ON LINEAR FUNCTIONALS IN ABELIAN GROUPS

BY

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We shall prove a theorem from which a lemma of Alexiewicz [1]) and an other application may be deduced.

1. Let E be a set with given convergence of Fréchet type, i. e. such that, x_n and x being elements of E,

1^0 $x_n = x$ implies $x_n \to x$;

2^0 $x_n \to x$ implies $x_{k_n} \to x$ for any increasing sequence $\{k_n\}$ of positive integers.

A subset G of E will be called *totally dense* if, given any sequence of elements $x_n \neq x$ of E, convergent to $x \epsilon E$, there is a double sequence $\{x_{nk}\}$ of elements of G such that

(i) $x_{nk} \overset{k}{\to} x_n$,

(ii) for every sequence $\{l_i\}$ of positive integers there exist two other $\{n_i\}$ and $\{k_i\}$ such that $n_{i+1} > n_i$, $k_i > l_{n_i}$ and $x_{n_i k_i} \overset{i}{\to} x$.

Each totally dense set is dense. Denote generally by $\overline{X}$ the closure of X (i. e. the set of all points and limit points of X). If E satisfies 1^0, 2^0, and moreover

3^0 $\overline{\overline{G}} = \overline{G}$ for any $G \subset E$,

then each dense subset G of E (i. e. such that $\overline{G} = E$) is evidently totally dense.

Now, suppose E to be an additive abelian group such that $x_n \to x$ and $y_n \to y$ imply $(x_n - y_n) \to (x - y)$; then also $(x_n + y_n) \to (x + y)$.

A functional $F(x)$ defined on E is *additive* if

$$F(x_1 + x_2) = F(x_1) + F(x_2);$$

it is *continuous* if $F(x_n) \to F(x)$ for every sequence $\{x_n\}$ convergent to x; it is *linear* if it is additive and continuous.

[1]) A. Alexiewicz, *Linear functionals on Denjoy-integrable functions*, this fascicle p. 289-293.

Theorem. Each linear functional in a totally dense subgroup G of E may be linearly extended on E; this extension is unique.

Proof. Let G be a totally dense subgroup of E. If $g_n, h_n \epsilon G$ and $g_n, h_n \to x \epsilon E$, then

$$F(g_n) - F(h_n) = F(g_n - h_n) \to 0;$$

in particular it holds if $\{h_n\}$ is any subsequence of $\{g_n\}$. Thus, the limit of $F(g_n)$ exists and is unique.

Put $F(x) = \lim\limits_n F(g_n)$. The additivity of $F(x)$ in E is evident. To prove that $F(x)$ is linear in E, it is sufficient to show that, given any sequence of elements $x_n \neq x$ of E, convergent to x, there exists a subsequence $\{x_{n_i}\}$ such that $F(x_{n_i}) \overset{i}{\to} F(x)$.

There is a double sequence $K = \{x_{nk}\}$ such that $x_{nk} \overset{k}{\to} x_n$ and

$$|F(x_{nk}) - F(x)| < 1/n \qquad (n, k = 1, 2, \ldots).$$

Since $x \epsilon \overline{K}$, there is a sequence $x_{n_i k_i}$ of elements of K such that $n_i \to \infty$ and $x_{n_i k_i} \overset{i}{\to} x$. Then $F(x_{n_i}) \overset{i}{\to} F(x)$.

2. Now, let E be the space of the measurable and bounded functions in a given closed interval $\langle a, b \rangle$ with the usual definition of addition. Let a sequence $\{x_n\}$ be called *convergent to x* if it is uniformly bounded, asymptotically convergent to x and convergent in the usual sense at b. Then the set G of the ACG-functions [2]) in $\langle a, b \rangle$ vanishing at a is totally dense in E, and the lemma of Alexiewicz and consequently his theorem 2 (ibidem) follow from the above theorem.

3. If the same convergence is considered, but without any supposition on the convergence at b, the set G is also totally dense. In this case the general form of linear functionals in E is, by the theorem of Fichtenholz [3]),

$$F(y) = \int_a^b y(t) g(t) dt,$$

[2]) See S. Saks, *Theory of the Integral*, Monografie Matematyczne, Warszawa 1937, p. 223.

[3]) G. Fichtenholz, *Sur les fonctionnelles linéaires continues au sens généralisé*, Recueil Mathématique de l'Académie des Sciences de l'URSS 4 (1938), p. 193-214, especially p. 200.

where $y \epsilon E$, and g is a Lebesgue-integrable function. Thus it is also the general form of linear functionals in G. The integration by parts gives

$$F(y) = (D)\int_a^b y'(t)[\dot{g}(b) - \dot{g}(t)]\,dt,$$

where $\dot{g}(t) = \int_a^t g(t)\,dt$ is an absolutely continuous function.

Let (D) be the space of Denjoy-integrable functions in $\langle a, b \rangle$ with the usual definition of addition. A sequence $\{x_n\}$ of elements of (D) will be called *convergent to* x if the sequence $\left\{(D)\int_a^t x_n(t)\,dt\right\}$ is uniformly bounded and asymptotically convergent to x. Then we obtain the following theorem:

The general form of linear functionals in (D) *is*

$$F(x) = (D)\int_a^b x(t)h(t)\,dt,$$

where $h(t)$ *is any absolutely continuous function vanishing at* b.

SUR LES PRESQUE-PÉRIODES DES FONCTIONS PÉRIODIQUES

PAR

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I. Le problème (S. Hartman). Désignons par $\delta_f(\varepsilon)$ le *ε-module de continuité uniforme* de la fonction $f(x)$ uniformément continue, c'est-à-dire la borne supérieure des nombres $\delta > 0$ pour lesquels

$$|x' - x''| < \delta \quad \text{entraîne} \quad |f(x') - f(x'')| < \varepsilon.$$

Appelons *ε-presque-période* d'une fonction continue $f(x)$ — et désignons par $\tau_f(\varepsilon)$ — tout nombre pour lequel on a

$$|f[x + \tau_f(\varepsilon)] - f(x)| < \varepsilon$$

quel que soit x.

Les fonctions $f(x)$ sont supposées définies pour tout x réel.

La notion de presque-période est étudiée d'habitude pour les fonctions presque périodiques, mais elle se montre utile dans l'étude des fonctions périodiques, où elle permet, par exemple, de simplifier les démonstrations de certains théorèmes. Tel est, entre autres, le théorème suivant de Gottschalk [1]):

Quels que soient les nombres réels t, $a_1, \ldots, a_k$ non nuls et δ positif, il existe un ensemble relativement dense [2]) *N de nombres entiers tel que $n \epsilon N$ entraîne l'existence des entiers $m_1, \ldots, m_k$ pour lesquels on a*

$$|nt - m_i a_i| < \delta \qquad (i = 1, \ldots, k). \tag{1}$$

Faisons d'abord deux remarques qui trouveront leur application aussi dans la partie III de ce travail.

Soit $f(x)$ une fonction non-constante, périodique et continue. Désignons par ω_f sa *période minimum*, c'est-à-dire la plus petite période positive.

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[1]) W. H. Gottschalk, *Almost-periodicity, Equi-continuity and Total-boundedness*, Bulletin of the American Mathematical Society 52 (1946), p. 633-636.

[2]) c'est-à-dire que l'ensemble des différences entre ses éléments voisins est borné.