11. 4

icm[©]

Application. Supposons qu'on mesure l'angle α en secondes. Pour quelles valeurs entières de α obtiendra-t-on les valeurs de $\cos \alpha$ exprimables par des radicaux réels?

L'angle 2π étant égal à $2^7 \cdot 3^4 \cdot 5^8$ secondes, on voit que α doit être un multiple de $2^7 \cdot 3^4 \cdot 5^8/n_0$ où n_0 est le plus grand des diviseurs de $2^7 \cdot 3^4 \cdot 5^8$ pour lesquels $\varphi(n_0)$ est une puissance de 2. Évidemment $n_0 = 2^7 \cdot 3 \cdot 5$, donc les α cherchés sont des multiples de 675", c'est-à-dire de 11'15".

A PROOF THAT em IS IRRATIONAL

BY

Z. BUTLEWSKI (POZNAŃ)

Niven has given a simple proof that n is irrational 1). Using the same method 2) we give here a similarly simple proof that e^m is irrational for any positive integer m 3).

We define the polynomials:

$$f(x) = \frac{(m^2 - x^2)^n}{n!},$$

$$F(x) = f(x) - f'(x) + f''(x) - \dots + (-1)^n f^{(2n)}(x),$$

where n is a positive integer. Then $f(m) = f'(m) = \dots = f^{(n-1)}(m) = 0$, and $f^{(n)}(x), f^{(n+1)}(x), \dots, f^{(2n)}(x)$ have integral values for x = m. Since f(x) = f(-x) and $f^{(i)}(x) = (-1)^{i}f(-x)$, we have $f(-m) = f'(-m) = \dots = f^{(n-1)}(-m) = 0$, and $f^{(n)}(x), f^{(n+1)}(x), \dots, f^{(2n)}(x)$ assume integral values for x = -m. Thus, F(m) and F(-m) are integers. By elementary calculus we have:

$$\frac{d}{dx}[e^{x}F(x)] = e^{x}[F(x) + F'(x)] = e^{x}f(x).$$

Suppose that $e^m = \frac{a_m}{b_m}$ (for some positive integer m), where a_m and b_m are positive integers.

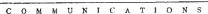
We have:

(*)
$$a_m b_m \int_{-m}^{+m} e^x f(x) dx = a_m b_m [e^x F(x)]_{-m}^{+m} = a_m^2 F(m) - b_m^2 F(-m).$$

¹) Ivan Niven, A simple proof that π is irrational, Bulletin of the American Mathematical Society 53 (1947), p. 509.

²⁾ This method is analogous to that used by Hurwitz for the proof that e is transcendental. Cf. E. Goursat, Cours d'Analyse Mathématique, I, 3-me édition, Paris, 1917 p. 210.

³⁾ G. H. Hardy and E. M. Wright write in An Introduction to the Theory of Numbers (Oxford 1938, p. 47): "There are other special proofs of the irrationality of e, e^2 and e^3 , but it is not much easier to prove e^m irrational, for arbitrary integral m, than it is to prove the full theorem of 11.13". This theorem says: "e is transcendental".



198

Now $a_m^2 F(m) - b_m^2 F(-m)$ is an integer. But for -m < x < m:

$$0 < a_m b_m e^x f(x) < \frac{a_m^2 m^{2n}}{n!},$$

so that the integral in (*) is positive, but arbitrarily small for n sufficiently large. Thus (*) is false, hence e^m (for m=1,2,3,...) is irrational.



CONCERNING THE SYMMETRIC DIFFERENCE IN THE THEORY OF SETS AND IN BOOLEAN ALGEBRAS

R.

E. MARCZEWSKI (WROCŁAW)

The set (A-B)+(B-A)=A+B-AB, i. e. the set of all points which belong to one and only one of the sets A and B, is called the symmetric difference of A and B and will denote in this note by A ildes B. The classical books on the General Theory of Sets by Hausdorff, Sierpiński etc. did not treat the symmetric difference 1), but this operation has proved useful in a number of recent papers. It has been applied especially to two fields: 1° in measure theory, the distance of two sets can be defined as the measure of their symmetric difference (Nikodym and Aronszajn) 2), 2° the class of all subsets of a set (or, more generally, any Boolean algebra) forms a ring in the algebraic sense under the operations symmetric difference and multiplication 3); in particular, symmetric difference is a group operation.

Some investigations in the measure theory suggest a partial analogy between the symmetric difference of sets and the absolute value of the difference of numbers. This idea leads (§ 1) to a new formulation of Kantorovitch-Livenson definition of quasi-analytical operations 4).

In § 2 I show that the symmetric difference is the only group operation among the binary quasi-analytical operations.

¹⁾ Hausdorff treats the symmetric difference only in the last edition of his book (F. Hausdorff, Mengenlehre, Berlin-Leipzig 1935, Ergänzungen, p. 276-278).

²) See e. g. O. Nikodym, Sur une généralisation des intégrales de M. J. Radon, Fundamenta Mathematicae 15 (1930), p. 131-179, especially p. 137, Definition 4.

³⁾ See e. g. G. Birkhoff, Lattice Theory, American Mathematical Society Colloquium Publications 25, New York 1940, p. 96.

⁴⁾ L. Kantorovitch and E. Livenson, Memoir on the Analytical Operations and Projective Sets I, Fundamenta Mathematicae 18 (1932), p. 214-279, especially p. 239, Definition 1'.