Duality theorems.

By

W. Mayer (Princeton N. J.).

§ 1. Introductory remarks and the Poincaré Duality Theorem.

The duality theorems *for a manifold* will be derived from the corresponding ones *for nets and co-nets of group systems* (Mayer [1], II, § 4)[1]. Of the two types of duality theorems — the Poincaré and the Alexander duality theorems — the first could be obtained more easily from the corresponding one for a *group system and its character system* (Mayer [1], I, § 5). But when applied to the Alexander type this method fails. Also in using the net and co-net theory for both types, the less complicated Poincaré theorem serves as an introduction for the Alexander theorem. We shall state the duality theorems in terms of *singular homology groups*.

Notations. $|\mathfrak{M}_N|$ denotes a manifold of dimension N with $\mathfrak{M}_N$ a simplicial subdivision of $|\mathfrak{M}_N|$. (Thus $|\mathfrak{M}_N| = |\mathfrak{M}'_N|$ for $\mathfrak{M}_N$, $\mathfrak{M}'_N$, simplicial subdivisions). Σ_λ indexed by $\Lambda = \{\lambda\}$, *is the symbol for a finite open covering* of $|\mathfrak{M}_N|$ (with elements (open sets) $\mathfrak{U}_{\lambda_1}, \ldots, \mathfrak{U}_{\lambda_\nu}$) *and also for the nerve of the covering.* The stars of the vertices of an $\mathfrak{M}_N$ constitute such a covering with $\mathfrak{M}_N$ itself as the nerve. We shall call it the *„star covering* $\mathfrak{M}_N$". We note the important fact that *the aggregate of the star coverings is co-final in* $\{\Sigma_\lambda\}$ *the set of all finite open coverings of* $|\mathfrak{M}_N|$.

Let H compact and G discrete be groups dually paired to K, the groups of reals modulo 1[2]). Then the nerve Σ_λ taken with H

[1]) The numbers in brackets refer to the bibliography at the end of the paper.

[2]) As an alternative let H and G be finite-dimensional vector spaces with linear topology over the discrete field K and dually paired to K. (Then both are discrete and linearly compact, Lefschetz [3] (11, (25.6), (27.7).)

as coefficient group is compact and if σ_i^p, $i=1,\ldots,\lambda_p$ are the p-simplexes of Σ_λ then $g\bar{\sigma}_i^{-p}$ with $g \epsilon G$ may symbolize the $(-p)$-character for which

$$(1.1)\qquad g\bar{\sigma}_i^{-p} \circ h\sigma_j^p = gh\delta_{ij}, \quad h \epsilon H, \quad gh \epsilon K.$$

The general element of the chain group of dimension $(-p)$ of Σ_λ^K (the K-character system of Σ_λ) is a sum: $\Sigma\, g_i \bar{\sigma}_i^{-p}$, $g_i \epsilon G$.

We are now prepared to use the result of Mayer [1], (II, § 2): The inverse and direct limit-groups

$$(1.2)\qquad B_i(\Sigma) = \lim_{\leftarrow} \{B_i(\Sigma_\lambda); \pi_\mu^\lambda\},$$

$$(1.3)\qquad B_{-i}(\Sigma^K) = \lim_{\rightarrow} \{B_{-i}(\Sigma_\lambda^K); \pi_\mu^\lambda\},$$

are dually paired to K (in intersection $\varphi^{-i} \circ a^i = \varphi_\lambda^{-i} \circ a_\lambda^i$)[3]).

Remark. $B_j(\ )$ at right in (1.2), (1.3) denotes *the ordinary homology group* of the nerve Σ_λ and its K-character system Σ_λ^K respectively, but at the left is symbolizes *limit groups* (i. e. Čech's homology groups) of those nerves.

The dual pairing of the preceding limit groups holds for the net and co-nets connected with any space $\mathfrak{S}$, the nerves of its finite open coverings taken with a compact coefficient group H. For $\mathfrak{S}$ a finite complex (as is well known) $B_i(\Sigma)$ the limit group (1.2) is strictly isomorphic with the homology group of dimension i of any of its simplicial subdivisions (again with H as coefficient group). Singular chain groups and hence singular homology groups are defined with discrete topology. *In the case of a finite complex* taken with a coefficient group of division-closure property (for instance compact), the singular homology group coincides algebraically with the corresponding homology group of any of its simplicial subdivisions. Let us agree to extend this algebraic isomorphism to an isomorphism in the strict sense, thus topologizing the singular homology groups of a complex accordingly. Then[4]) the limit group $B_i(\Sigma)$ and the singular homology group of dimension i of a complex are strictly isomorphic.

[3]) The requirements N (1, 2, 3) of Mayer [1], II, § 2 are satisfied.

[4]) In the alternative case of H and G finite-dimensional vector spaces over the discrete field K, $B_i(\Sigma)$ (being of finite dimension) is automatically discrete (Lefschetz II (25.6)) and the isomorphism between $B_i(\Sigma)$ and the singular homology group of dimension i thus is an isomorphism in the strict sense.

This well known result may be derived also in the following intuitive way: Let Σ and Σ' be simplicial subdivisions of the finite complex $\mathfrak{S}$ with Σ a refinement of Σ' $(\Sigma > \Sigma')$, and f a simplicial mapping

$$f: \Sigma \to \Sigma' \tag{1.4}$$

which satisfies the „star condition", Mayer [2], (8.15). Under these conditions any cycle C^i of Σ maps into a cycle fC^i of Σ' such that

$$fC^i \sim_s C^i, \quad (\sim_s = \text{singularly homologous}). \tag{1.5}$$

Let now C_i be a singular cycle of $\mathfrak{S}$ representing a singular homology class. Then in each simplicial subdivision, say $\mathfrak{S}_\lambda$ of $\mathfrak{S}$ there is a cycle C^i_λ (of $\mathfrak{S}_\lambda$) such that $C^i \sim_s C^i_\lambda$. By (1.5) these $C^i_\lambda \in \mathfrak{S}_\lambda$ constitute an element of $\underleftarrow{\lim}\{B_i(\mathfrak{S}_\lambda); \pi^\lambda_\mu\}$ and thus of (1.2) (derived for $\mathfrak{S}$) since $\{\mathfrak{S}_\lambda\}$ is cofinal in $\{\Sigma_\lambda\}$. Thus $C^i \to C^i_\lambda$, $C^i_\lambda \in B_i(\mathfrak{S}_\lambda)$, defines a correspondence between the singular homology group of dimension i and the limit group (1.2) which as easily seen algebraically is an isomorphism. By topologizing singular homology groups as indicated above, this isomorphism becomes one in the strict sense.

There is a similar identification of the singular homology group of dimension $(N-i)$ and the limit group (1.3) *in the case of a manifold* $|\mathfrak{M}_N|$. Here in fact for $\Sigma_\lambda = \mathfrak{M}_N$, a simplicial subdivision, the K-character system Σ^K_λ of Σ_λ has a „realization" in the star system $\mathfrak{M}^*_N$ of $\mathfrak{M}_N$ (Mayer [2], § 9), when taken with G as coefficient group. Since $\Sigma^K_\lambda = \mathfrak{M}^K_N$ and $\mathfrak{M}^*_N$ are both discrete, this is proved if we establish an algebraic isomorphism

$$\mathfrak{M}^*_N \cong \mathfrak{M}^K_N \tag{1.6}$$

for a given subdivision $\mathfrak{M}_N$ of $|\mathfrak{M}_N|$.

As before let σ^p_i, $i = 1, \dots, \lambda_p$, denote the set of p-simplexes of $\mathfrak{M}_N$. Then

$$\sum_i g_i \bar{\sigma}_i^{-p} \quad \text{and} \quad \sum_i g'_i \vartheta^* \sigma^p_i, \quad g_i, g'_i \in G, \tag{1.6.1}$$

$(\vartheta^* \sigma^p_i = c^p_i \cap {}^* C^N)$, Mayer [2], (8.5)), are the general $(-p)$ and $(N-p)$ chains of $\mathfrak{M}^K_N$ and $\mathfrak{M}^*_N$ respectively. By

$$\varepsilon(p) \sum_i g_i \vartheta^* \sigma^p_i \longleftrightarrow \sum_i g_i \bar{\sigma}_i^{-p}, \quad \varepsilon(p+1) = (-1)^{N-p} \varepsilon(p), \quad \varepsilon(0) = 1, \tag{1.7}$$

we establish the isomorphism (1.6). Of course (1.7) is one-to-one and that it preserves the boundary follows from

$$\left\{ \begin{aligned} &\partial(\varepsilon(p)\vartheta^*\sigma^p_i) = \varepsilon(p)(-1)^{N-p}\vartheta^*\delta\sigma^p_i = \varepsilon(p+1)\vartheta^*\delta\sigma^p_i = \\ &= \varepsilon(p+1)\sum_j [\sigma^{p+1}_j : \sigma^p_i]\vartheta^*\sigma^{p+1}_j \longleftrightarrow \sum_j [\sigma^{p+1}_j : \sigma^p_i]\bar{\sigma}_j^{-p-1} = \partial\bar{\sigma}_i^{-p}. \end{aligned} \right. \tag{1.7.1}$$

Now as in (1.4) let Σ and Σ' be simplicial subdivisions of $|\mathfrak{M}_N|$ with $\Sigma > \Sigma'$. The simplicial mapping (1.4) we now write (Mayer [2], § 3)

$$f: (\Sigma, \partial) \to (\Sigma', \partial). \tag{1.8}$$

By f' we denote the induced mapping

$$f': (\Sigma'^K, \partial) \to (\Sigma^K, \partial), \tag{1.9.1}$$

which in terms of co-homology is equivalent to

$$f': (\Sigma', \delta) \to (\Sigma, \delta). \tag{1.9.2}$$

Except for notation, the mappings (1.9) are identical and defined by

$$f'\bar{A}'^{-p} \circ B^p = \bar{A}'^{-p} \circ fB^p, \quad \bar{A}'^{-p} \in (\Sigma'^K, \partial), \quad B^p \in (\Sigma, \partial), \tag{1.10.1}$$

$$f'A'^p \circ B^p = A'^p \circ fB^p, \quad A'^p \in (\Sigma', \delta), \quad B^p \in (\Sigma, \partial). \tag{1.10.2}$$

The relationship (1.8) and (1.9) is valid for any pair of nerves Σ and Σ' subjected to $\Sigma > \Sigma'$ (and for f satisfying the star condition). For $\Sigma = \mathfrak{M}_N$ and $\Sigma' = \mathfrak{M}'_N$ however, a third form for the induced mapping f' might be obtained. In

$$\begin{gathered} \varepsilon(p)\vartheta^*A^p \circ B^p = A^p \circ B^p = \bar{A}^{-p} \circ B^p, \\ A^p = \sum g_i \sigma^p_i; \quad \bar{A}^{-p} = \sum g_i \bar{\sigma}_i^{-p}, \quad g_i \in G, \end{gathered} \tag{1.11}$$

the „intersection" $\vartheta^*A^p \circ B^p$ of $\vartheta^*A^p \in \mathfrak{M}^*_N$ and $B^p \in \mathfrak{M}_N$ is introduced. Then

$$f': \mathfrak{M}'^*_N \to \mathfrak{M}^*_N \tag{1.9.3}$$

with

$$f'\vartheta^{*\prime}A'^p \circ B^p = \vartheta^{*\prime}A'^p \circ fB^p \tag{1.10.3}$$

is equivalent to (1.10.1) (with $\bar{A}'^{-p} \in \mathfrak{M}'^K_N$ replaced by $\varepsilon(p)\vartheta^{*\prime}A'^p \in \mathfrak{M}'^*_N$, its map in (1.6)). Since ((1.10.2) and (1.11))

$$\vartheta^* f' A'^p \circ B^p = \vartheta^{*\prime} A'^p \circ fB^p, \tag{1.12}$$

and considering that B^p is an arbitrary chain of $\mathfrak{M}_N$

$$f'\vartheta^{*\prime}A'^p = \vartheta^* f' A'^p, \tag{1.13}$$

follows from (1.10.3) and (1.12). In Mayer [2], (8.22)

$$\vartheta^* f' A'^p \sim_s \vartheta^{*\prime} A'^p \tag{1.14}$$

for a cycle $\vartheta^{*\prime}A'^p$ of $\mathfrak{M}_N^*$ has been proved. Thus we have

$$f'\vartheta^{*\prime}A'^p \sim_s \vartheta^{*\prime}A'^p, \tag{1.15}$$

which relation (a counterpart of (1.5)) now will serve to set up an isomorphism between the limit group (1.3) and the singular homology group of dimension $(N-i)$ with G as coefficient group.

Let C^{N-i} be a singular cycle of $|\mathfrak{M}_N|$. In $\mathfrak{M}_N^*$ then there is a representative of the singular class of C^{N-i}, say $\vartheta^* A^i$; hence

$$C^{N-i} \sim_s \vartheta^* A^i; \quad \vartheta^* A^i \in \mathfrak{M}_N^*. \tag{1.16}$$

The chain $\vartheta^* A^i$ by (1.6), (1.7) represents the K-character $(\varepsilon(p)\vartheta^* A^i \leftrightarrow)\bar{A}^{-i}$ of $\mathfrak{M}_N^K$ and thus defines the element of (1.3) whose representative in $\mathfrak{M}_N^K$ the homology class of the character $\bar{A}^{-i}$ is.

From (1.15) then follows the independence of the so defined element of (1.3) of the subdivision $\mathfrak{M}_N$ used in the process.

On the other hand the representatives of *a given element* of $B_{-i}(\Sigma^K) = \underset{\rightarrow}{\lim}\{B_{-i}(\Sigma_\lambda^K); \pi_\mu^\lambda\}$ in star coverings $\Sigma_\lambda = \mathfrak{M}_N$ are related in the isomorphism (1.6) to cycles of the respective $\mathfrak{M}_N^*$ *all of which* (by (1.15)) *belong to a definite singular homology class* of $|\mathfrak{M}_N|$. This class for its part by (1.16) determines the given element of $B_{-i}(\Sigma^K)$. *Hence to different classes in* (1.16) *different elements of* $B_{-i}(\Sigma^K)$ *correspond*. Since each element of $B_{-i}(\Sigma^K)$ has a representative in one of the star coverings (these coverings being confinal in the set of all finite open coverings), *each element* of $\dot{B}_{-i}(\Sigma^K)$ *is a map-element* in the correspondence (1.16) *(of a definite singular homology class)*. *Thus* (all groups involved being discrete) *the correspondence* (1.16) *is an isomorphism*. We summarize the result in Poincaré's Duality Theorem:

The homology groups of $|\mathfrak{M}_N|$ *of dimension* i *with coefficient group* H *and of dimension* $(N-i)$ *with coefficient group* G *are dually paired to the group of reals modulo* 1, *if* H *(compact) and* G *(discrete) are so paired.*

§ 2. Alexander's Duality Theorem.

Let $|\mathfrak{M}_N|$ be an N-sphere and M a subspace of $|\mathfrak{M}_N|$ be a topological polyhedron. Then Alexander's Duality Theorem asserts: *For* $i \neq 0,\ N-1$, *the singular homology group of dimension* i *of* M *(taken with the compact group* H*) and the singular homology group of dimension* $(N-i-1)$ *of the complement space* $|M_N| - M$ *(taken with the discrete group* G*) are dually paired to the group of reals modulo* 1, *if* H *and* G *are so paired.* The cases $i=0$ and $i=N-1$ will be dealt with separately. We derive the Alexander Duality Theorem for the sphere from Mayer [1], (II, § 4). For $\{\Sigma_\lambda; \pi_\mu^\lambda\}$ of that paper we take the net of the nerves Σ_λ of $|\mathfrak{M}_N|$ introduced in the preceding paragraph; for $\{M_\lambda; \pi_\mu^\lambda\}$, the sub-net of $\{\Sigma_\lambda; \pi_\mu^\lambda\}$, we define $M_\lambda \subset \Sigma_\lambda$ as follows: A simplex $(\mathfrak{U}_{\lambda_1}\ldots\mathfrak{U}_{\lambda_\nu})$ of Σ_λ belongs to M_λ if the set-closure of $\mathfrak{U}_{\lambda_1} \cap \mathfrak{U}_{\lambda_2} \cap \ldots \cap \mathfrak{U}_{\lambda_\nu}$ meets M, i. e.

$$(\mathfrak{U}_{\lambda_1}\ldots\mathfrak{U}_{\lambda_\nu}) \in M_\lambda \longleftrightarrow \overline{\mathfrak{U}_{\lambda_1} \cap \ldots \cap \mathfrak{U}_{\lambda_\nu}} \cap M \neq 0. \tag{2.1}$$

All requirements of Mayer [1], (II, § 3) [i. e. N (a, b and c) and M_λ *closed* in Σ_λ (i. e. M_λ is a subcomplex of Σ_λ and $L_i(M_\lambda)$ (the i-dimensional chain group of M_λ) is a closed subgroup of $L_i(\Sigma_\lambda)$)] are satisfied and thus the results of that paper may be applied. Since $|\mathfrak{M}_N|$ is an N-sphere the groups $B_i(M, \Sigma)$ for $i \neq 0$ and $B_{-i-1}((\Sigma^K, M), \Sigma^K)$ for $N-i-1 \neq 0$ of Mayer [1] coincide with

$$B_i M = \underset{\leftarrow}{\lim}\{B_i(M_\lambda); \pi_\mu^\lambda\}. \tag{2.2.1}$$

and

$$B_{-i-1}(\Sigma^K, M) = \underset{\rightarrow}{\lim}\{B_{-i-1}(\Sigma_\lambda^K, M_\lambda); \pi_\mu^\lambda\} \tag{2.2.2}$$

respectively (a fact which follows from the two remarks on p. 19 of Mayer [1], the relation (1.6) and the sphere property of $|\mathfrak{M}_N|$).

Thus by the result of Mayer [1], (II, § 4) the above groups are dually paired to the group K of reals modulo 1. If therefore we can prove: (a) the isomorphism of the limit group (2.2.1) with the singular homology group of dimension i of M and (b) the isomorphism of the limit group (2.2.2) with the singular homology group of dimension $(N-i-1)$ of $|\mathfrak{M}_N| - M$, then (for $i \neq 0$, $N-1$) the Alexander Duality Theorem will be established.

As to the proof of (a), we notice that the definition (2.1) of M_λ slightly deviates from that of the subcomplex of Σ_λ commonly used to prove (a). If $\widetilde{M}_\lambda$ denotes this subcomplex, then

$$(\mathfrak{U}_{\lambda_1}\ldots\mathfrak{U}_{\lambda_\nu}) \in \widetilde{M}_\lambda \longleftrightarrow \mathfrak{U}_{\lambda_1}\cap\ldots\cap\mathfrak{U}_{\lambda_\nu}\cap M \neq 0$$

its definition, shows the inclusion $\widetilde{M}_\lambda \subset M_\lambda$. But notwithstanding

$$\text{(2.3)} \qquad \varprojlim \{B_i(M_\lambda);\, \pi^\lambda_\mu\} \cong \varprojlim \{B_i(\widetilde{M}_\lambda);\, \pi^\lambda_\mu\}$$

holds. In fact, let $\{\Sigma_\lambda\}'$ denote the subset of $\{\Sigma_\lambda\}$ (the aggregate of all the nerves) defined by

$$\text{(2.3.1)} \qquad \Sigma_\lambda \in \{\Sigma_\lambda\}' \longleftrightarrow M_\lambda = \widetilde{M}_\lambda.$$

As will be shown at the end of this paragraph, $\{\Sigma_\lambda\}'$ is *cofinal* in $\{\Sigma_\lambda\}$. Since (2.3) is true for limit groups restricted to representatives in $\{\Sigma_\lambda\}'$ it thus is true without this restriction. Now as it is well known that the right member in (2.3) coincides with the Čech homology group of dimension i of M and since M is a polyhedron, it thus coincides with the singular homology group of M of this dimension (with the compact group H as coefficient group and topologized as in § 1). *Thus* (a) *is proved.*

Before starting with the proof of (b) some introductory remarks will be necessary. If $\Sigma_\lambda = \mathfrak{M}_N$ is the nerve of a star covering, we shall symbolize the above subcomplex M_λ by $M_\lambda = \langle M|\mathfrak{M}_N\rangle$.

A simplex $\sigma^\nu = (P_0, P_1, \ldots, P_{\nu+1})$ of the simplicial subdivision M_N belongs to $\langle M|\mathfrak{M}_N\rangle$ if

$$\overline{\mathrm{St}\, P_0 \cap \ldots \cap \mathrm{St}\, P_{\nu+1}} \cap M \neq 0;$$

but $\mathrm{St}\, P_0 \cap \ldots \cap \mathrm{St}\, P_{\nu+1} = \mathrm{St}\,(P_0 \ldots P_{\nu+1})$. Hence [5])

$$\text{(2.4)} \qquad \sigma^\nu \in \langle M|\mathfrak{M}_N\rangle \longleftrightarrow \overline{\mathrm{St}}\ \sigma_\nu \cap M \neq 0.$$

An equivalent definition is: $\sigma^\nu \in \langle M|\mathfrak{M}_N\rangle$ *if* σ^ν *is the face of a simplex* σ^μ, $\mu \geq \nu$, *such that there is some point of* M *which is an inner or a boundary point of* σ^μ.

[5]) St $(P_0 \ldots P_{\nu+1})$ is the set of all simplexes of $\mathfrak{M}_N$ with $(P_0 \ldots P_{\nu+1})$ as a face (including $(P_0 \ldots P_{\nu+1})$). The point set of any simplex is the set of its *inner* points. St σ^ν thus is an open set.

Since $\langle M|\mathfrak{M}_N\rangle$ contains the faces of any of its simplexes, it is a subcomplex of $\mathfrak{M}_N$ (sometimes called „closed" subcomplex). Moreover as a point set

$$\text{(2.4.1)} \qquad M \subset \langle M|\mathfrak{M}_N\rangle$$

since the carrier simplex of any point of M is in $\langle M|\mathfrak{M}_N\rangle$.

As before, let $\Sigma_\lambda = \mathfrak{M}_N$ be a star covering. Then $\Sigma_\lambda^K = \mathfrak{M}_N^K \simeq \mathfrak{M}_N^*$ by (1.6). Let $\mathfrak{N}_N^*$ denote the subcomplex of $\mathfrak{M}_N^*$ corresponding to the subsystem $(\Sigma_\lambda^K, M_\lambda)$ (the annihilator of $M_\lambda = \langle M|\mathfrak{M}_N\rangle$) of Σ_λ^K in this isomorphism. By (1.11)

$$\vartheta^*\sigma_i^p \circ \sigma_j^p = \varepsilon(p)\sigma_i^p \circ \sigma_j^p = \varepsilon(p)\delta_i\,;$$

thus $\vartheta^*\sigma_i^p$ (in this product) maps any simplex but σ_i^p into zero. Hence

$$\text{(2.5)} \qquad \vartheta^*\sigma_i^p \in \mathfrak{N}_N^* \longleftrightarrow \sigma_i^p \in \mathfrak{M}_N - \langle M|\mathfrak{M}_N\rangle.$$

By (2.4) this may also be written

$$\text{(2.5.1)} \qquad \vartheta^*\sigma_i^p \in \mathfrak{N}_N^* \longleftrightarrow \overline{\mathrm{St}}\ \sigma_i^p \cap M = 0.$$

Since $(\Sigma_\lambda^K, M_\lambda)$ is a subsystem of Σ_λ^K, $\mathfrak{N}_N^*$ is a subcomplex of $\mathfrak{M}_N^*$ (i. e. containing its boundary with any cell) and thus $\mathfrak{N}_N^*$ *is a closed subspace* of $\mathfrak{M}_N^*$. We now prove the pointset inclusion

$$\text{(2.6)} \qquad \langle M|\mathfrak{M}_N\rangle \subset \mathfrak{M}_N^* - \mathfrak{N}_N^*.$$

In fact a simplex $\tau^p \in \langle M|\mathfrak{M}_N\rangle$ and a cell $\vartheta^*\sigma^q \in \mathfrak{N}_N^*$ have an empty intersection: This intersection by Mayer [2], § 9, is $\sigma^q \cap {}^*\tau^p$ and empty unless τ^p has σ^q as a face. But $\tau^p \in \langle M|\mathfrak{M}_N\rangle$ and thus any face of τ^p is in $\langle M|\mathfrak{M}_N\rangle$. On the other hand $\vartheta^*\sigma^q \in \mathfrak{N}_N^*$ by (2.5) contradicts $\sigma^q \in \langle M|\mathfrak{M}_N\rangle$. This proves (2.6). We now have all the materials to prove (b).

Let C^n *be a singular cycle of* $|\mathfrak{M}_N| - M$, $C^n \in |\mathfrak{M}_N| - M$, $\partial C^n = 0$. *Then for a subdivision* $\mathfrak{M}_N$ *of a mesh sufficiently small,* C^n *will lie in* $\mathfrak{N}_N^*$ (as a point set of course). In fact C^n and M are closed and thus (in Euclidean realization of $|\mathfrak{M}_N|$) have a non-zero distance ε. If then mesh $\mathfrak{M}_N$ is sufficiently small, $\mathfrak{M}_N^* - \mathfrak{N}_N^*$ will lie in the ε-neighborhood of M and thus $C^n \in \mathfrak{N}_N^*$ [6]).

[6]) The above fact is true for any closed M; no other property of M henceforth will matter. So the proof of statement (b) *holds for any closed subspace* M of $|\mathfrak{M}_N|$.

With $C^n \in |\mathfrak{M}_N| - M$ given, we restrict ourselves to subdivisions $\mathfrak{M}_N$ such that $C^n \in \mathfrak{N}_N^*$. Let $\Sigma_\lambda = \mathfrak{M}_N$ be such subdivision. Then in $\mathfrak{N}_N^*$ there is a cycle $\vartheta^* A^{N-n}$ such that

(2.7) $$C^n \sim_s \vartheta^* A^{N-n} \quad \text{in} \quad \mathfrak{N}_N^*.$$

$\vartheta^* A^{N-n}$ by (1.6), (1.7) defines the K-character

$$\bar{A}^{N-n} (\longleftrightarrow \varepsilon(N-n) \vartheta^* A^{N-n}) \quad \text{in} \quad (\Sigma_\lambda^h, M_\lambda)$$

(for $\Sigma_\lambda = \mathfrak{M}_N$ and $M_\lambda = \langle M | \mathfrak{M}_N \rangle$), the subsystem of Σ_λ^h corresponding to $\mathfrak{N}_N^*$ in (1.6). Thus $\vartheta^* A^{N-n}$ defines the element of $\varinjlim \{B_{n-N}(\Sigma_\lambda^K, M_\lambda); \pi_\mu^\lambda\}$ whose representative in $B_{n-N}(\Sigma_\lambda^K, M_\lambda)$ (corresponding to the *chosen* $\Sigma_\lambda = \mathfrak{M}_N$) the homology class of $\bar{A}^{n-N}$ is.

Thus is defined a mapping of the singular homology class of dimension n of $|\mathfrak{M}_N| - M$ into $\varinjlim \{B_{n-N}(\Sigma_\lambda^K, M_\lambda); \pi_\mu^\lambda\}$ if only the independence of the result of the chosen subdivision is proved. To establish independence we shall prove that for $\vartheta^{*\prime} A'^p \in \mathfrak{N}_N'^*$, where $\mathfrak{M}_N$ is a refinement of $\mathfrak{M}_N'$, relation (1.15) is replaced by

(2.8) $$f' \vartheta^{*\prime} A'^p \sim_s \vartheta^{*\prime} A'^p \quad \text{in} \quad |\mathfrak{M}_N| - M.$$

By (2.5.1)

(2.9) $$\vartheta^{*\prime} A' \in \mathfrak{N}_N'^* \longleftrightarrow \overline{\mathrm{St}}\, A' \cap M = 0,$$

with $\overline{\mathrm{St}}\, A'^p = \bigcup_\alpha \overline{\mathrm{St}}\, \sigma_\alpha'^p$, $\sigma_\alpha'^p \in A'^p$. From the local properties of Whitney's cap products $\vartheta' A' = A' \cap' C'^N$ and $\vartheta^{*\prime} A' = A' \cap'^* C'^N$,

(2.10) $$\vartheta' A', \; \vartheta^{*\prime} A' \in \overline{\mathrm{St}}\, A'.$$

If $\cap'$ and $\cap'^*$ are „principal" products (Mayer [2], § 5), then $\vartheta^{*\prime} A'$ is deformable into $\vartheta' A'$ *in* $\overline{\mathrm{St}}\, A'$, such that ($\overline{\mathrm{St}}\, A' \in |\mathfrak{M}_N| - M$, by (2.9))

(2.11) $$\vartheta^{*\prime} A' \sim_s \vartheta' A' \quad \text{in} \quad |\mathfrak{M}_N| - M.$$

Moreover

(2.12) $$\vartheta^{*\prime} A' \in \mathfrak{N}_N'^* \rightarrow \vartheta^* f' A' \in \mathfrak{N}_N^*.$$

Indeed, $\mathrm{St}\, \sigma' \supset \mathrm{St}\, f'\sigma'$, for any simplex σ' of $\mathfrak{M}_N'$ by the star condition of the mapping $f: \mathfrak{M}_N \rightarrow \mathfrak{M}_N'$. Thus

(2.13) $$\mathrm{St}\, f' A' \subset \mathrm{St}\, A',$$

and (2.12) follows from (2.9) and (2.13). As (2.11)) is derived from (2.9)

(2.14) $$\vartheta^* f' A' \sim_s \vartheta f' A' \quad \text{in} \quad |\mathfrak{M}_N| - M,$$

follows from $\vartheta^* f' A' \in \mathfrak{N}_N^*$. Replacing B^{p+q} in Mayer [2] (7.16) by the basic N-cycle of the (oriented) $\mathfrak{M}_N$, we have

(2.15) $$f \vartheta f' A' - \vartheta' A' = \partial(A' \wedge C^N) \sim_s 0 \quad \text{in} \quad |\mathfrak{M}_N| - M,$$

since $A' \wedge C^N$ as $\vartheta' A' = A' \cap' C'^N$ and hence $f \vartheta f' A'$ all are in $\overline{\mathrm{St}}\, A'$.

Each point of $\vartheta f' A'$ is in $\mathrm{St}\, f' A'$ (2.10) and thus in $\overline{\mathrm{St}}\, A'$, (2.13). The carrier simplex in $\mathfrak{M}_N'$ of a point of $\vartheta f' A'$ thus belongs to $\overline{\mathrm{St}}\, A'$. In the deformation $f: \mathfrak{M}_N \rightarrow \mathfrak{M}_\Lambda'$ which carries $\vartheta f' A'$ into $f \vartheta f' A'$ each point stays in its carrier simplex and hence in $\overline{\mathrm{St}}\, A'$. Since $\overline{\mathrm{St}}\, A' \cap M = 0$,

(2.16) $$\vartheta f' A' \sim_s f \vartheta f' A' \quad \text{in} \quad |\mathfrak{M}_N| - M$$

follows. By (2.11), (2.15), (2.16), (2.14) and (1.13) then (2.8) is proved.

Let $\vartheta^{*\prime} A' \sim_s C^n$ in $\mathfrak{N}_N^*$. Then (2.8) and (2.12), $f' \vartheta^{*\prime} A' \in \mathfrak{N}_N^*$ and $\sim_s C^n$ in $|\mathfrak{M}_N| - M$. Since $(C^n - f' \vartheta^{*\prime} A')$ is in $\mathfrak{N}_N^*$ and $\sim_s 0$ in $|\mathfrak{M}_N| - M$, this cycle is $\sim_s 0$ in $\mathfrak{N}_N^*$ (Pontrjagin [4], Satz IV). Thus $f' \vartheta^{*\prime} A' = \vartheta^* f' A' \sim_s C^n$ in $\mathfrak{N}_N^*$ follows.

This proves the independence of our mapping (of the singular classes of $|\mathfrak{M}_N| - M$ of dimension n into $\varinjlim \{B_{n-N}(\Sigma_\lambda^K, M_\lambda); \pi_\mu^\lambda\}$ *of the choice of* $\mathfrak{M}_N$ *used for the mapping* [7]). By an argument similar to that at the close of § 1, the above mapping is seen to be an isomorphism. *So our statement* (b) *and thus the Alexander Duality Theorem for* $i \neq 0$, $N-1$, *is verified.*

Remark. If $C^n \sim_s 0$ in $|\mathfrak{M}_N|$ then its „approximation" $\vartheta^* A$ is likewise $\sim_s 0$ in $|\mathfrak{M}_N|$, since $\vartheta^* A \sim_s C^n$ in $\mathfrak{N}_N^*$. Thus $\vartheta^* A \sim_s 0$ in $\mathfrak{M}_N^*$. The map element of class $\{C^n\}$ in the above mapping thus belongs to $B_{n-N}((\Sigma^K, M), \Sigma^K)$ of Mayer [1], (II, § 4). The converse of this fact is likewise true.

We make use of this remark in discussing the cases $i = 0$ and $i = N-1$.

a) $i = 0$. Then Mayer [1] (II, § 4), $B_0(M, \Sigma)$ and $B_{-1}((\Sigma^K, M), \Sigma^K)$ are dually paired to the group of reals modulo 1. Let M be *connected*. Then $B_0(M, \Sigma)$ is the zero group since all $B_0(M_\lambda, \Sigma_\lambda)$ for subdivisions Σ_λ are zero groups ($\langle M | \mathfrak{M}_N \rangle$ is connected). Thus

[7]) Established above for $\mathfrak{M}_N > \mathfrak{M}_N'$. But for any pair $\mathfrak{M}_\Lambda'$ and $\mathfrak{M}_N''$ there is a *common* refinement $\mathfrak{M}_N$.

$B_{-1}((\Sigma^K, M), \Sigma^K)$ is a zero group. But by the above remark this group is isomorphic to the $(N-1)$-homology group of $|\mathfrak{M}_N|-M$, since any $(N-1)$-cycle in an N-sphere bounds. *Thus for $i=0$ and M connected, the above dual pairing reduces to the fact that the singular homology group of $|\mathfrak{M}_N|-M$ of dimension $(N-1)$ vanishes.*

b) $i=N-1$. Now $B_{N-1}(M, \Sigma)$ and $B_{-N}((\Sigma^K, M), \Sigma^K)$ are dually paired. But $B_{N-1}(M, \Sigma)=B_{N-1}(M)$. Moreover, by the preceding remark $B_{-N}((\Sigma^K, M), \Sigma^K)$ is isomorphic to the subgroup of those (singular) zero classes of $|\mathfrak{M}_N|-M$ whose elements bound in $|\mathfrak{M}_N|$. *Thus this subgroup of the zero-homology group of $|\mathfrak{M}_N|-M$ and $B_{N-1}(M)$, the $(N-1)$-homology group of M* (the first with G the latter with H as coefficient group) *are dually paired* (*to the group of reals mod.* 1).

We finish this paper with the *proof that $\{\Sigma_\lambda\}'$ defined by* (2.3.1) *is cofinal in* $\{\Sigma_\lambda\}$. This we do in showing that any finite open covering $\Sigma=\{\mathfrak{U}_i\}$, $i=1,\dots,a$, *is shrinkable* to a covering $\Sigma'=\{\mathfrak{U}'_i\}$ of $\{\Sigma_\lambda\}'$, which then refines Σ.

Let $\mathfrak{U}_1\cap\dots\cap\mathfrak{U}_\nu\cap M\neq0$. Then some point P of M is in $\mathfrak{U}_1,\dots,\mathfrak{U}_\nu$ and thus has a non-zero distance from the boundaries of $\mathfrak{U}_1,\dots,\mathfrak{U}_\nu$ respectively. Let $\varepsilon'>0$ be smaller than any of these distances.

For each $(\mathfrak{U}_1,\dots,\mathfrak{U}_\nu)$ with $\mathfrak{U}_1\cap\dots\cap\mathfrak{U}_\nu\cap M\neq0$ we choose such a point and such an ε' and then denote by $\varepsilon>0$ a number smaller than all these ε'. Since $|\mathfrak{M}_N|$ is normal we may shrink $\{\mathfrak{U}_i\}$ into a covering $\{\mathfrak{U}'_i\}$ such that $\mathfrak{U}'_i\subset\mathfrak{U}_i$ and at the same time we can achieve that each point of $\mathfrak{U}_i$ of a distance $\geqslant\varepsilon$ from its boundary shall be a point of $\mathfrak{U}'_i$. Then the chosen point P of $(\mathfrak{U}_1,\dots,\mathfrak{U}_\nu)$ with $\mathfrak{U}_1\cap\dots\cap\mathfrak{U}_\nu\cap M\neq0$ which belongs to this intersection, will also belong to $\mathfrak{U}'_i\cap\dots\cap\mathfrak{U}'_\nu\cap M$. Thus

(2.17) $$\mathfrak{U}_1\cap\dots\cap\mathfrak{U}_\nu\cap M\neq0\to\mathfrak{U}'_i\cap\dots\cap\mathfrak{U}'_\nu\cap M\neq0.$$

Obviously a simplex $(\mathfrak{U}_1\dots\mathfrak{U}_\nu)$ of the property

(2.18) $$\overline{\mathfrak{U}_1\cap\dots\cap\mathfrak{U}_\nu\cap M}\neq0 \quad\text{but}\quad \mathfrak{U}_1\cap\dots\cap\mathfrak{U}_\nu\cap M=0$$

is shrunk into $(\mathfrak{U}'_1\dots\mathfrak{U}'_\nu)$ with

(2.18.1) $$\overline{\mathfrak{U}'_1\cap\dots\cap\mathfrak{U}'_\nu\cap M}=0.$$

Now

(2.19) $$\overline{\mathfrak{U}'_1\cap\dots\cap\mathfrak{U}'_\nu\cap M}\neq0\to\overline{\mathfrak{U}_1\cap\dots\cap\mathfrak{U}_\nu\cap M}\neq0.$$

But the right member in (2.19) implies either $\mathfrak{U}_1\cap\dots\cap\mathfrak{U}_\nu\cap M\neq0$ and then (2.17) $\mathfrak{U}'_1\cap\dots\cap\mathfrak{U}'_\nu\cap M\neq0$ or $\mathfrak{U}_1\cap\dots\cap\mathfrak{U}_\nu\cap M=0$ and then (by (2.18)) (2.18.1), but this contradicts the left member (2.19). Therefore

(2.20) $$0\neq\overline{\mathfrak{U}'_1\cap\dots\cap\mathfrak{U}'_\nu\cap M}\to\mathfrak{U}'_1\cap\dots\cap\mathfrak{U}'_\nu\cap M\neq0$$

and $\{\mathfrak{U}'_i\}\in\{\Sigma_\lambda\}'$.

Appendix: A generalized Alexander Duality Theorem.

Using linking products for singular cycles $\mathfrak{E}^{N-i-1}\circ\vartheta^i$ (whose (pointset) intersection $\mathfrak{E}^{N-i-1}\cap\vartheta^i$ is empty) and taking account of the retract property of the polyhedron M imbedded in $|\mathfrak{M}_N|$ an N-manifold (and not necessarily a sphere), we shall present a second proof of Alexander's Duality Theorem which, with some modifications, will run along the lines of Alexandroff-Hopf [5], Chap. XI. So with $M\subset|\mathfrak{M}_N|$, a topological polyhedron, we denote as before by $\langle M|\mathfrak{M}_N\rangle$ the subcomplex of $\mathfrak{M}_N$ containing all N-simplexes of $\mathfrak{M}_N$ (and their faces) which have points of M as inner or boundary points.

$\mathfrak{U}^*(M)=\mathfrak{M}_N^*-\mathfrak{N}_N^*$ shall denote the *open* subcomplex of $\mathfrak{M}_N^*$ containing all cells $\vartheta^*\tau^i$ of $\mathfrak{M}_N^*$ with $\tau^i\in\langle M|\mathfrak{M}_N\rangle$. In (2.6) we have already proved that $\langle M|\mathfrak{M}_N\rangle\subset\mathfrak{U}^*(M)$. We choose mesh $\mathfrak{M}_N$ so small that the open set $\mathfrak{U}^*(M)$ is a retract neighborhood of M; *this means the existence of a mapping $g:\mathfrak{U}^*(M)\to M$, leaving points of M fixed.*

Let now $\vartheta^i\sim_s0$ in $(\mathfrak{M}_N)$ be a singular cycle of M but $\not\sim_s0$ in M:

(1.1) $$\vartheta^i=\partial\mathfrak{K}^{i+1},\quad \mathfrak{K}^{i+1}\in|\mathfrak{M}_N|,$$

(1.2) $$\vartheta^i\not\sim_s0\quad\text{in}\quad M.$$

The above chain $\mathfrak{K}^{i+1}$ we approximate in the smallest $\mathfrak{K}^{i+1}$ containing subcomplex of $\mathfrak{M}_N$; let $K^{i+1}\in\mathfrak{M}_N$ be this approximation. In this approximation-process $\vartheta^i=\partial\mathfrak{K}^{i+1}$ is approximated by $D^i=\partial K^{i+1}$ but this in the smallest ϑ^i containing a subcomplex of $\mathfrak{M}_N$ (Seifert-Threlfall [6], § 72). Thus (1.1)

(2.1) $$D^i=\partial K^{i+1},\quad K^{i+1}\in\mathfrak{M}_N,$$

(2.2) $$D^i\sim_s\vartheta^i\quad\text{in}\quad\langle M|\mathfrak{M}_N\subset\mathfrak{U}^*(M),$$

since the smallest ϑ^i containing a subcomplex of $\mathfrak{M}_N$ is contained in $\langle\vartheta^i|\mathfrak{M}_N\rangle$ and thus in $\langle M|\mathfrak{M}_N\rangle$. *We shall prove that the above approximation $D^i \in \mathfrak{M}_N$ does not bound in $\mathfrak{U}^*(M)$.* Otherwise, $D^i = \partial D^{i+1}$, $D^{i+1} \subset \mathfrak{U}^*(M)$, in the mapping $g: \mathfrak{U}^*(M) \to M$ leads to $g(D^i) = \partial g(D^{i+1})$, that is to $g(D^i) \sim_s 0$ in M. On the other hand $g(D^i) \sim_s g(\vartheta^i) = \vartheta^i$ in M (by (2.2) and $\vartheta^i \in M$). Combining, we should have $\vartheta^i \sim_s 0$ in M which contradicts (1.2). So *$D^i \in \langle M|\mathfrak{M}_N\rangle$ bounds in $\mathfrak{M}_N$ but not in $\langle M|\mathfrak{M}_N\rangle$* (which is in $\mathfrak{U}^*(M)$).

Hence by Mayer [1] (I, § 5) in $\mathfrak{M}_N^* - \mathfrak{U}^*(M)$ there is a cycle $\mathfrak{E}^{N-i-1} = \vartheta^* A^{i+1}$ bounding in $|\mathfrak{M}_N|$ and linking D^i, i.e.

$$\mathfrak{E}^{N-i-1} \circledcirc D^i \neq 0,$$

and thus linking ϑ^i which is homologous D^i in $\mathfrak{U}^*(M)$ [8]).

Remark. *$\mathfrak{E}^{N-i-1}$ does not bound in $|\mathfrak{M}_N| - M$.* Otherwise $\mathfrak{E}^{N-i-1} = \partial \mathfrak{E}^{N-1}$, $\mathfrak{E}^{N-1} \in |\mathfrak{M}_N| - M$. By definition $\mathfrak{E}^{N-i-1} \circledcirc \vartheta^i = \mathfrak{E}^{N-1} \circ \vartheta^i$, and since $\mathfrak{E}^{N-1} \in |\mathfrak{M}_N| - M$ and $\vartheta^i \subset M$ this linking product would be zero, but it is not zero.

We now state as our first result: *To any cycle ϑ^i of M bounding in $|\mathfrak{M}_N|$ but not in M there exists a cycle $\mathfrak{E}^{N-i-1}$ of $|\mathfrak{M}_N| - M$ bounding in $|\mathfrak{M}_N|$ but not in $|\mathfrak{M}_N| - M$ which links ϑ^i* [9]).

On the other hand let $\mathfrak{E}^{N-i-1} \subset |\mathfrak{M}_N| - M$ be a cycle bounding in $|\mathfrak{M}_N|$ but not in $|\mathfrak{M}_N| - M$:

$$\mathfrak{E}^{N-i-1} = \partial \mathfrak{E}^{N-1}, \quad \mathfrak{E}^{N-1} \subset |\mathfrak{M}_N|, \tag{3.1}$$

$$\mathfrak{E}^{N-i-1} \not\sim_s 0 \quad \text{in} \quad |\mathfrak{M}_N| - M. \tag{3.2}$$

Since the set intersection of $\mathfrak{E}^{N-i-1}$ and M is empty, $\mu(\mathfrak{M}_N)$, the mesh of $\mathfrak{M}_N$, can be chosen so small that the closure of the above $\mathfrak{U}^*(M)$ lies in some subcomplex P of $\mathfrak{M}_N$ whose intersection with $\mathfrak{E}^{N-i-1}$ is likewise empty:

$$\overline{\mathfrak{U}}^*(M) \subset P, \quad \mathfrak{E}^{N-i-1} \cap P = 0. \tag{4}$$

---

[8]) In using Mayer [1], loc. cit., $\langle M|\mathfrak{M}_N\rangle$ and $\mathfrak{M}_N^* - \mathfrak{U}^*(M)$ replace the complexes M and (Σ^K, M) respectively of this paper.

[9]) Instead of coefficient groups H and G we use *finite-dimensional vector spaces over the discrete field K* (as in footnote 2). The respective singular homology vector spaces are taken with discrete topology such that the above linking product (of the singular homology classes) is (trivially) bicontinuous.

This $\overline{\mathfrak{U}}^*(M)$ moreover can be taken to be a „Euclidean closure" (Alexandroff-Hopf, p. 346) of M with respect to P. *Then* (by A.-H., ibid., Satz IV) *to any cycle $\mathfrak{z}$ of $\mathfrak{U}^*(M)$ a cycle $\mathfrak{z}'$ of M exists such that $\mathfrak{z}' \sim_s \mathfrak{z}$ in P.* With such a choice of $\mu(\mathfrak{M}_N)$ we approximate $\mathfrak{E}^{N-i-1}$ by a cycle $\vartheta^* A^{i+1}$ of $\langle \mathfrak{E}^{N-i-1}|\mathfrak{M}_N^*\rangle$, where $\langle \mathfrak{E}^{N-i-1}|\mathfrak{M}_N^*\rangle$ is the subcomplex of $\mathfrak{M}_N^*$ containing all N-cells of $\mathfrak{M}_N^*$ (and their faces) which have points of $\mathfrak{E}^{N-i-1}$ as inner or boundary points. Thus (Seifert-Threlfall [6], § 72)

$$\vartheta^* A^{i+1} - \mathfrak{E}^{N-i-1} = \partial \mathfrak{R}^{N-i}, \quad \vartheta^* A^{i+1}, \mathfrak{R}^{N-i} \subset \langle \mathfrak{E}^{N-i-1}|\mathfrak{M}_N^*\rangle. \tag{5}$$

Since $\mathfrak{M}_N^*$ has a mesh $\leqslant 2\mu(\mathfrak{M}_N)$ (where $\mu(\mathfrak{M}_N) = \text{mesh } \mathfrak{M}_N$), $\langle \mathfrak{E}^{N-i-1}|\mathfrak{M}_N^*\rangle$ is in a $2\mu(\mathfrak{M}_N)$-neighborhood of $\mathfrak{E}^{N-i-1}$. Hence for $\mu(\mathfrak{M}_N)$ sufficiently small (such that (4) $\mathfrak{U}_{2\mu(\mathfrak{M}_N)}(\mathfrak{E}^{N-i-1}) \cap P = 0$) we have

$$\vartheta^* A^{i+1} \cap P = 0, \quad \mathfrak{R}^{N-i} \cap P = 0. \tag{5.1}$$

Thus $\vartheta^* A^{i+1}$ and $\mathfrak{R}^{N-i}$ are in

$$|\mathfrak{M}_N| - P \subset |\mathfrak{M}_N| - \mathfrak{U}^*(M) = \mathfrak{M}_N^* - \mathfrak{U}^*(M).$$

But $\vartheta^* A^{i+1}$ which bounds in $|\mathfrak{M}_N|$ ((3.1) and (5)) does not bound in $\mathfrak{M}_N^* - \mathfrak{U}^*(M)$. Otherwise $\mathfrak{E}^{N-i-1}$ by (5) would bound in $|\mathfrak{M}_N| - M$ contradicting (3.2). Hence by Mayer [1] (I, § 5) in $\langle M|\mathfrak{M}_N\rangle$ there is a cycle D^i bounding in $|\mathfrak{M}_N|$ and linking $\vartheta^* A^{i+1}$: $\vartheta^* A^{i+1} \circledcirc D^i \neq 0$. Since $D^i \in \langle M|\mathfrak{M}_N\rangle \subset \mathfrak{U}^*(M)$, there exists in M a cycle ϑ^i such that $D^i \sim_s \vartheta^i$ in P. But then

$$\vartheta^* A^{i+1} \circledcirc D^i = \vartheta^* A^{i+1} \circledcirc \vartheta^i \tag{6.1}$$

since $\vartheta^* A^{i+1} \subset |\mathfrak{M}_N| - P$. Likewise

$$\vartheta^* A^{i+1} \circledcirc \vartheta^i = \mathfrak{E}^{N-i-1} \circledcirc \vartheta^i \tag{6.2}$$

since $\vartheta^i \subset M \subset P$ and $\vartheta^* A^{i+1} \sim_s \mathfrak{E}^{N-i-1}$ in $|\mathfrak{M}_N| - P$ ((5), (5.1)). Thus $\mathfrak{E}^{N-i-1} \circledcirc \vartheta^i \neq 0$ and our second result: *To any cycle $\mathfrak{E}^{N-i-1}$ of $|\mathfrak{M}_N| - M$ bounding in $|\mathfrak{M}_N|$ but not in $|\mathfrak{M}_N| - M$ there exists a cycle ϑ^i of M bounding in $|\mathfrak{M}_N|$* (and not in M) *which links $\mathfrak{E}^{N-i-1}$. The linking product obviously pairs the classes of i-cycles of M bounding in $|\mathfrak{M}_N|$ and those of $(N-i-1)$-cycles of $|\mathfrak{M}_N| - M$ bounding in $|\mathfrak{M}_N|$ and by the foregoing two statements this pairing is orthogonal.* Of these two discrete vector spaces (footnote 9) that belonging to M, a finite complex, is of a finite dimension and thus linearly compact (Lefschetz II, (27.7)). Thus we have before us the situation of Lefschetz II (20.6) (i. e. its counterpart for vector spaces) which leads to the main result: *The subspace of the singular homology vector space of M of classes of i-cycles bounding in $|\mathfrak{M}_N|$ and the subspace of the singular homology vector space of $|\mathfrak{M}_N|-M$ of classes of $(N-i-1)$-cycles bounding in $|\mathfrak{M}_N|$ are dually paired to the field K in the linking product when taken with finite-dimensional coefficient vector spaces* (over K) *H and G respectively, with H and G dually paired to K.*

Bibliography used in this paper.

[1] Mayer, W. *The duality theory and the basic isomorphism of group systems and nets and co-nets of group systems*, Annals of Math. **46** (1945).
[2] Mayer, W. *On products in topology*, Annals of Math. **46** (1945).
[3] Lefschetz, S. *Algebraic topology*, Colloquium Publications, 1942.
[4] Pontrjagin, L. *Über den Inhalt topologischer Dualitätsätze*, Math. Annalen 105.
[5] Alexandroff-Hopf, *Topologie*, Berlin 1935.
[6] Seifert-Threlfall, *Topologie*, Leipzig und Berlin 1934.

Institute for Advanced Study.

Sur un paradoxe de M. J. von Neumann.

Par

Wacław Sierpiński (Warszawa).

E et H étant deux ensembles situés dans un espace métrique M, dans lequel la distance entre deux points est désignée par ϱ, nous dirons que l'ensemble H est *plus petit au sens de M. J. von Neumann* que l'ensemble E, ou, plus brièvement, que H est *plus petit* (N) que E, s'il existe une fonction $f(p)$ définie pour $p\in E$ qui transforme d'une façon biunivoque E en H et telle que

$$\varrho(f(p),f(q))<\varrho(p,q) \quad \text{pour} \quad p\in E,\ q\in E,\ p\neq q\,^{1}).$$

Nous dirons que l'ensemble H est *plus petit* (N) *par décomposition finie* que l'ensemble E, s'il existe des décompositions des ensembles E et H en le même nombre fini d'ensembles disjoints:

$$E=E_1+E_2+\ldots+E_n \quad \text{et} \quad H=H_1+H_2+\ldots+H_n,$$

telles que, pour $k=1,2,\ldots,n$, l'ensemble H_n est plus petit (N) que l'ensemble E_k [2]).

M. J. von Neumann a démontré (en utilisant l'axiome du choix) que tout segment d'une droite est plus petit (N) par décomposition finie que tout autre segment de cette droite [3]). La démonstration de cette proposition est assez longue.

Or, MM. Banach et Tarski ont démontré [4]) que deux ensembles de points, situés sur la surface de la même sphère et qui ne sont pas ensembles frontières (par rapport à cette sphère) sont équivalents par décomposition finie (c.-à-d. se décomposent en le même nombre fini d'ensembles disjoints respectivement congruents).

---

[1]) Voir J. v. Neumann, Fund. Math. **13**, p. 85. Cf. aussi D. Kirszbraun Fund. Math. **12**, p. 77 et autres citations au renvoi [3]) l. c.; aussi W. Sierpiński, Mathematica **11**, p. 222.

[2]) M. J. v. Neumann dit dans ce cas (l. c.) que l'ensemble H est par rapport à E „zerlegungskleiner".

[3]) l. c., p. 115.

[4]) Fund. Math. **6**, p. 267, Théorème 31.