal{Z}$, the symbol $D_{\mathcal{Z}}(Z)$ denotes the set of all points $z\in\mathcal{Z}$ at which the set Z is of second category [6]). The subscript $\mathcal{Z}$ in the symbol $D_{\mathcal{Z}}(Z)$ will be omitted when there is no doubt what space is under consideration. In this paper x and X denote always points and subsets of the space $\mathcal{X}$ and y and Y those of $\mathcal{Y}$. So $D(X)$, $D(Y)$, $D(X\times Y)$ denote the sets of all points of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}\times\mathcal{Y}$ respectively at which the sets X, Y and $X\times Y$ are of second category (in $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}\times\mathcal{Y}$ respectively).

Theorem 1. $D(X\times Y)=D(X)\times D(Y)$.

If (x,y) *non* $\epsilon\, D(X\times Y)$, there exist two open (in $\mathcal{X}$ and $\mathcal{Y}$ respectively) sets: $G\subset\mathcal{X}$ and $H\subset\mathcal{Y}$ such that $(x,y)\in G\times H$ and that the set $(X\times Y)(G\times H)=XG\times YH$ is of first category in the space $\mathcal{X}\times\mathcal{Y}$. Since $\mathcal{Y}$ is locally separable we may suppose that the set H is separable. Since $G\times H$ is open in $\mathcal{X}\times\mathcal{Y}$ the set $XG\times YH$ is also of first category in the space $G\times H$ [7]). As H is separable, either XG is of first category in G or YH is of first category in H [8]). In the first case the set XG is of first category in the space $\mathcal{X}$, hence x *non* $\epsilon\, D(X)$; in the second case YH is of first category in $\mathcal{Y}$, hence y *non* $\epsilon\, D(Y)$. Thus in each case (x,y) *non* $\epsilon\, D(X)\times D(Y)$. Therefore:

$$D(X)\times D(Y)\subset D(X\times Y).$$

On the other hand, if (x,y) *non* $\epsilon\, D(X)\times D(Y)$, then either x *non* $\epsilon\, D(X)$ or y *non* $\epsilon\, D(Y)$. In the first case there exists an open subset G of the space $\mathcal{X}$ such that $x\in G$ and GX is of first category in $\mathcal{X}$. The set $GX\times\mathcal{Y}$ being of first category in $\mathcal{X}\times\mathcal{Y}$ [9]), the set $(G\times\mathcal{Y})(X\times Y)$ as a subset of $GX\times\mathcal{Y}$ is of first category in $\mathcal{X}\times\mathcal{Y}$ too. Since $G\times\mathcal{Y}$ is open in $\mathcal{X}\times\mathcal{Y}$ and $(x,y)\in G\times\mathcal{Y}$, so (x,y) *non* $\epsilon\, D(X\times Y)$. Analogously we can prove that in the second case also (x,y) *non* $\epsilon\, D(X\times Y)$. Therefore:

$$D(X\times Y)\subset D(X)\times D(Y).$$

Thus theorem 1 is proved.

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[6]) See C. Kuratowski [1], p. 45.

[7]) See C. Kuratowski [1], p. 45.

[8]) If H is separable, a set $A\times B\subset G\times H$ is of first category in $G\times H$ if and only if one of the sets A and B is of first category in G or H respectively. See C. Kuratowski [1], p. 139.

[9]) See C. Kuratowski [1], p. 140.

Corollary. *The set $X\times Y$ is of first category in $\mathfrak{X}\times\mathcal{Y}$ if and only if one of the sets X and Y is of first category in $\mathfrak{X}$ or $\mathcal{Y}$ respectively.*

Theorem 2. *The set $X\times Y$ possesses the property of Baire*[10] *in the space $\mathfrak{X}\times\mathcal{Y}$ if and only if either X and Y possess the property of Baire or one of the sets X, Y is of the first category (in $\mathfrak{X}$ or $\mathcal{Y}$ respectively)*[11].

The set $X\times Y$ possesses the property of Baire in $\mathfrak{X}\times\mathcal{Y}$ if and only if the set

$$A=D(X\times Y)\cdot D(\mathfrak{X}\times\mathcal{Y}-X\times Y)$$

is nowhere dense in $\mathfrak{X}\times\mathcal{Y}$. From theorem 1 it follows easily that

$$\text{(i)}\qquad A=[D(X)\cdot D(\mathfrak{X}-X)]\times D(Y)+D(X)\times[D(Y)\cdot D(\mathcal{Y}-Y)].$$

If $X\times Y$ possesses the property of Baire in $\mathfrak{X}\times\mathcal{Y}$ and $D(X)\neq 0\neq D(Y)$, it follows from (i) that the sets $D(X)\cdot D(\mathfrak{X}-X)$ and $D(Y)\cdot D(\mathcal{Y}-Y)$ are nowhere dense in $\mathfrak{X}$ and $\mathcal{Y}$ respectively[12]. This means that X possesses the property of Baire in $\mathfrak{X}$ and Y in $\mathcal{Y}$. The condition of the theorem 2 is thus necessary.

It is also sufficient. For, if either X or Y is of first category in $\mathfrak{X}$ and $\mathcal{Y}$ respectively, then $A=0$; if X and Y possess the property of Baire, the sets $D(X)\cdot D(\mathfrak{X}-X)$ and $D(Y)\cdot D(\mathcal{Y}-Y)$ are nowhere dense in $\mathfrak{X}$ and $\mathcal{Y}$ and the set A is nowhere dense in $\mathfrak{X}\times\mathcal{Y}$.

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[10]) A subset Z of a topological space $\mathcal{Z}$ possesses *the property of Baire* in $\mathcal{Z}$ if it can be represented in the form $Z=U+M-N$ where the set U is open in $\mathcal{Z}$ and the sets M and N are of first category in $\mathcal{Z}$. The set Z possesses the property of Baire in $\mathcal{Z}$ if and only if $D_{\mathcal{Z}}(Z)\cdot D_{\mathcal{Z}}(\mathcal{Z}-Z)$ is nowhere dense in $\mathcal{Z}$. See C. Kuratowski [1], p. 51.

[11]) Kuratowski has formulated earlier a necessary condition and a sufficient one for the set $X\times Y$ to possess the property of Baire in $\mathfrak{X}\times\mathcal{Y}$. See C. Kuratowski [1], p. 140.

[12]) A cartesian product of two sets is nowhere dense if and only if one of the two sets is nowhere dense. If $D_{\mathcal{Z}}(Z)\neq 0$, it is not nowhere dense in $\mathcal{Z}$. See C. Kuratowski [1], p. 137 and p. 47.

Theorem 3. *Let P be a set of first category in $\mathfrak{X}$ $(P\subset\mathfrak{X})$. If $I\subset\mathfrak{X}\times\mathcal{Y}$ and if for every $x\in\mathfrak{X}-P$ the set $I\cdot(x\times\mathcal{Y})$ is of first category in $x\times\mathcal{Y}$, then*

$$D(\mathfrak{X}\times\mathcal{Y}-I)=D(\mathfrak{X}\times\mathcal{Y}).$$

If in addition I possesses the property of Baire, it is of first category in $\mathfrak{X}\times\mathcal{Y}$[13].

In order to prove the first part of the theorem it is sufficient to show that the condition:

$$(x,y)\ \text{non}\in D(\mathfrak{X}\times\mathcal{Y}-I)\ \text{implies:}\ (x,y)\ \text{non}\in D(\mathfrak{X}\times\mathcal{Y}).$$

If $(x,y)\ \text{non}\in D(\mathfrak{X}\times\mathcal{Y}-I)$, there exist two open subsets: G and H of $\mathfrak{X}$ and $\mathcal{Y}$ respectively such that $(x,y)\in G\times H$ and such that the set $(G\times H)(\mathfrak{X}\times\mathcal{Y}-I)=G\times H-I$ is of first category in $\mathfrak{X}\times\mathcal{Y}$, hence in $G\times H$. As $\mathcal{Y}$ is locally separable, we may suppose that H is separable. There exists[14] a set $Q\subset G$ which is of first category in G such that for every $x\in G-Q$ the set $(x\times H)\cdot(G\times H-I)=x\times H-I$ is of first category in $x\times H$, hence in $x\times\mathcal{Y}$ too. There are two cases to be considered. If $G=Q+PG$, the set G is of first category in $\mathfrak{X}$ and $(x,y)\ \text{non}\in D(\mathfrak{X}\times\mathcal{Y})$, as $G\times H$ is of first category in $\mathfrak{X}\times\mathcal{Y}$. If $G-(Q+P)\neq 0$, let $x_0\in G-(Q+P)$. Since $x_0\in G-Q$, the set $x_0\times H-I$ is of first category in $x_0\times\mathcal{Y}$ and since $x_0\in\mathfrak{X}-P$, the set $(x_0\times H)I$, as a subset of $(x_0\times\mathcal{Y})\cdot I$, is of first category in $x_0\times\mathcal{Y}$ too. Hence $x_0\times H$ is of first category in $x_0\times\mathcal{Y}$, as $x_0\times H=x_0\times H-I+(x_0\times H)\cdot I$. Consequently H is of first category in $\mathcal{Y}$ and $G\times H$ is of first category in $\mathfrak{X}\times\mathcal{Y}$. Thus in this case also $(x,y)\ \text{non}\in D(\mathfrak{X}\times\mathcal{Y})$.

The second part of the theorem follows from the first. Since

$$D(I)=D(I)\cdot D(\mathfrak{X}\times\mathcal{Y})=D(I)\cdot D(\mathfrak{X}\times\mathcal{Y}-I)$$

and I possesses the property of Baire, the set $D(I)$ is nowhere dense in $\mathfrak{X}\times\mathcal{Y}$ and thus is empty.

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[13]) On the other hand: *if $\mathcal{Y}$ is separable and I is of first category in $\mathfrak{X}\times\mathcal{Y}$, the set $I(x\times\mathcal{Y})$ is of first category in $x\times\mathcal{Y}$ for all points $x\in\mathfrak{X}$ except a set P of first category in $\mathfrak{X}$.* The assumption that $\mathcal{Y}$ is separable is essential. See C. Kuratowski et St. Ulam [1], pp. 248—249 and C. Kuratowski [1], p. 139.

[14]) This follows from the theorem cited in the footenote [13]).

Corollary [15]. *Let $y=f(x)$ be an arbitrary mapping of $\mathcal{X}$ into a subset of $\mathcal{Y}$ and let $I=\underset{xy}{E}[y=f(x)]$. If $\mathcal{Y}$ is dense in itself, then:*

$$D(\mathcal{X}\times\mathcal{Y}-I)=D(\mathcal{X}\times\mathcal{Y}).$$

If in addition I possesses the property of Baire, it is of first category in $\mathcal{X}\times\mathcal{Y}$.

The question whether the theorems 1, 2 and 3 are true also in the case where $\mathcal{Y}$ is an arbitrary metric space remains unsettled.

References.

C. Kuratowski [1]. *Topologie I.* Warszawa-Lwów (1933).

C. Kuratowski [2]. *Sur les fonctions représentables analytiquement et les ensembles de première catégorie.* Fund. Math. **5** (1924), pp. 75-86.

C. Kuratowski et St. Ulam [1]. *Quelques propriétés topologiques du produit combinatoire.* Fund. Math. **19** (1932), pp. 247-251.

[15]) See C. Kuratowski [2], p. 84, C. Kuratowski et St. Ulam [1], p. 250 and C. Kuratowski [1], p. 143.

Une mesurabilité moyenne pour les ensembles de points.

Par

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La note présente a pour l'objet de proposer en grandes lignes une mesurabilité moyenne pour les ensembles de l'espace euclidien R à k dimensions.

Avant d'aborder le développement même de cette théorie, nous allons exposer d'une manière approximative les intuitions et les notions sur lesquelles repose essentiellement la définition de la mesurabilité moyenne. En même temps, quelques notions auxiliaires et les notations seront introduites.

$P=P(\xi_i)$ désignant des points de R et ξ_i $(i=1,2,...,k)$ leurs coordonnées cartésiennes, soit W un cube (à k dimensions) formé des points $P(\xi_i)$ dont les coordonnées satisfont à la condition

$$0\leqslant\xi_i<\omega \qquad (i=1,2,...,k).$$

Nous entendrons par *translation X de W en lui-même* toute transformation biunivoque de $W=X(W)$ en lui-même définie par la formule

$$X[P(\xi_i)]=P(\xi_i+\tau_i)$$

où toutes les additions de coordonnées sont entendues mod ω. Les nombres τ_i, fixes pour la translation X, s'appelleront *composantes* de cette translation.

Or, soit $A\subset W$ un sous-ensemble de W mesurable (L) de mesure $L(A)$.

Par n translations X_ν $(\nu=1,2,...,n)$ de W en lui-même, on obtient de A n ensembles $A_\nu=X_\nu(A)$ qui en sont des images égales à la fois „translativement” et en mesure.